Fine Manipulation with Multifinger Hands

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Abstract

We show the existence of two and three finger grasps in the presence of arbitrarily small friction for two and three dimensional smooth objects using a simple new technique. No convexity of the objects is assumed. We also prove the existence of *finger gaits* for rotating a planar object using three and four fingers. Additional results for smooth convex objects are presented which describe two different finger gaits using four fingers.

1 Introduction

We show here the existence of some fine motions characteristic of dextrous hands. By dextrous hands we mean robot hands with three or more articulated fingers capable of exerting forces through the fingertips. We focus on a small subclass of fine motion problems referred to as gaiting problems.

One typical fine motion is the continuous rotation (or change of orientation in $\Re^3$) of an object by repositioning the grasping fingers while maintaining a grasp at all times. To understand the motion better, one could think of a human hand. Assume we are holding a disk with our fingers. We may rotate the disk by simply rotating the wrist. But at some point we will reach the limit of rotation of the wrist. However, we may continue to rotate the object in the same direction by first "walking" the fingers around the edge in the opposite direction until the wrist is again within its operating range. That is, we can relocate some fingers on the edge while we maintain a grasp with the remaining ones. (Of course one could completely release the object and relocate all the fingers, but the whole point of fine manipulation is to manipulate objects while maintaining a grasp at all times.) We define a *finger gait* to be any periodic sequence of such finger relocations.

When the object to be manipulated is planar the whole motion may be carried out in the plane. Furthermore, the finger motions may be restricted so that they do not "cross over" one another. In other words, the ordering of the fingers around the edge of the object should not change throughout the manipulation.

We will study the possibility of executing such motions for certain classes of objects. In order for such motions to exist, we will at least need:

(a) the existence of a grasp

(b) the possibility of exerting arbitrary forces and torques on the object so that any position and orientation of the object can be attained.

In the last section we suggest further research directions.

2 Related works

Mishra, Schwartz, and Sharir [8] obtained bounds on the number of fingers needed to achieve positive and force closure grasps on piecewise smooth objects. In particular they showed the existence of three-finger positive grasps for smooth planar objects. They assumed no friction but some of the results extend to arbitrarily small friction. Linear time algorithms for the synthesis of such grasps were also given in the case of polyhedral objects. Markenscoff and Papadimitriou [6] present a way to select optimal form closure grasps for planar polygons. Nguyen in [9] studied force closure grasps and their synthesis for polyhedral objects.

In this paper we restrict the class of objects to those with smooth boundaries, but we are able to show existence of grasps for arbitrarily small friction. The ideas used also suggest some extensions to the case of piecewise smooth boundaries.

In the above works, the fingers were not repositioned once they achieved their grasping configuration. In [5] Li considers nonholonomic grasping constraints and studies the motion planning problem for the contact point between finger and object.

Okada [10] implemented *finger gaits* for turning a sphere with a three finger robot hand by teaching the sequence of motions required. Hor [3] implemented similar gaits to turn a disk with a four finger planar manipulator but used force feedback to maintain the grasp. Fearing [2] implemented an object twirling sequence using the Salisbury Hand.

Here we use several "distance" functions in the proofs which provide intuitive arguments for and concise proofs of the existence of most of the grasps. They also suggest further results.

Additional references can be found in the papers listed above.

3 Formulation of the problem

We will start with the simpler planar case. We make the following assumptions (so that the problem is well defined). For the concepts of differential geometry we refer the reader to [1].

(A1) The object A is smooth (as opposed to piecewise smooth), i.e. no corners are allowed. More precisely, we require that the boundary be diffeomorphic to a circle.

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(A2) The "fingers" are modeled as isolated points. This corresponds to the idealization of hard point contacts as in [7]. These point fingers are assumed to be capable of exerting forces in any direction but individually cannot exert any torque.

(A3) There is a nonzero coefficient of friction between the object and the fingers. (We assume a Coulomb friction model.) This implies that the only forces the fingers can transmit to the object are those which lie inside the cone of friction at the corresponding contact point.

(A4) Let $\{B\}$ be a fixed coordinate frame. For any pair of fingers P_i, P_j, there exist angles θ_{ij}^u, θ_{ij}^l such that the angle θ between the vector $P_i - P_j$ and the x-axis of $\{B\}$ satisfies $\theta_{ij}^l \leq \theta \leq \theta_{ij}^u$. (Here we are allowing $\theta_{ij}^u - \theta_{ij}^l > 2\pi$ with the interpretation that in this case $P_i - P_j$ can rotate by more than one revolution.)

(A5) Once the initial ordering of the fingers along the boundary is chosen (by selecting one of them as the first and numbering the others, say, clockwise) then it will remain the same for the duration of the manipulation.

Remark 3.1 Condition (A4) is clearly needed, otherwise the manipulation problem is trivial. Once a grip is obtained, we could simply maintain it and rotate the fingers for as long as it is needed. No relocation of the fingers would be required. Condition (A4) captures the mechanical limitation that forces a change of grip in order to rotate the object an arbitrarily large amount. Condition (A5) is not really necessary. It is included here because it models an additional limitation of human hands and of many robot hands, and because the finger gaits we will present actually satisfy it.

4 Finger gaits with 3 fingers on the plane

The following result is not new but, to the best of our knowledge, the approach used in the proof was not used before. Furthermore, it illustrates the use of the distance function f.

Definition 4.1 A grasp on an object is *force-closure* if it can exert an arbitrary force and torque on the object through the set of contacts.

Proposition 4.1 *Under Assumptions (A1)–(A3), there exists a two finger force closure grasp of object A.*

Remark 4.1 The previous proposition does not assume that the object is convex. Moreover, the result is independent of the coefficient of friction at any point on the boundary of A as long as it is nonzero.

PROOF. To show force closure it is enough to find two points P_1, P_2 such that

$$\begin{aligned} (P_1 - P_2)^T t_1 &= 0 \\ (P_2 - P_1)^T t_2 &= 0 \\ n_1 + n_2 &= 0 \end{aligned} \qquad (1)$$

where t_i is the unit tangent to the boundary of A, ∂A, at P_i and n_i is the inward normal at P_i. The first two equations of system (1) say that the normal line to ∂A at P_1 (P_2) passes through P_2 (P_1).

Let $\gamma : S^1 \to \partial A$ be a diffeomorphism. Consider the function $f : S^1 \times S^1 \to \Re$ defined by,

$$f(s_1, s_2) = \|\gamma(s_1) - \gamma(s_2)\|^2.$$

Then, f is continuously differentiable and attains its maximum in $S^1 \times S^1$. Let $(\bar{s}_1, \bar{s}_2)$ be a maximum of f. Then, $\nabla f(\bar{s}_1, \bar{s}_2) = 0$. In local coordinates we have

$$\begin{aligned} \frac{\partial f}{\partial s_1}(s_1, s_2) &= 2(\gamma(s_1) - \gamma(s_2))^T \dot{\gamma}(s_1) = 0 \\ \frac{\partial f}{\partial s_2}(s_1, s_2) &= 2(\gamma(s_1) - \gamma(s_2))^T \dot{\gamma}(s_2) = 0 \end{aligned}$$

Since γ is a diffeomorphism, then $\dot{\gamma}(\bar{s}_1) \neq 0$ and $\dot{\gamma}(\bar{s}_2) \neq 0$. Therefore, $P_1 = \gamma(\bar{s}_1)$ and $P_2 = \gamma(\bar{s}_2)$ satisfy the first two equations in (1). Also n_1 and n_2 are collinear. The maximality of $(\bar{s}_1, \bar{s}_2)$ implies, via a simple geometric argument, that n_1 (n_2) points toward P_2 (P_1). ∎

The following proposition is now easy to prove (see Figure 1).

Proposition 4.2 *Assume (A1)–(A5) hold. Let* $\bar{\theta}_{ij} = \theta_{ij}^u - \theta_{ij}^l$. *Then, if* $\min \bar{\theta}_{ij}$ *is large enough, there exist three-finger gaits for rotating the object A an arbitrarily large amount while maintaining nonzero grasping forces (i.e. forces which result in zero force and torque on the object).*

Moreover, a more precise estimate of how large $\min \bar{\theta}_{ij}$ *must be can be obtained as follows (see example in Figure 1). Assume that the fingers* P_1, P_2 *can be located at the positions found in Proposition 4.1. Assume further that* P_3 *can be positioned arbitrarily close to* P_2 *so that the arc* $P_1P_2P_3$ *is oriented clockwise. If, from those starting positions, each of the vectors* $P_i - P_j$ *can rotate counterclockwise by at least* π, *then A can be rotated clockwise an arbitrary amount.*

Remark 4.2 The only reason for the generality of the first part of the proposition is to guarantee that we can put the fingers in a convenient initial position. As the second part shows, once that is achieved, the vectors $P_i - P_j$ need only be able to rotate by π in order for the desired finger gait to exist.

5 Finger gaits with 4 fingers on the plane

We show here that by using 4 fingers we may manipulate the object under more constrained situations (smaller $\bar{\theta}_{ij}$). Once we consider using four fingers we could have two essentially different gaits: using two pairs separately, or using a three finger grasp and replacing each gripping finger cyclically one at a time using the fourth finger. We show below that both options are always possible.

Remark 5.1 Clearly, under our assumption of point fingers, we can do finger gaits using any number of fingers by simply accumulating them near one of the gripping points P_1, P_2 as in the previous proposition and then sliding them one at a time around the edge. The following proposition shows different gaits also exist.

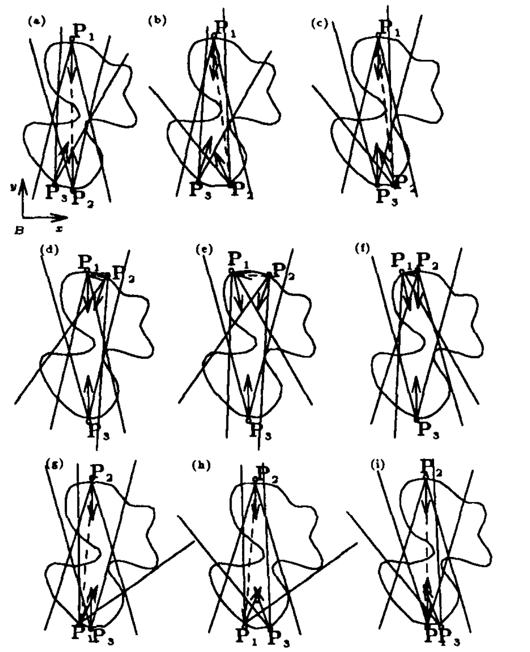

Figure 1: Finger gaits with 3 fingers on the plane.

The following definition will help simplify the notation.

Definition 5.1 Two points on the boundary of A are called *antipodal* if the positive ray through each point in the direction of the inward normal contains the other point.

These antipodal points correspond to the *double normals* discussed in [4] in the case of smooth convex objects.

5.1 A mountain pass theorem

The following theorem will allow us to show the existence of more than one pair of antipodal points. It will be presented in some generality so that we may then apply it to several different situations.

The proof uses arguments similar to the traditional mountain pass theorem [11] on the existence of critical points and will be omitted due to space limitations.

Theorem 5.1 *Let X be a compact differentiable manifold without boundary and $g : X \to \Re$ be a continuously differentiable function. Let $M, m \in \Re$ be the maximum and minimum values of g respectively. Define $Y = g^{-1}(M)$ and assume:*

(a) $X \setminus Y$ is connected,

(b) $g^{-1}(m)$ contains at least two points x_0, x_1.

Then, there exist a critical point x of g such that $x \notin Y \cup \{x_0, x_1\}$.

Remark 5.2 Notice that every point in $Y \cup \{x_0, x_1\}$ is a critical point of g. The theorem guarantees the existence of at least one other critical point.

Theorem 5.2 *Let A be a smooth object and $\gamma : S^1 \to \partial A$ be a diffeomorphism. Let $f : T^1 = S^1 \times S^1 \to \Re$ be defined by*

$$f(s_1, s_2) = \|\gamma(s_1) - \gamma(s_2)\|^2.$$

Then f has at least four critical points (s_1, s_2) with $s_1 \neq s_2$.

Remark 5.3 We are interested only on critical points outside the diagonal $\Delta = \{(s_1, s_1) : s_1 \in S^1\}$ since the diagonal corresponds to choosing $P_1 = P_2$, and would not result in a grasp.

As we already pointed out, since T^1 is a compact set f attains its maximum at some point $(\bar{s}_1, \bar{s}_2)$. Such a point is then a critical point of f. Moreover, f is symmetric (i.e., $f(s_1, s_2) = f(s_1, s_2)$ for all s_1, s_2). Hence $(\bar{s}_2, \bar{s}_1)$ is also a critical point. What the theorem is actually saying then is that there is at least one other critical point (and its symmetric) not on the diagonal. (Since $f(s_1, s_1) = 0$ and $f(s_1, s_2) \geq 0$ for all s_1, s_2, every point on the diagonal is a minimum and hence a critical point.) Notice that symmetric pairs actually give rise to the same pair of gripping points on the object.

In order to apply Theorem 5.1 we need the following lemma.

Lemma 5.1 *$T^1 \setminus \Delta$ is connected.*

To see this simply look at the torus T^1 as the unit square with the opposite sides properly identified.

We now return to the proof of Theorem 5.2. In view of the previous lemma we can apply Theorem 5.1 to the function $g = -f$ the compact space $X = T^1$ and the set $Y = \Delta$ where Δ is the diagonal of T^1. Since the critical points of g are the same as those of f we conclude the proof. ∎

Remark 5.4 Theorem 5.2 did not make any assumption about the convexity of the object A.

In order to show that the newly found critical point of f gives rise to a new pair of antipodal points we will assume that the object A is convex.

Proposition 5.1 *Assume the object A is convex. Let $\{P_1, P_2\}$ be the pair of antipodal points found in Proposition 4.1 and let (s_3, s_4) be the critical point of f found in Theorem 5.2. Let $P_3 = \gamma(s_3)$, $P_4 = \gamma(s_4)$. Then $\{P_3, P_4\}$ is a pair of antipodal points and P_1, P_2, P_3, P_4 are all distinct.*

Remark 5.5 This proposition says that in order to find antipodal points for convex objects all we have to do is find the critical points of the function f. The proof is not constructive. However, we could use standard numerical routines to search for the critical points.

PROOF. Since (s_3, s_4) is a critical point of f, Equation (1) holds. By the convexity assumption, the segment joining P_3 and P_4 is contained in A. So the vector $P_3 - P_4$ $(P_4 - P_3)$ is a positive multiple of the inward normal at P_4 (P_3). This shows that P_3, P_4 are antipodal.

We already know that $P_1 \neq P_2$ and $P_3 \neq P_4$. Assume, say, $P_3 = P_1$. Then the point P_4 must lie in the segment between P_1 and P_2. On convex objects a tangent line at a point on the boundary is a supporting line. That is, it leaves the convex set completely on one side. The tangent line to A at P_4 is perpendicular to the segment joining P_3 and P_4. But then the line must leave the whole segment on one side. This implies that $P_4 = P_2$. But this is a contradiction by the choice of critical point (s_3, s_4). ∎

Remark 5.6 The previous proposition suggest a finger gait on a smooth convex object using 4 fingers. First grasp the object at one pair of antipodal points $\{P_1, P_2\}$. Then rotate the object as much as possible. Now grasp the object at the other pair of antipodal points $\{P_3, P_4\}$ using the other two fingers. Replace the last two fingers with the first two fingers at the points $\{P_3, P_4\}$. This actually requires three steps (much like the three finger gait). See Figure 2.

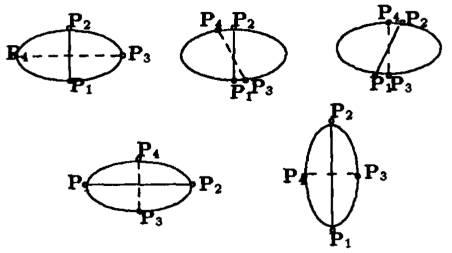

Figure 2: Four finger gait by pairs.

The procedure may now be repeated.

Remark 5.7 To show that $\{P_3, P_4\}$ are antipodal we only need that both ends of the segment $\overline{P_3P_4}$ lie inside A. More precisely, there should exist $\lambda_0 > 0$ such that $\lambda P_3 + (1-\lambda)P_4$ and $\lambda P_4 + (1-\lambda)P_3$ lie inside A for $0 < \lambda < \lambda_0$. To show that all four points are distinct it is only necessary for the segment $\overline{P_1P_2}$ to lie completely inside A.

One advantage in using this gait is that the total rotation required of the segments joining pairs of fingertips is smaller than with the three finger gait (this is rather intuitive). The amount required depends on the geometry of the object, but is at most $\pi/2$. This is so even if the lines joining the antipodal points are parallel (which might happen if the object has parallel sides).

Dropping Assumption (A5) and assuming no other constraints are imposed on the fingers a different gait could be used. Grasp the object with P_1, P_2 and keep P_3, P_4 away from the object. Rotate $P_1 - P_2$ an angle of π. Bring P_3, P_4 to their grasping positions and transfer the grip to them. Rotate $P_1 - P_2$ back an angle of π while away from the object. Transfer grip back to P_1, P_2. This can be achieved with human hands using the thumb and forefinger of the right and left hands. While this gait again requires a large rotation of the vectors $P_i - P_j$ it has fewer steps than the previous one.

As these examples show, for planar objects, the existence of these fine manipulations (finger gaits) is an easy consequence of the existence of a variety of "good" grasping points. The motion planning problem of relocation of the fingers is essentially one dimensional. Therefore more effort will be devoted in the sequel to showing the existence of suitable grasps than will be devoted to the details of the finger gaits. For 3 dimensional objects the motion planning for the relocation of the grasping fingers is not trivial and is the topic of current research.

Theorem 5.3 *Let A be a planar object satisfying (A1)-(A3). Then there exist three non collinear points P_1, P_2, P_3 on the boundary of A such that the positive rays at each of them in the direction of the inward pointing normals intersect at a point strictly inside the triangle determined by P_1, P_2, P_3. In particular, no two of them are antipodal.*

Remark 5.8 This theorem provides a new way of doing finger gaits with 4 fingers. The procedure is very similar to the 3 finger gait. First acquire the grasp suggested in the theorem. Then rotate the object as much as possible. To unwind the fingers now do the following. Replace the third finger with the fourth, the second with the third, the first with the second and the forth with the first. (Each replacement in fact requires three steps just as with the three finger gait.) The rotation can now proceed by repeating the sequence described.

PROOF. As in the proof of Proposition 4.1 we will find the desired points by looking at the critical points of some function. We want to guarantee that the resulting points P_i are all distinct.

Let $\psi : \Re^+ \to \Re^+$ be a continuous function such that

(a) $\psi : (0,\infty) \longrightarrow (0,\infty)$ is continuously differentiable.

(b) $\psi(x+y) \leq \psi(x) + \psi(y)$.

(c) If $x < y$, then $\psi(x) < \psi(y)$.

Examples of such functions ψ are $\psi_\alpha(x) = x^\alpha$ with $0 < \alpha \leq 1$. Notice that $\psi(x) = x^2$ does not satisfy (b) (this is the reason we cannot define the function f as in Proposition 4.1). We now define the function $f : S^1 \times S^1 \times S^1 \longrightarrow \Re$ by

$$\begin{aligned} f(s_1, s_2, s_3) &= \psi(\|\gamma(s_1) - \gamma(s_2)\|) + \psi(\|\gamma(s_2) - \gamma(s_3)\|) \\ &\quad + \psi(\|\gamma(s_3) - \gamma(s_1)\|) \end{aligned}$$

Let $(\bar{s}_1, \bar{s}_2, \bar{s}_3)$ be a maximum of f. Then we claim that $\overline{P}_i = \gamma(\bar{s}_i)$, for $i = 1, 2, 3$, are the desired points. The function f is not differentiable at points (s_1, s_2, s_3) which result in $P_i = P_j$ for some $i \neq j$. So, we first show that the maximum of f occurs at points where all three P_i's are distinct. We show more; namely, that the maximum occurs at noncollinear points. (It is for this step that we need assumptions (b) and (c).) For short we write

$$f(s_1, s_2, s_3) = \sum_{i<j} \psi(\|P_i - P_j\|)$$

where it is assumed that $P_i = \gamma(s_i)$. If at least two of the P_i's are distinct then $f > 0$. If, say, $P_2 = P_3$ then $f = 2\psi(\|P_1 - P_2\|)$. Let the P_i's be noncollinear. Then,

$$\|P_1 - P_2\| < \|P_1 - P_3\| + \|P_3 - P_2\|.$$

By assumptions (c) and (b) on ψ we get

$$\psi(\|P_1 - P_2\|) < \psi(\|P_1 - P_3\|) + \psi(\|P_3 - P_2\|).$$

Therefore,

$$2\psi(\|P_1 - P_2\|) < \psi(\|P_1 - P_2\|) + \psi(\|P_1 - P_3\|) + \psi(\|P_3 - P_2\|).$$

which shows that we get larger values of f when the points are noncollinear.

Then the gradient of f vanishes at a maximum of f. Since $\dot{\gamma}(t) \neq 0$ for all t, at a critical point of f a straightforward calculation shows that the normals to the object at the corresponding points are parallel to the vectors,

$$\begin{aligned} m_1 &= \left(\frac{\psi'(\|\gamma(s_1)-\gamma(s_2)\|)}{\|\gamma(s_1)-\gamma(s_2)\|}+\frac{\psi'(\|\gamma(s_1)-\gamma(s_3)\|)}{\|\gamma(s_1)-\gamma(s_3)\|}\right)\gamma(s_1) \\ &- \frac{\psi'(\|\gamma(s_1)-\gamma(s_2)\|)}{\|\gamma(s_1)-\gamma(s_2)\|}\gamma(s_2)-\frac{\psi'(\|\gamma(s_1)-\gamma(s_3)\|)}{\|\gamma(s_1)-\gamma(s_3)\|}\gamma(s_3) \\ m_2 &= \left(\frac{\psi'(\|\gamma(s_2)-\gamma(s_1)\|)}{\|\gamma(s_2)-\gamma(s_1)\|}+\frac{\psi'(\|\gamma(s_2)-\gamma(s_3)\|)}{\|\gamma(s_2)-\gamma(s_3)\|}\right)\gamma(s_2) \\ &- \frac{\psi'(\|\gamma(s_2)-\gamma(s_1)\|)}{\|\gamma(s_2)-\gamma(s_1)\|}\gamma(s_1)-\frac{\psi'(\|\gamma(s_2)-\gamma(s_3)\|)}{\|\gamma(s_2)-\gamma(s_3)\|}\gamma(s_3) \\ m_3 &= \left(\frac{\psi'(\|\gamma(s_3)-\gamma(s_2)\|)}{\|\gamma(s_3)-\gamma(s_2)\|}+\frac{\psi'(\|\gamma(s_3)-\gamma(s_1)\|)}{\|\gamma(s_3)-\gamma(s_1)\|}\right)\gamma(s_3) \\ &- \frac{\psi'(\|\gamma(s_3)-\gamma(s_2)\|)}{\|\gamma(s_3)-\gamma(s_2)\|}\gamma(s_2)-\frac{\psi'(\|\gamma(s_3)-\gamma(s_1)\|)}{\|\gamma(s_3)-\gamma(s_1)\|}\gamma(s_1) \end{aligned}$$

We hence need to look at the intersection of the lines $L_i = \{P_i + \lambda m_i : \lambda \in \Re\}$, $i = 1, 2, 3$. These lines are well defined as long as $m_i \neq 0$, which holds if the P_i's are not all equal. We assume this is the case. The crucial fact here is that for any choice of (s_1, s_2, s_3) (not necessarily a critical point of f) these lines intersect at a point which is a strict convex combination of the P_i's. To simplify the notation set

$$\alpha_{ij} = \alpha_{ji} = \frac{\psi'(\|\gamma(s_i)-\gamma(s_j)\|)}{\|\gamma(s_i)-\gamma(s_j)\|}$$

The point of intersection is then given by the formula,

$$P = \frac{1}{\Lambda}\left(\alpha_{12}\alpha_{13}P_1 + \alpha_{12}\alpha_{23}P_2 + \alpha_{13}\alpha_{23}P_3\right)$$

where

$$\Lambda = \alpha_{12}\alpha_{13} + \alpha_{12}\alpha_{23} + \alpha_{13}\alpha_{23}.$$

The following formulas show that the point P is in fact in the intersection of the three lines. For each $i = 1, 2, 3$,

$$P = P_i + \lambda_i m_i$$

where

$$\lambda_i = \frac{\alpha_{jk}}{\Lambda} \qquad i, j, k \text{ distinct.}$$

To conclude the proof we still need to show that the point P actually lies in the positive rays (with respect to the inward normals). To simplify the notation we define $g : \partial A \times \partial A \times \partial A \to \Re$ by

$$g(P_1, P_2, P_3) = f(\gamma(s_1), \gamma(s_2), \gamma(s_3))$$

where $\gamma(s_i) = P_i$.

We now show that if P is in the negative ray of the inward normal n_i then (P_1, P_2, P_3) is not a maximum of f. Say P is on the negative ray of n_1. We first show the existence of a vector v such that $v^T n_1 > 0$, $v^T(P_1 - P_2) > 0$, and $v^T(P_1 - P_3) > 0$ (see Figure 3). Then we show that any point of the form $\overline{P}_1 = P_1 + \lambda v$ with $\lambda > 0$ satisfies $g(\overline{P}_1, P_2, P_3) > g(P_1, P_2, P_3)$. Finally, since $v^T n_1 > 0$ there exists $\lambda > 0$ such that $P_1 + \lambda v \in \partial A$.

Since P_1, P_2, P_3 form a triangle we have

$$(P_1 - P_2)^T(P_1 - P_3) > -\|P_1 - P_2\|\|P_1 - P_3\| \qquad (2)$$

The following vector v then satisfies the requirements

$$v = \frac{1}{\|P_1 - P_2\|}(P_1 - P_2) + \frac{1}{\|P_1 - P_3\|}(P_1 - P_3)$$

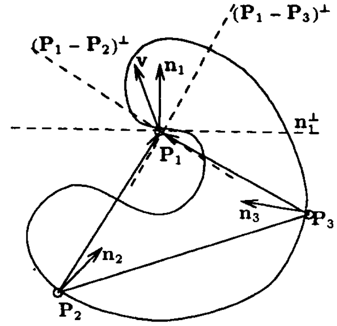

Figure 3: Point of concurrence outside object.

In fact, since P is in the negative ray of n_1 we can write

$$n_1 = \mu\frac{1}{\|P_1 - P_2\|}(P_1 - P_2) + \nu\frac{1}{\|P_1 - P_3\|}(P_1 - P_3)$$

with $\mu, \nu > 0$. The desired inequalities are then a consequence of (2). Since $v^T n_1 > 0$ the vector v points "inside" the object A. Hence it is easy to see (using local coordinates) that for small values of $\lambda > 0$ the point $P_1 + \lambda v$ lies in the interior of A. And then for some larger value $\lambda_0 > 0$ we have $P_1 + \lambda_0 v \in \partial A$. But then, replacing P_1 with $P_1 + \lambda_0 v$ we get a larger value of g contradicting the maximality of (P_1, P_2, P_3).

This shows that P must lie in the positive rays for all normals n_i. This concludes the proof of the theorem. ∎

Remark 5.9 It is easy to see that if the triangle formed by a maximal triplet (P_1, P_2, P_3) is not equilateral then there are infinitely many "good" triplets (i.e. triplets for which the normal inward rays intersect at one point). They can be obtained choosing different functions ψ. The example of an object with an ellipse as boundary illustrates this fact. Moreover, the same example also shows that in general we can not expect to find "good" four finger grasps except as two pairs of antipodal points. Finally, the example shows that five finger grasps with concurrent normals need not exist.

6 Three and four finger grasps for 3D objects

The following assumptions are the 3 dimensional equivalents of Assumptions (A1)–(A5).

(B1) The boundary ∂A is diffeomorphic to the sphere S^2. The case of ∂A diffeomorphic to a tours with n holes is completely analogous.

(B2) The fingers are modeled as isolated points. These point fingers are assumed capable of exerting forces in any direction but individually cannot exert any torque.

(B3) There is a nonzero coefficient of friction between the object and the fingers (we assume a Coulomb friction model). This means that the only forces the fingers can transmit to the object are those which lie inside the cone of friction at the corresponding contact point.

(B4) Let $\{B\}$ be a fixed coordinate frame. For any three fingers P_1, P_2, P_3 there exist three pairs of angles θ_x^l, θ_x^u, θ_y^l, θ_y^u, θ_z^l, θ_z^u such that, the ZYX-Euler angles α, β, γ of the frame $\{P_1 - P_2, P_1 - P_3, P_1 - P_2 \times P_1 - P_3\}$ with respect to $\{B\}$ satisfy, $\theta_x^l \leq \alpha \leq \theta_x^u$, $\theta_y^l \leq \beta \leq \theta_y^u$, and $\theta_z^l \leq \gamma \leq \theta_z^u$. (If P_1, P_2 and P_3 are collinear no conditions are imposed.)

Remark 6.1 In three dimensions there is no analogue to Assumption (A5).

Theorem 6.1 *Under Assumptions (B1)–(B3), there exist antipodal points on the boundary of A.*

PROOF. The proof is completely analogous to that of Proposition 4.1 except that the function f is defined on $S^2 \times S^2$ by

$$f(s_1, s_2) = \|\gamma(s_1) - \gamma(s_2)\|^2$$

where $\gamma : S^2 \to \partial A$ is a diffeomorphism. ∎

Theorem 6.2 *For a three dimensional smooth convex object there exist at least two disjoint sets consisting of antipodal points. Moreover, the points are not collinear.*

PROOF. The proof is analogous to that of Theorem 5.1. ∎

Remark 6.2 It is easy to show that with two antipodal pairs we get force closure. All forces and torques can be balanced by one antipodal pair except for a torque about the line joining the two points. This can be seen using the same calculations used in Proposition 4.1 by projecting on suitable planes containing the two points. The remaining torque can be balanced using the other pair of antipodal points (again with similar calculations).

Theorem 6.3 *There exist three-finger force closure grasps for smooth 3D objects in which no two of the gripping points are antipodal.*

PROOF. Use the function $f : S^2 \times S^2 \times S^2 \longrightarrow \Re$ defined by

$$\begin{aligned} f(s_1, s_2, s_3) &= \psi(\|\gamma(s_1) - \gamma(s_2)\|) + \psi(\|\gamma(s_2) - \gamma(s_3)\|) \\ &\quad + \psi(\|\gamma(s_3) - \gamma(s_1)\|) \end{aligned}$$

where here γ is a diffeomorphism between the sphere S^2 and the boundary of A and ψ is a function satisfying assumptions (a)–(c) from Theorem 5.3. The rest of the proof follows as in Theorems 5.3 and 6.1.

The existence of torque closure is a consequence of the fact that the point of intersection of the normals is in the plane passing through the grasping points. ∎

7 Conclusions and further work

We presented here several results on the existence of certain grasps suitable for fine manipulation. Additionally, some notions about fine manipulations were made precise. The results introduce new techniques for searching for suitable grasps and suggest other research directions. Among them are:

- Establishing and implementing specific algorithms for acquiring a grasp and planning a finger gait following the ideas in the proofs.
- Studying the sensitivity of such grasps to small errors in the location of the gripping points.
- Extending the techniques to piecewise smooth objects. A natural approach would be to allow the parametrizing functions γ to lose rank at some points. The critical point condition on f would then not give us information about the normal. Instead we may consider studying the limits of $\gamma(t)/\|\gamma(t)\|$ as t approaches a critical point. That information, together with a suitable definition of tangent cone (to replace that of tangent line and plane) could be used to formulate some existence results.

Acknowledgments

Work on this paper has been possible by Office of Naval Research Contracts #N00014-87-K-0129 and #00014-89-J3042, National Science Foundation CER Grant #DCR-8320085, and NASA grant #NAG 2-493. We also want to thank an anonymous reviewer for some helpful comments.

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