Social Networks

Lecture #4

Peterson Graph.

\[ \forall v \in V \quad d(v) = 3 = \overline{d} \]

Regular Graph.

\[ |V| = 10 \quad |E| = 15 \]

\[ \sum_{v \in V} d(v) = 3 \times 10 = 30 = 2 \times 15 \]

Given a vertex \( u \in V \),

define its neighbor set

\[ N(u) = \{ x \mid (x, u) \in E \} \]

= Set of vertices joined to \( u \)

\[ N^c(u) = \{ x \mid (x, u) \notin E \} \]

= Set of vertices distinct from \( u \) and not joined to \( u \).

\[ V = \{ u \} \cup \{ N(u) \} \cup N^c(u) \]

Def.: \( S \subseteq V \), define \( \langle S \rangle_G = \text{Subgraph of } G \)
induced by \( S \)

\[ \langle S \rangle_G = \langle S, E(G) \cap S \times S \rangle \]

= Graph with vertices \( S \) and with two vertices joined in \( \langle S \rangle_G \) iff they are joined in \( G \).
A subgraph of $G$ is a graph $H$ s.t. $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A spanning subgraph of $G$ is a subgraph $H$ s.t. $V(H) = V(G)$. Note $E(H) \subseteq E(G)$.

$\delta(G)$: The minimum degree of $G$

$= \min_{v \in V} d(v)$.

$\Delta(G)$: The maximum degree of $G$

$= \max_{v \in V} d(v)$.

$\sum_{v \in V} d(v) = 2|E| \Rightarrow$ The number of odd degree vertices in a graph is even.

$\forall v \in V \ d(v) = k \Rightarrow G$ is $k$-regular graph.

$k$-odd $\Rightarrow |V|$ is even.

$\Rightarrow$ Graph is of even order.

**Walk**: A walk in a graph consists of an alternating sequence of vertices and edges:

$x_0 \ e_1 \ x_1 \ ... \ e_t \ x_t$

$\forall 1 \leq i \leq t \ e_i = (x_{i-1}, x_i)$

$x_0 x_t$ - walk

A walk is closed, if $x_0 = x_t$; otherwise, open.
A path is an open walk with no repeated vertex.

A cycle is a closed walk with no repeated vertex.

The number of edges in a walk (path, cycle) is its length.

n: \( W_n, P_n, C_n \).

**Defn:** The girth of a graph \( G \),

\[ g(G) \]

is the minimum length of a cycle in a graph.

**Lemma:** (i) If \( G \) is a graph, then \( G \) contains a path of length \( s(G) \).

(ii) If \( s(G) \geq 2 \), then

\[ g(G) \geq s(G) + 1. \]

**Proof:**

(i) \( u_0, u_1, \ldots, u_r = P \) = A path of max. length \( r \) in \( G \).

\[ \Rightarrow N(u_r) \subseteq \{ u_0, u_1, \ldots, u_r \} \] (why?)

\[ \Rightarrow s(G) \leq r \]

(ii) Vertex joined to \( u_0 \) with,

\[ u_0 \ldots u_m \ldots u_r \ldots u_{r-1} \] min. index.

\[ u_m \ldots u_r \ldots u_{r-1} \]

= cycle of length \( \geq s(G) + 1 \).
Petersen Graph

\[ \delta(G) = \Delta(G) = 3 \]

\[ g(G) = 5 \quad g(G) \geq \delta(G) + 1. \]

\[ \text{A graph is connected if for each pair of vertices, there is a path connecting them} \]

\[ u \rightarrow u, \quad u \rightarrow v \rightarrow u \quad \& \]

\[ u \rightarrow v, \quad v \rightarrow w \Rightarrow u \rightarrow w. \]

\[ \text{The relation "connectivity" is an equivalence relation}. \]

\[ \Rightarrow \text{The equivalence classes are the connected components of } G. \]

\[ \text{The (geodesic)-distance between two vertices is the length of the shortest path connecting them}. \]

\[ d(u, v) = \text{Geodesic distance between } u \text{ and } v. \]

\[ \text{The maximal geodesic distance in a graph is its diameter } \delta(G). \]

\[ \forall u, v \quad d(u, v) \leq \delta(G). \]