A General Description of Evolutionarily Stable Strategies (ESS)

Two-player Two-strategy Game (Symmetric).

Condition for \((S, S)\) to be evolutionarily stable

\[ S = \text{ESS} \]

Let \(\epsilon > 0\)

\[ 1 - \epsilon \text{ fraction uses } S \] \(\Rightarrow\) Expected payoff for organism playing \(S\)

\[ = a(1 - \epsilon) + b \epsilon \]

\[ \epsilon \text{ fraction uses } T \] \(\Rightarrow\) Expected payoff for organism playing \(T\)

\[ = c(1 - \epsilon) + d \epsilon \]
We need the following inequality
\[ a(1-c) + bc > c(1-c) + de \]

\[ \lim_{\epsilon \to 0} a > c \]

\[ \therefore S = \text{ESS} \quad \text{if } a > c \]
\[ S \neq \text{ESS} \quad \text{if } a < c \]

\[ \text{if } a = c, \quad S = \text{ESS}, \quad \text{if } b > d. \]

\[ \text{Thm. In a two player, two-strategy, symmetric game, } S \text{ is evolutionarily stable} \]

\[ \text{when either (i) } a > c \]
\[ \text{or (ii) } a = c \text{ and } b > d. \]

\[ \therefore \]

Note: \((S, S) = \text{Nash Equilibrium}\)

\[ u_1(S, S) \geq u_1(T, S) \quad \boxed{\text{if } a > c} \]

\[ \& u_2(S, S) \geq u_2(S, T) \]

\[ \Rightarrow S : \text{ESS} \Rightarrow (S, S) = \text{N.E.} \]
Corollary:

If strategy $s$ (in a two-player, two-strategy symmetric game) is ESS (evolutionarily stable), then $(s,s)$ - Nash equilibrium (P.S.N.E.)

# Not all Nash Equilibria are E.S.S.

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Stag-Hunt Game.

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<table>
<thead>
<tr>
<th></th>
<th>Stag</th>
<th>Hare</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stag</td>
<td>4,4</td>
<td>0,3</td>
</tr>
<tr>
<td>Hare</td>
<td>3,0</td>
<td>3,3</td>
</tr>
</tbody>
</table>
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$a = 4$, $a > c$ \iff Two ESS.
$b = 0$
$c = 3$
$d = 3$

\(d > b\) \Rightarrow \text{Stag} \rightarrow \text{Hare.}
Modify \( c = 4 \)

\[
\begin{array}{cc}
S & H \\
S & 4,1 & 0,4 \\
H & 4,0 & 3,3 \\
\end{array}
\]

\( a = 4 \)
\( c = 4 \)
\( b = 0 \)
\( d = 3 \)

\( a = c, \ b < d \)

\[\Rightarrow\] Stag \# ESS.

But \( d > b \) \[\Rightarrow\] Hare = ESS

\((H,H) = \text{N.E.}\)

\(\sim\).

Note: ESS and N.E. are similar but underlying mechanisms differ

N.E \[\rightarrow\] Rationality (Fictitious Play)

ESS \[\rightarrow\] Replicator Dynamics

Reasoning vs. Learning.
Evolutionarily Stable Mixed Strategies (ESMS)

"How to handle games in which no strategy is ESS."

**Hawk-Dove Game**

- **Symmetric 2-player 2-strategy**
- **Animal 2**
  - D
  - H
- **Animal 1**
  - D: (3,3) (4,1)
  - H: (5,1) (0,0)

\[
\begin{align*}
\text{Animal 2} & : D \quad H \\
\text{Animal 1} & : \\
& D: (3,3) \quad (4,1) \\
& H: (5,1) \quad (0,0)
\end{align*}
\]

- **Hawk = Aggressive**
  - No ESS
  - But it has two P.S.N.E.
  - \(\{(D,H),(H,D)\}\)
  - Neither D nor H is a best response to itself.
  - Evolutionary Game in which strategy can be "mixed."

(by any individual)
Probability, \( p \in [0,1] \)
\[ q = 1 - p \]

Play \( S \) with probability \( p \)
\[ E \quad T \quad \{ \begin{array}{c} S \sim \text{Ber}(p) \end{array} \] \[ T \sim \text{Ber}(q) \]

Organism 1 chooses \( S \) with prob \( p \)

Organism 2 chooses \( S \) with prob \( p' \)

<table>
<thead>
<tr>
<th></th>
<th>( X )</th>
<th>( Y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X )</td>
<td>( pp' )</td>
<td>( pq' )</td>
</tr>
<tr>
<td>( (a,a) )</td>
<td>( (b,c) )</td>
<td></td>
</tr>
<tr>
<td>( Y )</td>
<td>( q\cdot p' )</td>
<td>( qq' )</td>
</tr>
<tr>
<td>( (c,b) )</td>
<td>( (d,d) )</td>
<td></td>
</tr>
</tbody>
</table>

Expected Pay-off for the player 1:

\[
V(p, p') = pp' a + pq' b + q\cdot p' c + qq' d
\]
\[
\begin{cases}
V(p, p) = p^2 a + 2pq(b+c) + q^2 d \\
V(p', p) = pp' a + pq' c + q\cdot p' b + qq' d \\
V(p', p') = p'^2 a + pq' (b+c) + q'^2 d
\end{cases}
\]
\[ p \not> p' \]

\[ \forall p, p' \exists \alpha, x, y \quad (1-x) V(p, p) + x V(p, p') > (1-x) V(p', p) + x V(p', p') \]

In the general symmetric game, \( p \) is an evolutionarily stable mixed strategy, if there is a (small) positive number \( y \) such that, when any other mixed strategy \( p' \) invades \( p \) (\( p \not> p' \)) at any level \( x \), the fitness of the organism playing \( p \) is strictly greater than the fitness of an organism playing \( p' \).

\[ p = \frac{1}{3} = \text{Probability of playing Dove} \]

To be shown: \( p = \frac{1}{3} \equiv \text{E.S.M.S.} \)