Social Networks

Lecture #14

Ultimatum Game

- Two anonymous players bargain to divide a fixed amount between them.
- Player 1: (Proposer) offers a division of the amount.
- Player 2: (Responder) decides whether to accept the offer.

If accepted both players receive their agreed-upon shares; otherwise, they receive nothing.

Total Amount to divide = c

1 offers to 2 some amount a≤x≤c

If 2 accepts the payoff is (c-x, x)

If 2 rejects the payoff is (0, 0)
Subgame Perfect (Nash) Equilibrium:
(SPE)
"Optimal" not only at the beginning of the game, but also after every history.

Backward Induction:

Player 2
\[ \left\{ \begin{array}{cl}
(1) & x > 0, \text{ Accept} \\
(2) & x = 0, \text{ Indiff} \\
\end{array} \right. \]
\[ \left\{ \begin{array}{cl}
A & x > 0, A \\
B & x > 0, A \\
\end{array} \right. \]
between
\[ x = 0, R \]
Accept/Reject.

Player 1
\[ \left\{ \begin{array}{cl}
A & x = 0 \\
B & \sup_{x > 0} (c - x) \rightarrow x \quad (\text{or } \bar{x} \text{ if } x \text{ is not divisible})
\end{array} \right. \]
\[ x = 0 \]

Non-Credible Threat:

Player 2
\[ \left\{ \begin{array}{cl}
(1) & x > \bar{x}, \text{ Accept} \\
(2) & x \leq \bar{x}, \text{ Reject.} \\
\end{array} \right. \]

Player 1 offers \( \bar{x} \).
For every $x \in [0, c]$, there exists a NE in which $i$ offers $\bar{x}$.

**Strategic Form Games**

Notation:

$$S = \Pi_{i} S_{i} \Rightarrow \text{Strategy Profile.}$$

$$S_{i} \in S_{i} = \text{strategy of all player } i \in I.$$  

$$S_{-i} = \Pi_{j \neq i} S_{j} = \text{Strategy Profile for all players except } i \in I.$$  

$$S = S_{i} \times S_{-i}, \quad \forall i \in S_{i}.$$  

$$S_{i} = \langle s_{j} \rangle_{j \neq i} = \text{Vector of all strategies/actions for all players excluding player } i \in I.$$  

$$\langle S_{i}, S_{-i} \rangle = \text{Strategy Profile.}$$
PLAY OF THE GAME

Each player chooses a strategy $s_i$

$\Downarrow$

Strategy Profile: $(s_1, s_2, \ldots, s_k) \equiv S$

$\Downarrow$

Utility = $u_i(S)$.

If $S^* = (s_1^*, s_2^*, \ldots, s_k^*)$ = "BEST"

then

$\forall i \in I \quad \forall s_i \in S_i$

$u_i(s_i^*, s_i^*) \geq u_i(s_i, s_i^*)$

$\Downarrow$

BEST RESPONSE.

1 STABILITY: No player can profitably deviate from the chosen strategy, given the strategy of the other players.

2 FIXED POINT UNDER CKR:

Each player chooses a strategy $(s_i^*)$

expecting all other players to choose rationally.
Example Be S.

- Greedy Strategy $\not\rightarrow$ NE.
  - $S_F = \text{Opera}, S_M = \text{Football}$
  - $\Rightarrow$ Pay-off $= (0, 0)$.
  - Either $S_F$ should deviate to football (payoff $0 \rightarrow 2$)
  - Or $S_M$ should deviate to Opera (payoff $0 \rightarrow 2$)

- Ultra-Altruistic Strategy $\not\rightarrow$ NE.
  - $S_F = \text{Football}, S_M = \text{Opera}$
  - $\Rightarrow$ Pay-off $= (0, 0)$.
  - Either $S_F$ should deviate to Opera (payoff $0 \rightarrow 3$)
  - Or $S_M$ should deviate to Football (payoff $0 \rightarrow 3$).

Two Nash Equilibria (Pure Strategy).
- $<0, 0>$
- $<\text{opera, opera}>$
- $<\text{football, football}>$

F enjoys both the Opera & M's company.
M gains utility by being in F's company.
Rock - Paper - Scissors \( \approx \) Matching Pennies.

\[
\begin{array}{c|cccc}
 & P & R & S & S \\
\hline
P & (0,0) & (1,1) & (1,-1) & \text{P.S.N.E.} \\
R & (1,1) & (0,0) & (1,1) & \text{Zero-Sum Game.} \\
P & (1,1) & (0,0) & (1,1) & \\
S & (1,-1) & (1,1) & (0,0) & \\
\end{array}
\]

\[
\downarrow
\]

**Mixed Strategies**

\[ \sum_i \sigma_i = \text{Probability Measure over Pure Strategies } S_i \]

\[ \sigma_i = (p_{i1}, p_{i2}, \ldots, p_{ik}) \in \Sigma_i \]

\[ p_{ij} = \Pr[\delta_{ij} \in S_i \text{ is played}] \]

\[ \forall j \; p_{ij} \geq 0 \; \sum_j p_{ij} = 1. \]

\[ \Sigma = \prod_i \Sigma_i \cong \text{Mixed Strategy Profile.} \; \sigma \in \Sigma \]

\[ \text{Utility} = \text{Expected Pay-off} = \sum p_{ij} u_i(s_{ij}, \sigma_{-i}) = u_i(\sigma_i, \sigma_{-i}) \]
Mixed Strategy Nash Equilibrium

\[ \sigma^* \in \Sigma \equiv \text{M.S.N.E.} \]

\[ \text{iff} \]

\[ \forall i \in I \quad \forall \sigma_i \in \Sigma_i \quad u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*) \]

\[ \text{O Nash Thm. (\text{\textsuperscript{\tiny \textregistered} Kakutani Fixed Point Thm.})} \]

Every finite game has a Mixed Strategy Nash Equilibrium.