**STRONG TRIADIC CLOSURE.**

- In a social network, let $f_1$ and $f_2$ be two close friends of yours...
- They are connected by strong ties to you.
- Then they ($f_1$ and $f_2$) belong to a $k$-clan (for small $k$) with high prob.
- Then, it is likely that $f_1$ and $f_2$ are acquaintances and are connected to each other by weak ties.

[At least your social network may recommend that $f_1$ and $f_2$ explore connecting with each other.]

$$
Pr[(u,w) \in E | (u,v) \in E_s \land (v,w) \in E_s] > Pr[(u,w) \in E].
$$
TRIADIC CLOSURE PROPERTIES

- If $s_1$ and $s_2$ have a large subgroup of common friends (which include you),
  - Then it is possible probable that they are acquaintances.
  - The probability monotonically increases with the size of the set of mutual friends.

- In a connected social network, we expect to see lots of $K_3$'s (Cliques of size 3).

Empirical Results:

"Strength of Weak Ties" (1973)

Mark Granovetter: (American Sociologist; Currently at Stanford Univ.)

- "Weak ties enable reaching populations and audiences with much higher efficiency than what is achievable or accessible via strong ties."

WHY?
Granovetter's PhD dissertation:
(Dept of Social Relation, Harvard Univ)

"Getting a Job"

Experiment:
Location: Newton, MA.
Subjects: 282 professional, technical & managerial workers.

Result: N = 54 = # individuals (out of 282) who found jobs through personal contacts.

- Strong Ties (16.6%)
  - Weak Ties (83.4%)
    - Occasional Contacts (55.6%)
    - Rare Contacts (27.8%)
**Augmented Graph**

**Def:** Consider an "augmented" graph (undir.)
\[ G = (V, E, E_s) \]

in which
\[ E_s \subseteq E \subseteq V \times V. \]

\[ E = \text{The edges/ties.} \]
\[ E_s = \text{The strong ties.} \]
\[ E \setminus E_s = \text{The weak ties.} \]

\( (u, v) \in E \Rightarrow u \text{ and } v \text{ are friends (acquaintances + close friends)} \)

\( (u, v) \in E_s \Rightarrow u \text{ and } v \text{ are close friends.} \)

The strong triadic closure property

\( \text{if } (u, v) \in E_s \text{ and } (u, w) \in E_s \)

\( \text{then } (v, w) \in E \text{ a.s.} \)

\[ \Rightarrow \Pr[(v, w) \in E \mid (u, v) \in E_s \land (u, w) \in E_s] > \Pr [(v, w) \in E] \]

\( \Rightarrow \text{Probability Raising in Social Network.} \)

(Influence vs. Social Influence)
(I) \[ \Pr \left[ (u, v) \in E \land (u, v) \in E_s \mid (u, w) \in E_s \right] \]
\[ > \Pr \left[ (u, v) \in E \land (u, v) \in E_s \mid (u, w) \in E \setminus E_s \right] \]

(II) Consider a new relation \( R \)
\[ \{ (u, v) \in R \} = \text{Event } u \text{ obtained a job} \]
through a "recommender" \( v \).
\[ \{ v \text{ recommended } u \text{ to } w; \]
\[ w \text{ verified } u \text{ independently;} \]
\[ w \text{ determined whether } u \text{ is a suitable candidate.} \]
Weak Ties: 
\((u,v) \in E \land (v,w) \in E \implies (u,w) \in E\) will go by w's recommendation only.

Strong Ties: 
\((u,v) \in E \land (u,w) \in E\) can use information in addition to what's provided in w's recommendation.

\(\text{Applicant: } u\) 
\(\text{ Recommender: } w\) 
\(\text{ Verifier (Employee): } v\)
\( G = (V, E) \) = Undirected graph. Directed graph

Connected Component:
A connected component of a graph is defined as a maximal subgraph in which path exists from every node to every other.

- A path in a graph is closed if its start and end vertices coincide:
  \[ v_0, v_1, \ldots, v_n = v_0 \]
  \[ \forall i (v_i, v_{i+1}) \in E \]
  \[ i, j \neq v_i, v_j \]

- A cycle is defined as a closed path in which \( n \geq 3 \).

Strongly Connected Component
A strongly connected component of a graph is defined as a maximal subgraph in which cycle exists connecting every node to every other.

TREE: A tree is a connected graph that contains no cycle.
Adjacency Matrix \((\text{Undir})\)

Every graph \(G = (V, E)\) with \(|V| = n\) has associated with it a symmetric adjacency matrix \(A \in \mathbb{R}^{n \times n}\) in which

\[
    a_{ij} = \begin{cases} 
        1 & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\
        0 & \text{otherwise}.
    \end{cases}
\]

Since in an undirected graph \((v_i, v_j) \equiv (v_j, v_i)\), \(a_{ij} = a_{ji}\)

\(A^T = A\) \leftarrow A \text{ real-valued symmetric matrix.}

\(d_i = d(v_i) = |\{ v_j \mid (v_i, v_j) \in E \}| \text{ Degree}\)

\[
    D = \begin{bmatrix}
        d_1 & 0 & \cdots & 0 \\
        0 & d_2 & \cdots & 0 \\
        \vdots & \vdots & \ddots & \vdots \\
        0 & 0 & \cdots & d_n
    \end{bmatrix} \text{ Diagonal Matrix.}
\]

\[
    \text{Trace } D = \sum_{i=1}^{n} d(v_i) = 2m
\]
Boundary Matrix $B \in \mathbb{R}^{n \times n}$

Columns are indexed by the vertices of $G$.
Rows are indexed by the edges of $G$.

$B(e,v) = \begin{cases} +1 & \text{if } v \text{ is the head of } e \\ -1 & \text{if } v \text{ is the tail of } e \\ 0 & \text{otherwise} \end{cases}$

$B^T B = L = D - A$.

Choose edge directions arbitrarily if $G$ is undir.

$L = D - A$.

Random Surfing on $G$.

$P(u,v) = \begin{cases} \frac{1}{d_u} & \text{if } (u,v) \in E \\ 0 & \text{otherwise} \end{cases}$

$P = D^{-1} A$.

$L = D(I - P) = DA$; $\Delta = I - P$.

$\Delta = I - P$ is (Discrete) Laplace operator.
Your position within the social network assigns a social value.  
⇒ Rank (e.g. page rank).

\[ f(x) \]

\[ f: V \rightarrow R \text{ a scalar rank function.} \]

Dirichlet Sum of \( G \)

\[ \sum_{u \neq v} (f(u) - f(v))^2 \]

You'd like to choose ranks so that the sum is minimized:
Avoid the trivial rank: \( f(x) = 1 \forall x \).

⇒ Focusing on the relative values.

\[ \Delta f(x) = \frac{1}{d_x} \sum_{(y, x) \in E} (f(x) - f(y)) \]

\[ \downarrow \]

\[ (I - P)f. \]
Random Surfer Model.

Imagine a web surfer bouncing along randomly following the graph (hyperlink graph of the web.)

When the surfer arrives at a node he chooses at random hyper-links (directed edge) to a new node.

Asymptotically, the proportion of time the random surfer spends on a given node/page is a measure of

Relevance (Relative Importance)

\[ \text{Dangling Nodes - Sinks} \]
\[ \text{Periodicity in the graph} \]
\[ \text{Stochastic Teleportation} \]