Lecture #7

March 25 2014

0-1 Laws

Describe a phenomenon, where an event
either occurs or does not occur a.s.
(almost surely)

TIPPING POINT (PHASE TRANSITION):

With a small change in a critical parameter
the event of interest very quickly goes from
probability 0 (ALMOST NEVER)
to probability 1 (ALMOST SURELY)

GAME OF "FRIENDING:"

◊ Imagine sending a friend request randomly to
(n-1) other individuals in a network (with total of
n individuals).

◊ Assume: (a) If the recipient is already a friend, he
simply ignores the request.

(b) Otherwise, he receives your request for the
first time and he accepts you as a friend.

(c) Under no circumstances, does he ignore,
decline or unfriend you.

◊ Tipping Point: After Θ(n ln n) requests, one will
have a.s. befriended all the other (n-1)
individuals.
Coupon Collector's Problem.

"Collect All Coupons And Win Contest"

Problem Statement

- There are \( n \) distinct coupons.
- Coupons can be collected with replacement.
  \[ x_1, x_2, \ldots, x_n \text{ iid r.v.} \]
  \[ x_i = 1 \quad \Rightarrow \quad \text{Event that you obtain the } i\text{th coupon } \sim \text{Bernoulli}(1/n) \]
- What is the probability that more than \( t \) sample trials are needed to collect all coupons?

\[ \Rightarrow \quad \text{Given } n \text{ coupons, how many coupons are expected to be drawn with replacement, before each coupon has been drawn at least once?} \]

Example \( n = 52, \ t = 225 \)

If you draw a card randomly (with replacement) from a full-deck, then after \( t = 225 \) draws you would have seen every card at least once almost surely. \( t = \Theta(n \ln n) \).

\( t_i \): Time to collect \( i\text{th coupon after collecting } (i-1)\text{th coupon.} \)
- \( t_i \)'s are independent.

\[ t = \sum_{i=1}^{n} t_i \quad \text{Time to collect all coupons.} \]
\[ p_i = \Pr \left[ \text{Collect a new coupon after } (i-1)^{th} \right] = \frac{n}{n-i+1} \]

\[ t_i \sim \text{Geometric} \left( \frac{1}{p_i} \right) \quad \Pr [t_i = k] = (1-p_i)^{k-1} p_i \]

\[ E(t_i) = \frac{1}{p_i} = \frac{n}{n-i+1} \quad \text{Var}(t_i) = \frac{1-p_i}{p_i^2} = \frac{(i-1)n}{(n-i+1)^2} \]

\[ E[T] = E(\sum t_i) \quad \text{Linearity of Expectation} \]

\[ = \frac{n}{n} + \frac{n}{n-1} + \cdots + \frac{n}{1} = n \log n \]

\[ = n \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} \right) \]

\[ = n \int_1^n \frac{1}{x} \, dx + \pi n + \frac{1}{2} + o(n) = n \log n + \Theta(n) \]

\[ n^* = 0.577 = \text{Euler's Constant}. \]

\[ \text{Var}[T] = \text{Var}(\sum t_i) \quad t_i \text{ 's i.i.d.} \]

\[ \leq \frac{n^2}{n^2} + \frac{n^2}{(n-1)^2} + \cdots + \frac{n^2}{1} = \frac{n^2}{6} \]

\[ \sigma(T) = \frac{n \sqrt{n}}{\sqrt{6}} \]

\[ \Pr \left( |T - n \log n| \geq c \cdot n \right) = \Pr \left( |T - n \log n| \geq \left( c \frac{n}{\sqrt{6}} \right) \sigma \right) \]

\[ \leq \frac{\pi^2}{6} c^2 \]

\[ \Pr \left( |T - n \log n| \geq 10 \cdot n \right) \leq \frac{\pi^2}{600} \approx \frac{1}{60} \]

If \( T < (1-c) n \log n \), you will a.m. get all the coupons.

If \( T > (1+c) n \log n \), you will a.s. get all the coupons.
GENERALIZATION

$T_k = \text{First time } k \text{ copies of each coupons are collected.}$

$T_k \sim n \log n + (k-1) n \log \log n + \Theta(n)$

$\begin{array}{c}
\text{Prob} = 0 \\
\text{Prob} = 1 \\
\end{array}$

$n \log n$

Tipping Point.

0-1 Law
(Phase Transition)