Lecture #11  April 22 2014

Strategic Form Games: \{ Normal-Form Matrix Games.

(A) Non-Cooperative:
All participants act simultaneously and without knowledge of other player's actions. (or Thoughts/intentions) \implies \text{REVELATION PRINCIPLE.}

(B) Rationality (or, Common knowledge of Rationality).

(i) $\mathcal{F}$: The set of players.
(ii) $S_i \in \mathcal{F}$: The strategies. \{ Select strategies to optimize payoffs.
(iii) $U_i: \mathcal{F}$: The payoffs.

(C) Information

(i) The game form: Captures order of play.

(ii) Information Set: Models asymmetric/incomplete information situations.

\[ S = \prod_{i \in \mathcal{F}} S_i; \quad S_i = \prod_{j \neq i} S_j \implies \langle s_i, s_{-i} \rangle \in S_i \times S_{-i} = \mathcal{S} \]

Best Response:

\[ B_i (s_{-i}) \in \arg\max_{s_i \in S_i} u_i (s_i, s_{-i}) \]

A strategy of $i \in \mathcal{F}$ that maximizes his utility, provided that all other players have selected $s_{-i}$.
EQUILIBRIUM:

Everyone should choose their best responses and not deviate from it: Select \( \sigma^* \)

\[
\forall i \in \mathcal{N} \quad B_i(\sigma^*_i) = \sigma^*_i
\]

Example: Partnership Game

<table>
<thead>
<tr>
<th></th>
<th>Friend 1</th>
<th>Friend 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Work</td>
<td>(2,2)</td>
<td>( B_1(\cdot, WH) = WH )</td>
</tr>
<tr>
<td>Hard</td>
<td>(1,1)</td>
<td>( B_2(WH, \cdot) = WH )</td>
</tr>
<tr>
<td>(WH)</td>
<td>Work</td>
<td>Shirking</td>
</tr>
<tr>
<td>Hard</td>
<td>Shirking</td>
<td>(1,-1)</td>
</tr>
<tr>
<td>(S)</td>
<td>Shirking</td>
<td>(0,0)</td>
</tr>
</tbody>
</table>

\( B_1(\cdot, S) = S \)

\( B_2(S, \cdot) = S \)

Dominant Strategy:

A strategy \( s_i \in S_i \) is dominant for player \( i \) if

\[
\forall s'_i \in S_i \quad \forall s_{-i} \in S_{-i} \quad u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})
\]

\( s_i \in S_i \) is strictly dominated if

\[
\exists s'_i \in S_i \quad \forall s_{-i} \in S_{-i} \quad u(s'_i, s_{-i}) > u(s_i, s_{-i})
\]

\( s_i \in S_i \) is weakly dominated if

\[
\exists s'_i \in S_i \quad \forall s_{-i} \in S_{-i} \quad u(s'_i, s_{-i}) \geq u(s_i, s_{-i})
\]

\[
\exists s_{-i} \in S_{-i} \quad u(s'_i, s_{-i}) > u(s_i, s_{-i})
\]
Iterated Elimination of Strictly Dominated Strategies

\[ j \leftarrow 0; \]
\[ \text{for } i \in \mathcal{I} \quad S_i^0 \leftarrow S_i; \]
\[ j \leftarrow j + 1; \]
\[ \text{Loop} \]
\[ \text{for } i \in \mathcal{I} \]
\[ S_i^j \leftarrow \{ s_i \in S_i^{j-1} \mid \forall s_i' \in S_i^{j-1} \quad \forall s_i'' \in S_i^{j-1} \]
\[ u_i(s_i', s_i'') > u_i(s_i, s_i) \} \]
\[ S_i^{\infty} \leftarrow \cap_{k=0}^{\infty} S_i^k; \]
\[ S^\infty \leftarrow \prod_i S_i^{\infty}. \]

Dominant Strategy Equilibrium:

A strategy profile \( s^* \) is the dominant strategy equilibrium if for each player \( i \in \mathcal{I} \)

\[ s_i^* = \text{A dominant strategy}. \]
**Example (Extended) Prisoner's Dilemma:**

<table>
<thead>
<tr>
<th>Prisoner 1</th>
<th>Confess</th>
<th>Silence</th>
<th>Suicide</th>
</tr>
</thead>
<tbody>
<tr>
<td>Confess</td>
<td>(-2, -2)</td>
<td>(0, -3)</td>
<td>(-2, -10)</td>
</tr>
<tr>
<td>Silence</td>
<td>(-3, 0)</td>
<td>(-1, -1)</td>
<td>(0, -10)</td>
</tr>
<tr>
<td>Suicide</td>
<td>(-10, -2)</td>
<td>(-10, 0)</td>
<td>(-10, -10)</td>
</tr>
</tbody>
</table>

(A) Suicide is **dominated** for both players:  
\[ \rightarrow \text{Eliminate Suicide} \]

(B) Next, Silence is dominated for both players:  
\[ \rightarrow \text{Eliminate Silence} \]

\[ S^\infty = \{ \text{(Confess, Confess)} \} \]

↑ **DOMINANT STRATEGY EQUILIBRIUM.**  
(Follows from CKR).
Example / Counter Example:

Rock-Paper-Scissors.

<table>
<thead>
<tr>
<th></th>
<th>R</th>
<th>P</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>(0,0)</td>
<td>(-1,1)</td>
<td>(1,-1)</td>
</tr>
<tr>
<td>R</td>
<td>(-1,1)</td>
<td>(0,0)</td>
<td>(-1,1)</td>
</tr>
<tr>
<td>S</td>
<td>(1,-1)</td>
<td>(1,-1)</td>
<td>(0,0)</td>
</tr>
</tbody>
</table>

← Zero-Sum Game.

 Doesn't have a

Pure-Strategy Nash Equilibrium.

Mixed Strategies: (Nash).

\[ \Sigma_i : \text{Probability measures over the Pure Strategies } S_i, \]
\[ \text{(for player } i). \]

\[ \sigma_i \in \Sigma_i : \text{Mixed Strategy of player } i. \]

\[ \begin{align*}
S_i &= \{ s_{i1}, s_{i2}, \ldots, s_{ik} \} \leftarrow i's \text{ strategies} \\
\sigma_i &= (p_{i1}, p_{i2}, \ldots, p_{ik}) \leftarrow \text{corresponding probabilities.} \\
p_{ij} &= \Pr [ s_{ij} \in S_i \text{ is played} ] \quad \left\{ \begin{array}{l} 
\sum p_{ij} = 1 \\
\end{array} \right. \\
\end{align*} \]

\[ \Sigma = \prod \Sigma_i : \text{Mixed Strategy Profiles.} \quad \left\{ \text{Players randomize independently} \right. \]

\[ \mu_i (\sigma) = \sum_{s} \mu_i (s) \sigma (s) = \sum p_{ij} \mu_i (s_{ij}, \sigma_{-i}) \]
**Mixed Strategy Nash Equilibrium.**

\[ \sigma^* \in \Sigma = \text{Mixed strategy profile.} \]

\[ \sigma^* = \text{Mixed Strategy Nash Equilibrium, if} \]

\[ \forall i \in \mathcal{I}, \forall \sigma_i \in \Sigma_i: \mu_i(\sigma_i^*, \sigma_{-i}^*) \geq \mu_i(\sigma_i, \sigma_{-i}^*) . \]

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**Example Matching Penny.**

<table>
<thead>
<tr>
<th>Matcher</th>
<th>H</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>(-1,1)</td>
<td>(1,-1)</td>
</tr>
<tr>
<td>T</td>
<td>(1,-1)</td>
<td>(-1,1)</td>
</tr>
</tbody>
</table>

\[ \left( \frac{1}{2}, \frac{1}{2} \right), \left( \frac{1}{2}, \frac{1}{2} \right) \]

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**Nash's Theorem.**

Every finite game has a mixed strategy Nash Eq. \( \square \).

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**Kakutani's Fixed Point Theorem.**

Let \( f: A \rightarrow A : x \in A \mapsto f(x) \subseteq A \) be a correspondence satisfying the following conditions:

1. \( A \) is compact, convex and nonempty (subset of finite dimensional Euclidean space \( \mathbb{R}^d \))
2. \( \forall x \in A \ f(x) \neq \phi \)
3. \( \forall x \in A \ f(x) \) is convex
4. \( f(x) \) has a closed graph: \( \{x_n, y_n\} \rightarrow \{x, y\} \) with \( y_n \in f(x_n) \) \( \Rightarrow y \in f(x) \).

Then \( \exists x^* \in A \ x^* \in f(x^*) \)

\( f \) has a fixed point = \( x^* \). \( \square \)

**Corollary:** MS NE = Fixed point \( f = \text{Best Response} \).
**Signaling Games.**

Two players: \( S = \text{Sender} \)
\( R = \text{Receiver} \)

1) Nature selects a **type** \( t_i \) from \( T = \{ t_1, \ldots, t_I \} \) with prob \( p(t_i) \)

2) Sender observes \( t_i \) and chooses a **message** \( m_j \) from \( M = \{ m_1, \ldots, m_J \} \)

3) Receiver observes \( m_j \) (but not \( t_i \)) and takes an **action** \( a_k \) from \( A = \{ a_1, \ldots, a_K \} \)

**Payoffs**:

\[ \{ U_S (t_i, m_j, a_k) \}
\{ U_R (t_i, m_j, a_k) \}

Signaling games have Nash equilibria:

- **Pooling Equilibrium**: All types of sender send the same message.
- **Separating Equilibrium**: All types of send send different messages.

Combination / Babbling.