GAME THEORY.

Study of strategic interactions.

Choices + Rational Decisions.

Privacy
Trust
Signal
Bargaining
Auction
Pricing.

° Static Games
° Dynamic Games.
° Signaling Games.
° Bargaining Games.
° Evolutionary Games.

Key Assumptions (often violated).

1) Rationality (Bounded Rationality)
2) CKR: Common Knowledge of Rationality.

Individuals (in a game/social network) act rationally

- Strategically select an option
- That optimizes their own utilities/payoffs.

a) Payoffs need not be just monetary

b) Rationality provides an idealization for developing

- Bounded Rationality
- Evolutionary Stable Strategies.
**Ordinal Information**

a) **Set of Strategies (Options/choices):** \( S = \{s_1, s_2, \ldots, s_n\} \)

b) **Utility Function:** (Real-valued)
   \[ u: S \rightarrow \mathbb{R} \]
   i) \( u(\cdot) \) represents a ranking of different options:
   \[ u(s_{\pi(1)}) \geq u(s_{\pi(2)}) \geq \cdots \geq u(s_{\pi(n)}) \]

c) **Every strategy induces a probability distribution over consequences:**
   \[ F^{s_i}(\cdot) \text{ or } P^{s_i}_{c_j} \]
   continuous pdf or discrete pm.

d) **Bernoulli Utility Function:** There is a utility function, called Bernoulli Utility Function, \( u(c) \), which gives utility of a consequence, \( c \).

**Expected Utility Under Uncertainty:**

\[
u(s_i) = \begin{cases} 
\sum P^{s_i}_{c_j} u(c_j) \\
\int u(c) f^{s_i}(c) \, dc = \int u(c) \, dF^{s_i}(c)
\end{cases}
\]
**Multiplayer Situation:**

\[ \{ \text{John von Neumann} \}
\{ \text{Oskar Morgenstern} \} \]

1) Rational Decision Making (under uncertainty)
   → Proposed a set of "reasonable" axioms.

2) Expected Utility Theory
   → Under uncertainty, every choice induces a "lottery".
   (Probability Distribution over different outcomes.)

Rationality → Optimize Expected Utility:

\[
\begin{align*}
\text{Two actions } & \left\{ S_a, S_b \right\} \rightarrow \text{Probability Dists } \left\{ F_{S_a}(c), F_{S_b}(c) \right\} \\
\rightarrow & \text{Expected utilities.} \left\{ u(S_a) = \int u(c) dF_{S_a}(c), u(S_b) = \int u(c) dF_{S_b}(c) \right\} \\
\text{Choose } a & \text{ over } b \text{ iff } u(S_a) \geq u(S_b) \text{.}
\end{align*}
\]

**Strategic Form Games. (Defn)**

A strategic form game is a triplet:
\[ \langle I, (S_i)_{i \in I}, (u_i)_{i \in I} \rangle \]

such that

1) **INDEX SET:** \( I = \text{Finite set of players} \)
   \( = \{ 1, 2, 3, \ldots, n \} \)

2) **STRATEGY SET:** \( S_i, i \in I \) = Set of available actions
   for player \( i \in I \).

3) **STRATEGY PROFILE:** \( S = \prod_{i \in I} S_i \)

4) **UTILITY FUNCTION:** \( u_i : S \rightarrow \mathbb{R} = \text{The payoff function of player } i \in I \).
Notations:

\[ S = \prod_i S_i = \text{Set of all action profiles.} \]

\[ s_i \in S_i = \text{An action available to player } i. \]

\[ S_{-i} = \prod_{j \neq i} S_j = \text{Set of all strategy profiles for all players except player } i. \]

\[ S = S_i \times S_{-i} \]

\[ \beta_i \in S_i; \quad \beta_i = <s_j>_{j \neq i} = \text{Vector of actions for all players excluding } i. \]

\[ <s_i, \beta_{-i}> = \text{A strategy profile} \]

Each player chooses a strategy \( s_i \rightarrow \)

Generates a strategy profile \( \rightarrow s = <s_1, s_2, ..., s_n> \)

Obtains a utility \( \rightarrow u_i(s) \)

How should each player make a strategic choice?

\[ s^* = <s^*_1, s^*_2, ..., s^*_n> = ? \]

Answer:

\[ u_i(s^*_i, \beta_{-i}) \geq u_i(s_i, s^*_i) \quad \forall i \in I \quad \forall s_i \in S_i \]

\( \rightarrow \text{BEST RESPONSE} \)

1) No player can profitably deviate given the strategy of the other players. (STABILITY)

2) Each player chooses a strategy \( (s^*_i) \) expecting all other players to choose corresponding "best" strategies. (FIXED POINT UNDER CKR)
**Nash Equilibrium (Pure Strategy N.E.)**

A pure strategy Nash Equilibrium of a strategic game:

\[ \langle I, (s_i)_{i \in I}, (u_i)_{i \in I} \rangle \]

is a **Strategy Profile** \( s^* \in S \) such that

\[ \forall i \in I \ \forall s_i \in S_i: \ u_i(s_i^*, s_i^*) \geq u_i(s_i, s_i^*) \]

**BOS (Battle of the Sexes) = Two Player Game:**

Two players of opposite sex \( \{ M = \text{Male} \ \? \ \ F = \text{Female} \} \).

- \( F = \) Row player
- \( M = \) Column player.

\[
\begin{array}{c|cc}
& \text{Opera} & \text{Football} \\
\hline
\text{Opera} & 3,2 & 0,0 \\
\text{Football} & 0,0 & 2,3 \\
\end{array}
\]

\( I = \{ F, M \} \)

\( S_F = S_M = \{ \text{Opera, Football} \} \)

\( S = S_F \times S_M = \{ \langle \text{Opera, Opera} \rangle, \langle \text{Opera, Football} \rangle, \langle \text{Football, Opera} \rangle, \langle \text{Football, Football} \rangle \} \)

\( u_F: S \rightarrow \mathbb{R}, \ u_M: S \rightarrow \mathbb{R} \)

\( u_F(\langle \text{opera, opera} \rangle) = u_M(\langle \text{football, football} \rangle) = 3 \ \{ \text{o.w.} \} \)

\( u_F(\langle \text{football, football} \rangle) = u_M(\langle \text{opera, opera} \rangle) = 2 \ \{ = 0 \} \)
1. In the matrix, in each entry
   \[ \begin{cases} 
   \text{first number is the payoff to player 1} \\
   \text{second number is the payoff to player 2} 
   \end{cases} \]
   row player female.

2. Player 1 chooses a row
   \[ S_F \in \{ \text{opera, football} \} \]
   Player 2 chooses a column
   \[ S_M \in \{ \text{opera, football} \} \]
   Choices must be made simultaneously.

3. The payoffs are
   \[ u_F(S_F, S_M), \ u_M(S_F, S_M) \]

   - Greedy Strategy: \( \neq \) NE
     \[ S_F = \text{opera}, \quad S_M = \text{football} \Rightarrow \text{payoff} = (0,0) \]
     or
     \[ S_F \text{ should deviate to football } \Rightarrow \text{payoff}: 0 \rightarrow 2. \]
     \[ S_M \text{ should deviate to opera } \Rightarrow \text{payoff}: 0 \rightarrow 2. \]

   - Ultra Altruistic Strategy: \( \neq \) NE
     \[ S_F = \text{football}, \quad S_M = \text{opera} \Rightarrow \text{payoff} = (0,0) \]
     or
     \[ S_F \text{ should deviate to opera } \Rightarrow \text{payoff}: 0 \rightarrow 3 \]
     \[ S_M \text{ should deviate to football } \Rightarrow \text{payoff}: 0 \rightarrow 3 \]

   - Two Nash Equilibria:
     \[ \langle \text{opera, opera} \rangle \quad \text{or} \quad \langle \text{football, football} \rangle \]
     \[ \Rightarrow \]
     \[ \{ F \text{ enjoys both the opera & M's company.} \]
     \[ \{ M \text{ gains utility by being in F's company.} \]
Honesty

**Auction:** \{ Used by Google, eBay, etc. \}

- **Second Price Auction** (With Complete Information)
  - Can be further relaxed.

**Players:** \( I = \{ 1, 2, 3, \ldots, n \} \)
  - An object is to be assigned to a single player \( i \in I \).

- Each player \( i \in I \) has his own "private" valuation of the object:
  \[ v_i = \text{Player } i \text{'s valuation} \]

WLOG, assume
\[ v_1 > v_2 > \ldots > v_n > 0 \]

- **Complete Information Version:**
  - Assume everyone knows all the valuations:
    \[ V = \{ v_1, v_2, \ldots, v_n \} \]

  → The players simultaneously submit bids, \( b_i, i \in I \)
  \[ B = \{ b_1, b_2, \ldots, b_n \} \]

  → The object is assigned to the **Highest Bidder**
    (with random tie-breaking)

  → The winner pays the **Second Highest Bid**

The utility function =
\[ u_i(b_1, b_2, \ldots, b_n) = \begin{cases} v_i - b_j & \text{i is highest bidder} \\ v_i - b_i & \text{j = 2nd highest bidder} \\ 0 & \text{o.w.} \end{cases} \]
Honest Bidding.

Lemma: In the second price auction, Honest Bidding, i.e.

\[ b^* = (v_1, v_2, \ldots, v_n) \quad \text{i.e.} \quad b_i = v_i \]

is a Nash Equilibrium. \text{\textcircled{}} Truthful Equilibrium.

Proof:
Player 1 receives the object and pays \( v_2 \).

\[ u_1(b^*) = v_1 - v_2 \quad \forall j \neq 1 \quad u_j(b^*) = 0. \]

Player 1 has no incentive to deviate, since

a) if \( b_1 > v_2 \), it has no effect on \( u_1(b) = v_1 - v_2 \)
b) if \( b_1 < v_2 \), it decreases his payoff to \( u_1(b) = 0 \).

Player \( j \neq 1 \) has no incentive to deviate, since

a) if \( b_j > v_j \), it decreases his payoff to \( u_j(b) = v_j - b_j < 0 \)
b) if \( b_j < v_j \), it has no effect on \( u_j(b) = 0 \).

No player has any incentive to deviate. \( \square \)

Incomplete Information Case: (Bit more complex)

- Two additional Nash Equilibria:
  \[ b^* = (v_1, 0, 0, \ldots, 0) \]
  \[ b^{**} = (v_2, v_1, 0, \ldots, 0) \]

- In general, it can be shown that Honest Bidding \( \Rightarrow \) Results in a Weakly Dominant Nash Equilibrium.