Lecture #4
February 19, 2013.

Graph Theory:
- Combinatorial Structures
- Spectral Properties
- Random Graphs and their Evolution

Pairwise Interactions: → Game Theory (Strategic choice)
- Graph Theory (Interaction choice)

Actors, Players → V
Links, Connections → E ⊆ V × V (Symmetric Binary Relation)

In addition, we will need Strategy Spaces S_v
& Pay-off functions:
    u_v: \Pi S_v \rightarrow \mathbb{R}_v

Graphs: Def
A graph - usually denoted \( G(V, E) \) or \( G = (V, E) \) - consists of a set of vertices \( V \) together with a set of edges \( E \subseteq V \times V \).

A graph is a mathematical object describing an (irreflexive, symmetric) binary relation on a discrete set (not necessarily finite)
Friendship:
1) Irreflexive: One is not his own friend.
2) Symmetric: One is a friend to a friend.
3) Non-Transitive: A friend’s friend is not necessarily a friend.

Friendship is described by an (undirected) edge in the graph.

- An edge \( e \in \{u,v\} \in E \) (where \( E = V \times V \)) is defined by the unordered pair of vertices (players) that serve as its end points.

- Two vertices \( u \) and \( v \) are adjacent if there exists an edge \( e \in \{u,v\} \) connecting them.

- The number of vertices is usually denoted by \( |V| = n \) &
  number of edges by \( |E| = m \).

\[
m \leq \frac{n(n-1)}{2} = \binom{n}{2}
\]

- \( n \) is \# of ways to choose \( u \)
- \( n \) is \# of ways to choose \( v \)
- Identify edge \( (u,v) : (v,u) \).
Synonyms:
Vertices = {Nodes, Points} = {Actors, Agents, Players}

Edges = {Connections, Lines} = {Links, Ties, Encounters}

Strict Graphs:
No Self-Loop \((u, u) \notin E\)
No Multi-Edges \(e_1 = (u_1, v_1), e_2 = (u_2, v_2)\) \(\Rightarrow (u_1 = u_2) \Rightarrow (v_1 = v_2)\)
\(\lambda(u_1, v_1) \Rightarrow (u_1, v_2)\)

Two distinct social relationships that are empirically measured on the same group of people are represented by separate graphs.

Two vertices are adjacent if there is an edge connecting them.
Two edges are incident if there is exactly one vertex, shared by them.
Adjacency Matrix:

Every graph $G = (V, E)$ with $|V| = n$ has associated with it a symmetric adjacency matrix, which is a Binary $n \times n$ Matrix $A$

in which

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \text{ is adjacent to } v_j, \text{i.e.} (v_i, v_j) \in E; \\ 0, & \text{otherwise.} \end{cases}$$

Since $(v_i, v_j) = (v_j, v_i)$ $A_{ij} = a_{ji}$

$A^T = A$

The matrix $A$ is a real-valued symmetric matrix.

The number of vertices adjacent to a given vertex $v$ is called the degree of the vertex and is denoted

$$d(v) = \left| \{ (v, w) \in E \} \right|$$

$$\sum_{v \in V} d(v) = 2 |E| = 2m$$

Average degree of the graph

$$\bar{d} = \frac{\sum_{v \in V} d(v)}{|V|} = \frac{2m}{n}$$
A graph is complete if all of its vertices are adjacent to all others.

The extent to which a graph is complete is indicated by its density or sparsity.

Density = Number of edges divided by the number of possible total
\[
\text{Density} = \frac{|E|}{\binom{n}{2}} = \frac{2m}{n(n-1)} \quad \text{(in strict graphs)}
\]

In a social network, if many "well-connected people" then the network is "dense."

\[ d = \text{density} \]

By connecting to a "well-connected group of individuals," one can belong to a "sub-network" with high connectivity:
- CLIQUE
- CLAN
- CLUB.
A subgraph of a graph \( G \) is a graph whose vertices and edges are contained in \( G \).

\[
\begin{align*}
G, & \subseteq G \quad V, \subseteq V \\
E, & \subseteq E.
\end{align*}
\]

A clique is a maximal complete subgraph:

\[
\text{Clique } G, \subseteq G, \quad |V_c| = k, \quad |E_c| = \binom{k}{2}
\]

\[
\delta = k-1, \quad \text{(local) density } = 1.
\]

Thus, it's a maximal subgraph of \( G \) that has a density \( \delta = 1 \).

No extra node can be added to a clique without density decreasing below \( 1 \).


How can we capture the notion of an interesting subgraph, which, while not a clique, is still very "cliquish."

Professional Society, Social Clubs, ...
Not every pair of vertices is adjacent.
Yet, one could construct "short walks" of vertices between every pair of vertices.
WALK: A sequence of adjacent vertices $v_1, v_2, ..., v_n$ $[(v_i, v_{i+1}) \in E]$ is called a walk.

A walk can also be described by a sequence of incident edges.

PATH: A walk in which no vertex occurs more than once is called a (simple) path.

TRAIL: A walk in which no edge occurs more than once is called a (simple) trail.

\[ \text{PATH} \rightarrow \text{TRAIL} \rightarrow \text{WALK} \]

Every path is a trail, and every trail is a walk.

A walk is closed if $v_1 = v_n$, i.e., its start and end vertices coincide.

CYCLE: A cycle is defined as a closed path in which $n \geq 3$. 
CONNECTED COMPONENT.
A connected component of a graph is defined as a maximal subgraph in which path exists from every node to every other.

STRONGLY CONNECTED COMPONENT.
A strongly connected component of a graph is defined as a maximal subgraph in which cycle exists connecting every node to every other.

TREE: A tree is connected graph that contains no cycle.
- Family Tree, Phylogeny Tree.
- In a tree every pair of vertices is connected by a unique path.
- Connected, but strongly connected components are trivial since vertices.
The length of a walk (resp. a trail or a path) is defined as the number of edges it contains.

A walk between two vertices whose length is as short as possible is called a geodesic (or shortest path).

The (geodesic) distance between two vertices is the length of the shortest path connecting them.

\[ d(u,v) \] Geodesic Distance Between \( u \) and \( v \).

The maximum geodesic distance in a graph is its Diameter.

Cohesive Subsets:

An \( n \)-clique \( S \) of a graph is a maximal set of nodes in which \n\[ \forall u,v \in S, d(u,v) = n \]\n
1-clique \( \subseteq \) Clique.

\( \{ a,b,d_3 \} \) is a 1-clique.

\( \{ a,b,c,d_e \} \) is a 2-clique.

Note: \( d(c,e) = 2 \) thin edge, even if \( c \& e \) is 2-clique.
An n-claw is an n-clique in which the diameter of the subgraph $G(S)$ induced by $S$ is less than or equal to $n$.

(Note: The subgraph $G(S)$ induced by the set of nodes $S$ is defined as the maximal subgraph of $G$ that has vertex set $S$.)

An $n$-claw $S$ is an $n$-clique in which all pairs of vertices are at distance less than or equal to $n$, even when the good edges/paths are restricted to and involve only members of $S$.

\{b, c, d, e\} = 2-claw (\& n-clique)
\{a, b, c, d\} = n-clique (but \& n-claw)

$n$-clique \& $n$-claw.

An $n$-club is a subset $S$ of nodes such that in the subgraph $G(S)$ induced by $S$, the diameter is $n$ or less.

\{a, b, c, d, e\} = 2-clique (but \& 2-claw)
\{a, b, c, d\} = 2-club (but \& 2-clique, not maximal)
Cohesive Subsets.

n-clique, n-club, n-clan

LS- Sets.
Let \( H \) be a set of nodes in \( G = (V,E) \)
and let \( K \) be a proper subset of \( H \).
Let \( \alpha(k) \) denote the number of
edges linking members of \( K \) and \( V \setminus K \).
Then \( H \) is an LS-set of \( G \) if

\[
\forall K \subseteq H \quad \alpha(K) > \alpha(H).
\]

Members of \( H \) have more ties with
the “insiders” (other members of \( H \)) than
“outsiders” (members of \( V \setminus H \)).
Class Project
"Start up" Model
→ LEAN STRUCTURE.
Create Value; Change the World
Create Social Goods
Disrupt.

5-WHYS
For want of a nail the shoe is lost.
shoe — horse — rider
rider — battle
battle — kingdom

Team
CEO: Overall Vision/Executive Summary/
& Business Plan/Elevator Pitch
Investor — Social Good
CSO: Science & Ideas/Intellectual Property/
Protection/Profit Art
(Patent Valuation)
CTO: Minimal Viable Product/Prototype/
Protos (Product Valuation)
C00: Collect Data/Analyze
(Clients Valuation)