Strategic Form Games.
(Normal-Form or Matrix Games).

(A) All participants act simultaneously and without knowledge of other players' actions (or thoughts/intentions, etc.)

(B) Main Ingredients:
- The set of players \((I)\)
- The strategies \((S_i \in I)\)
- The payoffs \((U_i \in I)\)

(C) In general, we may also need
- The game form.
  (Captures order of play)
- Information set.
  (Models asymmetric information or incomplete information situations)
STRATEGIC FORM GAME.
A strategic form game is a triplet
\[ \langle I, (S_i)_{i \in I}, (\pi_i)_{i \in I} \rangle \]
such that
- \( I \) is a finite set of players
- \( S_i \) is the set of actions (strategies) available to player \( i \in I \).
- \( s_i \in S_i \) is an action (strategy) for player \( i \).
- \( u_i : S \rightarrow \mathbb{R} \) \( \{ S_i \ni s_i \} \)
is the payoff function of player \( i \).
- \( S \) is the set of all strategy profiles.
Notation:

\[ \mathbf{s}_i = [s_j]_{j \neq i} \]  
\{ vectors of actions for all players except \( i \). \}

\[ S_{-i} = \Pi_{j \neq i} S_j \]  
\{ set of strategy profiles for all players except \( i \). \}

\[ (s_i, s_{-i}) \in S \]  
\{ a strategy profile or outcome. \}

Best Response:

\[ B_i(\mathbf{s}_i) \in \text{arg max } u_i(s_i, s_{-i}) \]  
\( s_i \in S_i \)

A strategy of \( i \) that maximizes his utility provided that other players have selected \( s_{-i} \).

We would like for everyone to choose their best responses and not deviate from it:

Choose \( s^* \) pt.

\[ \forall i \in I, B_i(s_i^*) = s_i^* \]
**Partnership Game**

<table>
<thead>
<tr>
<th></th>
<th>Work Hard</th>
<th>Shirk</th>
</tr>
</thead>
<tbody>
<tr>
<td>Friend1</td>
<td>(2, 2)</td>
<td>(1, -1)</td>
</tr>
<tr>
<td>Friend2</td>
<td>(-1, 1)</td>
<td>(1, 1)</td>
</tr>
</tbody>
</table>

- \( B_1(-, WH) = WH \)
- \( B_1(-, S) = S \)
- \( B_2(WH, -) = WH \)
- \( B_2(S, -) = S \)

**More Complex Game**

**Matching Penny**

<table>
<thead>
<tr>
<th></th>
<th>H</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>Matcher</td>
<td>(1, 1)</td>
<td>(-1, 1)</td>
</tr>
<tr>
<td>Mismatcher</td>
<td>(1, -1)</td>
<td>(-1, 1)</td>
</tr>
<tr>
<td>T</td>
<td>(-1, 1)</td>
<td>(1, -1)</td>
</tr>
</tbody>
</table>
Neither of these two games (both 0-sum) has a pure-strategy Nash Eq.
Dominant Strategy.

A strategy $s_i \in S_i$ is dominant for player $i$ if:

$$u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$$

$$\forall s'_i \in S_i \land \forall s_{-i} \in S_{-i}.$$ 

Strictly Dominated Strategy.

A strategy $s_i \in S_i$ is strictly dominated for player $i$ if:

$$u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})$$

$$\exists s'_i \in S_i \land \forall s_{-i} \in S_{-i}.$$ 

Weakly Dominated Strategy.

A strategy $s_i \in S_i$ is weakly dominated for player $i$ if:

$$\exists s'_i \in S_i: \forall s_{-i} \in S_{-i}: u_i(s'_i, s_{-i}) \geq u_i(s_i, s_{-i})$$

$$\land \exists s_i \in S_i: u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})$$
CKR+P (Common knowledge of Rationality & Payoffs)

\[ \Rightarrow \text{Iterated Elimination of Strictly Dominated Strategies} \]

\[ j := 0 \]
for each \( i \in I \)
\[ S_i^0 := S_i^j \]
\[ j := j + 1 \]
look
for each \( i \in I \)
\[ S_i^j := \{ s_i \in S_i^{j-1} \mid \forall s_i' \in S_i^{j-1}, \forall s_{-i} \in S_{-i}^{j-1}, \mu_i(s_i', s_{-i}) > \mu_i(s_i, s_{-i}) \} \]
\[ S_i^\infty := \bigcap_{k=0}^\infty S_i^k \]
\[ S^\infty := \prod_i S_i^\infty \]

\textbf{Dominant Strategy Equilibrium.}

A strategy profile \( S^* \) is the dominant strategy equilibrium if for each player \( i \),
\[ S_i^* = \text{is a dominant strategy.} \]
**PRISONER'S DILEMMA**

<table>
<thead>
<tr>
<th>Prisoner 2</th>
<th>Confess</th>
<th>Don't confess</th>
<th>Suicide</th>
</tr>
</thead>
<tbody>
<tr>
<td>Confess</td>
<td>(-2, -2)</td>
<td>(0, -3)</td>
<td>(-2, -10)</td>
</tr>
<tr>
<td>Don't Confess</td>
<td>(-3, 0)</td>
<td>(-1, -1)</td>
<td>(0, -10)</td>
</tr>
<tr>
<td>Suicide</td>
<td>(-10, -2)</td>
<td>(-10, 0)</td>
<td>(-10, -10)</td>
</tr>
</tbody>
</table>

- Suicide is dominated for both players
  ⇒ Eliminate Suicide.

- Don't Confess is dominated for both players
  ⇒ Eliminate Don't Confess.

$S^\infty = \{ (\text{confess, confess}) \}$

**DOMINANT STRATEGY EQUILIBRIUM**

(Follows from CKR.)
Mixed Strategies (Nash)

\( \Sigma_i : \) Probability Measures over all Pure Strategies \( S_i \) (for player \( i \))

\( \sigma_i \in \Sigma_i = \text{Mixed strategy of player } i \)

\( \sigma_i = (p_{i1}, p_{i2}, \ldots, p_{ik}) \)

corresponding to \( i \)'s strategies \( s_{i1}, s_{i2}, \ldots, s_{ik} \)

\( \Pr [s_{ij} \in S_i \text{ is played}] = p_{ij} \)

\( p_{ij} \geq 0 \quad \sum p_{ij} = 1 \)

\( \Sigma = \prod_{i \in I} \Sigma_i = \text{Mixed strategy profiles} \)

\( \sigma \in \Sigma \)

Players Randomize Independently:

\( u_i(\sigma) = \int_S u_i(s) \, d\sigma_i(s) \)

\[ = \sum p_{ij} u_i (s_{ij}, \sigma_{-i}) \]
Mixed Strategy Nash Eq.

Defn:
A mixed strategy profile $\sigma^*$ is a (mixed strategy) Nash Equilibrium, if

$$\forall i \in I, \forall \sigma_i \in \Sigma_i, u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*)$$

Nash's Theorem:
Every finite game has a mixed strategy Nash Equilibrium.

Matching Pennies:

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It has a unique Mixed Strategy Nash Equilibrium: $((x_2, y_2), (x_2, y_2))$
BoS (Battle of Sexes Game)

<table>
<thead>
<tr>
<th></th>
<th>Female</th>
<th>Male</th>
</tr>
</thead>
<tbody>
<tr>
<td>Opera</td>
<td>(1, 4)</td>
<td>(0, 0)</td>
</tr>
<tr>
<td>Football</td>
<td>(0, 0)</td>
<td>(1, 1)</td>
</tr>
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</table>

Two Pure Nash Equilibria

* One Mixed Nash Equilibrium:

\[ \left( \frac{1}{5}, \frac{1}{5} \right), \left( \frac{1}{5}, \frac{4}{5} \right) \]
KAKUTANI'S FIXED POINT THEOREM

Let $f : A \to A : x \in A \mapsto f(x) \in A$ be a correspondence satisfying the following conditions:

(A) $A = \text{Compact, convex and non-empty subset of a finite dimensional Euclidean space.}$

(B) $\forall x \in A \quad f(x) \neq \emptyset$

(C) $\forall x \in A \quad f(x) = \text{Convex Set}$

(D) $f(x)$ has a closed graph

$sx, y^* \to s(x, y)$ with $y^* \in f(x^*) \Rightarrow y \in f(x)$.

Then $f$ has a fixed point

$\exists x^* \in A \quad x^* \in f(x^*)$ □

Corollary: Nash Mixed Strategy Equilibria:

$f = \text{Best responses}$...