There exist relatively simple problems of the theory of ordinary whole numbers that cannot be decided on the basis of the axioms.
Gödel, 1931.

Holds true for an extensive class of mathematical systems: \{ Numbers, Measure Theory, Sets, Geometry, Probability, Topology \}

First order logic underlies all of these branches of mathematics.

What Gödel showed:
For each such system, there HAD TO BE A SENTENCE that asserted its unprovability in the system...

\[
\begin{align*}
S & \text{ is true} \\
\text{iff } & \\
S & \text{ is not provable in the system.} \\
\Rightarrow & \\
(S, \text{ true} & \land \neg \text{ provable}) \\
\lor & \\
(S, \text{ false} & \land \text{ provable}) \\
\lor & \\
(\neg S & \land \neg S) \\
\lor & \\
(\neg S & \land S) \\
\lor & \\
S & \not= S \\
\text{But } & \\
\vdash & \\
(S, \text{ soundness}) \\
\vdash & \\
\neg S & \land \neg S \\
\Rightarrow & \\
S & \text{ true and not provable.}
\end{align*}
\]
(A) \textit{S\潄 Not Liar's Paradox.}

Consider \textit{S' \潄 This sentence is unprovable}
\[\uparrow (\text{Soundness \& Completeness:)}
\text{All proved statements are true \& vice versa)}\]
\[\Rightarrow "S': \text{This sentence is false}".\]

\textbf{Suppose S' is false} \Rightarrow S' can be proved
\[\Rightarrow S' \text{ is true} \ (\& \ S' \text{ is proved to be true})\]
\[\Rightarrow S' \text{ is false}\]
\[\Rightarrow \# \ (\text{Paradox})\]

(B) \textbf{Proof has to be well-defined!}

\textit{(System vs Meta System)}

Within a given mathematical system, \(\Sigma\),
the notion of a proof within that system, \(\Sigma\),
\textit{must be well-defined.}

(c) To avoid paradox, we use the following:

\textit{S: This sentence is unprovable in system \(\Sigma\)}

\[\text{S is true} \Rightarrow \text{S is not provable in } \Sigma \Rightarrow \Sigma \vdash \neg S \ (\text{No contradiction})\]
\[\text{S is false} \Rightarrow \text{S is provable in } \Sigma \Rightarrow \Sigma \vdash 1 \Rightarrow \Sigma \text{ = inconsistent}\]

\[S = \text{true} \land \Sigma \vdash S \Rightarrow \text{Incompleteness (but not a paradox).}\]
Gödelian System

Language $L$ 

{Captures a wide class of mathematical objects, e.g. Natural Numbers, $\mathbb{N}$.}

$H$ = Predicate in $L$, e.g. $H \subseteq \mathbb{N}$, prime, perfect, etc.

$H(n) =$ sentence, defining a set.

$H(n) \iff n \in H$

(I) There is a well-defined set of sentences called TRUE SENTENCES.

For each sentence, $S$, is associated a sentence $\overline{S}$

$\overline{S} =$ Negation of $S \equiv \neg S$

For each predicate, $P$, is associated a predicate $\overline{P}$

$\overline{P} =$ Negation of $P \equiv \neg P$

$H =$ Predicate, $H(n) =$ sentence

$\overline{H(n)} \equiv \overline{H(n)} \equiv \{n \mid n \in \mathbb{N} \setminus H\}$

(II) To each expression $X$ a natural number $n$ can be assigned. $n =$ Gödel number of $X$

Each distinct expression $X$ has a distinct Gödel number $n$.

Call $n =$ sentence number iff it is the Gödel number of a sentence $S$

Call $m =$ predicate number iff it is the Gödel number of a predicate $P$.

(III) There exists a WELL-DEFINED procedure for proving a sentence; the system is SOUND iff every provable sentence is true.
A Map $N \to N$

$n$ = Predicate Number
$H_n$ = Corresponding Predicate
$H_n(n) =$ Sentence, which is true, iff $n \in H_n$

$S_{n^*}$

$n^* \in H_n$ or $n^* \in N \setminus H_n$

**DIAGONALIZER**

A predicate $K$ diagonalizes the predicate $H$ if $K(n)$ iff $H(n^*)$ is true.

**GÖDELIAN**

A system $\Sigma$ is said to be Gödelian iff it satisfies the following two conditions...

**G1** Every predicate $H$ has a diagonalizer $K$.

**G2** There is a "provability" predicate $P$ in $\Sigma$ such that for any sentence number $n$, the sentence $P(n)$ is true iff $S_n$ is provable.
Start with the provability predicate \( P (\therefore G_2) \)

Find its negation \( \overline{P} \) (obeys \( G_1 \))

\[ P(n) \iff \overline{P}(n) \]

\( \overline{P} \) has a diagonalizer \( \mathcal{K} (\therefore G_1) \)

\[ \mathcal{K}(n) = \text{true} \iff \overline{P}(n^* ) = \text{true} \iff P(n^* ) = \text{false} \iff S_{n^*} \text{ is not provable.} \]

Let \( k = \) the Gödel number of \( \mathcal{K} \), \( \mathcal{K} = H_k \)

\[ H_k(n) = \text{true} \iff S_{n^*} = \text{not provable \hspace{1em} \forall n}. \]

\[ H_k(k) = \mathcal{K}(k) = \overline{P}(k^* ) = \boxed{S_{k^*} = \text{true} \iff S_{k^*} \notin \text{provble}.} \]

\[ \therefore S_{k^*} \text{ is true & not provable} \hspace{1em} (\because \hspace{1em} \mathcal{K} \subseteq \mathcal{K}) \]

**Theorem GT (Gödel-Tarski):**

For every sound Gödelian system, there must be a sentence of the system that is true, but not provable in the system.
**Fixed Point**

A sentence $S$ is called a fixed point of a predicate $H$ iff

$S = \text{true iff } H(n)$, where $n$ is G"odel number of $S$.

$H(n) = \text{true iff } S_n = \text{true.}$

$\neg P(n) = \text{true iff } S_n = \text{true}$ \hspace{1cm} (P = Provability Predicate)

$\downarrow$

$S_n = \text{true iff } S_n \not\equiv \text{provable.}$ \hspace{1cm} (G"odel Sentence)

**Theorem F1**

In any system satisfying G1, each predicate of the system has a fixed point.

**Proof:** $H$ = Predicate $\rightarrow K^H$ = Diagonalizer of $H$. (\(\therefore\) G1)

$K(k) = \text{Sentence } = S_{k^*}$

$H(k^*) = \text{true iff } K(k) = H(k^*) = \text{true}$

$\iff S_{k^*} = \text{true.}$