Analyzing Tatonnement Dynamics in Economic Markets

by

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This dissertation is dedicated to

all eggs that break against the high and cold wall.

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We are visitors as have been visited, we come as we leave. But truth stays in our hearts.

Abstract

The impetus for this dissertation is to explain why well-functioning markets might be able to stay at or near a market equilibrium. We argue that tatonnement, a natural, simple and distributed price update dynamic in economic markets, is a plausible candidate to explain how markets might reach their equilibria.

Tatonnement is broadly defined as follows: if the demand for a good is more than the supply, increase the price of the good, and conversely, decrease the price when the demand is less than the supply. Prior works show that tatonnement converges to market equilibrium in some markets while it fails to converge in other markets. Our goal is to extend the classes of markets in which tatonnement is shown to converge. The prior positive results largely concerned markets with substitute goods. We seek market constraints which enable tatonnement to converge in markets with complementary goods, or with a mixture of substitutes and complementary goods. We also show fast convergence rates for some of these markets.

This dissertation is divided into two parts:

- In Part I, we will focus on properties of the aggregate demand rather than properties of individual buyers' demands. We show that when demand and income elasticities are suitably bounded, tatonnement converges quickly in certain markets with complementary goods. We also introduce a new type of elasticity, *adverse market elasticity*, and show that tatonnement converges quickly in markets with a mixture of substitutes and comple-

ments, when this elasticity is suitably bounded.

To have a realistic market setting for a price adjustment mechanism, out-of-equilibrium trade must be allowed so as to generate demand imbalances that then induce price adjustments. The *ongoing market model* is a fairly new market model which enables out-of-equilibrium trade, and also captures the distributed nature of markets by allowing independent and asynchronous price updates. Our analysis in Part I handles both the classical market setting and the ongoing market setting.

We introduce an amortized analysis technique to handle asynchronous events — in our case asynchronous price updates. We devise a potential function that decreases substantially and continuously when there is no price update, and in addition the potential function does not increase upon a price update. This amortized analysis technique may be of independent interest.

- In Part II, we define a new class of markets called *Convex Potential Function Markets* (CPF markets), in which tatonnement is equivalent to gradient descent. The equivalence opens up the entire toolbox developed to analyse gradient descent and provides a principled approach to show convergence of the tatonnement process.

  We show that Eisenberg-Gale markets, a fairly new class of markets, are contained in CPF markets. This allows us to prove that tatonnement converges in many interesting classes of markets, including Fisher markets with Leontief, complementary-CES or nested-CES utility functions.

For Fisher markets with Leontief or complementary-CES utilities, we bound the convergence rates, either by established tools for analysing gradient descent, or by showing that the potential function demonstrates *strong sandwiching property*, a new property we introduce that enables us to show rapid convergence. This property may be of independent interest.

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Chapter 1

Introduction

The impetus for this dissertation comes from the following question: why might well-functioning markets be able to stay at or near equilibrium prices? This raises two issues: what are plausible price adjustment mechanisms, and in what types of markets are they effective?

The concept of market equilibrium was first proposed by Walras [46] in 1874. Since then, for over a century, general equilibrium theory, the study of markets and their equilibria, has been a core topic in economics.[1] Two central questions in general equilibrium theory are whether market equilibria exist, and if so how to compute them. The issue of existence was settled for a very general setting by Arrow and Debreu [2] in 1954 by means of Katutani's fixed point theorem. Their proof is an existence proof, which does not provide insight on how the market equilibria can be computed/approximated (quickly).

The question of computing market equilibria has already been worked on for

[1] A fairly recent account of this classic perspective in economics is given by McKenzie [34].

more than a century. This question has received much attention in economics, operational research, and most recently in computer science. While researchers have been able to devise (efficient) algorithms that compute market equilibria for subclasses of general markets, this question is not yet completely settled.

We argue here for the relevance of this question from a computer science perspective. Much justification for looking at the market problem in computer science stems from the following argument: if economic models and statements about equilibrium and convergence are to make sense as being realizable in economies, then they should be concepts that are computationally tractable. Our viewpoint is that it is not enough to show that the problems are computationally tractable; it is also necessary to show that they are tractable in a model that might capture how a market works.

In addition to proposing the concept of market equilibrium, Walras [46] also proposed a natural, simple, distributed price update process which he named *tatonnement*. Tatonnement is broadly defined in terms of the following criteria: if the demand for a good is more than the supply, increase the price of the good, and conversely, when the demand is less than the supply, decrease the price. The price adjustment for each good is in the direction of its own excess demand and is independent of the demand for other goods. Tatonnement is a natural candidate that might capture how some markets work. This dissertation investigates when a tatonnement-style price update in a dynamic market setting can lead to (fast) convergence toward a market equilibrium.

As it is not possible to devise a tatonnement-style price dynamic which converges for general markets [40, 39, 36], the goal has been to devise plausible constraints on markets that enable rapid convergence. Prior positive results largely concerned markets in which the goods were substitutes. In this dissertation, we extend the class of markets for which tatonnement is shown to converge. We show that there are some classes of markets, either with complementary goods, or with a mixture of substitute and complementary goods, in which tatonnement converges (quickly).

1.1 Algorithms, Dynamics and Tatonnement

As we have just discussed, our viewpoint is that if market equilibria are to make sense as being realizable, the problem of computing market equilibria (or in short, the market problem) should not only be computationally tractable, but also tractable in a model that captures how markets work.

The issue of computational tractability concerns the possibility of devising an algorithm for the market problem, and furthermore, the possibility of devising an efficient one. In the most general settings, the answer is likely to be negative: Richter and Wong [37] proved that there exist markets in which all market components are Turing computable but the market equilibrium is not; a sequence of works [15, 11, 8, 45, 10] proved that some market problems are PPAD-hard, a complexity class for which is widely believed (albeit not proved) that no efficient algorithm exists for the problems it contains.

Despite the negative results for the general market problem, for many market problems in more restrictive settings, (efficient) algorithms have been devised. The market problem was already being worked on in the 1890s by Fisher, who built a hydraulic apparatus for this task (see [6] for a description). The recent activities in computer science have led to a considerable number of polynomial time algorithms for finding approximate and exact equilibria in a variety of markets with divisible goods; we cite a selection of these works [14, 19, 22, 21, 26, 29, 35, 43, 44, 47, 48, 4].[2]

However, these algorithms perform overt global computation with global information, which is implausible in markets with many interacting traders. Thus, these works do not address our question regarding how a market equilibrium can be reached "within the market". In a macroscopic sense, our question concerns the tractability of market equilibria in a computing environment (the market) with restricted information and computation resources. The tatonnement dynamic was proposed based on an observation that price setters in realistic markets know only limited information: they know the information (demand) about their own goods but not information about other goods, and they cannot know how a price change to their own good affects other goods. Saari and Simon [39] showed that if the market has sufficiently more information available, a market dynamic converges for a general class of markets, while tatonnement had been shown to fail to converge in these markets in general; however, as noted by these authors, the informational requirement is not realistic.

Classically, tatonnement has been thought of as a *continuous* process, with price adjustments and demand responses happening continuously and instanta-

[2]The works [47, 48, 4] analyse a market dynamic called the *proportional response dynamic* (PRD), in which prices are implicit; briefly speaking, goods are allocated among traders in proportion to what these traders spend. To the best of our knowledge, other than tatonnement, PRD is the only simple, distributed and arguably realizable dynamic which is shown to attain positive results for some market problems.

neously. A computer science approach is to consider updates at *discrete* time intervals and to bound the number of updates required (though discrete updates were also considered in the economics literature as early as the 1960s [42]). An early positive result, due to Arrow, Block and Hurwitz [1], showed that a continuous version of tatonnement converges to an equilibrium for markets satisfying the weak gross substitutes (WGS) property, namely that increasing the price of one good does not decrease the demand for any other good. However, the hope that tatonnement would converge for all markets was dashed by Scarf [40], who showed an example of a market where tatonnement does not converge; in fact, it exhibits cyclic behaviour. Thus one can hope to show that tatonnement converges only for specific classes of markets.

There are other positive results on tatonnement (e.g. [13, 17]), but they largely concern markets in which the goods are substitutes. In this dissertation, we extend the class of markets in which tatonnement is shown to converge. We show that there are some classes of markets, either with complementary goods, or with a mixture of substitute and complementary goods, in which the tatonnement dynamic converges (quickly). There are relatively few efficient algorithms devised [16, 12, 9] for markets with non-substitute goods, and to the best of our knowledge, none for the tatonnement dynamic.

We note that tatonnement is often considered to be an algorithmic process and not a market process; for example, [25] states: "such a model of price adjustment ... describes nobody's actual behaviour" (referring to the classic auctioneer explanation of tatonnement). The main point of [25], however, is to give an alternate and more plausible basis for tatonnement. More recently, Cole and

Fleischer [17] gave another plausible basis for tatonnement by introducing the *ongoing market model*, in which tatonnement and other price update processes can naturally be viewed as in-market processes. The continued interest in the plausibility of tatonnement is also reflected in some experiments by Hirota [28] which show the predictive accuracy of tatonnement in a non-equilibrium trade setting. In both the classical market setting and the ongoing market setting, we will address our main question: under what conditions can tatonnement-style price updates lead to convergence?

1.2 Market Models

Arrow-Debreu Markets

An Arrow-Debreu market has n divisible goods and a number of agents. Each agent i comes to market with an initial endowment of w_{ij} amount of good j; the supply of good j is $w_j := \sum_i w_{ij}$. Each agent has a utility function $u_i(x_{i1}, \cdots, x_{in})$ expressing its preferences: if agent i prefers a basket of goods with quantities $\vec{x}^I$ to the basket of goods with quantities $\vec{x}^{II}$, then $u_i(\vec{x}^I) > u_i(\vec{x}^{II})$.

Trades are driven by prices. At prices $p = (p_1, \cdots, p_n)$, each agent i sells all its endowment of goods at the prices p to get a budget of size $\sum_{j=1}^n p_j w_{ij}$. Then the agent purchases a basket of goods that maximizes its utility, subject to not exceeding its budget. If the utility function of an agent is strictly concave, then there is a unique utility-maximizing bundle when the prices are all positive, so we can talk of *the* demand of the agent. x_j, the demand for good j, is the quantity of good j sought by all agents in the market.

Prices $p = (p_1, \cdots, p_n)$ are said to be a *market equilibrium* if the demand for each good is bounded by the supply: $x_j \leq w_j$ and if $p_j > 0$ then $x_j = w_j$. Throughout this work, we use $p^* = (p_1^*, \cdots, p_n^*)$ to denote a market equilibrium. Arrow and Debreu [2] proved that market equilibria exists under quite mild conditions.

Let $z_j := x_j - w_j$ denote the excess demand for good j. $s_j = p_j x_j$ is the total spending on good j by all agents. Note that while w is part of the specification of the market, x and z are functions of the vector of current prices.

The One-Time Fisher Market

This setting is often simply called the Fisher market. The Fisher market is a special case of the Arrow-Debreu market.

A Fisher market comprises a set of n goods and two sets of agents, buyers and sellers. The sellers bring the goods to the market and the buyers bring money with which to buy goods. For simplicity, we assume that there is a distinct seller for each good; further it suffices to have one seller per good. The seller of good j brings a supply w_j of this good to the market; the seller seeks to sell its good for money at the price p_j. Each buyer i brings money of amount e_i to the market and has a utility function u_i.

As in an Arrow-Debreu market, each buyer purchases a basket of goods that maximizes its utility, subject to the budget constraint. Note that in a Fisher market, the budget of buyer i is a fixed constant e_i, while in an Arrow-Debreu market, the budget of agent i is a function of its initial endowment of goods and prices p.

The notions of demand, excess demand, total spending on good j and market equilibrium are the same as for Arrow-Debreu markets.

The Ongoing Fisher Market [17]

In order to have a realistic setting for a price adjustment algorithm, it would appear that out-of-equilibrium trade must be allowed, so as to generate the demand imbalances that then induce price adjustments. But then there needs to be a way to handle excess supply and demand. To this end, we suppose that for each good there is a *warehouse* which can meet excess demand and store excess supply. Each seller has a warehouse of finite capacity to enable it to cope with fluctuations in demand. It changes prices as needed to ensure its warehouse neither overfills nor runs out of goods.

The setting of an ongoing Fisher market is similar to a one-time Fisher market, but it enables out-of-equilibrium trade as follows. The seller of good j has a warehouse of capacity χ_j. The market repeats over an unbounded number of time intervals called *days*. Each day, the seller of good j receives w_j new units of good j, and each buyer i is given e_i amount of money. Each day, each buyer i purchases a basket of goods of cost at most e_i which maximizes its utility function. The resulting excess demand or surplus, $z_j = x_j - w_j$, is taken from or added to the warehouse stock.

Given initial prices p°, warehouse stocks v°, where $0 < v_j^\circ < \chi_j$ for $1 \le j \le n$, and ideal warehouse stocks v^*, where $0 < v_j^* < \chi_j$ for $1 \le j \le n$, the task is to repeatedly adjust prices so as to converge to a market equilibrium with the warehouse stocks converging to their ideal values; for simplicity, we suppose that $v_j^* = \chi_j/2$. Let v_j denote the current content of warehouse j.

We suppose that the sellers adjust the prices of their goods. In order to have progress, we require them to change prices at least once a day. However, we impose no upper bound on the frequency of price changes. This entails measuring demand on a finer scale than day units. Accordingly, we assume that each buyer spends their money at a uniform rate throughout the day. If one supposes there is a limit to the granularity, this imposes a limit on the frequency of price changes.

We recall from [17] that the goal of the ongoing market model is to capture the distributed nature of markets and the possibly limited knowledge of individual price setters. One important aspect is that the price updates for distinct goods are allowed to occur independently and *asynchronously*.

1.3 Roadmap

This dissertation is divided into two parts. While we define the market in terms of a set of buyers, all that matters for the tatonnement dynamic is the aggregate demand these buyers generate. In Part I, we will focus on properties of the aggregate demand rather than properties of individual buyers' demands. We seek conditions on the aggregate demand which ensure that tatonnement converges. Briefly speaking, we impose bounds which limit how the aggregate demand changes when the price changes. These bounds are called *elasticity* parameters of markets, which are common measures used by economists in the study of markets.

In Part II, we define a new class of markets called *Convex Potential Function markets* (CPF markets), in which tatonnement is equivalent to gradient descent on a convex function, with any minimum point of the convex function corresponding to a market equilibrium. The equivalence allows us to show that tatonnement converges to a market equilibrium in some sub-classes of CPF markets, and further allows us to bound the convergence rate.

We define a few basic notions below.

Substitutes and Complements

Two goods are substitutes if increasing the price of one good always leads to an increase in the demand for the other good; for instance, rice and pasta are substitutes. Two goods are complements if increasing the price of one good always leads to a decrease in the demand for the other good; for instance, pasta and pasta sauce are complements. It is possible that two goods are neither substitutes nor complements.

Leontief Utility Functions

When a buyer has a Leontief utility function, the buyer always purchases the goods in a fixed proportion; the goods are said to be *perfect complements* for the buyer. The formula for a Leontief utility function is

$$u(x_1, \cdots, x_n) = \max_{1 \leq j \leq n} \frac{x_j}{b_j},$$

where b_j are positive constants. Suppose the buyer has budget e. Recall that a buyer always purchases a utility-maximizing basket of goods. Given price vector p, it is easy to deduce that the buyer's demand for good j is

$$x_j = \frac{eb_j}{\sum_{j=1}^{n} p_j b_j}.$$

Constant Elasticity of Substitution (CES) Utility Functions

If a buyer has a linear utility function of the form $u(x_1, \cdots, x_n) = \sum_{j=1}^{n} a_j x_j$, the goods are said to be *perfect substitutes* for the buyer. CES utility functions are a class of utility functions which *interpolate* between perfect substitutes and perfect complements. The formula for a CES utility function is

$$u(x_1, \cdots, x_n) = \left(\sum_{j=1}^{n} a_j (x_j)^{\rho} \right)^{1/\rho},$$

where ρ is a constant satisfying $-\infty < \rho \leq 1$. When $\rho \searrow -\infty$, a CES utility function becomes a Leontief utility function in the limit.

When $\rho \to 0$, in the limit the buyer spends a fixed proportion of the budget on each good. This is called a *Cobb-Douglas* utility function. When $\rho > 0$, the goods are substitutes; when $\rho < 0$, the goods are complements.

Nested-CES (NCES) Utility Functions

Nested-CES utility functions, or NCES utility functions, are a generalization of CES utility functions proposed by Keller [32]. An intuitive way to describe a NCES utility function of a buyer to visualize it as a *utility tree.* In the utility tree, each leaf represents a good in the market, while each internal node represents a *utility component.* If an internal node I has q children, its utility component is of the form

$$u_I(x) = \left(\sum_{k=1}^{q} a_k (u_k(x))^{\rho(I)} \right)^{1/\rho(I)},$$

where $u_k(x)$ is the utility component of the k-th child of I. If a node is a leaf, we define its utility component to be the quantity of the good it represents. The overall utility function is the utility component at the root.

For example, the following specifies a NCES utility function $u(x)$, which can be visualized as the utility tree in Figure 1.1:

$$\begin{aligned}
u_A(x) &= \left[(x_1)^{1/2} + 9(x_2)^{1/2}\right]^2 \\
u_B(x) &= \left[8(x_3)^{-4} + 9(x_4)^{-4}\right]^{-1/4} \\
u_C(x) &= \left[6(x_5)^{2/7} + 4(x_6)^{2/7} + 25(x_7)^{2/7}\right]^{7/2} \\
u_D(x) &= \left[6(u_B(x))^{-1/2} + 4(u_C(x))^{-1/2} + 25(x_8)^{-1/2}\right]^{-2} \\
u(x) &= \left[(u_A(x))^{-1} + 2(u_D(x))^{-1} + 3(x_9)^{-1}\right]^{-1}
\end{aligned}$$

In 1-level CES utility functions, the ρ-parameter determines whether the goods form substitutes or complements. We generalize these notions for NCES utility function.

Definition 1.1. *A node in a NCES utility tree is called a* substitute node *if its parameter $\rho \geq 0$; if $\rho \leq 0$, the node is called a* complement node*.*

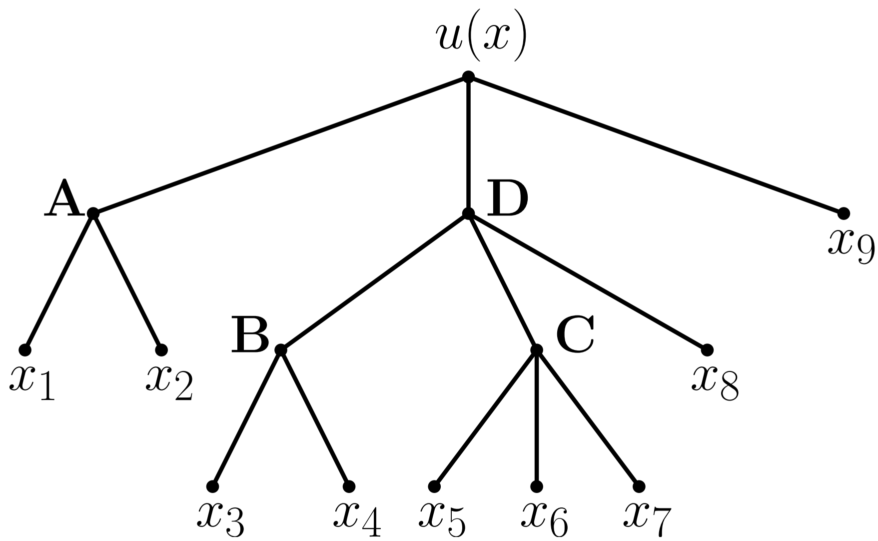

Figure 1.1: A NCES utility tree.

Part I

Markets with Suitably Bounded Elasticity Parameters

Chapter 2

Preliminaries

In Part I, we seek conditions on the aggregate demand of a market which ensure that tatonnement dynamic converges to a market equilibrium. Cole and Fleischer [17] gave such constraints for markets with goods that are substitutes. These constraints take the form of bounds on the elasticities of demand and income. Curiously, the best performance occurs at the boundary between substitutes and complements (with the buyers having so-called Cobb-Douglas utilities). Despite this, their result did not extend into the complements domain. We carry out such an extension, handling markets with goods that are complements, or markets with a mixture of substitutes and complements. The one-time and ongoing markets for which we show convergence include the following scenarios.

1. All the goods are complements with suitably bounded demand and income elasticities. A particular instance of this setting occurs when each buyer has a CES utility function with the parameter ρ satisfying $-1 < \rho \leq 0$.

2. A market with a mixture of substitutes and complements with suitably bounded *adverse market elasticity*, a new type of elasticity we introduce. There are two particular instances of this setting of interest. The first instance is when the goods are partitioned into groups, while goods in the same group are substitutes but goods from different groups are complements. This occurs when each buyer has a suitable 2-level NCES utility function. The second instance is when each buyer has a suitable arbitrary depth NCES utility function.

Tatonnement Price Update Rule

The basic tatonnement price update rule for the one-time market, proposed in [17], is given by

$$p_j \leftarrow p_j \left(1 + \lambda \cdot \min\left\{1, \frac{z_j}{w_j}\right\}\right), \tag{2.1}$$

where $0 < \lambda < 1$ is a suitable parameter whose value depends on the market elasticities.

In an ongoing market, we modify (2.1) by replacing z_j with the average excess demand for good j since the last update to good j. We allow prices to be updated independently and asynchronously, i.e. different prices may be updated at different times, and we allow multiple updates within a day. However, this would be difficult to analyse if some prices update too frequently (and hence change too aggressively). Hence, we make the update proportional to the length of time since the last price update. Plus, we need to take account of the warehouse excess with prices dropping if there is too much stock in the warehouse, and increasing if too little.

Precisely, let τ_j be the time of the last price update to good j and let t be the current time. Recall that we assume each seller adjusts the price of its good at least once each day, so $t - \tau_j \leq 1$. We define the *target demand* $\widetilde{w}_j$ to be $\widetilde{w}_j := w_j + \kappa(v_j - v_j^*)$, where $\kappa > 0$ is chosen to ensure that $|\kappa(v_j - v_j^*)| \leq \delta w_j$, for a suitable positive number δ. We define the *target excess demand* to be $\bar{z}_j := \frac{1}{t-\tau_j} \int_{\tau_j}^{t} z_j(t)\, dt - \widetilde{w}_j$. The tatonnement price update rule for the ongoing market is given by

$$p_j \leftarrow p_j \left(1 + \lambda \cdot \min\left\{1, \frac{\bar{z}_j}{w_j}\right\} \cdot (t - \tau_j)\right). \tag{2.2}$$

Warehouse Bound

It is impractical to assume that the warehouses have infinite capacity, so we need to show that finite warehouse capacities χ_j suffice. Also, we need to address the question of whether the warehouses could run out of stock.

It turns out that if the warehouses start with stocks that are close to their ideal values and if χ_j/w_j is finite but sufficiently large, the warehouses will neither overflow nor run out of stock in the tatonnement dynamic. For simplicity, we assume χ_j/w_j are equal for all j; we denote this ratio by r.

Amortized Analysis on Asynchrony

As we have discussed in Chapter 1.2, one important aspect of our analysis is that we allow asynchronous price updates in ongoing markets. We introduce an amortized analysis technique to handle asynchronous events, which may be of independent interest. We use a potential function ϕ which satisfies two properties:

- $\frac{d\phi}{dt} \leq -\Theta(\kappa) \cdot \phi$ for a suitable parameter $\kappa > 0$ whenever there is no event.
- ϕ is non-increasing when an event occurs (a price update in our application).

One can then conclude that $\phi(t) \leq \exp\left(-\Theta(\kappa)t\right) \cdot \phi(0)$, and so ϕ decreases by at least a $1 - \Theta(\kappa)$ factor daily (for $\kappa = O(1)$).

Comparison to Prior Work

Cole and Fleischer [17] introduced the notion of ongoing markets and analysed a class of ongoing Fisher markets with goods that are substitutes. The current work extends this analysis to classes of ongoing Fisher markets with respectively, only complementary goods, and with a mixture of complements and substitutes. The present work also handles the warehouses in the ongoing model more realistically.

This entails a considerably changed analysis and some modest changes to the price update rule. As in [17], the analysis proceeds in two phases. The new analysis for Phase 1, broadly speaking, is similar to that in [17], though a new understanding was needed to extend it to the new markets.

The techniques used in the analysis for Phase 2 and bounding warehouse sizes were first given in a preliminary manuscript [18], where markets with substitute goods were analysed. This manuscript included other results too (on managing with approximate values of the demands, and on extending the results to markets of indivisible goods). The current work subsumes the analysis techniques in [18].

In Chapter 3, we handle markets with complementary goods. In Chapter 4, we handle markets with a mixture of substitutes and complements.

Chapter 3

Markets with Complementary Goods

We first review the definitions of income and demand elasticities.

Definition 3.1. *The* income elasticity *of good j by a buyer with income (money) e is given by*

$$\frac{dx_j}{de} \Big/ \frac{x_j}{e}$$

We let γ denote the least upper bound on the income elasticities over all goods.

If all buyers are always spending their budgets in full, then $\gamma \geq 1$.

Definition 3.2. *The* demand elasticity *of good j is given by*

$$-\frac{dx_j}{dp_j} \Big/ \frac{x_j}{p_j}$$

We let α denote the largest lower bound on the demand elasticities over all *goods.*

In a market of complementary goods, $0 \le \alpha \le 1$; in the markets we consider, $\alpha > \gamma/2 \ge 1/2$. It is convenient to set $\beta := 2\alpha - \gamma$; note that by assumption, $0 < \beta \le 1$.

Our results will depend on the initial imbalances in the prices. To specify this, we define the notion of f-bounded prices.

Definition 3.3. *For a fixed market equilibrium p^* with positive prices, let*

$$f_j(p) := \max\left\{\frac{p_j}{p_j^*}, \frac{p_j^*}{p_j}\right\},$$

and

$$f(p) := \max_{1 \le j \le n} f_j(p).$$

The prices are f-bounded *if $f(p) \le f$.*

Clearly, $f(p) \ge 1$ and $f(p) = 1$ if and only if $p = p^*$. When there is no ambiguity, we use f as a shorthand for $f(p)$. We let f_{I} denote the maximum value f takes on during the first day, which will also bound f thereafter, as we will see.

Our results will require λ and κ, the parameters in the tatonnement rules (2.1) and (2.2), to obey the following conditions.

$$\frac{24}{r} \le \lambda \le \min\left\{\frac{3}{7}, \frac{29}{63}\ln\frac{1}{2(1-\alpha)}, \sqrt{\frac{\kappa r}{16}}\right\}. \tag{3.1}$$

$$\kappa \le \frac{2}{r} \cdot \min\left\{\frac{1}{8}, \frac{\beta}{4\gamma+\beta}, \frac{\beta}{16\beta+8\gamma}\ln\frac{1}{2(1-\alpha)}\right\} \tag{3.2}$$

3.1 Results

We state our main results in this chapter here, and defer their proofs to Chapter 3.2.

The following theorem bounds the convergence rate.

Theorem 3.4. *Suppose that $\beta > 0$, the prices are f-bounded throughout the first day, and in addition that (3.1) and (3.2) hold. Let $M = \sum_i e_i$ be the daily supply of money to all the buyers. Then for any $\eta > 0$, the prices become $(1+\eta)$-bounded after $O\left(\frac{1}{\lambda}\ln f + \frac{1}{\lambda\beta}\ln\frac{1}{\delta} + \frac{1}{\kappa}\log\frac{M}{\eta\beta\min_k p_k^* w_k}\right)$ days.*

We also bound the needed warehouse sizes. We say that warehouse j is *safe* if $v_j \in [\frac{1}{4}\chi_j, \frac{3}{4}\chi_j]$. We define $d(f) = \max_j x_j/w_j$ when the prices are f-bounded. In a market with complementary goods, $d(f) \le f^\gamma$.

The analysis of Theorem 3.4 proceeds in two phases. We need to specify some parameters relative to Phase 1. We define $D(f)$ to be the duration of Phase 1 in days. As we will see, $D(f) = O\left(\frac{1}{\lambda}\ln f + \frac{1}{\lambda\beta}\ln\frac{1}{\delta}\right)$. We define $v_j(f_{\mathrm{I}})$ to be the total net change to v_j during Phase 1. As we will see, $v_j(f_{\mathrm{I}}) = O(\frac{1}{\lambda}d(f_{\mathrm{I}}) + \frac{1}{\lambda\beta}d(2^{1/\beta})\ln\frac{1}{\delta})w_j$. Now we are ready to state our bounds on the warehouse sizes.

Theorem 3.5. *Suppose that in a market with complementary goods, the followings hold:*

1. $r = \frac{\chi_j}{w_j} \ge \frac{512}{\beta}$ *and* $r \ge \frac{8v_j(f_I)}{w_j}$ *for all* j.

2. $\delta = \frac{\kappa r}{2}$.

3. $\lambda^2 \le \frac{\kappa r}{16}$.

Further, suppose that the prices are always f-bounded. Also suppose that each price is updated at least once a day and initially the warehouses are all safe. Finally, suppose that (3.1) *and* (3.2) *hold. Then the warehouses never overflow nor run out of stock. Furthermore, after $D(f) + \frac{128}{\beta} + \frac{8}{\kappa} + 1$ days the warehouses will be safe thereafter.*

Example Scenario: All buyers have CES Utility Functions with $-1 < \rho \le 0$

In a Fisher market with buyers having CES utility functions, when the ρ parameter is negative for all buyers, the goods are complements. We will focus on the case $-1 < \rho \le 0$. It is well known that if a buyer has budget e and CES utility function $u(x_1, \cdots, x_n) = \left(\sum_{j=1}^n a_j x_j^\rho\right)^{1/\rho}$, the buyer's demands are given by $x_j = e \cdot (a_j)^{E(\rho)} (p_j)^{-E(\rho)} \left(\sum_{k=1}^n (a_k)^{E(\rho)} (p_k)^{1-E(\rho)}\right)^{-1}$, where $E(\rho) = 1/(1-\rho)$. A further calculation yields that $\frac{\partial x_j / \partial p_j}{x_j / p_j} \le -E(\rho)$. Let $\rho_{\min} = \min_i \rho_i$, where i runs over all buyers. Then α, the demand elasticity of the market, is $E(\rho_{\min})$. In addition, it is easy to show that for CES utility functions, the income elasticity $\gamma = 1$. Consequently, when $\rho > -1$, $\beta = 2E(\rho_{\min}) - 1 > 0$. Thus Theorems 3.4 and 3.5 apply.

3.2 Convergence Analysis

The largest challenge in the analysis is to handle the effect of warehouses. In [17], the price updates increased in frequency as the warehouse limits (completely full or empty) were approached, which ensured these limits were not breached. It was still a non-trivial matter to demonstrate convergence. In the present paper, the only constraint is that each price is updated at least once every full day. This seems more natural, but entails a different and new analysis.

The analysis partitions into two phases, the first one handling the situation when at least some of the prices are far from equilibrium, and for these prices, the warehouse excesses have a modest impact on the updates.

In the second phase, the warehouse excesses can have a significant effect. For this phase, we use amortized analysis. The imbalance being measured and reduced during Phase 2 is the *misspending* (roughly speaking, $\sum_j [p_j|z_j| + p_j|\widetilde{w}_j - w_j|]$). It is only when prices are reasonably close to their equilibrium values that we can show the misspending decreases, which is why two phases are needed. Interestingly, in a market with substitute goods, regardless of the prices, the misspending is always decreasing, so for these markets one could carry out the whole analysis within Phase 2.

3.2.1 Phase 1

For simplicity, we begin by considering the one-time market. In Phase 1, we show that each day $(f-1)$ shrinks by a factor of at least $1 - \Theta(\lambda\beta)$.

Suppose that currently the prices are exactly f-bounded, and that there is a good, WLOG good 1, with price $p_1 = p_1^*/f$. We will show that $x_1 \geq f^\beta w_1$ regardless of the prices of the other goods, so long as they are f-bounded. This ensures that the price update for p_1 will be an increase, by a factor of at least $1 + \lambda \min\{1, (f^\beta - 1)\}) \doteq 1 + \mu$.

To demonstrate the lower bound on x_1, we identify the following scenario as the one minimizing x_1: every other good j has price fp_j^*.

A symmetric observation applies when $p_1 = fp_1^*$, and then the price decreases by a factor of at most $1 - \lambda(1 - f^{-\beta}) \doteq 1 - \nu$.

We can show that the same market properties imply that after a day of price updates every price will lie within the bounds $[p^*(1+\mu)/f, fp^*(1-\nu)]$, thereby ensuring a daily reduction of the term $(f-1)$ by a factor of at least $1 - \Theta(\lambda\beta)$.

We use a similar argument for the ongoing market. First, we observe that if the price updates occurred simultaneously exactly once a day, then exactly the same bounds would apply to $\bar{x}_j$, the average demand for good j since the last price update to good j. So the rate of progress would be the same, aside from the contribution of the warehouse excess to the price update. So long as this contribution is small compared to $(f^\beta - 1)w_1$ or to $(1 - f^{-\beta})w_1$, say at most half this value, then the price changes would still be by a factor of at least $1 + \frac{\lambda}{2}\min\{1, (f^\beta - 1)\})$ and $1 - \frac{\lambda}{2}(1 - f^{-\beta})$, respectively.

To take account of the possible variability in price update frequency, we demonstrate progress as follows: we can show that if the prices have been f-bounded for a full day, then after two more days have elapsed, the prices will have been f'-bounded for a full day, for $(f' - 1) = (1 - \Theta(\lambda\beta))(f - 1)$. The reason we look at the f-boundedness over the span of a day is that the price updates are based on the average excess demand over a period of up to one day. A second issue we need to handle is that the price updates may have a variable frequency; the only guarantee is that each price is updated within one full day of its previous update. The net effect is that it takes one day to guarantee that

f shrinks and hence two days for the shrinkage to have lasted at least one full day.

It follows that Phase 1 lasts $O(\frac{1}{\lambda\beta}\ln[(f_{\mathrm{I}}-1)/(f_{\mathrm{II}}-1)])$ days, where f_{I} is the initial value of f, and f_{II} is its value at the start of Phase 2. We will choose the value of f_{II} later.

As already argued, the behaviour of the ongoing and one-time markets are within a constant factor of each other in Phase 1 (the ongoing market progresses in cycles of two days rather than one day, and reduces $(f-1)$ by a factor in which λ is replaced by $\lambda/2$). So in this phase we analyse just the one-time market.

We state several inequalities.

Lemma 3.6. *(a) If $0 \le \lambda \le 1$, then $\frac{1}{1+\lambda} \le 1 - \frac{\lambda}{2}$.*

(b) If $0 \le \lambda \le 1$ and $1 \le x \le 2$, then $1 - \lambda(1 - 1/x) \le x^{-\lambda/2}$.

(c) If $0 \le \lambda \le 1$ and $1 \le x \le 2$, then $\frac{1}{1+\lambda(x-1)} \le x^{-\lambda}$.

Proof: In Appendix A. □

Using the definitions of γ and α in Definitions 3.1 and 3.2, one can easily prove the following lemma.

Lemma 3.7. *(a) If the prices of all goods are raised from p_j to $p'_j = qp_j$, where $q > 1$, then $x'_j \ge x_j/q^{\gamma}$.*

(b) If the prices of all goods are reduced from p_j to $p'_j = qp_j$, where $q < 1$, then $x'_j \le x_j/q^{\gamma}$.

(c) If p_j is raised to $p'_j = qp_j$, where $q > 1$, and all other prices are fixed, then $x'_j \leq x_j/q^\alpha$.

(d) If p_j is reduced to $p'_j = qp_j$, where $q < 1$, and all other prices are fixed, then $x'_j \geq x_j/q^\alpha$.

Lemma 3.8. *When the market is f-bounded,*

1. *if $p_j = qp^*_j/f$ where $1 \leq q \leq f^2$, then $x_j \geq w_j f^\beta q^{-\alpha}$;*

2. *if $p_j = fp^*_j/q$ where $1 \leq q \leq f^2$, then $x_j \leq w_j f^{-\beta} q^\alpha$.*

Proof: We prove the first part; the second part is symmetric. By the definition of complements, x_j is smallest when $p_j = fp^*_j$ for all $j \neq i$. Consider the situation in which $p_k = fp^*_k$ for all goods k. By Lemma 3.7(a), $x_j \geq \frac{w_j}{f^\gamma}$. Now reduce $p_j = fp^*_j$ to $p_j = qp^*_j/f$. By Lemma 3.7(d), $x_j \geq \frac{w_j}{f^\gamma (r/f^2)^\alpha} = w_j f^{2\alpha-\gamma} q^{-\alpha} = w_j f^\beta q^{-\alpha}$. □

Lemma 3.9. *Suppose that $\beta = 2\alpha - \gamma > 0$. Further, suppose that the prices are updated independently using price update rule (2.1), where $0 < \lambda < 1$. Let p denote the current price vector and let p' denote the price vector after one day.*

(i.) If $f(p)^\beta \geq 2$, then $f(p') \leq \left(1 - \frac{\lambda}{2}\right) f(p)$.

(ii.) If $f(p)^\beta \leq 2$, then $f(p') \leq f(p)^{1-\lambda\beta/2}$.

Proof: Suppose that $p_j = q\frac{p^*_j}{f(p)}$, where $1 \leq q \leq f(p)^2$. By Lemma 3.8, $x_j \geq w_j f(p)^\beta q^{-\alpha}$ and hence $\frac{z_j}{w_j} \geq f(p)^\beta q^{-\alpha} - 1$. When p_j is updated with rule (2.1), the new price p'_j satisfies

$$p'_j \geq \frac{p^*_j}{f(p)} \cdot q\left[1 + \lambda \cdot \min\left\{1, f(p)^\beta q^{-\alpha} - 1\right\}\right].$$

Let $h_1(q) := q\left[1+\lambda\cdot\min\left\{1,(f(p)^{\beta}q^{-\alpha}-1)\right\}\right]$. If $f(p)^{\beta}q^{-\alpha} > 2$, then $h_1(q) = q(1+\lambda)$, so $\frac{d}{dq}h_1(q) = 1+\lambda > 0$. If $f(p)^{\beta}q^{-\alpha} < 2$, then $h_1(q) = q\left[1+\lambda(f(p)^{\beta}q^{-\alpha}-1)\right]$, so

$$\frac{d}{dq}h_1(q) = 1-\lambda+(1-\alpha)\lambda f(p)^{\beta}q^{-\alpha} \geq 1-\lambda \geq 0.$$

Thus $q = 1$ minimizes $h_1(q)$, and hence

$$p'_j \geq \frac{p^*_j}{f(p)}h_1(1) = \frac{p^*_j}{f(p)}\left[1+\lambda\cdot\min\left\{1, f(p)^{\beta}-1\right\}\right]. \tag{3.3}$$

Similarly, suppose that $p_j = \frac{1}{q}f(p)p^*_j$, where $1 \leq q \leq f(p)^2$. By Lemma 3.8, $x_j \leq w_j f(p)^{-\beta}q^{\alpha}$ and hence $\frac{z_j}{w_j} \leq f(p)^{-\beta}q^{\alpha}-1$. When p_j is updated with rule (2.1), the new price p'_j satisfies

$$p'_j \leq \frac{1}{q}f(p)p^*_j\left[1+\lambda\cdot\min\left\{1, f(p)^{-\beta}q^{\alpha}-1\right\}\right].$$

Let $h_2(q) := \frac{1}{q}\left[1+\lambda\cdot\min\left\{1, f(p)^{-\beta}q^{\alpha}-1\right\}\right]$. If $f(p)^{-\beta}q^{\alpha} > 2$, then $h_2(q) = \frac{1+\lambda}{q}$, so $\frac{d}{dq}h_1(q) \leq 0$. If $f(p)^{-\beta}q^{\alpha} > 2$, then $h_2(q) = \frac{1}{q}\left[1+\lambda(f(p)^{-\beta}q^{\alpha}-1)\right]$, so

$$\frac{d}{dq}h_2(q) = \frac{1}{q^2}\left(\lambda-1-(1-\alpha)\lambda f(p)^{-\beta}q^{\alpha}\right) \leq \frac{\lambda-1}{q^2} \leq 0.$$

Thus $q = 1$ maximizes $h_2(q)$, and hence

$$p'_j \leq f(p)p^*_j h_2(1) = f(p)p^*_j\left[1+\lambda(f(p)^{-\beta}-1)\right]. \tag{3.4}$$

By (3.3) and (3.4), after one day, a period in which each good updates its price at least once, we can guarantee that $f(p')$ is at most

$$f(p) \cdot \max\left\{1 - \lambda(1 - f(p)^{-\beta}), \frac{1}{1 + \lambda \cdot \min\{1, f(p)^{\beta} - 1\}}\right\}.$$

If $f(p)^{\beta} \geq 2$, by Lemma 3.6(a), $f(p')$ is at most $\left(1 - \frac{\lambda}{2}\right) f(p)$. If $f(p)^{\beta} \leq 2$, by Lemma 3.6(b), $1 - \lambda(1 - f(p)^{-\beta}) \leq f(p)^{-\lambda\beta/2}$, and by Lemma 3.6(c), $\frac{1}{1+\lambda(f(p)^{\beta}-1)} \leq f(p)^{-\lambda\beta}$. Then $f(p') \leq f(p)^{1-\lambda\beta/2}$. $\square$

As will be clear in Chapter 3.2.2, we set $f_{\text{II}} = \min\{(1-2\delta)^{-1/\beta}, (2-\delta)^{1/\gamma}\}$. As it turns out, the calculations simplify if we choose a sufficiently small δ to enforce that $(1-2\delta)^{-1/\beta} \leq (2-\delta)^{1/\gamma}$, i.e. $f_{\text{II}} = (1-2\delta)^{-1/\beta}$. Note that

$$\text{if } \frac{\delta}{\beta} \leq \frac{1}{4}, \text{ then } 1 + \frac{2\delta}{\beta} \leq f_{\text{II}} \leq 1 + \frac{4\delta}{\beta} \leq 2. \tag{3.5}$$

Theorem 3.10. *Suppose that $\beta > 0$, $\lambda < 1$, and that the prices are initially f-bounded. If $\frac{\delta}{\beta} \leq \frac{1}{4}$, Phase 1 will complete within $O\left(\frac{1}{\lambda}\ln f + \frac{1}{\lambda\beta}\ln\frac{1}{\delta}\right)$ days.*

Proof: Suppose that initially $f > 2^{1/\beta}$.[1] By Lemma 3.9(i), after n_1 days, where n_1 satisfies the inequality $f\left(1 - \frac{\lambda}{2}\right)^{n_1} \leq 2^{1/\beta}$, the market is $2^{1/\beta}$-bounded. It suffices that:

$$n_1 \geq \frac{\ln f - \frac{1}{\beta}\ln 2}{-\ln\left(1 - \frac{\lambda}{2}\right)} = O\left(\frac{1}{\lambda}\ln f\right).$$

Since $2^{1/\beta} > 1 + \frac{2\delta}{\beta}$, the market takes additional time to become f_{II}-bounded. By Lemma 3.9(ii), after an additional n_2 days, where n_2 satisfies the inequality

[1] The case when the initial f-value is less than $2^{1/\beta}$ is the better case, so we may ignore it.

$(2^{1/\beta})^{(1-\lambda\beta/2)^{n_2}} \leq 1 + 2\delta/\beta$, the market is $(1 + 2\delta/\beta)$-bounded. It suffices that:

$$n_2 = \frac{\ln\beta + \ln\ln(1 + 2\delta/\beta) - \ln\ln 2}{\ln(1 - \lambda\beta/2)} = O\left(\frac{1}{\lambda\beta}\left(\ln\frac{1}{\beta} + \ln\frac{1}{\delta}\right)\right) = O\left(\frac{1}{\lambda\beta}\ln\frac{1}{\delta}\right).$$

The last two equalities hold as $\frac{\delta}{\beta} \leq \frac{1}{4}$, hence $\ln\ln(1 + 2\delta/\beta) = \ln(\delta/\beta) + O(1)$ and $\ln\frac{1}{\beta} \leq \ln\frac{1}{\delta}$.

The sum $n_1 + n_2$ bounds the number of days Phase 1 lasts. □

Comment. If we wish to analyze the one-time market or the ongoing market without taking account of the warehouses, then arbitrarily accurate prices can be achieved in Phase 1, and the time till prices are $(1+\eta)$-bounded, for any η is given by the bound in Theorem 3.10, on replacing the term $\frac{1}{\beta}\ln\frac{1}{\delta}$ by $\frac{1}{\beta}\ln\frac{1}{\eta}$.

To apply this analysis of Phase 1 to other markets, it suffices to identify conditions that ensure $x_1 \geq f^{\beta}w_1$ when $p_1 = p_1^*/f$, and $x_1 \leq f^{-\beta}$ when $p_1 = fp_1^*$.

3.2.2 Phase 2

Once the warehouse excesses may have a large impact on the price updates, we can no longer demonstrate a smooth shrinkage of the term $(f-1)$. Instead, we use an amortized analysis. We associate the following potential ϕ_j with good j.

$$\phi_j := p_j\left[\text{span}\{\bar{x}_j, x_j, \widetilde{w}_j\} - c_1\lambda(t-\tau_j)|\bar{x}_j - \widetilde{w}_j| + c_2|\widetilde{w}_j - w_j|\right],$$

where $\text{span}\{\theta_1,\theta_2,\theta_3\} = \max\{\theta_1,\theta_2,\theta_3\} - \min\{\theta_1,\theta_2,\theta_3\}$ and $1 \geq c_1 > 0$, $c_2 > 1$ are suitably chosen constants. We define $\phi := \sum_j \phi_j$.

The term $-c_1\lambda(t-\tau_j)|\bar{x}_j - \widetilde{w}_j|$ ensures that ϕ decreases smoothly when no price update is occurring, as shown in the following lemma.

Lemma 3.11. *Suppose that $4\kappa(1+c_2) \le \lambda c_1 \le 1/2$. At any time when no price update is occurring,*

- *if $|\widetilde{w}_j - w_j| \le 2 \cdot span(x_j, \bar{x}_j, \widetilde{w}_j)$, then $\frac{d\,\phi_j}{d\,t} \le -\frac{\kappa(1+c_2)}{1+2c_2}\phi_j$;*
- *if $|\widetilde{w}_j - w_j| > 2 \cdot span(x_j, \bar{x}_j, \widetilde{w}_j)$, then $\frac{d\,\phi_j}{d\,t} \le -\frac{\kappa(c_2-1)}{2c_2}\phi_j$.*

Proof: Let K denote $\kappa(x_j - w_j)$ and let S denote $\text{span}(x_j, \bar{x}_j, \widetilde{w}_j)$.

Note the following equalities: $\frac{d\,x_j}{d\,t} = \frac{d\,w_j}{d\,t} = 0$, $\frac{d\,\widetilde{w}_j}{d\,t} = -K$ and $\frac{d\,\bar{x}_j}{d\,t} = \frac{x_j - \bar{x}_j}{t-\tau_j}$. Then $\frac{d\,c_2|\widetilde{w}_j - w_j|}{d\,t} = -c_2K \cdot \text{sign}(\widetilde{w}_j - w_j)$. Then

$$\begin{aligned}\frac{d\,\phi_j}{d\,t} &= p_j\left[\frac{d\,S}{d\,t} - c_1\lambda|\bar{x}_j - \widetilde{w}_j| - c_1\lambda(t-\tau_j)\frac{d\,|\bar{x}_j - \widetilde{w}_j|}{d\,t} - c_2K\cdot\text{sign}(\widetilde{w}_j - w_j)\right] \\ &= p_j\left[\frac{d\,S}{d\,t} - c_1\lambda(x_j - \widetilde{w}_j)\cdot\text{sign}(\bar{x}_j - \widetilde{w}_j) - c_1\lambda(t-\tau_j)K\cdot\text{sign}(\bar{x}_j - \widetilde{w}_j)\right. \\ &\qquad \left. - c_2K\cdot\text{sign}(\widetilde{w}_j - w_j)\right].\end{aligned}$$

Next by means of a case analysis, we show that

$$\frac{d\,\phi_j}{d\,t} \le p_j\left[|K| - c_1\lambda S - c_2K\cdot\text{sign}(\widetilde{w}_j - w_j)\right]. \tag{3.6}$$

Case 1(a): $x_j \ge \bar{x}_j \ge \widetilde{w}_j$ or $\widetilde{w}_j \ge \bar{x}_j \ge x_j$. Hence $\frac{dS}{dt} = K\cdot\text{sign}(x_j - \widetilde{w}_j)$.

$$\begin{aligned}\frac{d\,\phi_j}{d\,t} &= p_j\left[(K - c_1\lambda(x_j - \widetilde{w}_j) - c_1\lambda(t-\tau_j)K)\text{sign}(x_j - \widetilde{w}_j) - c_2K\cdot\text{sign}(\widetilde{w}_j - w_j)\right] \\ &= p_j\left[K(1 - c_1\lambda(t-\tau_j))\text{sign}(x_j - \widetilde{w}_j) - c_1\lambda|x_j - \widetilde{w}_j| - c_2K\cdot\text{sign}(\widetilde{w}_j - w_j)\right] \\ &\le p_j\left[|K| - c_1\lambda S - c_2K\cdot\text{sign}(\widetilde{w}_j - w_j)\right].\end{aligned}$$

Case 1(b): $x_j \geq \widetilde{w}_j \geq \bar{x}_j$ or $\bar{x}_j \geq \widetilde{w}_j \geq x_j$. Hence $\frac{dS}{dt} = \frac{\bar{x}_j - x_j}{t-\tau_j} \cdot \mathrm{sign}(x_j - \bar{x}_j)$.

$$\begin{aligned}
\frac{d\phi_j}{dt} &= p_j \left[\left(\frac{\bar{x}_j - x_j}{t-\tau_j} + c_1\lambda(x_j - \widetilde{w}_j) + c_1\lambda(t-\tau_j)K \right) \mathrm{sign}(x_j - \bar{x}_j) \right. \\
&\qquad \left. - c_2K \cdot \mathrm{sign}(\widetilde{w}_j - w_j) \right] \\
&\leq p_j \left[-|\bar{x}_j - x_j| + c_1\lambda|x_j - \widetilde{w}_j| + |K| - c_2K \cdot \mathrm{sign}(\widetilde{w}_j - w_j) \right] \\
&\leq p_j \left[|K| + (c_1\lambda - 1)|\bar{x}_j - x_j| - c_2K \cdot \mathrm{sign}(\widetilde{w}_j - w_j) \right] \\
&\leq p_j \left[|K| - c_1\lambda S - c_2K \cdot \mathrm{sign}(\widetilde{w}_j - w_j) \right]. \qquad (\lambda c_1 \leq \tfrac{1}{2})
\end{aligned}$$

Case 1(c): $\widetilde{w}_j \geq x_j \geq \bar{x}_j$ or $\bar{x}_j \geq x_j \geq \widetilde{w}_j$. Hence $\frac{dS}{dt} = \left(\frac{x_j - \bar{x}_j}{t-\tau_j} + K \right) \cdot \mathrm{sign}(\bar{x}_j - \widetilde{w}_j)$.

$$\begin{aligned}
\frac{d\phi_j}{dt} &= p_j \left[\left(\frac{\bar{x}_j - x_j}{t-\tau_j} - K + c_1\lambda(x_j - \widetilde{w}_j) + c_1\lambda(t-\tau_j)K \right) \mathrm{sign}(\widetilde{w}_j - \bar{x}_j) \right. \\
&\qquad \left. - c_2K \cdot \mathrm{sign}(\widetilde{w}_j - w_j) \right] \\
&\leq p_j \left[-|\bar{x}_j - x_j| - c_1\lambda|x_j - \widetilde{w}_j| - K(1 - c_1\lambda(t-\tau_j))\mathrm{sign}(\widetilde{w}_j - \bar{x}_j) \right. \\
&\qquad \left. - c_2K \cdot \mathrm{sign}(\widetilde{w}_j - w_j) \right] \\
&\leq p_j \left[-c_1\lambda|\bar{x}_j - x_j| - c_1\lambda|x_j - \widetilde{w}_j| + |K| - c_2K \cdot \mathrm{sign}(\widetilde{w}_j - w_j) \right] \\
&= p_j \left[|K| - c_1\lambda S - c_2K \cdot \mathrm{sign}(\widetilde{w}_j - w_j) \right].
\end{aligned}$$

Finally, we use the bound (3.6) to derive the bounds on the derivatives stated in the lemma. There are two cases: $|\widetilde{w}_j - w_j| \leq 2S$ and $|\widetilde{w}_j - w_j| > 2S$.

Case 2(a): $|\widetilde{w}_j - w_j| \leq 2S$.

Then $|x_j - w_j| \le |x_j - \widetilde{w}_j| + |\widetilde{w}_j - w_j| \le S + 2S = 3S$. Hence

$$\begin{aligned}
\frac{d\,\phi_j}{d\,t} &\le -c_1\lambda p_j S + (1+c_2)p_j|K| \\
&\le -c_1\lambda p_j S + 3(1+c_2)p_j\kappa S \\
&= -(c_1\lambda - 3(1+c_2)\kappa)p_j S \\
&\le -\frac{c_1\lambda - 3(1+c_2)\kappa}{1+2c_2} p_j(S + c_2|\widetilde{w}_j - w_j|) \\
&\le -\frac{c_1\lambda - 3(1+c_2)\kappa}{1+2c_2}\phi_j \\
&\le -\frac{\kappa(1+c_2)}{1+2c_2}\phi_j.
\end{aligned}$$

Case 2(b): $|\widetilde{w}_j - w_j| > 2S$.

Then $|\widetilde{w}_j - w_j| \le |\widetilde{w}_j - x_j| + |x_j - w_j| \le S + |x_j - w_j| < \frac{|\widetilde{w}_j - w_j|}{2} + |x_j - w_j|$ and hence $|\widetilde{w}_j - w_j| < 2|x_j - w_j|$. Note that $\operatorname{sign}(x_j - w_j) = \operatorname{sign}(\widetilde{w}_j - w_j)$, so $-c_2 K \cdot \operatorname{sign}(\widetilde{w}_j - w_j) = -c_2\kappa|x_j - w_j|$.

$$\begin{aligned}
\frac{d\,\phi_j}{d\,t} &\le -c_1\lambda p_j S + \kappa p_j|x_j - w_j| - c_2\kappa p_j|x_j - w_j| \\
&= -c_1\lambda p_j S - (c_2 - 1)\kappa p_j|x_j - w_j| \\
&< -c_1\lambda p_j S - \frac{c_2 - 1}{2}\kappa p_j|\widetilde{w}_j - w_j| \\
&< -4\kappa(1+c_2)p_j S - \frac{c_2 - 1}{2}\kappa p_j|\widetilde{w}_j - w_j| \\
&< -\frac{c_2 - 1}{2}\kappa p_j S - \frac{c_2 - 1}{2}\kappa p_j|\widetilde{w}_j - w_j| \\
&= -\frac{c_2 - 1}{2c_2}\kappa p_j(c_2 S + c_2|\widetilde{w}_j - w_j|) \\
&\le -\frac{\kappa(c_2 - 1)}{2c_2}\phi_j.
\end{aligned}$$

$\square$

The following lemma provides an upper bound on the change to the potential function when there is a price update. Subsequently, this lemma will be used to show that at a price update, the potential function stays the same or decreases under suitable conditions. Recall that s_j denotes the spending on good j.

Lemma 3.12. *Suppose p_j is updated. Let $S_{inc} := \sum_{k \neq j,\, \Delta s_k > 0} |\Delta s_k|$ and $S_{dec} := \sum_{k \neq j,\, \Delta s_k < 0} |\Delta s_k|$.*

1. *If $sign(x_j - \widetilde{w}_j)$ is not flipped by the update and x_j moves towards $\widetilde{w}_j$, the change to ϕ is at most*

$$-\widetilde{w}_j|\Delta p_j| + sign(\Delta p_j) \cdot \Delta s_j + S_{inc} + S_{dec} + c_1 \lambda p_j |\bar{x}_j - \widetilde{w}_j|(t - \tau_j) + c_2 \delta w_j |\Delta p_j|.$$

2. *If $sign(x_j - \widetilde{w}_j)$ is not flipped by the update and x_j moves away from $\widetilde{w}_j$, or if $sign(x_j - \widetilde{w}_j)$ is flipped by the update, the change to ϕ is at most*

$$\begin{aligned} -p_j|\bar{x}_j - \widetilde{w}_j| + \widetilde{w}_j|\Delta p_j| - sign(\Delta p_j) \cdot \Delta s_j + S_{inc} + S_{dec} \\ + c_1 \lambda p_j |\bar{x}_j - \widetilde{w}_j|(t - \tau_j) + c_2 \delta w_j |\Delta p_j|. \end{aligned}$$

Proof: Let p'_j, x'_j and $s'_j = p'_j x'_j$ denote, respectively, the price of good j, the demand for good j and the spending on good j just after the price update. Note that $\text{sign}(\Delta p_j) = \text{sign}(\bar{x}_j - \widetilde{w}_j)$ and $\text{sign}(\Delta p_j) = \text{sign}(x_j - x'_j)$ always. We separate the proof into three cases.

Case 1: $\text{sign}(x_j - \widetilde{w}_j)$ is not flipped by the update and x_j moves towards $\widetilde{w}_j$.

Then $\text{sign}(x_j - \widetilde{w}_j) = \text{sign}(x'_j - \widetilde{w}_j)$. This, together with the assumption that x_j moves towards $\widetilde{w}_j$, imply that x'_j is in between x_j and $\widetilde{w}_j$, and hence $\text{sign}(x'_j - \widetilde{w}_j) = \text{sign}(x_j - x'_j) = \text{sign}(\Delta p_j)$.

Before the update to p_j,

$$p_j \cdot \text{span}\{\bar{x}_j, x_j, \widetilde{w}_j\} \geq p_j|x_j - \widetilde{w}_j| = (s_j - p_j\widetilde{w}_j) \cdot \text{sign}(\Delta p_j).$$

Immediately after the update, $\bar{x}'_j = x'_j$, so

$$\begin{aligned} p'_j \cdot \text{span}\{\bar{x}'_j, x'_j, \widetilde{w}_j\} &= (p_j + \Delta p_j)|x'_j - \widetilde{w}_j| \\ &= (s'_j - p_j\widetilde{w}_j - \widetilde{w}_j\Delta p_j) \cdot \text{sign}(\Delta p_j). \end{aligned}$$

Hence, the change to the term following the update is at most

$$(s'_j - s_j - \widetilde{w}_j\Delta p_j) \cdot \text{sign}(\Delta p_j) = \text{sign}(\Delta p_j) \cdot \Delta s_j - \widetilde{w}_j|\Delta p_j|.$$

For the terms $-p_j c_1 \lambda(t - \tau_j)|\bar{x}_j - \widetilde{w}_j| + c_2 p_j|\widetilde{w}_j - w_j|$, an update on p_j resets τ_j to t. Also, recall that $|\widetilde{w}_j - w_j| \leq \delta w_j$. Hence the change to these two terms is at most $c_1\lambda p_j|\bar{x}_j - \widetilde{w}_j|(t - \tau_j) + c_2\delta w_j|\Delta p_j|$.

For any other good k, the terms $-p_k c_1 \lambda(t - \tau_k)|\bar{x}_k - \widetilde{w}_k| + c_2 p_k|\widetilde{w}_k - w_k|$ do not change; the term $p_k \cdot \text{span}\{\bar{x}_k, x_k, \widetilde{w}_k\}$ changes, but by at most Δs_k. In the worst case, the change of this term, summing over all k, is at most $S_{inc} + S_{dec}$.

Case 2: $\text{sign}(x_j - \widetilde{w}_j)$ is not flipped by the update and x_j moves away from $\widetilde{w}_j$. Then $\text{sign}(x_j - \widetilde{w}_j) = \text{sign}(x'_j - \widetilde{w}_j)$. This, together with the assumption that x_j moves away from $\widetilde{w}_j$, imply that x_j is in between x'_j and $\widetilde{w}_j$, and hence

$\mathrm{sign}(x_j - \widetilde{w}_j) = \mathrm{sign}(x'_j - \widetilde{w}_j) \neq \mathrm{sign}(x_j - x'_j) = \mathrm{sign}(\Delta p_j) = \mathrm{sign}(\bar{x}_j - \widetilde{w}_j)$. This implies that $\widetilde{w}_j$ is in between x_j and $\bar{x}_j$.

Before the update to p_j,

$$\begin{aligned} p_j \cdot \mathrm{span}\{\bar{x}_j, x_j, \widetilde{w}_j\} &= p_j|x_j - \widetilde{w}_j| + p_j|\widetilde{w}_j - \bar{x}_j| \\ &= (s_j - p_j\widetilde{w}_j) \cdot (-\mathrm{sign}(\Delta p_j)) + p_j|\widetilde{w}_j - \bar{x}_j|. \end{aligned}$$

Immediately after the update,

$$p'_j|x'_j - \widetilde{w}_j| = (s'_j - p_j\widetilde{w}_j - \widetilde{w}_j\Delta p_j) \cdot (-\mathrm{sign}(\Delta p_j)).$$

Hence the change to the term following the update is at most

$$-\mathrm{sign}(\Delta p_j)\Delta s_j + \widetilde{w}_j|\Delta p_j| - p_j|\widetilde{w}_j - \bar{x}_j|.$$

As in Case 1, there are further changes, but bounded above by $S_{inc} + S_{dec} + c_1\lambda p_j|\bar{x}_j - \widetilde{w}_j|(t - \tau_j) + c_2\delta w_j|\Delta p_j|$.

Case 3: $\mathrm{sign}(x_j - \widetilde{w}_j)$ is flipped by the update.

As $\mathrm{sign}(x_j - \widetilde{w}_j)$ is flipped, $\widetilde{w}_j$ is in between x_j and x'_j. Hence $\mathrm{sign}(\bar{x}_j - \widetilde{w}_j) = \mathrm{sign}(\Delta p_j) = \mathrm{sign}(x_j - x'_j) = \mathrm{sign}(x_j - \widetilde{w}_j) \neq \mathrm{sign}(x'_j - \widetilde{w}_j)$. Let $\widetilde{x}_j = \mathrm{argmax}_{x \in \{x_j, \bar{x}_j\}}|x - \widetilde{w}_j|$. Before the update to p_j,

$$p_j \cdot \mathrm{span}\{\bar{x}_j, x_j, \widetilde{w}_j\} = p_j|\widetilde{x}_j - \widetilde{w}_j| = (p_j\widetilde{x}_j - p_j\widetilde{w}_j) \cdot \mathrm{sign}(\Delta p_j).$$

Immediately after the update,

$$p'_j|x'_j - \widetilde{w}_j| = (s'_j - p_j\widetilde{w}_j - \widetilde{w}_j\Delta p_j) \cdot (-\mathrm{sign}(\Delta p_j)).$$

Hence the change to the term following the update is at most

$$\begin{aligned} &\widetilde{w}_j|\Delta p_j| - (\Delta s_j + s_j - p_j\widetilde{w}_j + p_j\widetilde{x}_j - p_j\widetilde{w}_j) \cdot \mathrm{sign}(\Delta p_j) \\ &= \widetilde{w}_j|\Delta p_j| - \mathrm{sign}(\Delta p_j)\Delta s_j - p_j|x_j - \widetilde{w}_j| - p_j|\widetilde{x}_j - \widetilde{w}_j|. \end{aligned}$$

As $-p_j|x_j - \widetilde{w}_j| \le 0$ and $-p_j|\widetilde{x}_j - \widetilde{w}_j| \le -p_j|\bar{x}_j - \widetilde{w}_j|$, we obtain the same upper bound on the change of $p_j \cdot \mathrm{span}\{\bar{x}_j, x_j, \widetilde{w}_j\}$ as in Case 2. The rest of this case is exactly as in Case 2. $\square$

The remaining task is to show that ϕ is non-increasing when a price update occurs. This entails showing that the decrease to the term $p_j \cdot \mathrm{span}\{\bar{x}_j, x_j, \widetilde{w}_j\}$ is at least as large as the increase to the term $p_j c_2|\widetilde{w}_j - w_j|$ plus the value of the term $p_j c_1 \lambda(t - \tau_j)|\bar{x}_j - \widetilde{w}_j|$, which gets reset to 0. The following lemma states several inequalities we need for the task.

Lemma 3.13. *(a) If $0 \le \epsilon < 1$ and $0 \le x \le 1$, then $(1+\epsilon)^x - 1 \le \epsilon x$.*

(b) If $0 \le \epsilon < 1$ and $0 \le x \le 1$, then $1 - (1-\epsilon)^x \le \left(1 + \frac{\epsilon}{2(1-\epsilon)}\right)\epsilon x$.

(c) If $E \ge 1$, $0 \le \epsilon < 1$ and $y := \max\left\{\frac{E\epsilon}{2}, \epsilon\right\} < 1$, then $(1-\epsilon)^{1-E} - 1 \le \frac{E-1}{1-y}\epsilon$.

(d) If $E \ge 1$ and $0 \le \epsilon < 1$, then $1 - (1+\epsilon)^{1-E} \le (E-1)\epsilon$.

(e) If $x \ge 1$ and $0 \le \epsilon < 1/x$, then $(1-\epsilon)^{-x} \le 1 + \frac{x}{1-\epsilon x}\epsilon$.

Proof: In Appendix A. □

Lemma 3.14. *Let $\beta = 2\alpha - \gamma$. Suppose that $\beta > 0$ and the following conditions hold:*

C1. $f \leq (1-2\delta)^{-1/\beta} \leq (2-\delta)^{1/\gamma}$ *since the last price update to* p_j*;*

C2. $\bar{\alpha} + c_1 + c_2\delta \leq 1 - \delta$*, where* $\bar{\alpha} := 2(1-\alpha)(1-2\delta)^{-\gamma/\beta}\left(1 + \frac{\lambda(1+\delta)}{2(1-\lambda(1+\delta))}\right)$*;*

C3. $(1 + \delta + c_1 + c_2\delta)\lambda \leq 1$.

Then, when a price p_j *is updated using rule* (2.2)*, the value of* ϕ *stays the same or decreases.*

Proof: Condition C1 and Lemma 3.8 ensure that $\bar{x}_j \leq (2-\delta)w_j$ and hence $\frac{\bar{z}_j}{w_j} \leq 1$. Then, by price update rule (2.2), $|\Delta p_j| = \frac{\lambda p_j |\bar{x}_j - \widetilde{w}_j|(t-\tau_j)}{w_j}$.

Step 1: This step shows that the amount of spending transferred due to a price change is bounded by $\bar{\alpha} w_j |\Delta p_j|$.

By Condition C1 and Lemma 3.8, $x_j \leq (1-2\delta)^{-\gamma/\beta} w_j$. Hence by the definition of $\bar{\alpha}$, $2(1-\alpha)\left(1 + \frac{\lambda(1+\delta)}{2(1-\lambda(1+\delta))}\right) x_j \leq \bar{\alpha} w_j$.

Case 1(a): Price p_j is increased to qp_j, where $q > 1$, i.e. $\Delta p_j = (q-1)p_j$.

By Lemma 3.7(c), the spending increase on good j due to this price increase is at most $(qp_j)\left(\frac{x_j}{q^\alpha}\right) - x_j p_j = (q^{1-\alpha} - 1)x_j p_j$. By Lemma 3.13(a), $q^{1-\alpha} - 1 \leq (1-\alpha)(q-1)$. Hence $2(q^{1-\alpha} - 1)p_j x_j$, which is twice the upper bound on the spending drawn from other goods due to the price increase, satisfies

$$\begin{aligned}2(q^{1-\alpha}-1)p_jx_j &\le 2(1-\alpha)(q-1)p_jx_j\\ &\le \bar{\alpha}w_j\left(1+\frac{\lambda(1+\delta)}{2(1-\lambda(1+\delta))}\right)^{-1}|\Delta p_j|\\ &\le \bar{\alpha}w_j|\Delta p_j|.\end{aligned}$$

Case 1(b): Price p_j is reduced to qp_j, where $q < 1$, i.e. $\Delta p_j = (q-1)p_j$.

By Lemma 3.7(d), the spending decrease on good j due to this price decrease is at most $x_jp_j - (qp_j)\left(\frac{x_j}{q^\alpha}\right) = (1-q^{1-\alpha})x_jp_j$.

By price update rule (2.2), $1 > q \ge 1-\lambda(1+\delta)$. Then by Lemma 3.13(b), $(1-q^{1-\alpha}) \le \left(1+\frac{\lambda(1+\delta)}{2(1-\lambda(1+\delta))}\right)(1-\alpha)(1-q)$. Hence $2(1-q^{1-\alpha})p_jx_j$, which is twice the upper bound on the spending lost to other goods due to the price reduction, satisfies

$$\begin{aligned}2(1-q^{1-\alpha})p_jx_j &\le 2\left(1+\frac{\lambda(1+\delta)}{2(1-\lambda(1+\delta))}\right)(1-\alpha)(1-q)p_jx_j\\ &\le \bar{\alpha}w_j|\Delta p_j|.\end{aligned}$$

Step 2: Apply Lemma 3.12 with the result of Step 1 to show that the potential function ϕ stays the same or decreases after the price update. Let $\Delta\phi$ denote the change to ϕ after the update.

We assume that $\Delta p_j > 0$. The proof is symmetric for $\Delta p_j < 0$. As the goods are complements, when $\Delta p_j > 0$, $S_{inc} = 0$ and $\Delta s_j = S_{dec}$.

Case 2(a): $\mathrm{sign}(x_j - \widetilde{w}_j)$ is not flipped by the update and x_j moves towards

$\widetilde{w}_j$.

By Case 1(a), $2|\Delta s_j| \le \bar{\alpha} w_j |\Delta p_j|$. Also, note that $\widetilde{w}_j / w_j \ge 1 - \delta$. Then by Lemma 3.12,

$$\begin{aligned}\Delta\phi &\le -\widetilde{w}_j|\Delta p_j| + 2|\Delta s_j| + c_1\lambda p_j|\bar{x}_j - \widetilde{w}_j|(t-\tau_j) + c_2\delta w_j|\Delta p_j| \\ &\le [\bar{\alpha} + c_1 + c_2\delta - (1-\delta)]\,\lambda p_j|\bar{x}_j - \widetilde{w}_j|(t-\tau_j)\end{aligned}$$

Condition C2 implies that $\Delta\phi$ is zero or negative.

Case 2(b): $\mathrm{sign}(x_j - \widetilde{w}_j)$ is not flipped by the update and x_j moves away from $\widetilde{w}_j$, or $\mathrm{sign}(x_j - \widetilde{w}_j)$ is flipped.

Note that $\widetilde{w}_j / w_j \le 1 + \delta$ and $t - \tau_j \le 1$. By Lemma 3.12,

$$\begin{aligned}\Delta\phi &\le -p_j|\bar{x}_j - \widetilde{w}_j| + \widetilde{w}_j|\Delta p_j| + c_1\lambda p_j|\bar{x}_j - \widetilde{w}_j|(t-\tau_j) + c_2\delta w_j|\Delta p_j| \\ &\le [(1+\delta+c_1+c_2\delta)\lambda - 1]\, p_j|\bar{x}_j - \widetilde{w}_j|.\end{aligned}$$

Condition C3 implies that $\Delta\phi$ is zero or negative. $\square$

Comment. If δ and λ are sufficiently small,

$$\begin{aligned}\bar{\alpha} &= (2-2\alpha)\left(1 + O\left(\frac{\gamma\delta}{\beta}\right)\right)(1+O(\lambda)) \\ &= 2 - 2\alpha + O\left(\frac{\gamma\delta}{\beta}\right) + O(\lambda),\end{aligned}$$

and Condition C2 becomes $2 - 2\alpha + c_1 + O(\delta(1+\gamma/\beta)) + O(\lambda) \le 1$, which is satisfied by setting $c_1, \lambda, \delta(1+\gamma/\beta) = O(2\alpha - 1)$. Condition C3 is then satisfied by having $\lambda = O(1)$. More precise bounds are given later.

To demonstrate a continued convergence of the prices during Phase 2, we need to relate the prices to the potential ϕ. We show the following bound.

Theorem 3.15. *Suppose that the conditions in Lemmas 3.11 and 3.14 hold. Let* $M = \sum_i e_i$ *be the daily supply of money to all the buyers. Then, in Phase 2, the prices become* $(1+\eta)$*-bounded after* $O\left(\frac{1}{\kappa}\ln\frac{M}{\eta\beta\min_j w_j p_j^*}\right)$ *days.*

Proof: During Phase 2, $p_j \le 2p_j^*$ and $x_j \le 2w_j$. Consequently,

$$\phi = O\left(\sum_j p_j^* w_j\right) = O(M).$$

If the prices are not $(1+\eta)$-bounded, then there exists a good k with price beyond the $(1+\eta)$-bounded boundary. By Lemma 3.8 $|x_k - \widetilde{w}_k| \ge \left(\Omega(\eta\beta) - \frac{|\widetilde{w}_k - w_k|}{w_k}\right) w_k$. Then

$$\phi \ge \phi_k \ge p_k\left[(1-c_1\lambda)\left(\Omega(\eta\beta) - \frac{|\widetilde{w}_k - w_k|}{w_k}\right) w_k + c_2|\widetilde{w}_k - w_k|\right] \ge \Omega\left(\eta\beta p_k^* w_k\right).$$

Hence, when ϕ has shrunk to $O\left(\eta\beta\min_k p_k^* w_k\right)$, all prices are $(1+\eta)$-bounded. Finally, Lemmas 3.11 and 3.14 imply that ϕ shrinks by a $(1-\Theta(\kappa))$ factor each day. $\square$

If the updates in Phase 2 start with an initial value for the potential of $\phi_{\mathrm{I}} \ll M$, then in the bound on the number of days one can replace M with ϕ_{I}.

Summing the bounds from Theorems 3.10 and 3.15, yields Theorem 3.4, modulo showing that (3.1) and (3.2) suffice to ensure the conditions in Theorems 3.10 and 3.15.

3.3 Bounds on Warehouse Sizes

Recall that f_{I} is the maximum f-value over the first day. Let $d(f) := \max_j x_j/w_j$ when the prices are f-bounded. In a market of complementary goods, by Lemma 3.8, $d(f) \leq f^{\gamma}$.

Lemma 3.16. *In Phase 1, the total net change to v_j is bounded by* $O\left(\frac{1}{\lambda}d(f_I) + \frac{d(2^{1/\beta})}{\lambda\beta}\ln\frac{1}{\delta}\right)w_j$.

Proof: In one day, v_j shrinks by at most $(d(f)-1)w_j$; it can grow by at most w_j. By Lemma 3.9, f shrinks by a $1-\Theta(\lambda)$ factor every $O(1)$ days while $f \geq 2^{1/\beta}$. During this part of Phase 1, v_j can shrink by at most

$$O\left(\sum_{\ell\geq 0}\left[d\left(f_{\text{I}}[1-\Theta(\lambda)]^{\ell}\right)-1\right]w_j\right) = O\left(\frac{w_j}{\lambda}d(f_{\text{I}})\right);$$

the equality holds because $d(f)$ grows at least linearly. In this part of Phase 1, v_j grows by at most $w_j \log_{1-\Theta(\lambda)} f_{\text{I}} = o\left(\frac{w_j}{\lambda}d(f_{\text{I}})\right)$.

The remainder of Phase 1 yields a further possible change of at most $d(2^{1/\beta})w_j$ to v_j per day. By Theorem 3.10, this remainder of Phase 1 will last $O\left(\frac{1}{\lambda\beta}\ln\frac{1}{\delta}\right)$ days. So the total further change to v_j is at most $O\left(\frac{d(2^{1/\beta})w_j}{\lambda\beta}\ln\frac{1}{\delta}\right)$. $\square$

To bound the warehouse sizes for Phase 2, we observe that in Phase 2 the price adjustments are always strictly within the bounds of $1 \pm \lambda\Delta t$, where Δt is the time since the previous update to p_j. If $v_j \leq \chi_j/2 - bw_j$, then an update of p_j by a factor $1-\lambda\mu\Delta t$, implies that $w_j - \bar{x}_j \geq (\mu w_j + \kappa b w_j)\Delta t$, and $(w_j - \bar{x}_j)\Delta t$ is exactly the amount by which v_j decreases between these two price updates. As the prices are $((1-2\delta)^{-\beta})$-bounded, the difference between the price increases and decreases is bounded, and consequently, over time the change to the warehouse stock will be dominated by the sum of the $\kappa b w_j \Delta t$ terms. An analogous result applies if $v_j \geq \chi_j/2 + bw_j$. These results are made precise in the following two lemmas.

Lemma 3.17. *Let $a_1, a_2, m > 0$. Suppose that $v_j \leq v_j^* - a_1 w_j$ and that $\kappa a_1 \geq 2\lambda^2$. Let τ be the time of a price update of p_j to $p_{j,1}$. Suppose that henceforth $p_j \leq e^{\bar{f}} p_{j,1}$ for some $\bar{f} \geq 0$. If $m \geq \frac{2}{\kappa a_1}(\bar{f} + a_2)$, then by time $\tau + (m+1)$ the warehouse stock will have increased to more than $v_j^* - a_1 w_j$, or by at least $a_2 w_j$, whichever is the lesser increase.*

Proof: Suppose that $v_j \leq v_j^* - a_1 w_j$ throughout (or the result holds trivially).

Each price change by a multiplicative $(1 + \mu\Delta t)$ is associated with a target excess demand $\bar{z}_j = \bar{x}_j - w_j - \kappa(v_j - v_j^*)$, where $\bar{z}_j = \mu w_j$. Furthermore, the increase to the warehouse stock since the previous price update is exactly $-(\overline{x}_i - w_j)\Delta t = [-\mu w_j - \kappa(v_j - v_j^*)]\Delta t \geq (-\mu + \kappa a_1) w_j \Delta t$.

Note that $1+x \geq \exp(x - x^2)$ for $|x| \leq \frac{1}{2}$. Thus $1+\mu\Delta t \geq \exp(\mu\Delta t - \lambda^2 \Delta t)$ (recall that all price changes are bounded by $1 \pm \lambda\Delta t$, so $\mu^2 \leq \lambda^2$).

Suppose that over the next m days there are $l-1$ price changes; let the next l price changes be by $1 + \mu_1 \Delta t_1, 1 + \mu_2 \Delta t_2, \cdots, 1 + \mu_l \Delta t_l$. Note that the total price change satisfies $e^{\bar{f}} \geq \Pi_{k=1}^{l}(1 + \mu_k \Delta t_k) \geq \exp\left(\sum_{k=1}^{l}(\mu_k \Delta t_k - \lambda^2 \Delta t_k))\right)$. Thus $\sum_{k=1}^{l} \Delta t_k(\mu_k - \lambda^2) \leq \bar{f}$.

We conclude that when the l-th price change occurs, which is after more than m days, the warehouse stock will have increased by at least

$$\sum_{1\le k\le l}(-\mu_k+\kappa a_1)w_j\Delta t_k \ge -\bar{f}w_j + m(-\lambda^2+\kappa a_1)w_j \ge \left(-\bar{f}+\frac{1}{2}m\kappa a_1\right)w_j;$$

the last inequality holds because $2\lambda^2 \le \kappa a_1$. If $m \ge \frac{2}{\kappa a_1}(\bar{f}+a_2)$, then the warehouse stock increases by at least $a_2 w_j$. □

Lemma 3.18. *Let $a_1, a_2, m > 0$. Suppose that $v_j \ge v_j^* + a_1 w_j$. Let τ be the time of a price update of p_j to $p_{j,1}$. Suppose that henceforth $p_j \ge e^{-\bar{f}}p_{j,1}$ for some $\bar{f} \ge 0$. If $m \ge \frac{1}{\kappa a_1}(\bar{f}+a_2)$, then by time $\tau+(m+1)$ the warehouse stock will have decreased to less than $v_j^*+a_1w_j$, or by at least a_2w_j, whichever is the lesser decrease.*

Proof: Suppose that $v_j \ge v_j^* + a_1 w_j$ throughout (or the result holds trivially).

Then each price change by a multiplicative $(1+\mu\Delta t)$ is associated with a target excess demand $\bar{z}_j = \bar{x}_j - w_j - \kappa(v_j - v_j^*)$, where $\bar{z}_j = \mu w_j$. Furthermore, the decrease to the warehouse stock since the previous price update is exactly $(\bar{x}_j - w_j)\Delta t = [\mu w_j + \kappa(v_j - v_j^*)]\Delta t \ge (\mu + \kappa a_1)w_j\Delta t$.

Note that $1+x \le \exp(x)$ for $|x| \le 1$. Thus $1+\mu\Delta t \le \exp(\mu\Delta t)$.

Suppose that over the next m days there are $l-1$ price changes; let the next l price changes be by $1+\mu_1\Delta t_1, 1+\mu_2\Delta t_2, \cdots, 1+\mu_l\Delta t_l$. Note that the total price change satisfies $\exp\left(-\bar{f}\right) \le \Pi_{k=1}^{l}(1+\mu_k\Delta t_k) \le \exp\left(\sum_{k=1}^{l}\mu_k\Delta t_k\right)$. Thus $\sum_{k=1}^{l}\mu_k\Delta t_k \ge -\bar{f}$.

We conclude that when the l-th price change occurs, the warehouse stock will have decreased by at least $\sum_{k=1}^{l}(\mu_k+\kappa a_1)w_j\Delta t_k \ge (-\bar{f}+m\kappa a_1)w_j$. If $m \ge \frac{1}{\kappa a_1}(\bar{f}+a_2)$, then the warehouse stock decreases by at least a_2w_j. □

Comment. The relationship between the change in capacity and the size of the price update is crucial in proving this lemma, and this depends on having the factor Δt in the price update rule.

To complete the analysis of Phase 2, we view each warehouse as having 8 equal-sized zones of fullness, with the goal being to bring the warehouse into its central four zones. The role of the outer zones is to provide a buffer to cope with initial price imbalances.

Definition 3.19. *The four zones above the half way target are called the* high *zones, and the other four are the* low *zones. Going from the center outward, the zones are called the* central *zone, the* inner buffer*, the* middle buffer*, and the* outer buffer*. The warehouse is said to be* safe *if it is in one of its central zones or one of its inner buffers.*

Recall that we assume the ratios χ_j/w_j are all the same and equal to r, i.e. every warehouse can store the same maximum number of days supply. This will be without loss of generality, for if the smallest warehouse can store only $2d$ days supply, Theorem 3.5 in effect shows that every warehouse remains with a stock within dw_j of $\chi_j/2$. An alternative approach is to suppose that each seller S_j has a separate parameter κ_j (replacing κ). The only effect on the analysis is that the convergence rate is now controlled by $\kappa = \min_j \kappa_j$.

Proof of Theorem 3.5: We will consider warehouse j. We will say that v_j lies in a particular zone to specify how full or empty the warehouse is.

Let $D(f_{\text{I}})$ bound the duration of Phase 1 and let $v(f_{\text{I}})$ be chosen so that $v(f_{\text{I}})w_j$ bounds the change to v_j, for all j, during Phase 1. We gave a bound on

$v(f_{\mathrm{I}})$ in Lemma 3.16.

After $D(f_{\mathrm{I}})$ days, Phase 2 has been reached. By the first condition, in this period of time the warehouse stock can change by at most $v(f_{\mathrm{I}})w_j \le \chi_j/8$, so v_j can have moved out by at most one zone; thus it lies in the middle buffer or a more central zone.

We show that henceforth the tendency is to improve, i.e. move toward the central zone, but there can be fluctuations of up to one zone width. The result is that every warehouse remains within its outer zone, and after a suitable time they will all be in either their inner or central zone.

In Phase 2, the prices are $(1-2\delta)^{-1/\beta}$-bounded, so we can conclude that $f_j \in [1-2\delta/\beta, 1+4\delta/\beta]$ if $\delta/\beta \le \frac{1}{4}$. (See (3.5).) Further, this is contained in the range $[\exp(-4\delta/\beta), \exp(4\delta/\beta)]$. Hence p_j can change by at most a factor of $\exp(\pm 8\delta/\beta)$.

Let t be a time in which v_j lies in the inner or outer zone. First we show that v_j can move outward by at most one zone width. By Lemma 3.18 (taking a_1 such that $a_1 w_j$ is the width of one zone, i.e. $a_1 = \frac{1}{8}\chi_j/w_j$, $a_2 = 0$ and $\bar{f} = 8\delta/\beta$), after $8\delta/(\beta\kappa a_1)+1$ days the value of v_j will have returned to value $v_j(t)$ or remained below this value. During this period of time, the stock can increase by at most $8\delta w_j/(\beta\kappa a_1)$. Note that as $\kappa\chi_j/2 = \delta w_j$, $a_1 = \frac{1}{8}\chi_j/w_j$. And by the first condition, $8\delta w_j/(\beta\kappa a_1) = 32 w_j/\beta \le \frac{1}{16}\chi_j$, which is half the width of a zone. This guarantees that the stock will never be overflow.

By Lemma 3.18 (taking $a_1 = \frac{1}{8}\chi_j/w_j$, $a_2 = \frac{1}{4}\chi_j/w_j$ and $\bar{f} = 8\delta/\beta$), v_j reaches the upper central zone after at most $2(8\delta/\beta + a_2)/(\kappa a_1) + 1 = \frac{64}{\beta} + \frac{4}{\kappa} + 1$ days. Applying the argument in the last paragraph anew shows that henceforth

v_j remains within the upper inner buffer.

We apply the same argument to the low zones using Lemma 3.17 (this is where the third condition of Theorem 3.5 is needed). The same results are achieved, but they take up to twice as long, and the possible increase in stock is twice as large as the possible decrease in the previous case, but still only one zone's worth. □

Comment. We note that were the price update rule to have the form $p'_j \leftarrow p_j \cdot \exp(\lambda \min\{1, \bar{z}_j/w_j\}\Delta t)$ rather than $p'_j \leftarrow p_j(1 + \lambda \min\{1, \bar{z}_j/w_j\}\Delta t)$ then the third condition in Theorem 3.5 would not be needed (this constraint comes from setting a_1 in Lemma 3.17 to the width of a zone). However, we prefer the form of the rule we have specified as it strikes us as being simpler and hence more natural.

3.4 Parameter Constraint Summary

Recall that $r = \chi_j/w_j$ and $r \leq 2\delta/\kappa$ from Chapter 2. As we will show, all conditions required by the above lemmas and theorems can be satisfied when (3.1) and (3.2) hold. Note that r needs to be sufficiently large, or in other words χ_j for every j needs to be sufficiently large, to ensure that there is a choice of λ which satisfies both the upper and lower bounds. Further note that the term $\sqrt{\frac{\kappa\tau}{16}}$, which is due to the third condition of Theorem 3.5, would not be needed were we to use the exponential price update rule.

Lemma 3.11, Lemma 3.14 and Theorem 3.5 require several constraints on the parameters $\kappa, \delta, \lambda, c_1, c_2$. We unwind these constraints to show how these parameters depend on the market parameters α, γ and β, and show that if (3.1) and (3.2) hold, then all these constraints are satisfied.

We list the constraints below:

1. $4\kappa(1 + c_2) \leq \lambda c_1 \leq 1/2$;

2. $\delta/\beta \leq 1/4$;

3. $(1 - 2\delta)^{-1/\beta} \leq (2 - \delta)^{1/\gamma}$;

4. $\bar{\alpha} + c_1 + c_2\delta \leq 1 - \delta$, where $\bar{\alpha} = 2(1 - \alpha)(1 - 2\delta)^{-\gamma/\beta}\left(1 + \frac{\lambda(1+\delta)}{2(1-\lambda(1+\delta))}\right)$;

5. $(1 + \delta + c_1 + c_2\delta)\lambda \leq 1$;

6. $r \geq \frac{512}{\beta}$ and $r \geq \frac{8v(f_I)}{w_j}$;

7. $\delta = \frac{\kappa r}{2}$;

8. $\lambda^2 \leq \frac{\kappa r}{16}$.

When $r \geq \max\left\{\frac{512}{\beta}, 8v(f_I)\right\}$, Constraint 6 is satisfied.

We first impose that

$$\delta \leq \min\left\{\frac{\beta}{4\gamma}, \frac{1}{8}\right\}, \ \lambda \leq \frac{3}{7}, \ c_1 = \delta, \ c_2 = 2. \tag{3.7}$$

Constraints 2 and 5 are then satisfied. Constraint 1 becomes

$$\frac{24}{r} \leq \lambda. \tag{3.8}$$

By Lemma 3.13(e), $(1-2\delta)^{-\gamma/\beta} \leq 1+\frac{4\gamma}{\beta}\delta$ as $2\gamma/\beta \leq 1/2$ and $\gamma/\beta \geq 1$. Thus Constraint 3 is satisfied when $1+\frac{4\gamma}{\beta}\delta \leq 2-\delta$, which is equivalent to

$$\delta \leq \frac{\beta}{4\gamma+\beta}. \tag{3.9}$$

Next, we show Constraint 4 is satisfied when $\left(8+\frac{4\gamma}{\beta}\right)\delta + \frac{63\lambda}{58} \leq \ln\frac{1}{2(1-\alpha)}$. As $\delta \leq 1/8$ and $\lambda \leq 3/7$, $\frac{1+\delta}{2(1-\lambda(1+\delta))} \leq \frac{63}{58}$. Then $\left(8+\frac{4\gamma}{\beta}\right)\delta + \frac{1+\delta}{2(1-\lambda(1+\delta))}\lambda \leq \ln\frac{1}{2(1-\alpha)}$ and hence $2(1-\alpha)\exp\left(\frac{4\delta\gamma}{\beta}+\frac{1+\delta}{2(1-\lambda(1+\delta))}\right) \leq \exp(-8\delta)$. Then

$$\begin{aligned}
\bar{\alpha} &= 2(1-\alpha)(1-2\delta)^{-\gamma/\beta}\left(1+\frac{1+\delta}{2(1-\lambda(1+\delta))}\lambda\right) \\
&\leq 2(1-\alpha)\exp(-4\delta)^{-\gamma/\beta}\cdot\exp\left(\frac{1+\delta}{2(1-\lambda(1+\delta))}\lambda\right) \\
&\leq \exp(-8\delta) \leq 1-4\delta.
\end{aligned}$$

Constraint 4 is satisfied since $\bar{\alpha}+c_1+c_2\delta = \bar{\alpha}+3\delta \leq 1-\delta$.

When we further impose that

$$(8+4\gamma/\beta)\delta \leq \frac{1}{2}\ln\frac{1}{2(1-\alpha)}, \tag{3.10}$$

Constraint 4 can be satisfied when

$$\lambda \leq \frac{29}{63}\ln\frac{1}{2(1-\alpha)}. \tag{3.11}$$

Using the bounds on δ in (3.7), (3.9) and (3.10) with Constraint 7 yields

$$\begin{aligned}\kappa &\leq \frac{2}{r}\cdot\min\left\{\frac{\beta}{4\gamma},\frac{1}{8},\frac{\beta}{4\gamma+\beta},\frac{1}{2(8+4\gamma/\beta)}\ln\frac{1}{2(1-\alpha)}\right\}\\ &= \frac{2}{r}\cdot\min\left\{\frac{1}{8},\frac{\beta}{4\gamma+\beta},\frac{\beta}{16\beta+8\gamma}\ln\frac{1}{2(1-\alpha)}\right\}.\end{aligned}$$

Using the bounds on λ in (3.7), (3.8) and (3.11) with Constraint 8 yields

$$\frac{24}{r}\leq\lambda\leq\min\left\{\frac{3}{7},\frac{29}{63}\ln\frac{1}{2(1-\alpha)},\sqrt{\frac{\kappa r}{16}}\right\}.$$

Note that $r=\chi_j/w_j$ needs to be sufficiently large to ensure that there is a choice of λ which satisfies both the upper and lower bounds.

The market is defined by the parameters α,γ,β. The parameters κ,λ,r are chosen to satisfy the constraints. The price update rule uses κ,λ, while the warehouse size χ_j is bounded above by rw_j. The parameters c_1,c_2 are needed only for the analysis.

Chapter 4

Markets with Mixtures of Substitutes and Complements

For market with a mixture of complements and substitutes, we need a generalized version of elasticity, which we call the *adverse market elasticity*. These are the extreme changes in demand that occur to one good, WLOG good 1, when its price changes, and other prices also change but by no larger a fraction than p_1. For suppose that p_1 were reduced with the goal of increasing x_1. But suppose that at the same time other prices may change by the same fractional amount (either up or down). How much can this undo the desired increase in x_1? The answer is that in general it can more than undo it. However, our proof approach depends on x_1 increasing in this scenario, which is why we introduce this notion of elasticity and consider those markets in which it is sufficiently bounded from below.

Definition 4.1. *Define $\bar{P}$ to be the following set of prices:*

$$\left\{ ((1+\delta)p_1, q_2, \cdots, q_n) \;\middle|\; \textit{for } j \geq 2,\ q_j \in \left[\frac{p_j}{1+\delta}, (1+\delta)p_j\right] \right\}.$$

The adverse market elasticity *for good* 1 *is*

$$-\max_{\bar{p}\in\bar{P}} \lim_{\delta\to 0} \frac{x_1(\bar{p}) - x_1(p)}{\delta x_1}$$

We let β be a lower bound on the adverse market elasticity over all goods and prices.

It is not hard to see that for the case in which all the goods are complements, $\beta \geq 2\alpha - \gamma$. In the markets we consider, the adverse market elasticity is positive.

The results in this chapter parallel those in the previous chapter, and their proofs share some similarities. Instead of repeating all the details, we will point readers to some of the proofs in the previous chapter and provide the necessary modifications.

4.1 Results

To understand the constraints needed in this setting, we first recap the essence of the analysis for markets with complementary goods, which we adapt to the current setting. The analysis proceeds in two phases. Recall that the prices are f-bounded. In Phase 1, $(f-1)$ reduces multiplicatively each day. Phase 1 ends when further such reductions can no longer be guaranteed. In Phase 2, an amortized analysis shows that the misspending, roughly $\sum_j p_j|z_j| +$

$p_j|\widetilde{w}_j - w_j|$, decreases multiplicatively each day.

Now that substitutes are present, we will need an upper bound on the price elasticity (see Definition 3.2), as in [17]. Let $E \geq 1$ denote this upper bound. For convergence we will need that $\lambda = O(1/E)$.

Let S_s denote the spending on all goods which are substitutes of good j and let S_c denote the spending on all goods which are complements of good j.[1] We need to introduce a further constraint. The reason is that the amortized analysis depends on showing that the misspending decreases. However, the current constraints do not rule out the possibility that when, say p_1 is increased, the spending decrease on complements of good 1, $|\Delta S_c|$, and the spending increase on substitutes of good 1, $|\Delta S_s|$, are both large compared with the reduction in misspending on good j. We rule it out with the following assumption.

Assumption 4.2. *Suppose that p_j changes by Δp_j. Then there is a constant $\alpha' < \frac{1}{2}$, such that $|\Delta S_c| \leq \alpha' x_j |\Delta p_j|$.*

We require that β, as defined in Definition 4.1, satisfy $\beta > 0$. Our results will require λ and κ, the parameters in the tatonnement update rule (2.2), to obey the following conditions.

$$\frac{24}{r} \leq \lambda \leq \min\left\{\frac{1}{8E + 4\alpha' - 6}, \sqrt{\frac{\kappa r}{16}}\right\} \tag{4.1}$$

$$\kappa \leq \frac{2}{r} \cdot \min\left\{\frac{\beta}{\beta + 4(2E - \beta)}, \frac{(1 - 2\alpha')\beta}{8\alpha'(2E - \beta) + 4\beta}\right\} \tag{4.2}$$

[1] In this setting, it is possible that a pair of goods are substitutes at some prices while they are complements at other prices. Hence, S_s and S_c may not be constant sets; in this work, the two sets are always determined w.r.t. a price change to good j.

Theorem 4.3. *Suppose that the adverse market elasticity $\beta > 0$, and the prices are f-bounded throughout the first day, and in addition that Assumption 4.2 and Equations* (4.1) *and* (4.2) *hold. Let $M = \sum_i e_i$ be the daily supply of money to all the buyers. Then the prices become $(1+\eta)$-bounded after $O\left(\frac{1}{\lambda}\ln f + \frac{1}{\lambda\beta}\ln\frac{1}{\delta} + \frac{1}{\kappa}\ln\frac{M}{\eta\beta\min_k p_k^* w_k}\right)$ days.*

Theorem 3.5 with (4.1)–(4.2) replacing (3.1)–(3.2) also continues to apply. The proof of Theorem 3.5 for markets with mixtures of substitutes and complements is identical to the one for markets with complementary goods, except that we use $d(f) \le f^{2E-\beta}$ instead.

4.2 Convergence Analysis

4.2.1 Phase 1

As with the case of markets of complementary goods, it suffices to analyse the one-time markets in Phase 1.

Lemma 4.4. *When the market is f-bounded,*

1. *if $p_j = qp_j^*/f$ where $1 \le q \le f^2$, then $x_j \ge w_j f^{\beta} q^{-E}$;*

2. *if $p_j = fp_j^*/q$ where $1 \le q \le f^2$, then $x_j \le w_j f^{-\beta} q^{E}$.*

Proof: We prove the first part; the second part is symmetric. Let $(p'_{-j}, qp_j^*/f)$ be the f-bounded prices maximizing x_j when $p_j = qp_j^*/f$. First consider adjusting the prices from p^* to $(p'_{-j}, p_j^*/f)$ by smooth proportionate multiplicative changes (or equivalently, proportionate linear changes to the terms $\ln p_j$ for all

j). From the definition of β in Definition 4.1, it is easy to show that x_j is at least $w_j f^\beta$. Now increase p_j by a factor of q. As E is the upper bound on the demand elasticity, the increase in the value of p_j reduces x_j by at most q^{-E}, yielding the bound $x_j \geq w_j f^\beta q^{-E}$. □

Lemma 4.5. *Suppose that $\beta > 0$. Further, suppose that the prices are updated independently using price update rule* (2.1)*, and that $0 < \lambda < \frac{1}{2E-1}$. Let p denote the current price vector and p' denote the price vector after one day.*

(i.) If $f(p)^\beta \geq 2$, then $f(p') \leq \left(1 - \frac{\lambda}{2}\right) f(p)$.

(ii.) If $f(p)^\beta \leq 2$, then $f(p') \leq f(p)^{1-\lambda\beta/2}$.

Proof: Suppose that $p_j = q\frac{p_j^*}{f(p)}$, where $1 \leq q \leq f(p)^2$. By Lemma 4.4, $x_j \geq w_j f(p)^\beta q^{-E}$ and hence $\frac{z_j}{w_j} \geq f(p)^\beta q^{-E} - 1$. When p_j is updated using price update rule (2.1), the new price p_j' satisfies

$$p_j' \geq \frac{p_j^*}{f(p)} q \left[1 + \lambda \cdot \min\left\{1, f(p)^\beta q^{-E} - 1\right\}\right].$$

Let $h_3(q) := q\left[1 + \lambda \cdot \min\left\{1, (f(p)^\beta q^{-E} - 1)\right\}\right]$. If $f(p)^\beta q^{-E} \geq 2$, then $h_3(q) = q(1+\lambda)$, so $\frac{d}{dq}h_3(q) = 1+\lambda > 0$. If $f(p)^\beta q^{-E} \leq 2$, then $h_3(q) = q\left[1 + \lambda(f(p)^\beta q^{-E} - 1)\right]$, so

$$\frac{d}{dq}h_3(q) = 1 - \lambda\left(1 + (E-1)f(p)^\beta q^{-E}\right) \geq 1 - \lambda(2E-1) \geq 0.$$

Thus $q = 1$ minimizes $h_3(q)$, and hence

$$p_j' \geq \frac{p_j^*}{f(p)} h_3(1) = \frac{p_j^*}{f(p)}\left[1 + \lambda \cdot \min\left\{1, f(p)^\beta - 1\right\}\right].$$

Similarly, suppose that $p_j = \frac{1}{q} f(p) p_j^*$, where $1 \le q \le f(p)^2$. By Lemma 4.4, $x_j \le w_j f(p)^{-\beta} q^E$ and hence $\frac{z_j}{w_j} \le f(p)^{-\beta} q^E - 1$. When p_j is updated using price update rule (2.1), the new price p_j' satisfies

$$p_j' \le \frac{1}{q} f(p) p_j^* \left(1 + \lambda \cdot \min\left\{1, f(p)^{-\beta} q^E - 1\right\}\right).$$

Let $h_4(q) := \frac{1}{q}\left[1 + \lambda \cdot \min\left\{1, f(p)^{-\beta} q^E - 1\right\}\right]$. If $f(p)^{-\beta} q^E \ge 2$, then $h_4(q) = \frac{1+\lambda}{q}$, so $\frac{d}{dq} h_4(q) \le 0$. If $f(p)^{-\beta} q^E \le 2$, then $h_4(q) = \frac{1}{q}\left[1 + \lambda(f(p)^{-\beta} q^E - 1)\right]$, so

$$\frac{d}{dq} h_4(q) = \frac{1}{q^2}\left[-1 + \lambda\left(1 + (E-1) f(p)^{-\beta} q^E\right)\right] \le \frac{1}{q^2}\left(-1 + \lambda(2E - 1)\right) \le 0.$$

Thus $q = 1$ maximizes $h_4(q)$, and hence

$$p_j' \le f(p) p_j^* h_4(1) = f(p) p_j^* \left[1 + \lambda(f(p)^{-\beta} - 1)\right].$$

The remainder of this proof is identical to the final part of the proof of Lemma 3.9. $\square$

Theorem 3.10 continues to apply to with an almost identical proof, except that Lemma 4.5 replaces Lemma 3.9.

4.2.2 Phase 2

We use the same potential function as in Chapter 3.2.2. Lemmas 3.11 and 3.12 can be reused with no modification needed.

Lemma 4.6. *Suppose that Assumption 4.2 holds,* $\lambda E \leq 1$ *and* $\lambda \leq \frac{1}{2}$. *Then* $|\Delta S_s| \leq (\alpha' + 2E - 2)x_j|\Delta p_j|$.

Proof: There are two cases.

Case 1. The price of good j is reduced from p_j to $p'_j = qp_j$, where $q < 1$. Note that by the price update rule, $q \geq 1 - \lambda$.

Then $x'_j \leq q^{-E}x_j$ and hence $\Delta s_j \leq (q^{1-E} - 1)p_j x_j$. Then

$$\begin{aligned} |\Delta S_s| = |\Delta S_c| + \Delta s_j &\leq \alpha' x_j|\Delta p_j| + (q^{1-E} - 1)p_j x_j \\ &\leq \alpha' x_j|\Delta p_j| + 2(E-1)(1-q)p_j x_j. \end{aligned}$$

The last inequality holds by applying Lemma 3.13(c) with $\max\left\{\frac{E(1-q)}{2}, 1-q\right\} \leq 1/2$ and $q \geq 1 - \lambda$. Noting that $(1-q)p_j = |\Delta p_j|$, we are done.

Case 2. The price of good j is raised from p_j to $p'_j = qp_j$, where $q > 1$.

Then $x'_j \geq q^{-E}x_j$ and hence $\Delta s_j \geq (q^{1-E} - 1)p_j x_j$. Then

$$\begin{aligned} |\Delta S_s| = |\Delta S_c| - \Delta s_j &\leq \alpha' x_j|\Delta p_j| + (1 - q^{1-E})p_j x_j \\ &\leq \alpha' x_j|\Delta p_j| + (E-1)(q-1)p_j x_j. \end{aligned}$$

The last inequality holds by applying Lemma 3.13(d). Noting that $(q-1)p_j = |\Delta p_j|$, we are done. □

Lemma 4.7. *Suppose that Assumption 4.2 holds and the adverse market elasticity* $\beta > 0$. *Suppose that the following conditions hold:*

M1. $f \leq (1-2\delta)^{-1/\beta} \leq (2-\delta)^{1/(2E-\beta)}$ *since the last price update to* p_j*;*

M2. $2\alpha'(1-2\delta)^{-(2E-\beta)/\beta} + c_1 + c_2\delta \leq 1-\delta$*;*

M3. $\left[2(\alpha'+2E-2)(1-2\delta)^{-(2E-\beta)/\beta} + 1 + \delta + c_1 + c_2\delta\right]\lambda \leq 1$.

Then, when a price p_j *is updated using rule* (2.2)*, the value of* ϕ *stays the same or decreases.*

Proof: Condition M1 and Lemma 4.4 ensure that $x_j, \bar{x}_j \leq (1-2\delta)^{-(2E-\beta)/\beta} w_j \leq (2-\delta)w_j$, and hence that $\frac{\bar{z}_j}{w_j} \leq 1$. By price update rule (2.2), $|\Delta p_j| = \frac{\lambda p_j |\bar{x}_j - \widetilde{w}_j|(t-\tau_j)}{w_j}$.

We assume $\Delta p_j > 0$. The proof is symmetric for $\Delta p_j < 0$. When $\Delta p_j > 0$, $S_{inc} = |\Delta S_s|$ and $S_{dec} = |\Delta S_c|$. Also, note that $|\Delta S_s| = |\Delta S_c| - \Delta s_j$.

Case 1: $\text{sign}(x_j - \widetilde{w}_j)$ is not flipped by the update and x_j moves towards $\widetilde{w}_j$.

By Lemma 3.12, the change to ϕ is at most

$$-\widetilde{w}_j|\Delta p_j| + \Delta s_j + |\Delta S_c| + |\Delta S_s| + c_1\lambda p_j|\bar{x}_j - \widetilde{w}_j|(t-\tau_j) + c_2\delta w_j|\Delta p_j|,$$

which equals $-\widetilde{w}_j|\Delta p_j| + 2|\Delta S_c| + c_1\lambda p_j|\bar{x}_j - \widetilde{w}_j| + c_2\delta w_j|\Delta p_j|$. Noting $\widetilde{w}_j/w_j \geq 1-\delta$ and $x_j \leq (1-2\delta)^{-(2E-\beta)/\beta} w_j$, and applying Assumption 4.2, the change to ϕ is at most

$$\left[2\alpha'(1-2\delta)^{-(2E-\beta)/\beta} + c_1 + c_2\delta - (1-\delta)\right] w_j|\Delta p_j|.$$

Condition M2 implies that the change is zero or negative.

Case 2: $\text{sign}(x_j - \widetilde{w}_j)$ is not flipped by the update and x_j moves away from $\widetilde{w}_j$, or $\text{sign}(x_j - \widetilde{w}_j)$ is flipped by the update.

By Lemma 3.12, the change to ϕ is at most

$$-p_j|\bar{x}_j - \widetilde{w}_j| + \widetilde{w}_j|\Delta p_j| - \Delta s_j + |\Delta S_c| + |\Delta S_s| + c_1\lambda p_j|\bar{x}_j - \widetilde{w}_j|(t - \tau_j) + c_2\delta w_j|\Delta p_j|,$$

which equals $-p_j|\bar{x}_j - \widetilde{w}_j| + \widetilde{w}_j|\Delta p_j| + 2|\Delta S_s| + c_1\lambda p_j|\bar{x}_j - \widetilde{w}_j| + c_2\delta w_j|\Delta p_j|$. Noting that $\widetilde{w}_j/w_j \le 1 + \delta$, $t - \tau_j \le 1$ and $x_j \le (1 - 2\delta)^{-(2E-\beta)/\beta} w_j$, and applying Lemma 4.6, the change to ϕ is at most

$$\left[(2(\alpha' + 2E - 2)(1 - 2\delta)^{-(2E-\beta)/\beta} + 1 + \delta + c_1 + c_2\delta)\lambda - 1\right] p_j|\bar{x}_j - \widetilde{w}_j|.$$

Condition M3 implies that the change is zero or negative. □

Theorem 4.8. *Suppose that the conditions in Lemmas 3.11 and 4.7 hold. Let $M = \sum_i e_i$ be the daily supply of money to all the buyers. Then, in Phase 2, the prices become $(1 + \eta)$-bounded after $O\left(\frac{1}{\kappa} \ln \frac{M}{\eta\beta \min_k p_k^* w_k}\right)$ days.*

Theorem 4.3 follows on summing the bounds from Theorems 3.10 and 4.8, and on showing that (4.1) and (4.2) imply the constraints in Lemmas 3.11 and 4.7.

4.3 Example Scenarios with NCES Utility Functions

Example Scenario: 2-Level NCES Utility Functions

A natural type of market with a mixture of substitutes and complements occurs when the goods are partitioned into different groups, with two goods in the same group being substitutes and two goods from different groups being complements. One example is the market of pasta and sauces. Different brands of pasta/sauces are substitutes, while any pasta and any sauce are complements. We use NCES utility functions to provide a class of utility functions that yields such type of markets.

We focus on the demands of a single buyer i. Suppose that there are k groups of goods $G_1, G_2, \cdots, G_k$. For each group G_ℓ, the buyer has a utility component

$$u_{i\ell}(x) := \left(\sum_{g \in G_\ell} a_{ig} (x_g)^{\rho_{i\ell}} \right)^{1/\rho_{i\ell}},$$

where $1 > \rho_{i\ell} \geq 0$. The overall utility function of the buyer is

$$u_i(x) := \left(\sum_{\ell=1}^{k} a_{i\ell} (u_{i\ell}(x))^{\rho_i} \right)^{1/\rho_i},$$

where $-1 < \rho_i \leq 0$. The bounds on $\rho_{i\ell}$ and ρ_i are needed to allow us to show convergence.

We will show that $E = \max_{i,\ell} \frac{1}{1-\rho_{i\ell}}$, $\beta = \min_i \frac{2}{1-\rho_i} - 1$ and $\alpha' = \max_i \frac{-\rho_i}{1-\rho_i}(1-\lambda)^{-E}$. Thus, Theorems 4.3 and 3.5 can apply.

We will use index j to denote a good, $G(j)$ to denote the set of goods in the group that contains good j, index k to a denote a good in $G(j) \setminus \{j\}$ and index h to denote a good in any other group. Let e_i denote the budget of buyer i, and let $s_{i\ell}$ denote the total spending on all the goods in G_ℓ. Keller [32] derived that

$$\frac{\partial x_{ij}/\partial p_j}{x_{ij}/p_j} = -\frac{1}{1-\rho_{i,G(j)}}\left(1-\frac{s_{ij}}{s_{i,G(j)}}\right) - \frac{1}{1-\rho_i}\left(\frac{s_{ij}}{s_{i,G(j)}} - \frac{s_{ij}}{e_i}\right) - \frac{s_{ij}}{e_i}$$

$$\frac{\partial x_{ij}/\partial p_k}{x_{ij}/p_k} = \frac{s_{ik}}{e_i}\left(\frac{1}{1-\rho_{i,G(j)}}\frac{e_i}{s_{i,G(j)}} - \frac{1}{1-\rho_i}\left(\frac{e_i}{s_{i,G(j)}} - 1\right) - 1\right)$$

$$\frac{\partial s_{ih}}{\partial p_j} = \frac{s_{ih}}{e_i}\frac{\rho_i}{1-\rho_i}x_{ij}.$$

As $1 > \rho_{i\ell} \geq 0$, $e_i \geq s_{i\ell}$ for all ℓ, and $\rho_i < 0$, it follows that $\frac{\partial x_{ij}}{\partial p_k} \geq 0$ and $\frac{\partial x_{ij}/\partial p_j}{x_{ij}/p_j} \geq -\frac{1}{1-\rho_{i,G(j)}}$; i.e. every pair of goods in the same group are substitutes and $E = \max_{i,\ell} \frac{1}{1-\rho_{i,\ell}}$.

As $\rho_i < 0$, $\frac{\partial s_{ih}}{\partial p_j} < 0$, or equivalently $\frac{\partial x_{ih}}{\partial p_j} < 0$; i.e. two goods from different groups are complements.

To compute β, suppose that p_j is raised by a factor q, where $q > 1$, and all other prices may be changed by a factor chosen in $[1/q, q]$. The demand for good j is maximized when the prices of all its substitutes, i.e. the goods in $G(j)$, are raised by the same factor q, and the prices of all other goods are reduced by the factor $1/q$. As $\rho_i < 0$,

$$\frac{\partial x_{ij}/\partial p_j}{x_{ij}/p_j} + \sum_{k\in G(j)\setminus\{j\}} \frac{\partial x_{ij}/\partial p_k}{x_{ij}/p_k} = -\frac{1}{1-\rho_i} - \frac{s_{i,G(j)}}{e_i}\left(1-\frac{1}{1-\rho_i}\right) \leq -\frac{1}{1-\rho_i}.$$

Hence, when the prices of all goods are reduced by the factor $1/q$ and the prices of the goods in $G(j)$ are raised by a factor q^2, then $x'_j \leq x_j q^{1-2/(1-\rho_i)}$. Thus

$\beta = \min_i \frac{2}{1-\rho_i} - 1$.

We compute α' as follows. Note that $\sum_h \frac{\partial s_{ih}}{\partial p_j} = \frac{\sum_h s_{ih}}{e_i} \frac{\rho_i}{1-\rho_i} x_{ij} \geq \frac{\rho_i}{1-\rho_i} x_{ij}$ and $|\Delta S_c| = \sum_h |\Delta s_h|$. When p_j is raised, the demand for good j decreases, and hence $|\Delta S_c| \leq \frac{-\rho_i}{1-\rho_i} x_{ij} |\Delta p_j|$; when p_j is reduced, the price is reduced by a factor of at most $(1-\lambda)$, so by demand elasticity argument, $x'_{ij} \leq x_{ij}(1-\lambda)^{-E}$. Hence $|\Delta S_c| \leq \frac{-\rho_i}{1-\rho_i}(1-\lambda)^{-E} x_{ij} |\Delta p_j|$. Thus Assumption 4.2 is satisfied with $\alpha' = \max_i \frac{-\rho_i}{1-\rho_i}(1-\lambda)^{-E}$. A sufficiently small λ ensures that $\alpha' < 1/2$.

Example Scenario: Arbitrary-Level NCES Utility Functions

We extend the above example scenario to arbitrary levels of NCES utility functions. Recall that we provide a visualization of a NCES utility function as a utility tree in Chapter 1.3.

We focus on one particular good j. Let $A_0, A_1, \cdots, A_N$ be the nodes along the path from the leaf node of good j to the root of the utility tree (A_0 is the leaf node, A_N is the root), and let $\rho_1, \rho_2, \cdots, \rho_N$ be the associated ρ-values. Let $\sigma_q = \frac{1}{1-\rho_q}$ for $1 \leq q \leq N$. Let $\beta_j := \sigma_1 - \sum_{q=1}^{N-1} |\sigma_q - \sigma_{q+1}| - |\sigma_N - 1|$, $E_j := \max_{1 \leq q \leq N} \{1, \sigma_q\}$ and $\alpha'_j := (1-\lambda)^{-E} \left(\sum_{q=1}^{N-1} |\sigma_q - \sigma_{q+1}| + |\sigma_N - 1| \right)$. We will show that the market parameters are given by $\beta = \min_j \beta_j$, $E = \max_j E_j$ and $\alpha' = \max_j \alpha'_j$, in which the maximum/minimum is taken over all buyers in the market.

Let G_q denote the set of goods which are in the subtree rooted at A_q. Let s_q denote the total spending on all goods in G_q. For any good $k \neq j$, if the least common ancestor of goods j and k is A_ℓ, let $\Lambda(k) = \ell$.

Keller [32] derived the following:

$$\frac{\partial x_j/\partial p_k}{x_j/p_k} = \sum_{q=\Lambda(k)}^{N-1} \frac{s_k}{s_q}(\sigma_q - \sigma_{q+1}) + \frac{s_k}{s_N}(\sigma_N - 1)$$
$$\frac{\partial x_j/\partial p_j}{x_j/p_j} = -\sigma_1 + \sum_{q=1}^{N-1} \frac{s_j}{s_q}(\sigma_q - \sigma_{q+1}) + \frac{s_j}{s_N}(\sigma_N - 1).$$

We now compute the adverse market elasticity of good j. When the price of good j is reduced by a factor of $(1-\delta)$, raise the prices of all the complements of good j by a factor of $1/(1-\delta)$ and reduce the prices of all the substitutes of good j by a factor of $(1-\delta)$.[2] By the above formulae, $x'_j \geq x_j(1-\delta)^{-\beta_j}$, where

$$\begin{aligned}
\beta_j &= -\frac{\partial x_j/\partial p_j}{x_j/p_j} - \sum_{k\neq j}\left|\frac{\partial x_j/\partial p_k}{x_j/p_k}\right| \\
&= \sigma_1 - \sum_{q=1}^{N-1}\frac{s_j}{s_q}(\sigma_q - \sigma_{q+1}) - \frac{s_j}{s_N}(\sigma_N - 1) \\
&\qquad - \sum_k \left|\sum_{q=\Lambda(k)}^{N-1}\frac{s_k}{s_q}(\sigma_q - \sigma_{q+1}) + \frac{s_k}{s_N}(\sigma_N - 1)\right| \\
&\geq \sigma_1 - \sum_{q=1}^{N-1}\frac{s_j}{s_q}|\sigma_q - \sigma_{q+1}| - \frac{s_j}{s_N}|\sigma_N - 1| \\
&\qquad - \sum_k \left(\sum_{q=\Lambda(k)}^{N-1}\frac{s_k}{s_q}|\sigma_q - \sigma_{q+1}| + \frac{s_k}{s_N}|\sigma_N - 1|\right) \\
&= \sigma_1 - \sum_{q=1}^{N-1}\left(|\sigma_q - \sigma_{q+1}|\sum_{k\in G_q}\frac{s_k}{s_q}\right) - |\sigma_N - 1|\left(\sum_{k\in G_N}\frac{s_k}{s_N}\right) \\
&= \sigma_1 - \sum_{q=1}^{N-1}|\sigma_q - \sigma_{q+1}| - |\sigma_N - 1|.
\end{aligned}$$

Set β, as defined in Definition 4.1, to $\min_j \beta_j$.[3]

[2] Think of δ as being very small, so that there is no ambiguity about substitutes and complements.

[3] The lower bound on β_j is almost tight when $\frac{s_N}{s_{N-1}}, \frac{s_{N-1}}{s_{N-2}}, \cdots, \frac{s_2}{s_1}, \frac{s_1}{s_j}$ are all very large.

Next, note that

$$
\begin{aligned}
\frac{\partial x_j/\partial p_j}{x_j/p_j} &= \frac{s_j}{s_N}\left[-1-\sum_{q=1}^{N}\sigma_q\left(\frac{s_N}{s_{q-1}}-\frac{s_N}{s_q}\right)\right] \\
&\geq \max_{1\leq q\leq N}\{1,\sigma_q\}\left(-\frac{s_j}{s_N}-\sum_{q=1}^{N}\left(\frac{s_j}{s_{q-1}}-\frac{s_j}{s_q}\right)\right) \\
&= -\max_{1\leq q\leq N}\{1,\sigma_q\}.
\end{aligned}
$$

Let $E_j = \max_{1\leq q\leq N}\{1,\sigma_q\}$ and set E, the upper bound on demand elasticity, to $\max_j E_j$.

Keller derived that

$$
\frac{\partial s_k}{\partial p_j} = x_j\left(\sum_{q=\Lambda(k)}^{N-1}\frac{s_k}{s_q}(\sigma_q-\sigma_{q+1})+\frac{s_k}{s_N}(\sigma_N-1)\right).
$$

This yields

$$
\begin{aligned}
\sum_{k\neq j}|\Delta s_k| &\leq (1-\lambda)^{-E}x_j|\Delta p_j|\sum_{k\neq j}\left(\sum_{q=\Lambda(k)}^{N-1}\frac{s_k}{s_q}|\sigma_q-\sigma_{q+1}|+\frac{s_k}{s_N}|\sigma_N-1|\right) \\
&\leq x_j|\Delta p_j|(1-\lambda)^{-E}\left(\sum_{q=1}^{N-1}|\sigma_q-\sigma_{q+1}|+|\sigma_N-1|\right).
\end{aligned}
$$

Let $\alpha'_j = (1-\lambda)^{-E}\left(\sum_{q=1}^{N-1}|\sigma_q-\sigma_{q+1}|+|\sigma_N-1|\right)$. Assumption 4.2 is satisfied with $\alpha' = \max_j \alpha'_j$. Theorems 4.3 and 3.5 apply when $\beta > 0$ and $\alpha' < 1/2$.

4.4 Parameter Constraint Summary

Lemma 3.11, Lemma 4.7 and Theorem 3.5 require several constraints on the parameters $\kappa, \delta, \lambda, c_1, c_2$. We unwind these conditions to show how these parameters depend on the market parameters β, E and α'. We list the constraints below:

1. $4\kappa(1+c_2) \le \lambda c_1 \le 1/2$;
2. $\frac{\delta}{\beta} \le \frac{1}{4}$;
3. $(1-2\delta)^{-1/\beta} \le (2-\delta)^{1/(2E-\beta)}$;
4. $2\alpha'(1-2\delta)^{-(2E-\beta)/\beta} + c_1 + c_2\delta \le 1-\delta$;
5. $\left[2(\alpha'+2E-2)(1-2\delta)^{-(2E-\beta)/\beta} + 1 + \delta + c_1 + c_2\delta\right]\lambda \le 1$;
6. $r \ge \frac{512}{\beta}$ and $r \ge \frac{8v(f_1)}{w_j}$;
7. $\delta = \frac{\kappa r}{2}$;
8. $\lambda^2 \le \frac{\kappa r}{16}$.

We first impose that

$$\delta \le \min\left\{\frac{\beta}{4(2E-\beta)}, \frac{1}{4}\right\}, \lambda \le 1, c_1 = \delta, c_2 = 2. \tag{4.3}$$

Constraint 2 is satisfied. Constraint 1 becomes

$$\lambda \ge \frac{24}{r}. \tag{4.4}$$

As $\frac{2(2E-\beta)\delta}{\beta} \leq 1/2$ and $\frac{2E-\beta}{\beta} \geq 1$, by Lemma 3.13(e), $(1-2\delta)^{-(2E-\beta)/\beta} \leq 1 + \frac{4(2E-\beta)}{\beta}\delta$. Thus Constraints 3 and 4 are satisfied when $1 + \frac{4(2E-\beta)}{\beta}\delta \leq 2 - \delta$ and $2\alpha'\left(1 + \frac{4(2E-\beta)}{\beta}\delta\right) + 4\delta \leq 1$ respectively, which are equivalent to

$$\delta \leq \frac{\beta}{\beta + 4(2E-\beta)}, \delta \leq \frac{(1-2\alpha')\beta}{8\alpha'(2E-\beta) + 4\beta}. \tag{4.5}$$

Next, we show that Constraint 5 is satisfied when

$$\left[2(\alpha' + 2E - 2)\left(1 + \frac{4(2E-\beta)}{\beta}\delta\right) + 1 + 4\delta\right]\lambda \leq 1.$$

The bounds on δ in (4.3) gives $\frac{4(2E-\beta)}{\beta}\delta \leq 1$ and $4\delta \leq 1$, hence Condition 5 is satisfied when

$$\lambda \leq \frac{1}{4(\alpha' + 2E - 2) + 2} = \frac{1}{8E + 4\alpha' - 6}. \tag{4.6}$$

Using the bounds on δ in (4.3) and (4.5) with Constraint 7 yields

$$\kappa \leq \frac{2}{r} \cdot \min\left\{\frac{\beta}{\beta + 4(2E-\beta)}, \frac{(1-2\alpha')\beta}{8\alpha'(2E-\beta) + 4\beta}\right\}.$$

Using the bounds on λ in (4.3), (4.4) and (4.6) with Constraint 8 yields

$$\frac{24}{r} \leq \lambda \leq \min\left\{\frac{1}{8E + 4\alpha' - 6}, \sqrt{\frac{\kappa r}{16}}\right\}.$$

The market is defined by the parameters E, β and α'. The parameters κ, λ, r are chosen to satisfy the constraints. The price update rule uses κ, λ, while the warehouse sizes are lower bounded by rw_j. The parameters c_1, c_2 are needed only for the analysis.

Part II

Convex Potential Function Markets

Chapter 5

Preliminaries and Results

In Part II, we relate the tatonnement process to another simple and natural algorithmic process: *gradient descent*. Gradient descent is a family of algorithms used to minimize convex functions. It works by starting at some point and moving in the direction of the negative of the gradient. We consider the class of markets for which the tatonnement process is formally equivalent to performing gradient descent on a convex function. In particular, we define the class of Convex Potential Function (CPF) markets to be those markets for which there is a convex potential function whose gradient[1] is always equal to the negative of the excess demand.

The equivalence with gradient descent opens up the entire toolbox developed to analyse gradient descent and provides a principled approach to show convergence of the tatonnement process. For a large class of CPF markets, we

[1] More generally, the potential function need not be differentiable and the demand need not be unique, in which case the equivalence is between the sub-gradient of the potential function and the set of excess demand vectors.

show that a continuous version of tatonnement converges to an equilibrium.

The subclass of CPF markets for which the demand is differentiable can be characterized in terms of the Jacobian[2] of the demand function. These are exactly the markets for which the Jacobian of the demand function is always symmetric and negative semi-definite.[3] We call this class the *Convex Conservative Vector Field* (CCVF) markets, since functions that have a symmetric Jacobian are called conservative vector fields. CCVF markets contain Fisher markets with Leontief, CES or NCES utilities; for these markets, we show that a discrete version of tatonnement converges to an equilibrium.

We summarize the main results in Part II:

- The class of Eisenberg-Gale (EG) markets contains all Fisher markets for which the equilibrium allocation is captured by a certain type of convex program called the Eisenberg-Gale-type (EG-type) convex program. We show that EG markets are CPF markets by explicitly constructing a convex potential function (Theorem 6.3). In fact, the potential function is the objective function of the dual of the corresponding EG-type convex program.

- We show that a family of continuous versions of the tatonnement process converges to the equilibrium for a large class of CPF markets. This family is derived by considering gradient descent with respect to any *Bregman di-*

[2]Recall that the Jacobian of a differentiable function from $\mathbb{R}^n$ to $\mathbb{R}^n$ is the matrix whose (i, j) entry is the rate of change of the i^{th} component of the function with respect to a change in the j^{th} coordinate.

[3]By contrast, if the off-diagonal entries of the Jacobian are all positive, then the market satisfies the weak gross substitutes property.

vergence and taking the limit as the step size goes to zero (Theorem 7.21). In addition, the process based on KL-divergence (a particular Bregman divergence) converges for an even larger class of CPF markets. This mirrors the classic result of Arrow, Debreu and Hurwitz [1] which shows a similar result for gross substitutes markets.

- For Fisher markets with Leontief utility functions, we show a fairly fast rate of convergence for a discrete version of the tatonnement process, namely, the number of time steps required to reduce the distance from the equilibrium to an ϵ fraction of its initial value, as measured by the potential function, is $O(1/\epsilon)$ (Theorem 8.1).[4] This follows from a small modification of a general result of Chen and Teboulle [7] that shows convergence of gradient descent with Bregman divergences at this rate whenever the convex function satisfies an *upper sandwiching property*. Actually we observe that a slightly weaker version of the property suffices. We show that the potential function for Leontief Fisher markets satisfies the upper sandwiching property for an appropriate choice of parameters with respect to the KL-divergence.

  We also show that, in the worst case, tatonnement uses $\Omega(1/\sqrt{\epsilon})$ iterations with Leontief utilities. Consequently, the linear bounds achieved for CES utilities (see below) cannot extend to Leontief utilities.

- For Fisher markets with complementary-CES utility functions we show a linear convergence for a discrete version of the tatonnement process,

[4]The $O()$ hides market dependent parameters.

i.e. the number of time steps required to reduce the distance from the equilibrium to an ϵ fraction of its initial value, again as measured by the potential function, is $O(\log(1/\epsilon))$ (Theorem 9.5). This is obtained by showing that the potential function in this case satisfies both *upper and lower sandwiching properties*, which together we name the *strong sandwiching property*. The strong sandwiching property is reminiscent of strong convexity but to the best of our knowledge, this particular property has not been used before. The analysis of gradient descent process with the strong sandwiching property may be of independent interest.

We note that when reasonably near to equilibrium, the potential function has value $\Theta(\sum_j p_j(z_j)^2)$, where z_j is the excess demand for good j at prices p (Lemmas 9.2 and 9.3).

- For Fisher markets with NCES utility functions we show that a discrete version of the tatonnement process converges. For complementary-CES Fisher markets, we need, and we prove, the fact that all prices are bounded away from zero throughout the tatonnement process; we need, and we prove, the same fact for NCES Fisher markets too. The proof for the NCES case is less intuitive than that for the complementary-CES case; roughly speaking, the lower bound on prices for the complementary-CES case is *static* — when p_j is low enough, the demand for good j must be huge which then prevents p_j from dropping further; for the NCES case, it is possible to find a set of price vectors with arbitrarily low p_j but with its demand not blowing up. We need to go into the tatonnement dynamic to derive a *dynamical* lower bound on all prices.

This analysis handles substitute-CES Fisher markets as a special case, thereby providing an alternate analysis for the results in [17].

Generalized Gradient Descent

We present a generalized version of gradient descent and a convergence result for this version. For any strictly convex differentiable function h, the *Bregman divergence* with kernel h is defined as

$$d_h(p,q) = h(p) - h(q) - \nabla h(q) \cdot (p - q). \tag{5.1}$$

For example, the square of the Euclidean distance is obtained as a Bregman divergence, $\frac{1}{2}\|p-q\|^2 = d_h(p,q)$, if $h(p) = \frac{1}{2}\|p\|^2$. Another well-known example is the KL-divergence, $\sum_j p_j \ln p_j$, which is obtained when

$$h(p) = \sum_j p_j \ln p_j - p_j. \tag{5.2}$$

For a convex function ϕ, define the tangent hyperplane at a given point q, thought of as a linear approximation to the function, as

$$\ell_\phi(p;q) = \phi(q) + \nabla\phi(q) \cdot (p - q),$$

where $\nabla\phi(q)$ denotes an arbitrary subgradient of ϕ at q. The *generalized gradient descent* w.r.t. a Bregman divergence d_h on the convex function ϕ is a sequence $p^0, p^1, \ldots, p^t \ldots$, defined inductively (for any given starting point p^0)

by

$$p^{t+1} = \arg\min_p \{\ell_\phi(p; p^t) + d_h(p, p^t)\}. \tag{5.3}$$

Note that if the subgradient is not unique, then this sequence need not be unique either.

For the quadratic kernel, $h(p) = \frac{1}{2}\|p\|^2$, the above update rule reduces to the usual gradient descent rule:

$$p^{t+1} = p^t - \nabla\phi(p^t).$$

If the kernel is the weighted entropy, $h(p) = \sum_j \gamma_j(p_j \log p_j - p_j)$ for some weights γ_j, the update rule is

$$p_j^{t+1} = p_j^t \exp\left(\tfrac{-\nabla_j \phi(p^t)}{\gamma_j}\right), \qquad \text{for all } j. \tag{5.4}$$

Birnbaum, Devanur and Xiao [4] showed the following convergence result for gradient descent (5.3).

Theorem 5.1 ([4])**.** *Suppose that the convex function ϕ and the kernel h satisfy: for all p, q,*

$$\phi(p) \leq \ell_\phi(p; q) + d_h(p, q). \tag{5.5}$$

Let p^* be a minimizer of ϕ. Then for all t,

$$\phi(p^t) - \phi(p^*) \leq \frac{d_h(p^*, p^0)}{t}.$$

We need a slightly more general version of this theorem where we require

(5.5) to hold only for consecutive pairs p^t, p^{t+1} for all t, instead of requiring it for all pairs p, q. It is easy to see that their proof needs only this weaker condition, yielding the following theorem.

Theorem 5.2. *Suppose that the sequence of prices p^t obey the following condition:*

$$\phi(p^{t+1}) \leq \ell_\phi(p^t; p^{t+1}) + d_h(p^t, p^{t+1}). \tag{5.6}$$

Let p^* be a minimizer of ϕ. Then for all t,

$$\phi(p^t) - \phi(p^*) \leq \frac{d_h(p^*, p^0)}{t}.$$

The discrete version of the tatonnement process we consider will be equivalent to the gradient descent (5.3) where h is the weighted entropy function, i.e. the update (5.4) for a suitable choice of weights γ_j. The potential function ϕ will satisfy $\nabla_j \phi = -z_j$. The continuous versions we consider are obtained by introducing a multiplier $1/\epsilon$ to the divergence term d_h and taking the limit as $\epsilon \to 0$. This will be presented in more detail in Chapter 7.

Chapter 6

Eisenberg-Gale Markets, Convex Potential Function Markets and Convex Conservative Vector Field Markets

From now on, WLOG, the supply of any good is normalized to be 1 unit.

An *Eisenberg-Gale-type* (EG-type) convex program is a convex program of the form

$$\begin{aligned} &\text{maximize } \sum_i e_i \log u_i(x_{i1}, x_{i2}, \ldots, x_{in}) \\ &\text{s.t. } \forall\, j, \quad \sum_i x_{ij} \le 1, \ \text{(supply constraints)} \\ &\qquad \forall\, i, j, \quad x_{ij} \ge 0. \end{aligned}$$

The base of the log does not matter for the maximization in the convex program. However, later in the paper some calculations are simplified if we assume that the natural logarithm is intended, and so we assume this henceforth.

We note that the above program satisfies Slater's conditions for strong duality (see [5], p. 226, for example) and consequently an optimal solution to the dual problem yields the same optimizing value as the primal program.

Eisenberg-Gale markets (EG markets) were defined by Jain and Vazirani [30], after observing that many markets in the Fisher model have EG-type convex programs that captured the equilibrium, i.e. the optimal solution and the (corresponding) Lagrange multipliers of the supply constraints in the above convex program are respectively equilibrium demands and prices for the market; and conversely, equilibrium demands and prices are respectively an optimal solution and Lagrange multipliers of the supply constraints to the above convex program. The following is a brief list of such markets: Eisenberg and Gale [24] gave a convex program for the linear utilities case, generalized by Eisenberg [23] to the case of homothetic utilities. Jain et al. [31] gave one for homothetic utilities with production and Kelly and Vazirani [33] gave one for certain network-flow markets. Also, it is known that NCES Fisher markets are EG markets. Jain and Vazirani [30] showed many algorithmic and structural properties of EG markets.

We now define the new classes of markets being introduced in this paper.

Definition 6.1. *A market is said to be a* Convex Potential Function *(CPF) market if there is a convex potential function ϕ of the prices such that for all prices p, $\nabla\phi(p) = -z(p)$. By abuse of notation, we let $\nabla\phi$ denote the set of* *sub-gradients when ϕ is not differentiable*[1] *and we let $z(p)$ denote the set of excess demand vectors when the demand is not unique.*

The subclass of CPF markets for which the demand function is differentiable is called the Convex Conservative Vector Field *(CCVF) markets.*

Markets with Leontief utility functions or CES utility functions are both CCVF markets. By contrast, markets with linear utilities are not CCVF. The following characterization of CCVF markets follows essentially immediately from Green's Theorem [27].

Lemma 6.2. *A market with a differentiable demand function is CCVF if and only if the Hessian of its demand function is always a negative semi-definite symmetric matrix.*

Proof: For a CCVF market, the potential function satisfies $\nabla\phi(p) = -z(p)$. As $x(p)$ and hence $z(p)$ are differentiable, it is now easy to check that the Hessian is symmetric. Negative semi-definiteness follows because the potential function ϕ associated with the CCVF market is convex, and hence the Hessian of $-z(p)$ is positive semi-definite.

If the Hessian of $x(p)$ is symmetric, by Green's Theorem [27], there is a function $f : \mathbb{R}^n \to \mathbb{R}$ such that $\nabla f = x$. Let $\phi = \sum_j p_j - f(p)$. Then $\nabla\phi(p) = \mathbb{1} - x(p) = -z(p)$. $\phi(p)$ is convex as its Hessian is positive semi-definite, and as $\nabla\phi(p) = -z(p)$, it follows that the market is a CPF market with a differentiable demand, i.e. it is a CCVF market. □

The main result in this chapter is that all EG markets are CPF markets.

[1] We assume throughout that ϕ is continuous.

Theorem 6.3. *All EG markets are CPF markets.*

Proof: We will give an explicit construction of a convex potential function ϕ for which $\nabla\phi(p) = -z(p)$. ϕ is actually the dual of the corresponding EG-type convex program. Recall that the EG-type convex program has variables x_{ij} for all i and j. We let X denote the set of all these variables. Also recall that the optimum solution gives the equilibrium allocation and the optimal Lagrangian multipliers of the supply constraints in the program are the equilibrium prices. The *KKT conditions* characterize the optimal solution to a convex program and the corresponding Lagrange multipliers. We now write the KKT conditions in terms of the *Lagrangian* function, which is obtained by multiplying the supply constraints by the prices and adding them to the objective function.

$$L(X,p) := \sum_i e_i \log(u_i) - \sum_{i,j} p_j x_{ij} + p \cdot \mathbb{1},$$

on the domain $\{X, p\colon \forall i, j,\ x_{ij} \geq 0;\ \forall j,\ p_j \geq 0\}$. X^* and p^* are said to satisfy the KKT conditions if

1. $X^* \in \arg\max_{X \geq 0} L(X, p^*)$ and
2. $p^* \in \arg\min_{p \geq 0} L(X^*, p)$, which is equivalent to

$$\text{for all } j, \quad p_j^* \cdot (1 - \sum_i x_{ij}^*) = 0. \tag{6.1}$$

We define the potential function to be the dual objective of the EG-type convex program.

$$\phi(p) := \max_{X \geq 0} L(X, p).$$

ϕ is convex by construction. The theorem follows by showing that the gradient of ϕ is equal to the negative of the excess demand (Lemma 6.5). However, the key property of EG markets is captured by the following lemma.

Lemma 6.4. *For an EG market, for all* p*, the demand set* $x(p)$ *is exactly equal to* $\arg\max_{X \geq 0} L(X, p)$*, whenever they are both finite.*

Proof:

Step 1: $x(p) \subseteq \arg\max_{X \geq 0} L(X, p)$: We first argue that if $x(p)$ is the demand at price p then it must also maximize $L(X, p)$. In fact, we first argue it for the special case when the price and the demand form an equilibrium, denoted by p^* and $x(p^*)$. Since this is an EG market, by its definition, the pair $(p^*, x(p^*))$ must correspond to an optimal solution of the corresponding convex program. They must therefore satisfy the corresponding KKT conditions (6.1), which imply that $x(p^*) \in \arg\max_{X \geq 0} L(X, p^*)$ as desired. This immediately shows the same for any price p and every demand $x(p)$, since the pair forms an equilibrium when the supply is equal to $x(p)$. Thus the above holds for all prices and for all demand vectors.

Step 2: $\arg\max_{X \geq 0} L(X, p) \subseteq x(p)$: The argument is similar to Step 1. Consider any p and an X that maximize $L(X, p)$. Consider the market instance with supply equal to $\sum_i x_{ij}$ for good j. Note that the KKT conditions (6.1) are then satisfied with p and X for this instance and therefore they form an optimal solution to the corresponding EG-type convex program. Since any optimal solution to the convex program must also be an equilibrium, it follows that X must be a demand at price p as desired. $\square$

In fact it is easy to see that the converse of Lemma 6.4 is also true, that if for all p the demand set is equal to $\arg\max_{X \geq 0} L(X, p)$ then the market is an EG market. The KKT conditions (6.1) are then exactly the same as the equilibrium conditions.

Lemma 6.5. $\nabla\phi(p) = \mathbb{1} - x(p) = -z(p)$.

Proof: It is well known that if a convex function is defined as the maximum of many linear functions then the gradient is given by the gradient of the linear function providing this maximum. ϕ is indeed defined in this way and by Lemma 6.4 the "arg max'es" are given by the demands, or in other words the maximizing linear function $L(X, p)$ is the one defined using the demands. Thus $\nabla\phi(p) = \mathbb{1} - x(p) = -z(p)$. $\square$

$\square$

The following convenient form for $\phi(p)$ was shown in [20], and will be used in the analyses of the markets with Leontief and CES utilities.

Lemma 6.6. *For EG markets with linear utilities, Leontief utilities, CES utilities or NCES utilities, the dual objective can be written as*

$$\phi(p) = \sum_j p_j - \sum_i e_i \log(\nu_i) + \text{ a constant independent of } p$$

where ν_i is the ratio of e_i to the optimal utility of i at price p, i.e. the minimum cost for obtaining one unit of utility.

Proof: Recall that

$$\phi(p) := \max_{X \geq 0} L(X, p) = \max_{X \geq 0} \left\{ \sum_j p_j + \sum_i e_i \log u_i(x_i) - \sum_{i,j} p_j x_{ij} \right\},$$

where x_i denotes the demands of buyer i. From Lemma 6.4, for each i, an x_{ij} in the $\arg\max$ above is buyer i's demand for good j and therefore $\sum_j p_j x_{ij}$ must be equal to e_i. Hence $\sum_{i,j} p_j x_{ij} = \sum_i e_i$ is a constant. We also rewrite $\sum_i e_i \log u_i(x_i) = \sum_i -e_i \log[e_i/u_i(x_i)] + \sum_i e_i \log e_i$; then setting $\nu_i = e_i/u_i(x_i)$ gives ϕ in the desired form. $\square$

Chapter 7

Convergence of Continuous Time Tatonnement

A continuous version of tatonnement is a trajectory in the price space which, to be notationally consistent with the discrete version, is denoted by p^t for all $t \in \mathbb{R}_+$. Classically, the trajectory is defined by specifying a differential equation $\frac{dp}{dt} = F(t, p(t))$ for all t, which we also call the "update rule". We define a family of update rules derived from gradient descent. As before, let h be a strictly convex differentiable function. The natural way to specify the differential equation is

$$\begin{aligned} p(\epsilon) &:= \arg\min_p \left\{ \nabla\phi(p^t) \cdot (p - p^t) + \tfrac{1}{\epsilon} d_h(p; p^t) \right\}. \\ \tfrac{dp_j}{dt} &:= \lim_{\epsilon \to 0} \frac{p_j(\epsilon) - p_j^t}{\epsilon}. \end{aligned}$$

However, there are three issues we need to address with this specification.

The first issue is that in the markets we consider, the demand function of an agent can be multi-valued at a price vector[1], and hence $\nabla\phi(p^t)$ can also be a set of multiple elements, namely the set of sub-gradients of ϕ at p^t. Since $\nabla\phi(p^t)$ can be multi-valued, $p(\epsilon)$ and hence $\frac{dp_j}{dt}$ can be too. To resolve this, we need the notion of *differential inclusion*, which is a generalization of differential equations. In brief, a differential inclusion is a system which allows $\frac{dp}{dt}$ to take any value from a set. We specify our class of differential inclusions *in the domain* $\mathbb{R}^n_+$, as follows:

$$p^t(\vec{v},\epsilon) := \arg\min_p \left\{\vec{v}\cdot(p-p^t) + \tfrac{1}{\epsilon}d_h(p;p^t)\right\} \tag{7.1}$$

$$F(p^t) := \left\{ \lim_{\epsilon\to 0} \frac{p^t(\vec{v},\epsilon)-p^t}{\epsilon} \;\middle|\; \vec{v}\in\nabla\phi(p^t)\right\} \tag{7.2}$$

$$\tfrac{dp}{dt} :\in F\left(p^t\right). \tag{7.3}$$

The existence of a solution to (7.3) requires F to be non-empty, convex, compact and *upper semi-continuous*. (We will give precise definitions and state the relevant results in Chapter 7.1; we refer the readers to Smirnov's text [41] for more detail on this topic.) In fact, going to set-valued maps also helps us handle some discontinuities, since upper semi-continuity for set-valued maps is in a sense a weaker requirement than continuity for functions.

The second issue is related to the requirement that F be non-empty and compact. Lemma 7.6 shows that if $\nabla\phi$ is finite and bounded, then $\lim_{\epsilon\to 0}\frac{p(\vec{v},\epsilon)-p}{\epsilon}$ exists and is bounded. The main difficulty in showing $\nabla\phi$ is bounded occurs

[1] An example: if a buyer has utility function $u(x_1,x_2)=x_1+3x_2$ and budget 40, then at prices $(p_1,p_2)=(2,6)$, the buyer optimizes her utility by purchasing $(x_1,x_2)=(20-3y,y)$, for any $y\in[0,20/3]$.

when one of the prices tends to zero. This is also related to the next issue. Prices tending to ∞ create a similar difficulty.

The third issue is that we do not allow prices to be negative. This imposes a boundary on the price domain. Classically, existence theorems for differential inclusions/equations guarantee the existence of a solution up to the boundary, i.e. a solution may only be guaranteed for a finite time span. Yet we want *global existence*, i.e. a solution for $t \in [0, +\infty)$. To help resolve this, we will extend $F(p)$ to the negative price domain so as to remove the boundary while ensuring that any solution remains in $\mathbb{R}^n_+$.

As we mentioned, the second and the third issues are connected to the main impediment to proving the convergence of tatonnement, which is the possible presence of zero-valued prices on the tatonnement path. If we are using a rule such as the multiplicative update rule, $\frac{dp_j^t}{dt} = z_j^t p_j^t$, the price p_j will not change if it is equal to zero, precluding convergence to an equilibrium with $p_j^* \neq 0$. One way to avoid this difficulty is to limit the update rule so as to ensure that if a price starts out positive, it will remain positive. Note that this does not preclude a price converging to zero as $t \to \infty$, if that is its value at equilibrium. In this case, the price domain boundary is never reached.

By contrast, if the price update rule is additive, e.g. $\frac{dp_j^t}{dt} = z_j^t$, a price might take on a zero value despite being positive initially. However, this will still be viable so long as the tatonnement avoids price vectors p with $p_j = 0$ and $z_j = \{\infty\}$, which we call *unbounded demand price vectors.* As we shall see, for linear, Leontief, CES or NCES Fisher markets, if they start from a non-unbounded (i.e. from a *bounded*) demand price vector, they will reach only bounded demand price vectors, regardless of which tatonnement rule is used. In this case, we do need to ensure that the price domain boundary is not crossed.

A solution to the differential inclusion (7.3) could seek to leave the domain $\mathbb{R}^n_+$ if $F_j(p^t)$ contains a negative value when $p^t_j = 0$, as it may.[2] But, when $p^t_j = 0$, $F_j(p^t)$ can always be made to contain items with $z_j \geq 0$, providing the hope of a solution that remains in $\mathbb{R}^n_+$. To this end, we observe that at a price vector p with $p_k = 0$, as good k is free, an agent may purchase an infinite amount of good k even if it does not increase her utility. We will use this freedom of being able to purchase additional quantities of zero-priced goods to modify the definition of F so that it is non-empty, compact and includes non-negative excess demands for the goods with zero prices. In addition, we will extend the domain of F to all of $\mathbb{R}^n$ in such a way that the only solutions are those with prices that stay in the domain $\mathbb{R}^n_+$.

7.1 Differential Inclusion and Semi-Continuity of Sets

Definition 7.1. *A differential inclusion is an equation of the form $\frac{dp}{dt} \in F(t, p(t))$, where $F(t, p)$ is a non-empty set for all t and p. This generalizes standard differential equations of the form $\frac{dp}{dt} = f(t, p(t))$, where $f(t, p)$ is single-valued.*

In our setting, F is a function of p alone.

Let $\mathbb{P}(A)$ denote the power set of the set A. Let $\Omega(a)$ denote an open

[2] An example: in a Leontief Fisher market, it is possible that at an equilibrium some $p_j = 0$ with a negative excess demand for good j.

neighborhood of a point a.

Definition 7.2. *A set-valued map $F : Z \to \mathbb{P}(Y)$ is* upper semi-continuous *at $z_0 \in Z$ if for any open set $M \in \mathbb{P}(Y)$ which contains $F(z_0)$, there exists $\Omega(z_0)$ such that for all $z \in \Omega(z_0)$, $F(z) \subset M$. A set-valued map F is upper semi-continuous if it is so at every $z_0 \in Z$.*

A set-valued map $F : Z \to \mathbb{P}(Y)$ is lower semi-continuous *at $z_0 \in Z$ if for any $y_0 \in F(z_0)$ and any neighborhood $\Omega(y_0)$, there exists a neighborhood $\Omega(z_0)$ such that for all $z \in \Omega(z_0)$, $F(z) \cap \Omega(y_0) \neq \varnothing$. A set-valued map F is lower semi-continuous if it is so at every point $z_0 \in Z$.*

A set-valued map $F : Z \to \mathbb{P}(Y)$ is continuous *at $z_0 \in Z$ if it is both upper and lower semi-continuous at z_0. A set-valued map F is continuous if it is so at every $z_0 \in Z$.*

For any sets $A_1, A_2, \cdots, A_k$, let their *sumset* be $\left\{\sum_{i=1}^{k} a_i \,|\, a_i \in A_i\right\}$. We state the following basic facts, which will be useful later.

Lemma 7.3. *(a) If $A_1, A_2, \cdots, A_k$ are convex and compact, then their sumset is convex and compact.*
(b) If $A_1, A_2, \cdots, A_k : Z \to \mathbb{P}(Y)$ are upper semi-continuous at $z \in Z$, then their sumset is upper semi-continuous at z.
(c) If $F_1, F_2 : Z \to \mathbb{P}(Y)$ are two set-valued maps which are upper semi-continuous at $z \in Z$, the map $F^{\cap} : Z \to \mathbb{P}(Y)$, defined as $F^{\cap}(z) = F_1(z) \cap F_2(z)$, is also upper semi-continuous at $z \in Z$.

The following Maximum Theorem is well-known in mathematical economics. It provides results on set-valued map semi-continuity, which are among the required conditions for the existence of a solution to our differential inclusions.

Theorem 7.4 (Maximum Theorem, [3, p. 116])**.** *Let $u : P \times X \to \mathbb{R}$ be a continuous function, and $C : P \to \mathbb{P}(X)$ be a compact set-valued map. Let $C^*(p) = \arg\max_{x \in C(p)} u(p, x)$ and $u^*(p) = \max_{x \in C(p)} u(p, x)$. If C is continuous at some p, then u^* is continuous at p and C^* is non-empty, compact and upper semi-continuous at p.*

Let $B(p_0, \rho)$ denote the closed ball around p_0 with radius ρ.

Theorem 7.5 ([41, p. 96–103])**.** *Let $\frac{dp}{dt} \in F(p(t))$ be a differential inclusion, where $F : P \to \mathbb{P}(\mathbb{R})$ is upper semi-continous at every $p' \in B(p_0, \rho)$ for some $\rho > 0$. Suppose that $F(p')$ is convex and compact for every $p' \in B(p_0, \rho)$, and there exists a finite κ such that $\sup_{\vec{z} \in F(p')} ||z|| \leq \kappa$ for every $p' \in B(p_0, \rho)$. Then for $0 \leq t \leq \rho/\kappa$, there exists an absolutely continuous solution $p(t)$ to the differential inclusion with $p(0) = p_0$.*

7.2 Existence of Trajectory

We will limit the study to the special case where h is a *separable* function, i.e., it is of the form $\sum_j h(p_j)$, for a 1-dimensional function $h : \mathbb{R} \to \mathbb{R}$. Now the minimization in (7.1) separates out into independent minimization problems for each good j. We will use $d_h(p_j, q_j)$ to denote $h(p_j) - h(q_j) - h'(q_j)(p_j - q_j)$, the one dimensional version of Bregman divergence. Note that as h is convex,

$$d_h(p_j, q_j) \geq 0, \tag{7.4}$$

and, by the strict convexity of h,

$$\text{if } p_j \neq q_j, \quad d_h(p_j, q_j) > 0. \tag{7.5}$$

As we will see shortly in Lemma 7.6, $\frac{dp_j}{dt} = -\nabla_j \phi(p^t)/h''(p_j^t)$ if $\nabla_j \phi(p^t)$ and $h''(p_j^t)$ are finite. In order to apply Theorem 7.5 on a ball B around a price vector p, we need this term to be bounded on B. And, in order to make progress, we will also need that $h''(p_j^t) \neq \infty$. (Otherwise, we may "get stuck" at a non-equilibrium price since $\frac{dp_j}{dt}$ would be 0.) These lead us to make assumptions on the allowable h and on the behaviour of the tatonnement, namely that it is *controllable*, as defined in the subsequent subsections.

7.2.1 Allowable h

We will also need h to be twice differentiable. It may be that $h'(0) = -\infty$, but by the convexity of h, this is the only argument for which h' might be infinite. And if $h'(0) = -\infty$ then $h''(0) = \infty$.

Lemma 7.6. *For all j, if $\nabla_j \phi(p^t)$ and $h''(p_j^t)$ are finite, then*

$$\lim_{\epsilon \to 0} \frac{p(\epsilon) - p^t}{\epsilon} = \frac{-\nabla_j \phi(p^t)}{h''(p_j^t)}.$$

Proof: The minimizer in (7.1) must have a zero derivative:

$$\nabla_j \phi(p^t) + \frac{1}{\epsilon} \frac{d\big(d_h(p_j, p_j^t)\big)}{dp_j} = 0. \tag{7.6}$$

Since $\frac{d\left(d_h(p_j,p_j^t)\right)}{dp_j} = h'(p_j) - h'(p_j^t)$, substituting in (7.6) and solving for p_j gives

$$p_j(\epsilon) = h'^{-1}\left(h'(p_j^t) - \epsilon \nabla_j \phi(p^t)\right).$$

Note that since h is strictly convex, h' is strictly increasing and hence is invertible. For notational convenience, let $g(y) = h'^{-1}(y)$. Then $h'(g(y)) = y$, $h''(g(y)) \cdot g'(y) = 1$, therefore $g'(p_j) = \frac{1}{h''(g(p_j))}$. Also note that $g(h'(y)) = y$. Using these we obtain

$$g'(h'(p_j)) = \frac{1}{h''(p_j)}. \tag{7.7}$$

Strictly speaking, the above argument is not valid for $p_j = 0$ if $h'(0) = -\infty$. But in this case, we can check directly that (7.7) is still correct, for then $g'(-\infty) = 0$ and $h''(0) = \infty$. Now,

$$\begin{aligned}
\lim_{\epsilon \to 0} \frac{p_j(\epsilon) - p_j^t}{\epsilon} &= \lim_{\epsilon \to 0} \frac{g(h'(p_j^t) - \epsilon \nabla_j \phi(p^t)) - g(h'(p_j^t))}{\epsilon} \\
&= -\, g'(h'(p_j^t)) \cdot \nabla_j \phi(p^t) \\
&= -\, \nabla_j \phi(p^t) / h''(p_j^t) \quad \text{(by (7.7))}.
\end{aligned}$$

□

We make the following additional assumptions on h.

Definition 7.7. *$h(p)$ is allowable if h is twice differentiable and strictly convex (hence $h''(p) > 0$), $h''(p)$ is finite if $p > 0$, $1/h''$ is continuous, and either*

A1. The market is a Fisher market, or

A2. $\int_p^\infty h''(q)dq = \infty$ for all $p > 0$, *and in addition either*

B1. $h''(p)$ *is finite for all* p*, or*

B2. $\int_0^p h''(q)dq = \infty$ *for all* $p > 0$*; in this case, we say* h *is* controlling*.*

Henceforth, we assume that h is allowable.

We note that two of the most commonly used Bregman divergences satisfy the above assumptions. The first one uses $h(p_j) = \frac{1}{2}p_j^2$; thus $h''(p_j) = 1$; hence $\frac{dp_j}{dt} = -\nabla_j \phi(p)$. Also, for $p > 0$, $\int_p^\infty h''(q)dq = \infty$, so conditions A2 and B1 are satisfied. The second one, which is the KL-divergence, uses $h(p_j) = p_j \log p_j - p_j$, $h'(p_j) = \log p_j$ and $h''(p_j) = 1/p_j$. Hence $\frac{dp_j}{dt} = -p_j \nabla_j \phi(p)$. Also, $\int_0^p \frac{dq}{q} = \log p - \log 0 = \infty$ and for $p > 0$, $\int_p^\infty \frac{dq}{q} = \log \infty - \log p = \infty$, so conditions A2 and B2 are satisfied.

The reason for the condition B2 in Definition 7.7 is to ensure that if the tatonnement starts at a point with finite h'' it will never reach a point with infinite h''. When $h''(p_j) = \infty$, by Lemma 7.6, $\frac{dp_j^t}{dt} = 0$ no matter what the value of $\nabla \phi(p^t)$ is, i.e. p_j remains constant hereafter. This is an unreasonable tatonnement rule.

Lemma 7.8. *Suppose that* $h''(p_j^0)$ *is finite. If* h *is allowable then* $h''(p_j^t)$ *is finite for all* $t \geq 0$*.*

Proof: If condition B1 of Definition 7.7 holds then the result is immediate. So suppose that condition B2 holds. By assumption, $h''(p) = \infty$ only if $p = 0$. As $z_j \geq -1$ always, $\nabla_j \phi(p) \leq 1$ always. Consequently, by Lemma 7.6, $\frac{dp_j^t}{dt} \geq -1/h''(p_j^t)$. Suppose that $p_j^0 > 0$. Then let $\bar{t} > 0$ be the earliest time at which

p_j could be zero. We use condition B2 to justify the last equality below:

$$\bar{t} \geq -\int_0^{p_j^0} \frac{dp_j^t}{dp_j^t/dt} \geq \int_0^{p_j^0} h''(p)dp = \infty.$$

Thus only at time $t = \infty$ can p_j be 0, and hence only at time $t = \infty$ can $h''(p_j)$ be ∞. $\square$

The reason for the condition A2 in Definition 7.7 is to ensure that if the tatonnement starts at a point with finite value, no price will blow up to $+\infty$ in finite time.

Lemma 7.9. *Suppose that p^0 is finite. If h is allowable then p^t is finite for all $t \geq 0$.*

Proof: If the market is a Fisher market then prices remain bounded by the maximum of their initial value and the amount of money in the market. So suppose the market is not a Fisher market; then, by assumption, $\int_p^\infty h''(q)dq = \infty$ for all $p > 0$. Let $p_{\max} = \max p_j$. Define $M^t = \sum_j p_j^t \leq p_{max} \cdot n$. Then $z_{\max} \leq n$. So $\frac{d}{dt} p_{\max}^t \leq n/h''(p_{max}^t)$.

Let $\bar{t}$ be the earliest time at which $p_{\max}^t$ could be infinite. Let tmin $= \arg\min_{t<\bar{t}} p_{\max}^t$. If $p_{\max}^{\text{tmin}} > 0$, then by Condition A2,

$$\bar{t} \geq \frac{1}{n} \int_{p_{\max}^{\text{tmin}}}^\infty h''(p)dp = \infty,$$

and if $p_{\max}^{\text{tmin}} = 0$, then the same bound holds by Conditions A2 and B2. $\square$

The following example shows that a price may blow up to $+\infty$ in finite time if condition A2 is violated.

Example 7.10. *Consider an Arrow-Debreu market with one agent and two goods. The agent has one unit of each good as initial endowment. The agent wants only good* 1*. So the equilibrium price vector is* $(p_1^*, p_2^*) = (p, 0)$ *for any* $p > 0$*. At any* (p_1, p_2)*, the excess demand for good* 1 *is* $(p_1 + p_2)/p_1 - 1 = p_2/p_1$ *and the excess demand for good* 2 *is* -1*.*

Suppose the tatonnement starts at $(p_1, p_2) = (2, 1)$ *and* h *satisfies* $h''(p) = 1/p$ *for* $p \le 1$ *and* $h''(p) = 1/p^3$ *for* $p \ge 1$*. Then* $\frac{dp_2^t}{dt} = -p_2^t$ *and* $\frac{dp_1^t}{dt} = (p_1^t)^2 p_2^t$*. The solution is* $p_1(t) = \frac{2}{2e^{-t}-1}$ *and* $p_2(t) = e^{-t}$*. Note that* $p_1(t)$ *blows up to* $+\infty$ *at* $t = \log 2$*.*

7.2.2 Local Existence

Next we show that there is a solution to (7.3) for some time interval $[0, \bar{t}]$, under additional assumptions. Later, we will show how to extend the solution to arbitrarily large t and remove the assumptions.

In order to apply Theorem 7.5 to (7.3), we need its right hand side $(-\nabla_j \phi(p)/h''(p_j)$ when $\nabla_j \phi(p)$ is finite) to be convex, compact and upper semi-continuous in any ball $B(p_0, \rho)$ we consider. The difficulty we face is that when some prices are zero, the corresponding demands can be infinite, and then compactness will not hold for such price vectors.

To restore compactness we modify F as follows. Let $b > 0$. We define $F_b(p^t) = F(p^t) \cap \{v \mid -b\mathbb{1} \le v \le b\mathbb{1}\}$. We then define the following differential inclusion on $\mathbb{R}^n_+$:

$$\frac{dp}{dt} = F_b(p^t). \tag{7.8}$$

This introduces the possibility that $F_b(p)$ is empty for some p which makes the differential inclusion trivially unsatisfiable. We assume for now that $F_b(p)$ is non empty in a small neighborhood around p, and remove this assumption later.

Definition 7.11. *We say that F is* bounded near p *if there exists some neighborhood $\Omega(p)$ of p and a finite positive number b such that for all $q \in \Omega(p) \cap \mathbb{R}^n_+$, $F_b(q)$ is non-empty and $h''(q)$ is finite.*[3]

Lemma 7.12. *Suppose that h is allowable and that F is bounded near p. Then $F_b(p)$ is convex-valued, compact-valued and upper semi-continuous at p.*

Proof: Let $\Omega(p)$ be the neighborhood of p given by the assumption that F is bounded near p (Definition 7.11), and let $B \subset \mathbb{R}^n_+$ be a compact neighborhood of p such that $B \subset \Omega(p)$ and every positive price in p is positive in B. By our choice of B, $h''(q_j)$ is positive and finite for all $q \in B$ and for all j, so there exists a positive number $\bar{h}$ such that $h''(q_j) \leq \bar{h}$ for all $q \in B$ and for all j. Then on B, $b \geq |z_j(q)/h''(q_j)| \geq |z_j(q)/\bar{h}|$, i.e. $x_j(q) = z_j(q) + 1 \leq b\bar{h} + 1$. Let $\bar{b}$ denote $b\bar{h} + 1$.

We apply Theorem 7.4 with $P = \Omega(p)$, $X = [0, \bar{b}]^n$. u is the utility function of an agent, which we assume to be continuous and concave. For any $q \in \Omega(p)$, $C(q)$ is the set of all affordable bundles in X of the agent at price q. It is well known that $C(q)$ is continuous, and since its range is confined to the compact set X, $C(q)$ is compact-valued. By Theorem 7.4, $C^*(p)$, the set of all affordable optimal bundles of the agent at price p contained in X, is compact and upper semi-continuous at p. By our assumption that $F_b(p)$ is non-empty, $C^*(p)$ is

[3] We remark that this condition is satisfied automatically when $p > \vec{0}$.

also a subset of all affordable optimal bundles of the agent at price p globally (i.e. without confinement to X). Also, since u is concave, $C^*(p)$ is convex.

By the definition of $C^*(p)$ and ϕ, $-\nabla\phi(p)$ is the sumset of $C^*(p)$ over all agents and the set $\{-\mathbb{1}\}$. As $C^*(p)$ is non-empty for each agent, $-\nabla\phi(p)$ is also non-empty. By Lemma 7.3(a) and (b), $-\nabla\phi(p)$ is convex and compact, and it is upper semi-continuous at any p. $(F_b)_j$ is $-\nabla_j\phi(p)$ divided by $h''(p_j)$, while $1/h''$ is continuous and positive at any $p \in P$. So the division by h'' will not affect convexity, compactness and upper semi-continuity. $\square$

The following lemma is immediate.

Lemma 7.13. *Any solution to system* (7.8) *over time interval* $[0, \bar{t}]$ *starting at a price vector* p^0 *such that* F *is bounded near* p^0 *is also a solution to system* (7.3)*.*

As discussed previously, we want to extend the domain for the differential inclusion to $\mathbb{R}^n$. We will work with F_b rather than F, however. To help specify the new differential inclusion system, for any price vector p, we introduce the following notation: letting $p = (p_j)$, we define $p_j^+ = \max\{0, p_j\}$ and $p^+ = (p_j^+)$. The new system is given by

$$\frac{dp^t}{dt} \in G_b(p^t), \tag{7.9}$$

with G_b defined as follows:

1. For $p \in \mathbb{R}^n_+$, $G_b(p) = F_b(p)$.

2. For $p \notin \mathbb{R}^n_+$, let $J(p) = \{j \mid p_j < 0\}$, then set $G_b(p) = G_b(p^+) \cap \{z \mid \forall j \in J(p),\, z_j \geq 0\}$.

Lemma 7.14. *Let $p \in \mathbb{R}^n$. Suppose that h is allowable and F is bounded near p^+. Then $G_b(p)$ is convex, compact and upper semi-continuous at p.*

Proof: As $G_b \equiv F_b$ in $\mathbb{R}^n_+$, by Lemma 7.12, the result is immediate for $p > \vec{0}$.

For the other p's, note that $G_b(p) = G_b(p^+) \cap \{z \mid \forall j \in J(p),\, z_j \geq 0\}$ is the intersection of two sets, the first being convex and compact and the second being convex and closed. So $G_b(p)$ is convex and compact. What remains is to check upper semi-continuity at these p's. There are two cases: $p \in \mathbb{R}^n_+$ but it has some zero prices, or $p \notin \mathbb{R}^n_+$.

Case 1: $p \in \mathbb{R}^n_+$ but it has some zero prices. For any open set M which contains $G_b(p) = F_b(p)$, by Lemma 7.12, we can take a sufficiently small neighborhood $B(p, \delta)$ of p such that for all $q \in B(p, \delta) \cap \mathbb{R}^n_+$, $F_b(q) \subset M$. Then, for any $q \in B(p, \delta) \setminus \mathbb{R}^n_+$, note that $q^+ \in B(p, \delta)$ since $||q^+, p|| \leq ||q, p||$, and, of course, $q^+ \in \mathbb{R}^n_+$. Thus $F_b(q^+) \subset M$; and $G_b(q) \subseteq G_b(q^+) = F_b(q^+) \subset M$. So G_b is upper semi-continuous at p.

Case 2: $p \notin \mathbb{R}^n_+$. For any $q \in \mathbb{R}^n$, let $V(q)$ denote the set $\{v \mid \forall j \in J(q),\, v_j \geq 0\}$. For any $q \notin \mathbb{R}^n_+$, $G_b(q) = G_b(q^+) \cap V(q)$. By Case 1 and our conditions on p, $G_b(p^+)$ is upper semi-continuous at p^+. p^+ is continuous in p. Hence $G_b(p^+)$ is upper semi-continuous at p. Next, we observe that there exists a small $\delta > 0$ such that for all $q \in B(p, \delta)$, if $p_j \neq 0$, then $\operatorname{sign}(q_j) = \operatorname{sign}(p_j)$ and consequently $V(q) \subseteq V(p)$; it immediately follows that $V(p)$ is upper semi-continuous at p. Now, by Lemma 7.3(c), $G_b(p)$ is upper semi-continuous at p. $\square$

Lemma 7.15. *Any solution to system* (7.9) *over time interval* $[0, \bar{t}]$ *starting at price vector* p^0 *is also a solution of* (7.3) *if* F *is bounded near* p^0.

Proof: We will show that any solution of (7.9) is a solution of (7.8). The result then follows from Lemma 7.13.

In the definition of G_b, at a price vector p with $p_j < 0$, $G_{b,j}(p)$ is always positive or zero, so it is impossible for any tatonnement trajectory satisfying (7.9) to enter the region $p_j < 0$. Hence, all prices remain positive or zero, i.e. $p^t \in \mathbb{R}^n_+$ for all t. In $\mathbb{R}^n_+$, (7.8) is identical to (7.9), so we are done. □

Lemma 7.16. *Suppose that h is allowable, and F is bounded near p^0. Then there is a time $\bar{t} > 0$ such that* (7.3) *has an absolutely continuous solution for time interval* $[0, \bar{t}]$ *with* $p(0) = p^0$.

Proof: By Lemma 7.14, $G_b(p)$ is convex, compact and upper semi-continuous at p in the interior of $\Omega(p^0)$. Now, by Theorem 7.5, (7.9) has an absolutely continuous solution with $p(0) = p^0$ for some time interval $[0, \bar{t}]$, where $\bar{t} > 0$. And by Lemma 7.15, this is also a solution to (7.3). □

Lemma 7.16 gives us a local solution (i.e., up to some time $\bar{t} > 0$) under the assumption that F is bounded near p^0. We need to remove this assumption and we need a solution for arbitrarily large $\bar{t}$. For these we need the notion of controllability.

7.2.3 Controllability

Given a starting price vector p^0 and any finite time $\bar{t} \geq 0$, we need to ensure that there is a sufficiently large $b = b(p^0, \bar{t})$ guaranteeing that the tatonnement remains in the domain with $F_b \neq \phi$ during the time interval $[0, \bar{t}]$ (so that the differential inclusion is defined for all points encountered during the tatonnement).

This will be ensured by the assumption of controllability. To understand this, we first need to characterize the set of optimal bundles of an agent at price vector p. There are two possibilities:

1. Every bundle includes at least one good having infinite demand. Then we say that p is an *unbounded demand* price vector. Note that this good must then have price zero, and by Lemma 7.8 this can occur only if $h''(0)$ is finite. Via the controllability requirement, we will ensure that in this case the tatonnement trajectory does not reach any unbounded demand price vector. (If $h''(0)$ is infinite then this is already ensured by Lemma 7.8).

2. All the demands in at least one bundle are finite. Then we say that p is a *bounded demand* price vector. Note that if p includes a zero price, $p_j = 0$ say, then an optimal bundle can have an infinite demand for good j; but p is a bounded demand price vector if for all such j, the demand for good j could be finite.

   For instance, in a Leontief Fisher market, an equilibrium price vector may include a zero price but it will be a bounded demand price vector; clearly, we want the tatonnement trajectory to be able to converge to it. Furthermore, in this case, as the tatonnement proceeds, we want the agent's sequence of optimal bundles to always have bounded demands, and further these bounds should apply throughout the tatonnement process.

We are now ready to define controllability.

Definition 7.17. *Let ϕ be a potential function and $\mathcal{T}$ a continuous tatonnement rule. The pair $(\phi, \mathcal{T})$ is* controlled*, if for any bounded demand starting price* *vector* p^0 *and any finite time* $\bar{t} \geq 0$*, there are finite bounds* $b(p^0, \bar{t})$ *and* $c(p^0, \bar{t})$ *such that for any tatonnement trajectory induced by* (7.8)*, there exists a neighborhood* Ω *of the trajectory in which for any* $p \in \Omega$ *and for any* j*,*

1. $|-\nabla_j \phi(p)/h''(p_j)| \leq b(p^0, \bar{t})$ *and* $p \leq c(p^0, \bar{t})$ *for all* $0 \leq t \leq \bar{t}$*;*

2. $\lim_{t \nearrow \bar{t}} b(p^0, t)$ *and* $\lim_{t \nearrow \bar{t}} c(p^0, t)$ *are finite* [4]*,*

i.e. both the prices and the rate of change of the prices remain bounded throughout the tatonnement process up to and including time $\bar{t}$*.*

We will show that if h is controlling (recall Definition 7.7) then $(\phi, \mathcal{T})$ is controlled. In this subsection, we will also show that controllability is obeyed by Fisher markets with linear, Leontief or CES utilities along with *any* tatonnement rule (i.e. even if h is not controlling). The same holds for Fisher markets with NCES utilities[5]; we will prove this in Chapter 10. But it is not clear if this applies to all markets or even to all EG markets.

Lemma 7.18. *If* h *is controlling then* $(\phi, \mathcal{T})$ *is controlled.*

Proof: As h is controlling, in finite time $\bar{t}$, the trajectory is both upper-bounded and bounded away from zero[6], say $0 < \underline{p}(\bar{t}) \leq p_j^t \leq \bar{p}(\bar{t}) < +\infty$, for all j and for all $0 \leq t \leq \bar{t}$. Then there exists a neighborhood Ω of the

[4] Without loss of generality, we may assume that $b(p^0, t), c(p^0, t)$ are increasing functions of t, so the limits exist.

[5] In Chapter 10, we will prove that for a discrete tatonnement dynamic in NCES Fisher market, if the starting prices are all positive, then all prices are bounded away from zero throughout the tatonnement process. Since the total amount of money in the market is finite, so the demand for any good is upper bounded by a finite value. One can easily extend this result for continuous tatonnement.

[6] These follow easily from the proofs of Lemma 7.8 and Lemma 7.9.

trajectory up to time $\bar{t}$ such that all prices in Ω are between $\underline{p}(\bar{t})/2$ and $\bar{p}(\bar{t})+1$. Set $c(p^0, \bar{t}) = \bar{p}(\bar{t}) + 1$.

For all $p \in \Omega$, $0 < \underline{p}(\bar{t})/2 \leq p_j \leq \bar{p}(\bar{t}) + 1 < +\infty$, so $h''(p_j)$ is bounded away from 0.

As ϕ is convex, $\nabla\phi$ is finite except possibly at the boundary, i.e. when one or more prices is zero. When all prices are between $\underline{p}(\bar{t})/2$ and $\bar{p}(\bar{t}) + 1$, $\nabla\phi$ is bounded. Combined with the last paragraph, $|-\nabla\phi(p)/h''(p_j)|$ is bounded on Ω. Set $b(p^0, \bar{t})$ to be an upper bound of $|-\nabla\phi(p)/h''(p_j)|$ on Ω. □

Lemma 7.19. *Fisher markets with linear, Leontief or CES utilities along with* any *tatonnement rule are all controlled.*

Proof: We first observe that in Fisher markets prices remain bounded. The following notation will be helpful. Let U be the maximum initial price and M the total money in the market, and let $\overline{U} = \max\{U, M\}$. Observe that for any j, if $p_j = \overline{U}$, then $x_j \leq 1$, and consequently any tatonnement rule will not increase p_j beyond $\overline{U}$.

We can now show that for Fisher markets $1/h''$ remains bounded. For $h'' > 0$ and consequently in the bounded region $\mathbb{R}^n_+ \cap \{p \leq \overline{U}\,\mathbb{1}\}$ the supremum of $1/h''$ is its maximum, which is therefore finite.

Thus to prove the result of the lemma it suffices to show that $-\nabla_j\phi(p) = z_j(p)$ remains bounded throughout the tatonnement.

We begin by considering substitute-CES utilities. Let $f = \min_j\{p_j/p_j^*, 1\}$. Cole and Fleischer [17] showed that if $p_j = f\,p_j^*$, then $x_j \geq 1$. Thus if p_j is ever reduced to $f\,p_j^*$, the tatonnement update will not decrease it further.

Consequently, for all j, $p_j \geq fp_j^*$ throughout the tatonnement process. Hence $x_j \leq M/(fp_j^*)$ throughout the tatonnement process, for all j, where M is the total money in the market. It follows that $z_j \leq M/(fp_j^*) - 1$, for all j. This analysis applies to linear utilities too.

We turn to complementary CES utilities. By Lemma 9.8,

$$p_j^t \geq p_j^* \cdot \min\{p_j^0/p_j^*, (\overline{U}/L^*)^{\min_i \rho_i}\},$$

where $L^* = \min_j\{p_j^*\}$. It follows that the demands are upper bounded by $x_j \leq \max\{p_j^*/p_j^0, (L^*/\overline{U})^{\min_i \rho_i}\}$, and hence $z_j \leq \max\{p_j^*/p_j^0, (L^*/\overline{U})^{\min_i \rho_i}\} - 1$.

Finally, we consider Leontief utilities. By Lemma 8.4, $x_j^t \leq x_j^\circ + \sum_i \max_k \frac{b_{ij}}{b_{ik}}$, and hence $z_j^t \leq z_j^\circ + \sum_i \max_k \frac{b_{ij}}{b_{ik}}$. $\square$

Now we are ready to complete the proof of the existence of a solution to the differential inclusion system (7.3).

Lemma 7.20. *Suppose that $h''(p^0)$ is finite, h is allowable and $(\phi, \mathcal{T})$ is controlled. Then for any bounded demand starting price vector p^0 there exists a solution p^t to (7.3) for time range $[0, \infty)$, with p^t an absolutely continuous function for any bounded time span, and $p^t(t = 0) = p^0$.*

Proof: We will prove the result for differential inclusion (7.9) and then the result follows from Lemma 7.15.

The controllability assumption allows us to pick $b = b(p^0, t)$ for some $t > 0$ and have F be bounded near p. We can therefore apply Lemma 7.16 to get a solution for some time interval $[0, t']$ with $t' > 0$. By Lemma 7.15, this is also a solution for (7.3). Once again, due to the assumption of controllability, the solution path cannot end at a point with $\nabla_j \phi / h'' = -\infty$ for any j. By the continuity of $\nabla_j \phi / h''$, there is then a ball around $p^{t'}$ in which $\nabla_j \phi / h''$ is bounded. So we can repeatedly extend the path by additional applications of Lemma 7.16. Suppose that this yields an open path ending at but possibly not reaching some time $\bar{t}$. We first argue that it can be extended to $\bar{t}$ and then can be extended yet further.

By the controllability assumption, for any $t \in [0, \bar{t})$, all prices in p^t are bounded by $\lim_{t \nearrow \bar{t}} c(p^0, t)$, which is finite; then the sequence $\{p^t\}_{0 \le t < \bar{t}}$ has a cluster point $\tilde{p}$. Then by the controllability assumption again, all $\frac{dp_j^t}{dt}$ are bounded by $\lim_{t \nearrow \bar{t}} b(p^0, t)$, which is again finite. Hence, $\{p^t\}_{0 \le t < \bar{t}}$ has at most one cluster point. So $\tilde{p}$ is the unique cluster point of the sequence $\{p^t\}_{0 \le t < \bar{t}}$. Setting $p^t(t = \bar{t}) = \tilde{p}$ extends the solution to $t = \bar{t}$.

Again by the controllability assumption, there exists a neighborhood of $\tilde{p}$ such that all q in the neighborhood has finite $\nabla \phi(q) / h''(q)$. By Lemma 7.16, we can extend the path P beyond time $\bar{t}$ by at least a positive time period. $\square$

7.3 Convergence of Trajectory

In Arrow-Debreu markets, it is well-known that if p^* is an equilibrium price vector, then cp^*, where c is any positive constant, is also an equilibrium price vector. It is standard to consider *normalized prices*, price vectors $\hat{p}$ such that $\sum \hat{p} = 1$. Note that for any price vector p with at least one positive price, the corresponding normalized price vector $\hat{p}$ is given by $\hat{p}_j = p_j / \left(\sum_\ell p_\ell\right)$.

We are ready to state the main result of this section.

Theorem 7.21. *Let $\phi : \mathbb{R}^n_+ \to \mathbb{R}$ and $p^t \in \mathbb{R}^n_+$ be defined by* (7.1)–(7.3)*. Suppose that ϕ is convex and h is allowable. Further suppose that ϕ together with the tatonnement rule given by* (7.3) *is controlled. Then, for any starting bounded demand price vector p^0 such that for all j, $h''(p^0_j)$ is finite, if the market is a Fisher market, then*

$$\lim_{t\to\infty} p^t = p^*$$

where p^* is a minimizer of ϕ.

In Arrow-Debreu markets, if in addition d_h is the KL-divergence then

$$\lim_{t\to\infty} \hat{p}^t = \hat{p}^*$$

where $\hat{p}^*$ is a normalized minimizer of ϕ.

For any CPF market, by definition, there exists a ϕ such that $-\nabla\phi(p) = z(p)$. Substituting z for $-\nabla\phi$ in (7.1)–(7.3) gives a tatonnement update rule for which, by Theorem 7.21, the potential converges to its equilibrium value.

Lemma 7.22. *For any Arrow-Debreu market in which ϕ exists, for any positive real number c, $\phi(p) = \phi(cp)$.*

Proof: By the Walras law, $p \cdot \nabla\phi(p) = 0$. By the definition of ϕ, $\nabla\phi(p) = \nabla\phi(cp)$. By the definition of subgradient,

$$\phi(p) \geq \phi(cp) + (p - cp) \cdot \nabla\phi(cp) = \phi(cp) + (1-c)p \cdot \nabla\phi(p) = \phi(cp)$$

and

$$\phi(cp) \geq \phi(p) + (cp - p) \cdot \nabla\phi(p) = \phi(p) + (c-1)p \cdot \nabla\phi(p) = \phi(p).$$

These two inequalities imply that $\phi(p) = \phi(cp)$. $\square$

Lemma 7.23. *Suppose that h is allowable and $h''(p_j^0)$ is finite for all j. Let p^* be any minimizer of ϕ. Then $d_h(p_j^*, p_j^t)$ is finite for all t and j.*

Suppose that ϕ is the potential function for a Fisher market. Then $\sum_j \frac{d}{dt} d_h(p_j^*; p_j^t) < 0$, unless p^t is a minimizer of ϕ.

Suppose that ϕ is the potential function for an Arrow-Debreu market. Let $\hat{p}^*$ be any normalized minimizer of ϕ, and suppose that d_h is the KL-divergence. Then $\hat{p}^t$, the normalized price vector corresponding to p^t, satisfies $\sum_j \frac{d}{dt} d_h(\hat{p}_j^*, \hat{p}_j^t) < 0$, unless p^t is a minimizer of ϕ.

Proof: By Lemma 7.8, $h''(p_j^t)$ is finite for all t and j, and hence so is $h'(p_j^t)$. As h is always finite, it follows that $d_h(p_j^*, p_j^t) = h(p_j^*) - h(p_j^t) - h'(p_j^t)(p_j^* - p_j^t)$ is finite.

To avoid clutter we write p_j for p_j^t. We first prove the result for Fisher markets. Recall that $d_h(p_j^*, p_j) = h(p_j^*) - h(p_j) - h'(p_j)(p_j^* - p_j)$. So,

$$\begin{aligned}
\frac{d}{dt} d_h(p_j^*, p_j) &= -\frac{dh(p_j)}{dt} - \frac{dh'(p_j)}{dt}(p_j^* - p_j) + h'(p_j)\frac{dp_j}{dt} \\
&= -h''(p_j) \cdot \frac{dp_j}{dt} \cdot (p_j^* - p_j) \qquad \left(\text{since } \frac{dh(p_j)}{dt} = h'(p_j)\frac{dp_j}{dt}\right) \\
&= \nabla_j \phi(p) \cdot (p_j^* - p_j) \qquad\qquad (\text{from Lemma 7.6}).
\end{aligned}$$

By the definition of the subgradient, $\phi(p^*) \geq \phi(p) + \nabla\phi(p) \cdot (p^* - p)$. Thus

$$\sum_j \tfrac{d}{dt} d_h(p_j^*, p_j) = \sum_j \nabla_j \phi(p) \cdot (p_j^* - p_j) \leq \phi(p^*) - \phi(p) < 0, \tag{7.10}$$

unless $p = p^*$.

Next we prove the result for Arrow-Debreu markets. Let $S = \sum_\ell p_\ell$. Then $\hat{p}_j = p_j / S$.

$$\begin{aligned}
&\frac{d}{dt} d_h(\hat{p}^*, \hat{p}) \\
&= \sum_j \frac{\partial d_h(\hat{p}_j^*, \hat{p}_j)}{\partial \hat{p}_j} \cdot \frac{\partial \hat{p}_j}{\partial t} = \sum_j \frac{\partial d_h(\hat{p}_j^*, \hat{p}_j)}{\partial \hat{p}_j} \sum_k \frac{\partial \hat{p}_j}{\partial p_k} \cdot \frac{\partial p_k}{\partial t} \\
&= \sum_j \frac{\partial d_h(\hat{p}_j^*, \hat{p}_j)}{\partial \hat{p}_j} \left[\frac{1}{S} \frac{\partial p_j}{\partial t} + \sum_k \frac{-p_j}{S^2} \frac{\partial p_k}{\partial t} \right] \\
&= \frac{1}{S^2} \sum_j h''(\hat{p}_j) \cdot (\hat{p}_j^* - \hat{p}_j) \left[S \frac{\nabla_j \phi(p)}{h''(p_j)} - p_j \sum_k \frac{\nabla_k \phi(p)}{h''(p_k)} \right] \\
&= \frac{1}{S} \sum_j \frac{h''(\hat{p}_j)}{h''(p_j)} \nabla_j \phi(p) \cdot (\hat{p}_j^* - \hat{p}_j) - \frac{1}{S^2} \left(\sum_k \frac{\nabla_k \phi(p)}{h''(p_k)} \right) \sum_j p_j h''(\hat{p}_j) \cdot (\hat{p}_j^* - \hat{p}_j).
\end{aligned}$$

When h is the kernel of the KL-divergence, $h''(\hat{p}_j) = \frac{1}{\hat{p}_j} = \frac{S}{p_j}$. Thus $p_j h''(\hat{p}_j) = S$ and $\frac{h''(\hat{p}_j)}{h''(p_j)} = S$. It follows that

$$\frac{d}{dt} d_h(\hat{p}^*, \hat{p}) = \sum_j \nabla_j \phi(p) \cdot (\hat{p}_j^* - \hat{p}_j) - \frac{1}{S} \left(\sum_k \frac{\nabla_k \phi(p)}{h''(p_k)} \right) \left(\sum_j (\hat{p}_j^* - \hat{p}_j) \right).$$

Since $\hat{p}^*$ and $\hat{p}$ are both normalized prices, the second term on the right hand side is zero. Noting that $\nabla_j \phi(p) = \nabla_j \phi(\hat{p})$, and by Lemma 7.22, we see that the rest of the argument is the same as for the Fisher markets. $\square$

Proof of Theorem 7.21: In a Fisher market the prices will be bounded by the maximum of their initial values and $\sum_i e_i$. In an Arrow-Debreu market, we consider only the normalized prices, and these too are bounded. Let B denote the bounded set of prices. We may assume that B is closed[7].

The proof comprises four steps:

1. As p^t lies in a bounded domain, it must have a convergent subsequence, which converges to a point q, say.

2. Let P^* denote the set of equilibrium prices for Fisher markets, or the set of normalized equilibrium prices for Arrow-Debreu markets. Recall that $d_h(p^*, p) = \sum_j d_h(p_j^*, p_j)$. Then, for any fixed $p^* \in P^*$, we can conclude from Lemma 7.23 that $d_h(p^*;, p^t)$ is monotonically decreasing. By (7.4), $d_h(p^*;, p^t) \geq 0$; consequently $\lim_{t\to\infty} d_h(p^*, p^t)$ exists, and it must equal $d_h(p^*; q)$, by the continuity of d_h.

3. Show that q is a minimizer of ϕ. (Proof below.)

4. By the second and the third steps, $d_h(q;, p^t) \to d_h(q, q) = 0$. Using this, show that $p^t \to q$. (Proof below.)

<u>Proof of Step 3.</u> Suppose that q were not a minimizer of ϕ.

Note that the set P^* is closed (due to the continuity of ϕ), so $P^* \cap B$ is compact. Let $d(q') = \min_{p' \in P^* \cap B} d_h(p', q')$; since $P^* \cap B$ is compact, the minimum is attained.

[7]If not, replace B by its closure.

Since $q \notin P^*$, $d(q) > 0$. Also, by Lemma 7.23, $d_h(p^*, q)$ is finite, hence $d(q) \leq d_h(p^*, q)$ is also finite. Let $Q = \{q' \mid d(q') \geq d(q)\} \cap B$. Since d_h is continuous and $P^* \cap B$ is compact, it follows that Q is compact. Let $\delta = \min_{q' \in Q} \phi(q') - \phi(p^*)$, since Q is compact, the minimum is attained. By definition, Q contains no minimizer of ϕ, so $\delta > 0$.

From Step 2, for any $p^* \in P^* \cap B$, for all $t \geq 0$, $d_h(p^*, p^t) \geq d_h(p^*, q)$ and $d_h(p^*, q) \geq d(q)$, so $p^t \in Q$ for all $t \geq 0$. By (7.10), $\frac{d}{dt} d_h(p^*, p^t) \leq -[\phi(p^t) - \phi(p^*)] \leq -\delta < 0$, which implies that $d_h(p^*, p^t)$ will eventually go below zero, a contradiction.

Proof of Step 4. Suppose that p^t does not converge to q. Then there exists an $\epsilon > 0$ such that for any T, there exists a $t(T) > T$ with $||p^{t(T)}, q|| \geq \epsilon$.

Let $A = \{p \mid ||q, p|| \geq \epsilon\}$, which is closed. Note that $A \cap B$ is compact. Since $d_h(q; p)$ is non-negative (but possibly $+\infty$), finite at some $p \in A \cap B$ (e.g. $p^{t(T)}$ for any T), and continuous at every $p \in A \cap B$ at which it is finite, $\inf_{p \in A \cap B} d_h(q; p) = \min_{p \in A \cap B} d_h(q; p) = \delta' > 0$, by (7.5). Since $p^{t(T)} \in A \cap B$, $d_h(q, p^{t(T)}) \geq \min_{p \in A \cap B} d_h(q; p) = \delta' > 0$, i.e. $d_h(q; p^t)$ does not converge to zero, a contradiction. $\square$

Chapter 8

Leontief Fisher Markets

In this chapter we consider Fisher markets in which every buyer has a Leontief utility. We analyse the update rule (5.3) with $d_h = 6 \cdot \gamma \cdot d_{\text{KL}}$ where d_{KL} is the KL-divergence, and γ is a market dependent parameter. This update rule, which is minimizing $\nabla\phi(p^t) \cdot (p - p^t) + \gamma d_h(p, p^t)$ or equivalently is minimizing $-z \cdot (p - p^t) + \gamma[p \log p - p - \log p^t \cdot (p - p^t)]$, amounts to

$$p_j^{t+1} = p_j^t \cdot \exp(z_j/\gamma). \tag{8.1}$$

We show an $O(1/\epsilon)$ convergence rate as specified in the next theorem.

Theorem 8.1. *For a Leontief market, for a sequence of price updates defined by* (8.1), *for all* t,

$$\phi(p^t) - \phi(p^*) \leq \frac{6\gamma d_{KL}(p^*, p^0)}{t}$$

where $\gamma = 5 \cdot \max_j\{x_j^\circ + 2 \cdot \sum_i \max_k \frac{b_{ij}}{b_{ik}}\}$.

The theorem follows by showing that the sandwiching property (5.6) required by Theorem 5.2 is satisfied, which is done in Lemma 8.3 below (recall that $d_h = 6 \cdot \gamma \cdot d_{\text{KL}}$ here).

We also show that in general the convergence rate is $\Omega(1/\sqrt{\epsilon})$ as specified in the next theorem. We defer its proof to Chapter 8.1.

Theorem 8.2. *There is a 2-good, 2-buyer Leontief market such that*

$$\phi(p^t) - \phi(p^*) = \Omega\left[\frac{\phi(p^0) - \phi(p^*)}{t^2}\right].$$

By Lemma 6.6, $\phi(p^t) = \sum_j p_j^t - \sum_i e_i \log \nu_i$, where ν_i is the minimum cost buyer i has to pay to obtain one unit of utility. The maximum utility obtainable by buyer i, as given in Chapter 1.3, is $e_i / \sum_j b_{ij} p_j$. This utility is obtained by spending e_i money; consequently, the minimum cost for one unit of utility is $\sum_j b_{ij} p_j$. Thus the potential function is given by

$$\phi(p^t) = \sum_j p^t - \sum_i e_i \log\left(\sum_j b_{ij} p_j\right).$$

Notation. We let x^t denote the demands following the price update at time t, and x° denote the initial demands. We also let $\Delta p_j = p_j^{t+1} - p_j^t$ for all j.

Lemma 8.3. *If* $|\Delta p_j| \leq p_j/4$, *then*

$$\phi(p^{t+1}) - \ell_\phi(p^{t+1}; p^t) \leq 6\gamma d_{KL}(p^{t+1}, p^t).$$

Thus the sandwiching property (5.6) holds if $|\Delta p_j| \leq p_j/4$. To ensure this, we require that $\gamma \geq 5 \cdot \max_{j,t}\{1, x_j^t\}$, where we are maximizing the x_j^t over all the time steps of the algorithm, for then $p_j^{t+1} \leq p_j^t \exp(1/5)$ and $|\Delta p_j|/p_j \leq \exp(1/5) - 1 \leq \frac{1}{4}$. Of course, γ has to be picked at the beginning, at which point one may not know the value of $\max_{j,t}\{1, x_j^t\}$. In the following lemma, we show that picking $\gamma = 5 \cdot \max_j\{x_j^\circ + 2 \cdot \sum_i \max_k \frac{b_{ij}}{b_{ik}}\}$ suffices. However, if a better bound were known, that could be used instead.

Proof of Theorem 8.1: The result follows by applying Theorem 5.2. To do this, it suffices to ensure that (5.6) holds for every price update. This is guaranteed by Lemma 8.3, for, as we have just seen, by construction $|\Delta p_j| \leq p_j/4$ for every price update. □

Lemma 8.4. *For any continuous tatonnement, $x_j^t \leq x_j^\circ + \sum_i \max_k \frac{b_{ij}}{b_{ik}}$, and for the discrete tatonnement with update rule* (8.1)*, $x_j^t \leq x_j^\circ + 2 \cdot \sum_i \max_k \frac{b_{ij}}{b_{ik}}$, for all goods j and all times t.*

Proof: We drop the superscript t when the meaning is clear from the context. Suppose that $x_{ij} = e_i \cdot b_{ij}/\sum_k b_{ik}p_k \geq 1$; then $x_j \geq 1$ and so p_j can only increase. If $\min_l e_i \cdot b_{i\ell}/\sum_k b_{ik}p_k \geq 1$, or equivalently if $e_i/\sum_k b_{ik}p_k \geq 1/\min_l b_{i\ell}$, then every p_k for which $b_{ik} \neq 0$ can only increase, and hence the x_{ik} for which $b_{ik} \neq 0$ can only decrease; i.e. if $x_{ij} = e_i \cdot b_{ij}/\sum_k b_{ik}p_k \geq b_{ij}/\min_l b_{i\ell} = \max_k b_{ij}/b_{ik}$, x_{ij} can only decrease. Hence, for any continuous tatonnement, x_{ij} is never larger than the maximum of this value and its initial value; i.e. $x_{ij} \leq \max_k\{x_{ij}^\circ, b_{ij}/b_{ik}\}$. Thus, in this case, $x_j^t \leq x_j^\circ + \sum_i \max_k \frac{b_{ij}}{b_{ik}}$. In the case of the discrete price updates, in one round of price changes, the prices drop by at most $\exp(\frac{1}{5})$, and hence the demands increase by at most $\exp(\frac{1}{5}) \leq 2$. Thus, unless initially larger, $x_{ij} < 2 \cdot \max_k b_{ij}/b_{ik}$[1]. Thus $x_{ij} \leq \max_k\{x^\circ_{ij}, 2 \cdot b_{ij}/b_{ik}\}$. Consequently, $x_j = \sum_i x_{ij} \leq x^\circ_j + 2 \cdot \sum_i \max_k \frac{b_{ij}}{b_{ik}}$. □

Before proving Lemma 8.3, we state the following claims, proved in the appendix. We let Δp_j denote $p^{t+1} - p^t$. In the following claims, the index t on the prices and demands is implicit.

Claim 8.5. *For all j,*

$$\frac{1}{e_i}\sum_{j,k} x_{ij}x_{ik}|\Delta p_j| \cdot |\Delta p_k| \leq \sum_l \frac{x_{i\ell}}{p_\ell}(\Delta p_\ell)^2.$$

Proof: This result follows by rewriting e_i as $\sum_k x_{ik}p_k$.

$$\begin{aligned}
e_i \sum_l \frac{x_{il}}{p_\ell}(\Delta p_\ell)^2 &= \sum_l \frac{x_{il}\left(\sum_k x_{ik}p_k\right)}{p_\ell}(\Delta p_\ell)^2 = \sum_{l,k} x_{il}x_{ik}\frac{p_k}{p_\ell}(\Delta p_\ell)^2 \\
&= \sum_l x_{il}^2(\Delta p_\ell)^2 + \sum_{k,l:k\neq l} x_{ik}x_{il}\frac{p_k}{p_\ell}(\Delta p_\ell)^2 \\
&= \sum_l x_{il}^2(\Delta p_\ell)^2 + \sum_{k<l} x_{ik}x_{il}\left(\frac{p_k}{p_\ell}(\Delta p_\ell)^2 + \frac{p_\ell}{p_k}(\Delta p_k)^2\right).
\end{aligned}$$

Now, we apply the AM-GM inequality:

$$\begin{aligned}
e_i \sum_l \frac{x_{il}}{p_\ell}(\Delta p_\ell)^2 &\geq \sum_l x_{il}^2(\Delta p_\ell)^2 + \sum_{k<l} x_{ik}x_{il} \cdot 2|\Delta p_\ell||\Delta p_k| \\
&= \sum_{j,k} x_{ij}x_{ik}|\Delta p_j||\Delta p_k|.
\end{aligned}$$

□

[1] A more careful argument shows the multiplier of 2 is not needed.

Claim 8.6. *Suppose that for all j, $|\Delta p_j| \le p_j/4$. Then*

$$\frac{(\Delta p_j)^2}{p_j} \le \frac{9}{2} d_{\scriptscriptstyle KL}(p_j + \Delta p_j, p_j).$$

Proof: We use the bound $\log x \ge x - \frac{11}{18}x^2$ for $|x| \le \frac{1}{4}$. By (5.1) and (5.2),

$$\begin{aligned}
&d_{\text{KL}}(p_j + \Delta p_j, p_j)\\
&= (p_j + \Delta p_j)\log(p_j + \Delta p_j) - (p_j + \Delta p_j) - p_j \log p_j + p_j - (\log p_j)\Delta p_j\\
&= -\Delta p_j + (p_j + \Delta p_j)\log\left(1 + \frac{\Delta p_j}{p_j}\right)\\
&\ge -\Delta p_j + (p_j + \Delta p_j)\left(\frac{\Delta p_j}{p_j} - \frac{11}{18}\frac{(\Delta p_j)^2}{p_j^2}\right)\\
&= \frac{7}{18}\frac{(\Delta p_j)^2}{p_j}\left(1 - \frac{11}{7}\frac{\Delta p_j}{p_j}\right) \ge \frac{7}{18}\frac{17}{28}\frac{(\Delta p_j)^2}{p_j} \ge \frac{2}{9}\frac{(\Delta p_j)^2}{p_j}.
\end{aligned}$$

$\square$

Proof of Lemma 8.3: We write $\phi(p^t)$ and $\phi(p^{t+1})$ as functions of the p_j, and then upper bound these terms using the inequalities $x(1+x)^{-1} \le x + \frac{4}{3}x^2$ for $|x| \le \frac{1}{4}$ and $\log(1+y) \le y$ for $|y| \le 1$, along with Claims 8.5 and 8.6.

$$\begin{aligned}
&\phi(p^{t+1}) - \ell_\phi(p^{t+1}; p^t)\\
=&\phi(p^{t+1}) - \phi(p^t) - \nabla\phi(p^t)\cdot(p^{t+1} - p^t)\\
=&\sum_j (p_j + \Delta p_j) - \sum_i e_i \log \sum_k b_{ik}(p_k + \Delta p_k) - \sum_j p_j + \sum_i e_i \log \sum_k b_{ik} p_k + \sum_j z_j \Delta p_j\\
=&\sum_j x_j \Delta p_j + \sum_i e_i \log\left[1 - \frac{\sum_k b_{ik}\Delta p_k}{\sum_k b_{ik} p_k}\left(1 + \frac{\sum_l b_{i\ell}\Delta p_\ell}{\sum_l b_{i\ell} p_\ell}\right)^{-1}\right].
\end{aligned}$$

Next we use the bound $x(1+x)^{-1} \le x + \frac{4}{3}x^2$ for $|x| \le \frac{1}{4}$, noting that $|\frac{\sum_l b_{i\ell}\Delta p_\ell}{\sum_l b_{ik} p_\ell}| \le \frac{1}{4}$, as every $|\Delta p_\ell| \le \frac{1}{4} p_\ell$ by assumption. Thus:

$$\phi(p^{t+1}) - \ell_\phi(p^{t+1}) \le \sum_j x_j \Delta p_j + \sum_i e_i \log\left[1 - \frac{\sum_k b_{ik}\Delta p_k}{\sum_k b_{ik} p_k} + \frac{4}{3}\frac{\sum_k b_{ik}\Delta p_k \sum_l b_{i\ell}\Delta p_\ell}{\sum_k b_{ik} p_k \sum_l b_{i\ell} p_\ell}\right].$$

Now we use the bound $\log(1+y) \le y$, which applies as the second and third terms in the log are each bounded by $\frac{1}{4}$ (note that $|\frac{\sum_l b_{i\ell}\Delta p_\ell}{\sum_l b_{ik} p_\ell}| \le \frac{1}{4}$). Hence

$$\begin{aligned}
&\phi(p^{t+1}) - \ell_\phi(p^{t+1}) \\
\le &\sum_j x_j \Delta p_j - \sum_i e_i \frac{\sum_k b_{ik}\Delta p_k}{\sum_k b_{ik} p_k} + \frac{4}{3} e_i \frac{\sum_k b_{ik}\Delta p_k \sum_l b_{i\ell} p_\ell}{\sum_k b_{ik} p_k \sum_l b_{ik} p_\ell} \\
\le &\sum_j x_j \Delta p_j - \sum_k x_k \Delta p_k + \frac{4}{3}\sum_i \frac{1}{e_i}\sum_k x_{ik}\Delta p_k \sum_l x_{i\ell}\Delta p_\ell \\
\le &\frac{4}{3}\sum_{i,j}\frac{x_{ij}}{p_j}(\Delta p_j)^2 \quad \text{(by Claim 8.5)} \\
= &\frac{4}{3}\sum_j \frac{x_j}{p_j}(\Delta p_j)^2 \le \ 6\sum_j x_j \cdot d_{\mathrm{KL}}(p_j + \Delta p_j, p_j) \quad \text{(by Claim 8.6).}
\end{aligned}$$

□

8.1 Lower Bound on Convergence Rate for Leontief Fisher Markets

We prove Theorem 8.2 here. We consider the following Leontief Fisher market with two buyers and two goods. Buyer 1 has budget $e_1 = 3$ and $b_{11} : b_{12} = 1 : 3$; buyer 2 has budget $e_2 = 2$ and $b_{21} : b_{22} = 2 : 1$. There is a unique market equilibrium $(p_1^*, p_2^*) = (0, 5)$, with equilibrium demands $(x_{11}^*, x_{12}^*, x_{21}^*, x_{22}^*) = (1/5, 3/5, 4/5, 2/5)$. We will show that if tatonnement starts at a carefully chosen price vector, (p_1, p_2), the potential function value is $\Theta((p_1)^2)$ but in the next time step the potential function drops by only $\Theta((p_1)^3)$.

Let $B = \left\{(p_1, p_2 \,|\, p_1 \leq \bar{\delta}, -\frac{2}{5}p_1^2 \leq p_1 + p_2 - 5 \leq \frac{2}{5}p_1^2\right\}$, where $\bar{\delta} > 0$ is a sufficiently small positive number which satisfies several conditions stated in the proofs below.

The price update rule of good j is $p_j^{t+1} = p_j^t \cdot \exp(z_j^t/\gamma)$.

Lemma 8.7. *If a tatonnement starts at a price vector in B, the set of prices remain in B throughout the whole tatonnement.*

Proof: Let (p_1, p_2) be a price vector in B and let $(p_1, p_2) = (\delta, 5 - \delta + C\delta^2)$, where $|C| \leq \frac{2}{5}$. Then the demands are

$$x_1 = \frac{3}{15 - 2\delta + 3C\delta^2} + \frac{4}{5 + \delta + C\delta^2} \qquad x_2 = \frac{9}{15 - 2\delta + 3C\delta^2} + \frac{2}{5 + \delta + C\delta^2}.$$

Let (p_1', p_2') denote the new prices after an update, i.e.

$$p_1' = \delta \exp((x_1 - 1)/\gamma) \qquad p_2' = (5 - \delta + C\delta^2) \exp((x_2 - 1)/\gamma)$$

The Taylor expansions of x_1, x_2, p_1', p_2' (with respect to δ) are

$$x_1 = 1 - \frac{2}{15}\delta + O(\delta^2), \qquad x_2 = 1 + \left(\frac{2}{75} - \frac{C}{5}\right)\delta^2 + O(\delta^3),$$

$$p_1' = \delta - \frac{2}{15\gamma}\delta^2 + O(\delta^3), \qquad p_2' = 5 - \delta + \left(C - \frac{C}{\gamma} + \frac{2}{15\gamma}\right)\delta^2 + O(\delta^3).$$

We choose $\bar{\delta}$ to be sufficiently small so that $p_1' < p_1$, and hence $p_1' < \bar{\delta}$.

The Taylor expansion of $\frac{p_1'+p_2'-5}{(p_1')^2}$ is

$$\frac{p_1'+p_2'-5}{(p_1')^2} = C\left(1-\frac{1}{\gamma}\right) + O(\delta).$$

We choose $\bar{\delta}$ to be sufficiently small so that

$$C\left(1-\frac{1}{\gamma}\right) - \frac{1}{10\gamma} \le \frac{p_1'+p_2'-5}{(p_1')^2} \le C\left(1-\frac{1}{\gamma}\right) + \frac{1}{10\gamma}.$$

Since $|C| \le \frac{2}{5}$ and $\gamma \ge 1$, $C\left(1-\frac{1}{\gamma}\right) - \frac{1}{10\gamma} \ge -\frac{2}{5}$ and $C\left(1-\frac{1}{\gamma}\right) + \frac{1}{10\gamma} \le \frac{2}{5}$. So (p_1', p_2') is in B. □

Lemma 8.8. *If (p_1^t, p_2^t) is in B, then $\phi(p^t) - \phi(p^{t+1}) = \Theta((p_1)^3)$ and $\phi(p^t) - \phi(p^*) = \Theta((p_1)^2)$.*

Proof: Let $(p_1^t, p_2^t) = (\delta, 5-\delta+C\delta^2)$. Since the potential function is convex,

$$\begin{aligned}
\phi(p^t) - \phi(p^{t+1}) &\le -\nabla\phi(p^t)\cdot(p^{t+1}-p^t) \\
&= (x_1-1)\left(\exp((x_1-1)/\gamma)-1\right)p_1 + (x_2-1)\left(\exp((x_2-1)/\gamma)-1\right)p_2 \\
&= O\left(\frac{p_1(x_1-1)^2}{\gamma}\right) + O\left(\frac{p_2(x_2-1)^2}{\gamma}\right).
\end{aligned}$$

Recall the Taylor expansions of x_1 and x_2. We choose $\bar{\delta}$ to be sufficiently small so that

$$|x_1-1| = \Theta(\delta), \qquad |x_2-1| = O(\delta^2).$$

Then

$$\phi(p^t) - \phi(p^{t+1}) = \delta \cdot \Theta\left(\frac{\delta^2}{\gamma}\right) + \Theta(1) \cdot O\left(\frac{\delta^4}{\gamma}\right) = \frac{1}{\gamma}\Theta(\delta^3).$$

Next, we will show that the potential function is $\Theta(\delta^2)$. The following derivation is similar to the one for the upper sandwiching bound. Let $\Delta^* p_\ell = p^*_\ell - p_\ell$. Recall that

$$\phi(p^*) - \ell_\phi(p^*; p) = \sum_j x_j \Delta^* p_j - \sum_i e_i \log\left(1 + \frac{\sum_\ell b_{i\ell}\Delta^* p_\ell}{\sum_\ell b_{i\ell} p_\ell}\right).$$

We choose $\bar{\delta}$ to be sufficiently small so that $\frac{3}{4} \leq \frac{\sum_\ell b_{i\ell}(p_\ell + \Delta^* p_\ell)}{\sum_\ell b_{i\ell} p_\ell} \leq \frac{5}{4}$. Then we can use (10.6) to obtain

$$\begin{aligned}
\phi(p^*) - \ell_\phi(p^*; p) &\leq \sum_j x_j \Delta^* p_j - \sum_i e_i \left[\frac{\sum_\ell b_{i\ell}\Delta^* p_\ell}{\sum_\ell b_{i\ell} p_\ell} - \frac{2}{3}\left(\frac{\sum_\ell b_{i\ell}\Delta^* p_\ell}{\sum_\ell b_{i\ell} p_\ell}\right)^2\right] \\
&= \frac{2}{3}\sum_i e_i \left(\sum_\ell \frac{x_{i\ell}}{e_i}\Delta^* p_\ell\right)^2 = \frac{2}{3}\sum_i \frac{1}{e_i}\left(\sum_\ell x_{i\ell}\Delta^* p_\ell\right)^2.
\end{aligned}$$

Note that $\Delta^* p_1 = -\delta$ and $\Delta^* p_2 = \delta - C\delta^2$. The Taylor expansions for the $\{x_{ij}\}$ are

$$x_{11} = \frac{1}{5} + O(\delta) \qquad x_{12} = \frac{3}{5} + O(\delta) \qquad x_{21} = \frac{4}{5} + O(\delta) \qquad x_{22} = \frac{2}{5} + O(\delta).$$

Hence, the Taylor expansions of $\frac{1}{e_i}\left(\sum_\ell x_{i\ell}\Delta^* p_\ell\right)^2$ are

$$\frac{1}{e_1}\left(x_{11}\Delta^* p_1 + x_{12}\Delta^* p_2\right)^2 = \frac{4}{75}\delta^2 + O(\delta^3), \qquad \frac{1}{e_2}\left(x_{21}\Delta^* p_1 + x_{22}\Delta^* p_2\right)^2 = \frac{2}{25}\delta^2 + O(\delta^3).$$

Thus

$$\phi(p^*) - \ell_\phi(p^*; p) \le \frac{4}{45}\delta^2 + O(\delta^3).$$

Then

$$\begin{aligned}\phi(p^*) - \phi(p) &\le \frac{4}{45}\delta^2 + O(\delta^3) - z_1\Delta^* p_1 - z_2\Delta^* p_2 \\ &= \frac{4}{45}\delta^2 - \left(-\frac{2}{15}\delta\right)(-\delta) - \left(\frac{2}{75} - \frac{C}{5}\right)\delta^2 \cdot (\delta - C\delta^2) + O(\delta^3) \\ &= -\frac{2}{45}\delta^2 + O(\delta^3).\end{aligned}$$

We can choose $\bar{\delta}$ sufficiently small so that $\phi(p) - \phi(p^*) = \Theta(\delta^2)$. □

Proof of Theorem 8.2: By Lemma 8.8, it takes $\Theta(1/p_1)$ steps for $\phi(p)$ to halve. So starting at $p_1 = \bar{\delta}$, to reduce $\phi(p)$ by a 2^i factor takes $\Theta([1 + \sqrt{2} + \ldots + \sqrt{2^i}] \cdot [1/\sqrt{\bar{\delta}}]) = \Theta(\sqrt{2^i/\bar{\delta}})$ steps. In other words,

$$\phi(p^t) - \phi(p^*) = \Theta\left(\frac{\phi(p^0) - \phi(p^*)}{t^2\bar{\delta}}\right) = \Theta\left(\frac{\phi(p^0) - \phi(p^*)}{t^2}\right).$$

□

Chapter 9

Complementary CES Fisher Markets

9.1 Convergence in Complementary CES Fisher Markets

In this section we consider the weighted update rule,

$$p_j^{t+1} = p_j^t \cdot \exp(z_j/\gamma_j^t)), \tag{9.1}$$

for markets in which every buyer has a complementary CES utility, i.e. the ith buyer has a parameter ρ_i in the range $-\infty < \rho_i < 0$. In addition, the weights γ_j^t are allowed to change from one time step to the next; our updates to price p_j

will use the weight $\gamma_j^t = 5 \cdot \max\{1, x_j^t\}$.[1] This seems a very natural distributed rule, and indeed a linearisation of this rule, $p_j^{t+1} = p_j^t[1 + \lambda \max\{1, z_j\}]$[2] was used in Part I.

For these markets we will show that $\phi(p^t) - \phi(p^*)$ reduces by at least a $1-\mu$ factor at each time step, where $0 < \mu < 1$ depends on the initial price and the market parameters we will specify.

Henceforth, the index t on all the parameters except prices will be implicit.

Notation. We set $\gamma = \max_j \gamma_j$, and again, we let Δp_j denote $p_j^{t+1} - p_j^t$. We define $c_i := \rho_i/(\rho_i - 1)$. Note that $c_i = \sigma_i - 1$, where $\sigma_i = 1/(1-\rho_i)$ is the demand elasticity of the associated CES utility function. Finally, let $c := \max_i c_i$.

As is well known, the demand for good j when buyer i optimizes her utility is given by

$$x_{ij} = e_i b_{ij} (p_j)^{c_i - 1} / S_i, \tag{9.2}$$

where $b_{ij} := a_{ij}^{1-c_i}$ and $S_i = \sum_\ell b_{i\ell}(p_\ell)^{c_i}$. It is easy to compute that the optimal utility equals $e_i S_i^{-1/c_i}$. It follows that the minimum cost for one unit of utility is S_i^{1/c_i}. Thus, by Lemma 6.6, ϕ is given by

$$\phi(p^t) = \sum_j p_j^t - \sum_i e_i \log S_i^{1/c_i}.$$

In the next two subsections we will show that the potential function in this case satisfies a stronger sandwiching property, as specified in Lemmas 9.2 and 9.3

[1] Any greater value for γ_j would work too, but would entail a proportionate change to the bound in Lemma 9.3.

[2] The λ replaces the constant of 5 used here, as a greater range of values for this parameter is needed in markets of substitutes.

(their proofs occur later in this chapter). This stronger property immediately yields the claimed bound on the convergence rate (Theorem 9.5).

Claim 9.1. $|p_j^{t+1} - p_j^t| \leq \frac{1}{4} p_j^t$.

Lemma 9.2. *Suppose that* $|p_j^{t+1} - p_j^t| \leq \frac{1}{4} p_j^t$ *for all* j*. Then*

$$\phi(p^t) - \phi(p^{t+1}) \geq \frac{1}{2} \sum_j \frac{z_j^2 p_j^t}{\gamma_j}.$$

Lemma 9.3.

$$\phi(p^t) - \phi(p^*) \leq \max_j \left\{ 10, \frac{5}{2m_j} \right\} \sum_j \frac{z_j^2 p_j^t}{\gamma_j},$$

where $m_j = (1-c)/2$ *for* $r_j \leq 1$ *and* $m_j = \frac{1 - r_j^c + c(r_j - 1)}{c(r_j-1)^2}$ *otherwise, and* $r_j = p_j^* / p_j^t$.

It is a simple calculation to check that the definitions of m_j coincide at $r_j = 1$. We defer the proof of the following claim to the end of this chapter.

Claim 9.4. *Let* $h_c(r_j) = m_j / c$*. Then*

i. For $0 < c < 1$, $h_c(r) := \frac{1 - r^c + c(r-1)}{(r-1)^2}$ *is a decreasing function of* r*.*

ii. $h_c(r)/c$ *is a decreasing function of* c*.*

iii.

$$m_j \geq \min \left\{ \frac{(2^{1/c} - 1)}{2^{1/c}\ (r_j - 1)}, \frac{(2^{1/c} - 1) c^{1/(1-c)}}{2^{2/c}} \right\}.$$

We can now prove our main result.

Theorem 9.5. *For all complementary-CES Fisher markets, for the sequence of prices p^t defined by the update rule (9.1), for all t,*

$$\phi(p^t) - \phi(p^*) \leq [(1 - \Theta(1)]^t d_{KL}(p^*, p^0).$$

In other words, for any $\lambda > 0$, $\phi(p^t) - \phi(p^*) \leq \lambda d_{KL}(p^*, p^0)$, if $t = \Omega(\log(1/\lambda))$.

Proof:

$$\begin{aligned}
\phi(p^{t+1}) - \phi(p^*) =& \phi(p^t) - \phi(p^*) - [\phi(p^t) - \phi(p^{t+1})] \\
\leq& \phi(p^t) - \phi(p^*) - \frac{1}{2} \sum_j \frac{z_j^2 p_j^t}{\gamma_j} \quad \text{(by Lemma 9.2)} \\
\leq& [\phi(p^t) - \phi(p^*)] \left[1 - \frac{1}{2} \left(\max_j \left\{ 10, \frac{5}{2m_j} \right\} \right)^{-1} \right] \quad \text{(by Lemma 9.3).}
\end{aligned}$$

Lemma 9.8, stated in Chapter 9.1.3, will show that $r_j = p_j^*/p_j$ remains bounded throughout the tatonnement process for all j, and hence m_j remains bounded away from zero. Consequently,

$$\phi(p^t) - \phi(p^*) = (1 - \Theta(1))[\phi(p^t) - \phi(p^*)].$$

□

9.1.1 The Upper Bound: Good Progress on a Price Update

The proof of Lemma 9.2 proceeds in two steps. First, we show that $\phi(p^{t+1}) - \phi(p^t) + \sum_j z_j[p_j^{t+1} - p_j^t] \leq 2\sum_j \frac{x_j}{p_j}[p_j^{t+1} - p_j^t]^2$. We then choose $\gamma_j = 5 \cdot \max\{1, x_j\}$. Finally, we deduce the bound in Lemma 9.2. Our first bound uses the following result.

Lemma 9.6. *Suppose that for all j, $|\Delta p_j|/p_j \leq \frac{1}{4}$. Then $\phi(p + \Delta p) - \ell_\phi(p + \Delta p; p) \doteq \phi(p + \Delta p) - \phi(p) + \sum_j z_j \Delta p_j \leq 2\sum_j \frac{x_j}{p_j}(\Delta p_j)^2$.*

Proof: As in the proof of Lemma 8.3, we use two bounds: First, a bound on $\log(1 + \lambda)$, namely:

$$\log(1 + \lambda) \geq \lambda - \frac{2}{3}\lambda^2, \text{ when } |\lambda| \leq \frac{7}{24}. \tag{9.3}$$

And second, a bound on the following polynomial, which follows from a simple power series expansion: if $|\Delta p_j/p_j| \leq 1/4$ and $0 \leq c \leq 1$,

$$(p_j + \Delta p_j)^c \geq p_j^c + cp_j^{c-1}(\Delta p_j) - \frac{2}{3}cp_j^{c-2}(\Delta p_j)^2. \tag{9.4}$$

We let D_ϕ denote $\phi(p + \Delta p) - \ell_\phi(p + \Delta p; p)$, for short. Recall that $S_i(p) =$

$\sum_\ell b_{i\ell} p_\ell^{c_i}$. Then:

$$\begin{aligned} D_\phi =& \phi(p+\Delta p) - \phi(p) + \sum_j z_j \Delta p_j \\ =& \sum_j \Delta p_j + \sum_j z_j \Delta p_j - \sum_i \frac{e_i}{c_i} \log \frac{S_i(p+\Delta p)}{S_i(p)}. \\ =& \sum_j x_j \Delta p_j - \sum_i \frac{e_i}{c_i} \log \left(\frac{\sum_\ell b_{i\ell}(p_\ell + \Delta p_\ell)^{c_i}}{S_i(p)} \right). \end{aligned}$$

As $\rho < 0$, $0 < c_i < 1$. So we can apply (9.4), yielding:

$$D_\phi \le \sum_j x_j \Delta p_j - \sum_i \frac{e_i}{c_i} \log \left(1 + \frac{\sum_\ell b_{i\ell} c_i p_\ell^{c_i-1}(\Delta p_\ell)}{S_i(p)} - \frac{\frac{2}{3}\sum_\ell b_{i\ell} c_i p_\ell^{c_i-2}(\Delta p_\ell)^2}{S_i(p)} \right).$$

Recalling from (9.2) that $x_{i\ell} = e_i b_{i\ell}(p_\ell)^{c_i-1}/S_i(p)$, yields:

$$D_\phi \le \sum_j x_j \Delta p_j - \sum_i \frac{e_i}{c_i} \log \left(1 + \sum_\ell c_i \frac{x_{i\ell}}{e_i}(\Delta p_\ell) - \frac{2}{3} \sum_\ell c_i \frac{x_{i\ell}}{p_\ell e_i}(\Delta p_\ell)^2 \right).$$

On applying (10.6), which we can do as $\sum_\ell x_{i\ell} p_\ell \le e_i$, $c_i \le 1$, and $|\Delta p_\ell|/p_\ell \le \frac{1}{4}$, we obtain the bound:

$$\begin{aligned} D_\phi \ \le \ & \sum_j x_j \Delta p_j - \sum_i \frac{e_i}{c_i} \left(\sum_\ell c_i \frac{x_{i\ell}}{e_i}(\Delta p_\ell) - \frac{2}{3} \sum_\ell c_i \frac{x_{i\ell}}{p_\ell e_i}(\Delta p_\ell)^2 \right) \\ & + \sum_i \frac{e_i}{c_i} \frac{2}{3} \left(\sum_\ell c_i \frac{x_{i\ell}}{e_i}(\Delta p_\ell) - \frac{2}{3} \sum_\ell c_i \frac{x_{i\ell}}{p_\ell e_i}(\Delta p_\ell)^2 \right)^2 \\ = \ & \frac{2}{3} \sum_\ell \frac{x_\ell}{p_\ell}(\Delta p_\ell)^2 + \frac{2}{3} \sum_i \frac{c_i}{e_i} \left(\sum_\ell x_{i\ell}(\Delta p_\ell) - \frac{2}{3} \sum_\ell \frac{x_{i\ell}}{p_\ell}(\Delta p_\ell)^2 \right)^2 \\ = \ & \frac{2}{3} \sum_\ell \frac{x_\ell}{p_\ell}(\Delta p_\ell)^2 + \frac{2}{3} \sum_i \frac{c_i}{e_i} \left(\sum_\ell x_{i\ell}(\Delta p_\ell) \left(1 - \frac{2\Delta p_\ell}{3 p_\ell} \right) \right)^2. \end{aligned}$$

Now recall that $\Delta p_\ell / p_\ell \leq \frac{1}{4}$, to give the bound:

$$\begin{aligned}
D_\phi &\leq \frac{2}{3}\sum_\ell \frac{x_\ell}{p_\ell}(\Delta p_\ell)^2 + \frac{2}{3}\sum_i \frac{c_i}{e_i}\left(\sum_\ell x_{i\ell}|\Delta p_\ell| \cdot \frac{7}{6}\right)^2 \\
&= \frac{2}{3}\sum_\ell \frac{x_\ell}{p_\ell}(\Delta p_\ell)^2 + \frac{49}{54}\sum_i \frac{1}{e_i}\left(\sum_\ell x_{i\ell}|\Delta p_\ell|\right)^2 \qquad (\text{as } c_i \leq 1) \\
&= \frac{2}{3}\sum_\ell \frac{x_\ell}{p_\ell}(\Delta p_\ell)^2 + \frac{49}{54}\sum_i \frac{1}{e_i}\sum_{j,k} x_{ij}x_{ik}|\Delta p_j||\Delta p_k| \\
&\leq \left(\frac{2}{3} + \frac{49}{54}\right)\sum \frac{x_\ell}{p_\ell}(\Delta p_\ell)^2 \quad (\text{by Claim } 8.5) \\
&\leq 2\sum \frac{x_\ell}{p_\ell}(\Delta p_\ell)^2.
\end{aligned}$$

□

Proof of Lemma 9.2: Recall that $\Delta p_j = p_j^{t+1} - p_j^t$ and that $p_j^{t+1} = p_j^t e^{(z_j/\gamma_j)}$. By Lemma 9.6,

$$\phi(p^t) - \phi(p^{t+1}) \;\geq\; \sum_j z_j[p_j^{t+1} - p_j^t] - 2\sum_j \frac{x_j}{p_j^t}[p_j^{t+1} - p_j^t]^2. \tag{9.5}$$

Next, using the formula for p^{t+1} and the fact that $\gamma_j \geq 5x_j$ gives the bound:

$$\begin{aligned}
\phi(p^t) - \phi(p^{t+1}) &\geq \sum_j z_j p_j^t [e^{(z_j/\gamma_j)} - 1] - \frac{2}{5} \sum_j \gamma_j p_j^t [e^{(z_j/\gamma_j)} - 1]^2 \\
&= \sum_j z_j p_j^t [e^{(z_j/\gamma_j)} - 1] \left(1 - \frac{2}{5} \frac{\gamma_j}{z_j} [e^{(z_j/\gamma_j)} - 1]\right) \\
&\geq \sum_{z_j \geq 0} \frac{z_j^2 p_j^t}{\gamma_j} \left(1 - \frac{2}{5} \cdot \frac{10}{9}\right) + \sum_{z_j < 0} \frac{z_j^2 p_j^t}{\gamma_j} \frac{9}{10} \left(1 - \frac{2}{5}\right) \\
&\geq \frac{1}{2} \sum_j \frac{z_j^2 p_j^t}{\gamma_j}.
\end{aligned}$$

□

9.1.2 An Upper Bound on the Distance to Equilibrium

Lemma 9.7. *Suppose that $p_j^*/p_j \leq r_j$ for all j, where $r_j \geq 1$. Let $c = \max_i c_i$. Then*

$$\phi(p^*) - \ell_\phi(p^*; p) \geq \sum_\ell \frac{h_c(r_\ell)}{c} x_\ell \cdot \frac{(p_\ell^* - p_\ell)^2}{p_\ell}.$$

Proof: As with previous lemmas, we use a bound on the polynomial $(p_j^* - p_j)^{c_i}$, but now we use the bound given by Claim 9.4 i. Specifically, if $p_j^*/p_j \leq r_j$ and $0 < c \leq 1$, $h_c(p_j^*/p_j) \geq h_c(r_j)$, i.e.

$$\frac{\frac{1}{p_j^c}[p_j^c - (p_j^*)^c + c p_j^{c-1}(p_j^* - p_j)]}{\frac{1}{p_j^2}(p_j^* - p_j)^2} \geq h_c(r_j),$$

so

$$(p_j^*)^c \leq p_j^c + c p_j^{c-1}(p_j^* - p_j) - h_c(r_j) p_j^{c-2}(p_j^* - p_j)^2. \tag{9.6}$$

We also use a simple bound on the log function, namely $\log(1+\lambda) \le \lambda$ for $\lambda \ge -1$. To avoid clutter, we omit the superscript t on the prices.

Let $\Delta^* p_j = p_j^* - p_j$. Then

$$\phi(p^*) - \ell_\phi(p^*;p) = \sum_j x_j \Delta^* p_j - \sum_i \frac{e_i}{c_i} \log\left(\frac{\sum_\ell b_{i\ell}(p_l^*)^{c_i}}{S_i(p)}\right).$$

Recalling that $S_i(p) = \sum_l b_{i\ell}(p_\ell)^{c_i}$ and using the upper bound on $(p_j^*)^{c_i}$ from (9.6) gives:

$$\begin{aligned}
&\phi(p^*) - \ell_\phi(p^*;p) \\
&\ge \sum_j x_j \Delta^* p_j - \sum_i \frac{e_i}{c_i} \log\left(1 + \frac{\sum_\ell b_{i\ell} c_i p_\ell^{c_i - 1}(\Delta^* p_\ell)}{S_i(p)} - \frac{\sum_\ell b_{i\ell} h_{c_i}(r_\ell) p_\ell^{c_i-2}(\Delta^* p_\ell)^2}{S_i(p)}\right) \\
&= \sum_j x_j \Delta^* p_j - \sum_i \frac{e_i}{c_i} \log\left(1 + \sum_\ell c_i \frac{x_{i\ell}}{e_i}(\Delta^* p_\ell) - \sum_\ell h_{c_i}(r_\ell) \frac{x_{i\ell}}{p_\ell e_i}(\Delta^* p_\ell)^2\right).
\end{aligned}$$

On noting that the argument for the log is positive (as it is an upper bound for $S_i(p^*)/S_i(p)$), we can apply the bound $\lambda \ge \log(1+\lambda)$ for $\lambda \ge -1$ to give:

$$\begin{aligned}
\phi(p^*) - \ell_\phi(p^*;p) &\ge \sum_j x_j \Delta^* p_j - \sum_i \frac{e_i}{c_i}\left(\sum_\ell c_i \frac{x_{i\ell}}{e_i}(\Delta^* p_\ell) - \sum_\ell h_{c_i}(r_\ell)\frac{x_{i\ell}}{p_\ell e_i}(\Delta^* p_\ell)^2\right) \\
&= \sum_i \sum_\ell \frac{h_{c_i}(r_\ell)}{c_i} x_{i\ell} \frac{(\Delta^* p_\ell)^2}{p_\ell} \\
&\ge \sum_i \sum_\ell \frac{h_c(r_\ell)}{c} x_{i\ell} \frac{(\Delta^* p_\ell)^2}{p_\ell} \qquad \text{(by Claim 9.4 ii.)} \\
&= \sum_\ell \frac{h_c(r_\ell)}{c} x_\ell \frac{(\Delta^* p_\ell)^2}{p_\ell}.
\end{aligned}$$

$\square$

Proof of Lemma 9.3: Note that $m_j = h_c(r_j)/c$. Then, by Lemma 9.7:

$$\begin{aligned}
\phi(p^t) - \phi(p^*) =& l_\phi(p^*, p^t) - \phi(p^*) - \nabla\phi(p^t) \cdot (p^* - p^t) \\
&\leq \sum_j z_j(p_j^* - p_j^t) - \sum_j m_j x_j \frac{(p_j^* - p_j^t)^2}{p_j^t} \\
&\leq \max_{p'} \sum_j \left(z_j(p'_j - p_j^t) - m_j x_j \frac{(p'_j - p_j^t)^2}{p_j^t} \right).
\end{aligned}$$

There are two cases.

Case 1: $0 \leq x_j \leq 1/2$.

Then $-1 \leq z_j \leq -1/2$ and hence $z_j \geq -2z_j^2$. Thus

$$z_j(p'_j - p_j^t) - m_j x_j \frac{(p'_j - p_j^t)^2}{p_j^t} \leq -z_j p_j^t \leq 2z_j^2 p_j^t = 2\gamma_j \frac{z_j^2 p_j^t}{\gamma_j}.$$

As $x_j \leq 1/2 < 1$, $2\gamma_j = 10$. Hence

$$z_j(p'_j - p_j^t) - m_j x_j \frac{(p'_j - p_j^t)^2}{p_j^t} \leq 10 \frac{z_j^2 p_j^t}{\gamma_j}.$$

Case 2: $x_j \geq 1/2$.

$z_j(p'_j - p_j^t) - m_j x_j \frac{(p'_j - p_j^t)^2}{p_j^t}$ is a quadratic function of $(p'_j - p_j^t)$. The quadratic function is maximized when $(p'_j - p_j^t) = \frac{z_j p_j^t}{2m_j x_j}$, with its maximum value being $\frac{z_j^2 p_j^t}{4m_j x_j} = \frac{\gamma_j}{4m_j x_j} \frac{z_j^2 p_j^t}{\gamma_j}$.

As $x_j \geq 1/2$ and $\gamma_j = 5 \cdot \max\{1, x_j\}$, $\gamma_j / x_j \leq 10$. Hence

$$z_j(p'_j - p_j^t) - m_j x_j \frac{(p'_j - p_j^t)^2}{p_j^t} \leq \frac{5}{2m_j} \frac{z_j^2 p_j^t}{\gamma_j}.$$

Combining the two cases yields the result. □

9.1.3 Bounding m_j

Let $p_{\text{U}} = \max_j\{p_j^\circ\}$, the maximum initial price, $U = \max\{p_{\text{U}}, M\}$, and $L^* = \min_j\{p_j^*\}$.

Lemma 9.8. *Let $\overline{U} = U$ for any continuous tatonnement, and let $\overline{U} = 2U$ for the discrete tatonnement with update rule* (9.1)*. For any continuous tatonnement, $p_j^*/p_j^t \leq \max\{p_j^*/p_j^\circ, (L^*/\overline{U})^{\min_i \rho_i}\}$, and for the discrete tatonnement , $p_j^*/p_j^t \leq 2 \cdot \max\{p_j^*/p_j^\circ, (L^*/\overline{U})^{\min_i \rho_i}\}$.*

Proof: We first note two observations.

Observation 1. No price will exceed $\overline{U}$ during the entire tatonnement.

Reason: Suppose not, then let $t = \tau$ be the first time when some price, say p_k, exceed $\overline{U}$. Then $p_k^\tau \geq M$ and $x_k^\tau \leq M/p_k^\tau \leq 1$. In the continuous tatonnement, the price update rule will not increase p_k any further.

For the discrete tatonnement we argue as follows. At $t = \tau - 1$, $p_k^{\tau-1} < \overline{U} = 2U$. But $p_k^{\tau-1} \geq U \geq M$, as p_k can at most double in one time unit. By the same argument as for x_k^τ, $x_k^{\tau-1} \leq 1$. By the price update rule, $p_k^\tau \leq p_k^{\tau-1} < 2U$, a contradiction. □

Observation 2. $p_k \geq \min\{p_k^\circ, (\overline{U}/L^*)^{\min_i \rho_i} p_k^*\}$ throughout the entire continuous tatonnement process, and half this value in the discrete case.

Reason: Suppose that for some k, $p_k \leq L^*(\overline{U}/L^*)^{\min_i \rho_i} p_k^*$. We claim that $x_k \geq 1$. At equilibrium prices, all demands equal 1. If the prices are all raised by a factor of $\frac{\overline{U}}{L^*}$, then all demands equal $\frac{L^*}{\overline{U}}$. Note that now all prices are at least $\overline{U}$.

Now reduce the price of p_k from $\frac{\overline{U}}{L^*}p_k^*$ to $\left(\frac{\overline{U}}{L^*}\right)^{\min_i \rho_i} p_k^*$, that is, reduce the price by a factor of $\left(\frac{\overline{U}}{L^*}\right)^{1-\min_i \rho_i}$. The price reduction can only decrease S_i. It then follows from (9.2) that the new demand x_k' for good k is bounded as follows

$$x_k' \geq x_k \cdot \left[\left(\frac{\overline{U}}{L^*}\right)^{1-\min_i \rho_i}\right]^{1/(1-\min_i \rho_i)} = \frac{L^*}{\overline{U}} \cdot \frac{\overline{U}}{L^*} = 1.$$

We just proved that when $p_k = \left(\frac{\overline{U}}{L^*}\right)^{\min_i \rho_i} p_k^*$ and all other prices are at values specified which are all at most $\overline{U}$, the demand for good k is at least 1. By Observation 1, no price exceeds $\overline{U}$ during the entire tatonnement process. In complementary markets, since the demand for one good increases when the prices of other goods decrease, we have shown that $x_k \geq 1$ if $p_k \leq \left(\frac{\overline{U}}{L^*}\right)^{\min_i \rho_i} p_k^*$.

In the case of the continuous tatonnement, it follows that no price can decrease below the minimum of this value and the initial value of this price. For the discrete case, we argue as follows. Let $\bar{L}_k = (1/2) \cdot \min\{p_k^\circ, (\overline{U}/L^*)^{\min_i \rho_i} p_k^*\}$. Suppose that Observation 2 were incorrect, then let $t = \tau$ be the first time when some price, say p_j, is below $\bar{L}_j$.

At $t = \tau - 1$, $p_j^{\tau-1} \geq \bar{L}_j$. But $p_j^{\tau-1} \leq 2\bar{L}_j$, as p_j can reduce by at most half in one time unit.

Then $x_j^{\tau-1} \geq 1$. By the price update rule, $p_j^\tau \geq p_j^{\tau-1} \geq \bar{L}_j$, a contradiction.

□

The lemma now follows from Observation 2. □

Proof of Claim 9.4: (i) and (ii) are readily checked by calculus. For (iii) we argue as follows. For $r_j \geq 2$,

$$\begin{aligned}
m_j =& \frac{1-(r_j-1)^c[1+1/(r_j-1)]^c + c(r_j-1)}{c(r_j-1)^2} \\
\geq& \frac{1-(r_j-1)^c[1+c/(r_j-1)-\frac{1}{2}c(1-c)/(r_j-1)^2]+c(r_j-1)}{c(r_j-1)^2} \quad \text{(as } r_j \geq 2\text{)} \\
\geq& \frac{c(r_j-1)+1-(r_j-1)^c - c/(r_j-1) - \frac{1}{2}c(1-c)/(r_j-1)^2)^{1-c}}{c(r_j-1)^2} \\
\geq& \frac{c(r_j-1)-(r_j-1)^c}{c(r_j-1)^2} \quad \text{(as } 1 \geq c[1+\tfrac{1}{2}(1-c)]\text{, for } c \leq 1\text{).}
\end{aligned}$$

If $r_j - 1 = c^{-1/(1-c)}$, $c(r_j-1) = (r_j-1)^c$. So when $r_j - 1 = 2^{1/c}c^{-1/(1-c)}$,

$$c(r_j-1)-(r_j-1)^c = (2^{1/c}-1)c \cdot c^{-1/(1-c)}.$$

And as $c(r_j-1)$ grows faster than $(r_j-1)^c$, for $r_j - 1 \geq 2^{1/c}c^{-1/(1-c)}$,

$$c(r_j-1)-(r_j-1)^c \geq (2^{1/c}-1)c(r_j-1)2^{-1/c}.$$

Then $m_j \geq (2^{1/c}-1)2^{-1/c}/(r_j-1)$.

m_j is a decreasing function of r_j. It follows that for $0 \leq r_j - 1 \leq 2^{1/c}c^{-1/(1-c)}$,

$$m_j \geq (2^{1/c}-1)2^{-2/c}c^{1/(1-c)}.$$

□

9.2 Revisiting Leontief Fisher Markets: a More Natural Tatonnement Update Rule

In Chapter 8, we prove that tatonnement converges toward market equilibrium in Leontief Fisher markets with the update rule (8.1), where the parameter γ used for updating p_j is proportional to the *maximum* demand for good j throughout the tatonnement process. However, it is not reasonable to assume that sellers know the maximum demands at the beginning of the tatonnement process. In contrast, the tatonnement update rule (9.1) for complementary CES Fisher markets only require γ to be proportional to the *current* demand for good j. A natural question arises: what is the limiting behaviour if sellers use the more natural update rule (9.1) in Leontief Fisher markets?

We observe that the potential function of Leontief Fisher markets and that of complementary CES Fisher markets both satisfy a similar upper sandwiching bound: $\phi(p^{t+1}) \leq \ell_\phi(p^t; p^{t+1}) + \sum_j Cx_j \cdot d_h(p_j^t, p_j^{t+1})$, where x_j is the demand for good j at time t and C is a constant. With this bound, by following the proof of Lemma 9.2, we see that the update rule (9.1) makes good progress in one time step in both types of Fisher markets, so it is reasonable to expect that the update rule (9.1) converges toward market equilibrium in Leontief Fisher markets. However, the proof for the Leontief case then proceeds with the use of Theorem 5.2; to suit the conditions required by the theorem, we were obliged to use a weaker version of the upper sandwiching bound — replacing x_j with the maximum demand for good j throughout the tatonnement. This weaker upper sandwiching bound leads to the less reasonable update rule (8.1).

In this section, we will prove that the tatonnement update rule (9.1) converges toward market equilibrium in Leontief Fisher markets. The proof does not use Theorem 5.2. We only show convergence but cannot bound the rate of convergence.

Notation. We consider a Leontief Fisher market with n goods. Let $d_1(p,p')$ denote the ℓ_1-distance between the price vectors p and p', i.e. $d_1(p,p') = \sum_j |p_j - p'_j|$. Let p° be the initial price vector, and the prices are updated with the rule (9.1) to generate the sequence $\{p^t\}$.

Claim 9.9. *The sequence $\{\phi(p^t)\}$ is decreasing; furthermore, $\phi(p^t) - \phi(p^{t+1}) \geq \frac{1}{2}\sum_j \frac{p_j^t (z_j^t)^2}{\gamma_j^t}$.*

Proof: □

Claim 9.10. *For all $\epsilon > 0$, there are only finitely many t such that satisfying $d_1(p^t, p^{t+1}) > \epsilon$.*

Proof: □

Let L denote the set of all limit points of the sequence $\{p^t\}$. Since the sequence $\{p^t\}$ is bounded by the compact set $[0, 2U]^n$,[3] L is non-empty.

Claim 9.11. *The potential values at all the price vectors in L are the same.*

Let ϕ_L denote the common potential value. Suppose $\phi_L > \phi^*$.

Claim 9.12. *If $p \in L$ and $p_k > 0$, then $z_k = 0$. Also, there exists ℓ such that $p_\ell = 0$ and $z_\ell > 0$.*

[3] U was defined in Section 9.1.3.

Proof: Suppose the contrary that there exists $p_k = \kappa > 0$ and $z_k = \zeta \neq 0$. By the continuity of the excess demand function, we can always choose a sufficiently small neighbourhood of p such that within the neighbourhood, the price of good k is at least $\kappa/2$ and the excess demand for good k is at between $\zeta/2$ and $3\zeta/2$. Every time p^t is in the neighbourhood, $d_1(p^t, p^{t+1}) \geq |p_k^t - p_k^{t+1}| \geq \frac{1}{2} \cdot \frac{\kappa}{2} \cdot \min\{(\zeta/2)/\Gamma, 1/\Gamma\}$. By the definition of limit points, the sequence $\{p^t\}$ is in the neighbourhood infinitely often. These contradict with Claim 9.10.

If there does not exist ℓ such that $p_\ell = 0$ and $z_\ell > 0$, then by the last paragraph, p must be a market equilibrium, a contradiction. $\square$

Lemma 9.13 ([16]). *In any Leontief Fisher market, there may be multiple market equilibrium price vectors, but the demands of each buyer are the same at all these equilibrium price vectors.*

For any $J \subset [1 \cdots n]$, let L_J denote the set $\{p \in L \,|\, p_j = 0 \Leftrightarrow j \in J\}$.

Claim 9.14. *For each $J \subset \mathcal{J}_\#$, the demands in L_J are unique.*

Proof: For any $p \in L_J$, for all $j \notin \mathcal{J}_0 \cup J$, p_j is positive, and hence $z_j = 0$. Consider a new Leontief Fisher market with same set of buyers and with goods that are not in $\mathcal{J}_0 \cup J$, in which each buyer has the same budget and utility function (but ignoring the goods in $\mathcal{J}_0 \cup J$). Then p (ignoring the zero prices) is a market equilibrium of the new Leontief Fisher market. By Lemma 9.13, the demands of each buyer are the same. Since the demands of each buyer in the original Leontief Fisher market are always in proportion, the demands for the zero-priced goods are forced to be the same also. $\square$

Let $z_{k,J}$ denote the excess demand for good k at any point in L_J.

Claim 9.15. *If at two price vectors p^1, p^2 the demands (of every buyer) are the same, then for any Δp, the demands at $p^1 + \Delta p$ are equal to the demands at $p^2 + \Delta p$.*

Consider the following graph. The vertices are $\{J \subset [1 \cdots n] \,|\, L_J \neq \varnothing\}$. For any two vertices J_1, J_2, there is an edge connecting them if and only if $d_1(L_{J_1}, L_{J_2}) = 0$.

Claim 9.16. *The graph is connected.*

Proof: Suppose not, i.e. there are at least two connected components. Let $\mathcal{C}$ be one of the components. By the definition of the graph, the ℓ_1-distances between $\mathcal{C}$ and all other connected components are positive; let $\delta > 0$ denote the minimum of all these distances. Let $\tilde{\delta} := \delta/3$. Let $\mathcal{L}$ denote the union of all $\tilde{\delta}$-neighbourhoods of the limit points in $\mathcal{C}$, and let $\mathcal{L}'$ denote the union of all $\tilde{\delta}$-neighbourhoods of the limit points in other connected components. Note that $d_1(\mathcal{L}, \mathcal{L}') = \tilde{\delta}$.

By definition of limit points, there exists a finite time T such that for all $t \geq T$, the sequence $\{p^t\}$ stays in $\mathcal{L} \cup \mathcal{L}'$, and furthermore, the sequence enters and leaves $\mathcal{L}$ infinitely often. Whenever the sequence leaves $\mathcal{L}$, it must enter $\mathcal{L}'$, so by the last paragraph, it moves for a distance of at least $\tilde{\delta}$. But this is impossible by Claim 9.10. $\square$

Claim 9.17. *The demands are equal for all L_J where J is a vertex of the graph.*

Chapter 10

NCES Fisher Markets

In this chapter, we prove that tatonnement converges in Fisher markets with NCES utility functions (see Section 1.3 for the definition and background). The proof comprises two key steps.

- First, we show that if the starting prices are all positive, then all prices are bounded away from zero throughout the tatonnement process; see Lemma 9.8, the analogous result for the complementary CES case.

- Second, we show that the potential function for a NCES Fisher market satisfies an upper sandwiching bound; see Lemma 9.6, the analogous result for the complementary CES case.

While the steps are analogous to the complementary CES case, it turns out that the proofs for NCES case are more sophisticated technically, especially the first step. Intuitively, one may think that when the price of a good is close to zero, its demand will blow up hugely, then by the tatonnement rule the price cannot drop further. This intuition is right for the complementary CES case (as shown in Lemma 9.8), but it can be wrong for the NCES case. Here is a quick example to see why the intuition is too naive. Consider a market with several substitute goods[1], with the prices of two goods both very close to zero. Depending on which price is relatively larger, it is possible that the demand for that good is very small, so by the tatonnement rule that price will get even smaller. In short, in markets with substitute goods, there may not exist an absolute bound $\epsilon_j > 0$ such that if $p_j \leq \epsilon_j$ then $x_j \gg 1$. In the NCES case, by allowing arbitrary levels of nesting of the utility components, for a price vector with many prices close to zero, the demand functions can become very hard to predict.

We briefly describe our approach for the first step. Consider the NCES utility tree of an arbitrary buyer. For each node in the tree, we define three quantities: *consolidated price*, *consolidated spending* and *consolidated demand*. Briefly speaking, the consolidated price of a node is the cost for the buyer to derive one unit of utility from the subtree rooted at that node; the consolidated spending of a node is the amount of money the buyer spends on the goods in the subtree rooted at that node. The consolidated demand, in an unintuitive way, reflects how the goods in the subtree rooted at that node are preferred *as a whole*. We prove that if the consolidated demand of a node is sufficiently large, then its consolidated price must increase significantly in the next time step of

[1]For markets with substitute goods, Cole and Fleischer [17] showed that the prices cannot drop arbitrarily close to zero, by proving that if the tatonnement dynamic starts out f-bounded (see Definition 3.3), all prices remain f-bounded throughout. We raise this example here solely to provide the readers a quick insight that knocks down the wrong intuition, but *not* to claim that prices really drop arbitrarily close to zero in the example.

tatonnement, although individual prices in the subtree might still drop significantly (Lemma 10.6). This will allow us to show that the consolidated spending at each node is bounded away from zero throughout the tatonnement; in particular, the spending on every good is bounded away from zero (Lemma 10.7). Then the price of every good cannot get arbitrarily close to zero (Corollary 10.8).

We point out that although there is a positive lower bound on the prices, the bound can be unexpectedly bad. We provide two concrete examples to show that prices can drop unexpectedly close to zero.

Example 10.1. *Consider the following one-buyer NCES Fisher market. The budget of the buyer is* 1*, and its utility function is*

$$u(x_1, x_2, x_3) = \left\{ \frac{1}{2} \left[\left((x_1)^{-4} + \frac{1}{5}(x_2)^{-4} \right)^{-1/4} \right]^{7/10} + (x_3)^{7/10} \right\}^{10/7}.$$

The unique market equilibrium occurs at $p^* = (p_1^*, p_2^*, p_3^*) \approx (0.2719, 0.0544, 0.6737)$*. Suppose the prices are updated with the tatonnement rule* (10.1)*, with* $\Gamma_1 = 32, \Gamma_2 = 89, \Gamma_3 = 64$*. Suppose the initial prices of the tatonnement process is* $p^\circ = (p_1^\circ, p_2^\circ, p_3^\circ) = (0.01, 0.8, 0.001)$*. Although* p_1° *is small compared to* p_1^**, it drops further significantly.* p_1 *reaches its minimum at* $t = 202$*, with value* 2.7752×10^{-5}*. We plot the graph* $\log_2(p_j^*/p_j)$ *against time in Figure 10.1(left).*

The drop of p_1 *can be made more significantly by modifying some of the* *parameters in the above example. Replace the buyer's utility function with*

$$u(x_1, x_2, x_3) = \left\{ \frac{1}{2} \left[\left((x_1)^{-99} + \frac{1}{5}(x_2)^{-99} \right)^{-1/99} \right]^{99/100} + (x_3)^{99/100} \right\}^{100/99}.$$

The unique market equilibrium occurs at approximately $(0.2774, 0.0555, 0.6671)$*. Replace the* Γ*-values with* $\Gamma_1 = 1024, \Gamma_2 = 8192, \Gamma_3 = 2048$*. The initial prices are changed to* $(0.01, 0.9, 0.0001)$*. Then* p_1 *reaches minimum at* $t = 16006$*, with value* 1.6581×10^{-9}*. We plot the graph* $\log_2(p_j^*/p_j)$ *against time in Figure 10.1(right).*

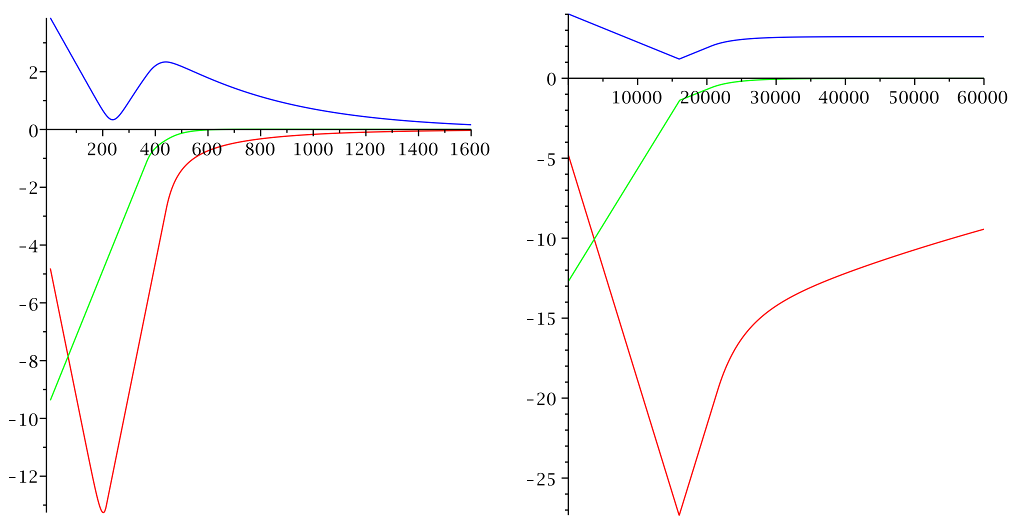

Figure 10.1: The graphs plot $\log_2(p_j^*/p_j)$ against time. They show how prices change in the tatonnement processes described in Example 10.1. The red curve is about good 1, the blue curve about good 2 and the green curve about good 3.

10.1 Consolidated Prices, Consolidated Spending and Consolidated Demands

Recall that an NCES utility can be viewed as a utility tree. We are about to define the consolidated price, the consolidated spending and the consolidated demand for every node in a NCES utility tree. In the definitions below, we assume that the utility component of an internal node $\mathbf{N}$ is given by the formula $\left(\sum_\ell a_\ell (u_\ell)^{\rho(\mathbf{N})}\right)^{1/\rho(\mathbf{N})}$. Let $c(\mathbf{N}) := \rho(\mathbf{N})/(\rho(\mathbf{N}) - 1)$ and $\tilde{a}_\ell := (a_\ell)^{1-c(\mathbf{N})}$. We assume that $-\infty < \rho(\mathbf{N}) < 1$ for all nodes. Note that $c(\mathbf{N})$ is positive if $\mathbf{N}$ is a complement node ($\rho(\mathbf{N}) < 0$) and $c(\mathbf{N})$ is negative if $\mathbf{N}$ is a substitute node ($\rho(\mathbf{N}) > 0$). Consolidated prices are defined in a bottom-up manner as follows.

Definition 10.2. *In an NCES utility tree, the* consolidated price *of a leaf is the actual price of the corresponding good. For any internal node* $\mathbf{N}$ *in the tree, the* consolidated price *of the node is defined to be*

$$\mathcal{P}_{bN} := \left(\sum_\ell \tilde{a}_\ell (\mathcal{P}_\ell)^{c(\mathbf{N})}\right)^{1/c(\mathbf{N})},$$

where the sum is over $\mathbf{N}$*'s children.*

Consolidated spending is defined in a top-down manner in Definition 10.3 below; the consolidated demand of a node is the consolidated spending of the node divided by its consolidated price.

Definition 10.3. *In an NCES utility tree, the* consolidated spending *of the root is the budget of the buyer. For any other node* $\mathbf{I}$*, let* $\mathbf{N}$ *be its parent, then the* consolidated spending *of* $\mathbf{I}$ *is*

$$\mathcal{S}_{\mathbf{I}} := \mathcal{S}_{\mathbf{N}} \cdot \frac{\tilde{a}_I(\mathcal{P}_I)^{c(\mathbf{N})}}{(\mathcal{P}_N)^{c(\mathbf{N})}}.$$

The consolidated demand *of any node* $\mathbf{I}$ *in the tree is*

$$\mathcal{X}_{\mathbf{I}} := \frac{\mathcal{S}_{\mathbf{I}}}{\mathcal{P}_{\mathbf{I}}}.$$

The following lemma relates the consolidated quantities to what the buyer purchases at a given price vector p.

Lemma 10.4. *At a given price vector, the optimal utility of a buyer with NCES utility function is its budget divided by the consolidated price at the root of its utility tree. The consolidated demand of a leaf in the tree equals the actual demand for the corresponding good by the buyer.*

Proof: Let e denote the budget of the buyer. We proceed by induction on the height of the utility tree h. When $h = 0$, there is only one good in the tree, so $\mathcal{P}_{\mathbf{N}}$ equals the actual price of the good at time t. The result is trivial.

Suppose $\mathbf{N}$ is the root of an NCES utility tree of height $h \geq 1$. Suppose $\mathbf{N}$ has k children $\mathbf{N}_1, \mathbf{N}_2, \cdots, \mathbf{N}_k$ and its utility component is $\left(\sum_{\ell=1}^k a_\ell (u_\ell)^{\rho(\mathbf{N})}\right)^{1/\rho(\mathbf{N})}$. The buyer distributes the spending e_ℓ on the goods in the tree rooted at $\mathbf{N}_k$, such that $\sum_{\ell=1}^k e_\ell = e$. By the inductive hypothesis, the optimal utility with this spending distribution equals

$$\left(\sum_{\ell=1}^{k} a_\ell \left(\frac{e_\ell}{\mathcal{P}_{\mathbf{N}_\ell}}\right)^{\rho(\mathbf{N})}\right)^{1/\rho(\mathbf{N})}.$$

It remains to determine which spending distribution is the best one. This is a standard constrained optimization problem. The optimal solution is attained when $e_\ell = \mathcal{S}_{\mathbf{N}_\ell}$, and the optimal utility is $e/\mathcal{P}_{\mathbf{N}}$.

By the inductive hypothesis again, the consolidated demand of a leaf in the tree equals the actual demand for the corresponding good by the buyer. $\square$

The remainder of this chapter is organized as follows. In Section 10.2, we prove some simple properties about consolidated prices, consolidated spendings and consolidated demands, which are then used to prove that the prices in any tatonnement process starting with positive prices are bounded away from zero. In Section 10.3, we show that the potential functions with NCES utility functions satisfy an upper sandwiching bound (similar to the one proved in Section 9.1.1). We use this to show that tatonnement must converge to the market equilibrium.

The tatonnement rule used in this case is same as the one used in complementary CES Fisher markets:

$$p_j^{t+1} = p_j^t \cdot \exp(z_j^t/\gamma_j^t), \quad \text{where } \gamma_j^t = \Gamma_j \cdot \max\{1, z_j^t\}. \tag{10.1}$$

We will determine Γ_j later.

10.2 Prices Are Bounded Away From Zero

In the following lemma, we state some simple properties about consolidated prices, consolidated spendings and consolidated demands. These properties can be proved easily by simple arithmetic or calculus. Recall that Γ_g is a parameter used in the tatonnement update rule of good g. Let

$$\hat{\Gamma}_{\mathbf{N}} = \max \{\Gamma_g \mid \ g \text{ is a good in the utility tree rooted at } \mathbf{N}\}$$

and

$$\breve{\Gamma}_{\mathbf{N}} = \min \{\Gamma_g \mid \ g \text{ is a good in the utility tree rooted at } \mathbf{N}\}.$$

Lemma 10.5. *Let* $\mathbf{N}$ *be a node in an NCES utility tree with* k *children* $\mathbf{N}_1, \mathbf{N}_2, \cdots, \mathbf{N}_k$.

(a) (Substitute Effect) Suppose that $\mathbf{N}$ *is a substitute node, and* $\mathcal{P}_{\mathbf{N}_2}, \mathcal{P}_{\mathbf{N}_3}, \cdots, \mathcal{P}_{\mathbf{N}_k}$ *are fixed. When* $\mathcal{P}_{\mathbf{N}_1}$ *drops,* $\mathcal{X}_{\mathbf{N}_2}, \mathcal{X}_{\mathbf{N}_3}, \cdots, \mathcal{X}_{\mathbf{N}_k}$ *decrease, and hence* $\mathcal{S}_{\mathbf{N}_2}, \mathcal{S}_{\mathbf{N}_3}, \cdots, \mathcal{S}_{\mathbf{N}_k}$ *decrease.*

(b) (Complement Effect) Suppose that $\mathbf{N}$ *is a complement node, and* $\mathcal{P}_{\mathbf{N}_2}, \mathcal{P}_{\mathbf{N}_3}, \cdots, \mathcal{P}_{\mathbf{N}_k}$ *are fixed. When* $\mathcal{P}_{\mathbf{N}_1}$ *drops,* $\mathcal{X}_{\mathbf{N}_2}, \mathcal{X}_{\mathbf{N}_3}, \cdots, \mathcal{X}_{\mathbf{N}_k}$ *increase, and hence* $\mathcal{S}_{\mathbf{N}_2}, \mathcal{S}_{\mathbf{N}_3}, \cdots, \mathcal{S}_{\mathbf{N}_k}$ *increase.*

(c) (Law of Consolidated Demand) Suppose that $\mathcal{P}_{\mathbf{N}_2}, \mathcal{P}_{\mathbf{N}_3}, \cdots, \mathcal{P}_{\mathbf{N}_k}$ *are fixed positive numbers. When* $\mathcal{P}_{\mathbf{N}_1}$ *drops,* $\mathcal{X}_{\mathbf{N}_1}$ *increases; further,* $\lim_{\mathcal{P}_{\mathbf{N}_1} \searrow 0} \mathcal{X}_{\mathbf{N}_1} = +\infty$.

(d) (Homogeneity of NCES Utility) Suppose that the consolidated prices of all children of $\mathbf{N}$ *are changed by the same factor* F. *Then the consolidated demand of each child of* $\mathbf{N}$ *is changed by a factor of* $1/F$.

(e) (Elasticity of Consolidated Demand) Suppose $\mathbf{N}$ *is a substitute node with parameter* $\rho(\mathbf{N}) > 0$, *and* $\mathcal{P}_{\mathbf{N}_2}, \mathcal{P}_{\mathbf{N}_3}, \cdots, \mathcal{P}_{\mathbf{N}_k}$ *are fixed. If* $\mathcal{P}_{\mathbf{N}_1}{}' = F \cdot \mathcal{P}_{\mathbf{N}_1}$,

where $F \geq 1$, *then* $\mathcal{X}_{\mathbf{N}_1}{}' \geq F^{-1/(1-\rho(\mathbf{N}))}\mathcal{X}_{\mathbf{N}_1}$.

(f) $\exp\left(1/\breve{\Gamma}_{\mathbf{N}}\right) \cdot \mathcal{P}_{\mathbf{N}}(t) \geq \mathcal{P}_{\mathbf{N}}(t+1) \geq \exp\left(-1/\breve{\Gamma}_{\mathbf{N}}\right) \cdot \mathcal{P}_{\mathbf{N}}(t)$.

Lemma 10.6. *For any NCES utility tree rooted at* $\mathbf{N}$ *with height* h*, there exists* $\mathring{\mathcal{X}}_{\mathbf{N}} \in \mathbb{R}^+$ *such that if* $\mathcal{X}_{\mathbf{N}}(t) \geq \mathring{\mathcal{X}}_{\mathbf{N}}$*, then* $\mathcal{P}_{\mathbf{N}}(t+1) \geq \exp\left(\frac{1}{2^h \hat{\Gamma}_{\mathbf{N}}}\right)\mathcal{P}_{\mathbf{N}}(t)$.

Proof: We prove the result by induction on the utility tree height h. When $h = 0$, there is only one good in the tree, so $\mathcal{X}_{\mathbf{N}}(t)$ equals to the actual demand for the good at time t and $\mathcal{P}_{\mathbf{N}}(t)$ equals to the actual price of the good at time t. Hence, when $\mathcal{X}_{\mathbf{N}}(t) \geq 2$, by the tatonnement update rule (10.1), $\mathcal{P}_{\mathbf{N}}(t+1) = \exp\left(\frac{1}{\hat{\Gamma}_{\mathbf{N}}}\right)\mathcal{P}_{\mathbf{N}}(t)$. So $\mathring{\mathcal{X}}_{\mathbf{N}} = 2$.

Suppose $\mathbf{N}$ is the root of an NCES utility tree of height $h \geq 1$ with $\mathcal{P}_{\mathbf{N}}(t) = \epsilon\mathcal{S}_{\mathbf{N}}(t)$. Suppose $\mathbf{N}$ has k children. Then each child of $\mathbf{N}$ is the root of an NCES utility tree of height at most $h-1$. By the inductive hypothesis, for $\ell = 1, 2, \cdots, k$, there exists $\mathring{\mathcal{X}}_{\ell}$ such that if $\mathcal{X}_{\ell}(t) \geq \mathring{\mathcal{X}}_{\ell}$, then $\mathcal{P}_{\ell}(t+1) \geq \exp\left(\frac{1}{2^{h-1}\hat{\Gamma}_{\ell}}\right)\mathcal{P}_{\ell}(t)$. Then there are two cases.

<u>Case 1: $\mathbf{N}$ is a complement node.</u>

Recall that $\mathcal{P}_{\mathbf{N}} = \left(\sum_{\ell=1}^{k} \tilde{a}_{\ell}(\mathcal{P}_{\ell})^{c(\mathbf{N})}\right)^{1/c(\mathbf{N})}$, where $c(\mathbf{N}) > 0$ in this case. Note that for $j = 1, 2, \cdots, k$, $\sum_{\ell=1}^{k} \tilde{a}_{\ell}(\mathcal{P}_{\ell})^{c(\mathbf{N})} \geq \tilde{a}_j(\mathcal{P}_j)^{c(\mathbf{N})}$, so

$$\epsilon\mathcal{S}_{\mathbf{N}} = \mathcal{P}_{\mathbf{N}} = \left(\sum_{\ell=1}^{k} \tilde{a}_{\ell}(\mathcal{P}_{\ell})^{c(\mathbf{N})}\right)^{1/c(\mathbf{N})} \geq (\tilde{a}_j)^{1/c(\mathbf{N})}\mathcal{P}_j$$

and hence

$$\mathcal{P}_j \leq (\tilde{a}_j)^{-1/c(\mathbf{N})}\epsilon\mathcal{S}_{\mathbf{N}}. \tag{10.2}$$

Given constraint (10.2), by the Complement Effect and the Law of Consolidated Demand, $\mathcal{X}_j$ is minimized when for all $\ell = 1, 2, \cdots, k$, $\mathcal{P}_\ell = (\tilde{a}_\ell)^{-1/c(\mathbf{N})}\epsilon\mathcal{S}_\mathbf{N}$. Since all these consolidated prices are proportional to ϵ, by Homogeneity of NCES Utility, $\mathcal{X}_j \geq v_j/\epsilon$ for some positive constant v_j. Then we choose a sufficiently small ϵ_1 such that if $\epsilon \leq \epsilon_1$, for all $j = 1, 2, \cdots, k$, $v_j/\epsilon \geq \mathring{\mathcal{X}}_j$. Then, if $\mathcal{P}_\mathbf{N}(t) \leq \epsilon_1 \mathcal{S}_\mathbf{N}(t)$, i.e. $\mathcal{X}_\mathbf{N}(t) \geq 1/\epsilon_1$,

$$\begin{aligned}
\mathcal{P}_\mathbf{N}(t+1) &= \left(\sum_{\ell=1}^{k} \tilde{a}_\ell (\mathcal{P}_\ell(t+1))^{c(\mathbf{N})}\right)^{1/c(\mathbf{N})} \\
&\geq \left(\sum_{\ell=1}^{k} \tilde{a}_\ell \exp\left(\frac{c(\mathbf{N})}{2^{h-1}\hat{\Gamma}_\ell}\right)(\mathcal{P}_\ell(t))^{c(\mathbf{N})}\right)^{1/c(\mathbf{N})} \qquad \text{(by the inductive hypothesis)} \\
&\geq \left(\exp\left(\frac{c(\mathbf{N})}{2^{h-1}\hat{\Gamma}_\mathbf{N}}\right)\sum_{\ell=1}^{k} \tilde{a}_\ell (\mathcal{P}_\ell(t))^{c(\mathbf{N})}\right)^{1/c(\mathbf{N})} \\
&= \exp\left(\frac{1}{2^{h-1}\hat{\Gamma}_\mathbf{N}}\right)\mathcal{P}_\mathbf{N}(t) \\
&> \exp\left(\frac{1}{2^{h}\hat{\Gamma}_\mathbf{N}}\right)\mathcal{P}_\mathbf{N}(t).
\end{aligned}$$

So $\mathring{\mathcal{X}}_\mathbf{N} = 1/\epsilon_1$ suffices.

<u>Case 2: **N** is a substitute node.</u>

Recall that $\mathcal{P}_\mathbf{N} = \left(\sum_{\ell=1}^{k} \tilde{a}_\ell (\mathcal{P}_\ell)^{c(\mathbf{N})}\right)^{1/c(\mathbf{N})}$, where $c(\mathbf{N}) < 0$ in this case. Note that for $j = 1, 2, \cdots, k$, $\sum_{\ell=1}^{k} \tilde{a}_\ell (\mathcal{P}_\ell)^{c(\mathbf{N})} \geq \tilde{a}_j (\mathcal{P}_j)^{c(\mathbf{N})}$, so

$$\epsilon\mathcal{S}_\mathbf{N} = \mathcal{P}_\mathbf{N} = \left(\sum_{\ell=1}^{k} \tilde{a}_\ell (\mathcal{P}_\ell)^{c(\mathbf{N})}\right)^{1/c(\mathbf{N})} \leq (\tilde{a}_j)^{1/c(\mathbf{N})}\mathcal{P}_j$$

and hence

$$\mathcal{P}_j \geq (\tilde{a}_j)^{-1/c(\mathbf{N})}\epsilon\mathcal{S}_\mathbf{N}. \tag{10.3}$$

Next, let $q := \arg\min_{1\le\ell\le k} \mathcal{P}_\ell$. Note that $\sum_{\ell=1}^k \tilde{a}_\ell (\mathcal{P}_\ell)^{c(\mathbf{N})} \le \left(\sum_{\ell=1}^k \tilde{a}_\ell\right)(\mathcal{P}_q)^{c(\mathbf{N})}$, so $\epsilon\mathcal{S}_\mathbf{N} = \mathcal{P}_\mathbf{N} = \left(\sum_{\ell=1}^k \tilde{a}_\ell (\mathcal{P}_\ell)^{c(\mathbf{N})}\right)^{1/c(\mathbf{N})} \ge \left(\sum_{\ell=1}^k \tilde{a}_\ell\right)^{1/c(\mathbf{N})} \mathcal{P}_q$, and hence

$$\mathcal{P}_q \le \left(\sum_{\ell=1}^k \tilde{a}_\ell\right)^{-1/c(\mathbf{N})} \epsilon\mathcal{S}_\mathbf{N}. \tag{10.4}$$

Given constraints (10.3) and (10.4), by the Substitute Effect and the Law of Consolidated Demand, $\mathcal{X}_q$ is minimized when $\mathcal{P}_q = \left(\sum_{\ell=1}^k \tilde{a}_\ell\right)^{-1/c(\mathbf{N})} \epsilon\mathcal{S}_\mathbf{N}$ and for all $\ell \ne q$, $\mathcal{P}_\ell = (\tilde{a}_\ell)^{-1/c(\mathbf{N})}\epsilon\mathcal{S}_\mathbf{N}$. Since all these consolidated prices are proportional to ϵ, by homogeneity of NCES utility, $\mathcal{X}_q \ge d_q/\epsilon$ for some positive constant d_q. We choose a sufficiently small ϵ_2 such that $d_q/\epsilon_2 \ge \mathring{\mathcal{X}}_q$. Hence if $\epsilon \le \epsilon_2$, $\mathcal{X}_q(t) \ge \mathring{\mathcal{X}}_q$.

For any $\ell \ne q$ with fixed $\mathcal{P}_\ell$, by the Substitute Effect, $\mathcal{X}_\ell$ is minimized when for all $r \ne \ell$, $\mathcal{P}_r = (\tilde{a}_r)^{-1/c(\mathbf{N})}\epsilon\mathcal{S}_\mathbf{N}$. If $\mathcal{P}_\ell = (\tilde{a}_\ell)^{-1/c(\mathbf{N})}\epsilon\mathcal{S}_\mathbf{N}$, all the consolidated prices are proportional to ϵ, so by Homogeneity of NCES Utility, $\mathcal{X}_\ell \ge b_\ell/\epsilon$ for some positive constant b_ℓ. We choose a sufficiently small ϵ_3 such that for all $\ell \ne q$, $b_\ell/\epsilon_3 \ge \mathring{\mathcal{X}}_\ell$.

By the Elasticity of Consolidated Demand, if $\mathcal{P}_\ell = F \cdot (\tilde{a}_\ell)^{-1/c(\mathbf{N})}\epsilon\mathcal{S}_\mathbf{N}$, where $F > 1$, then $\mathcal{X}_\ell \ge F^{-1/(1-\rho(\mathbf{N}))} \cdot b_\ell/\epsilon$. Hence, so long as

$$\mathcal{P}_\ell \le (\epsilon\mathring{\mathcal{X}}_\ell/b_\ell)^{-(1-\rho(\mathbf{N}))} \cdot (\tilde{a}_\ell)^{-1/c(\mathbf{N})}\epsilon\mathcal{S}_\mathbf{N} = \mathcal{S}_\mathbf{N}(\mathring{\mathcal{X}}_\ell)^{\rho(\mathbf{N})-1}(b_\ell)^{1-\rho(\mathbf{N})}(\tilde{a}_\ell)^{-1/c(\mathbf{N})}\epsilon^{\rho(\mathbf{N})},$$

$\mathcal{X}_\ell \ge \mathring{\mathcal{X}}_\ell$.

Let $L^\uparrow := \left\{\ell \ne q \,\middle|\, \mathcal{X}_\ell(t) \ge \mathring{\mathcal{X}}_\ell\right\}$ and $L^\downarrow := \left\{\ell \ne q \,\middle|\, \mathcal{X}_\ell(t) < \mathring{\mathcal{X}}_\ell\right\}$. A simple consequence of the last paragraph is that for all $\ell \in L^{\downarrow}$,

$$\mathcal{P}_\ell(t) > \mathcal{S}_{\mathbf{N}}(\mathring{\check{\mathcal{X}}}_\ell)^{\rho(\mathbf{N})-1}(b_\ell)^{1-\rho(\mathbf{N})}(\tilde{a}_\ell)^{-1/c(\mathbf{N})}\epsilon^{\rho(\mathbf{N})}.$$

Then, by Lemma 10.5(f),

$$\mathcal{P}_\ell(t+1) > \exp\left(-\frac{1}{\breve{\Gamma}_\ell}\right)\mathcal{S}_{\mathbf{N}}(\mathring{\check{\mathcal{X}}}_\ell)^{\rho(\mathbf{N})-1}(b_\ell)^{1-\rho(\mathbf{N})}(\tilde{a}_\ell)^{-1/c(\mathbf{N})}\epsilon^{\rho(\mathbf{N})}.$$

Combining this with (10.4), for all $\ell \in L^{\downarrow}$,

$$\frac{\mathcal{P}_\ell(t+1)}{\mathcal{P}_q(t)} > \exp\left(-\frac{1}{\breve{\Gamma}_\ell}\right)\cdot(\mathring{\check{\mathcal{X}}}_\ell)^{\rho(\mathbf{N})-1}(b_\ell)^{1-\rho(\mathbf{N})}(\tilde{a}_\ell)^{-1/c(\mathbf{N})}\left(\sum_{\ell=1}^{k}\tilde{a}_\ell\right)^{1/c(\mathbf{N})}\epsilon^{\rho(\mathbf{N})-1}.$$

Note that $\lim_{\epsilon\searrow 0}\frac{\mathcal{P}_\ell(t+1)}{\mathcal{P}_q(t)} = +\infty$.

Next,

$$
\begin{aligned}
\mathcal{P}_{\mathbf{N}}(t+1) &= \left(\sum_{\ell=1}^{k} \tilde{a}_\ell (\mathcal{P}_\ell(t+1))^{c(\mathbf{N})}\right)^{1/c(\mathbf{N})} \\
&= \left(\tilde{a}_q (\mathcal{P}_q(t+1))^{c(\mathbf{N})} + \sum_{\ell\in L^{\uparrow}} \tilde{a}_\ell (\mathcal{P}_\ell(t+1))^{c(\mathbf{N})} + \sum_{\ell\in L^{\downarrow}} \tilde{a}_\ell (\mathcal{P}_\ell(t+1))^{c(\mathbf{N})}\right)^{1/c(\mathbf{N})} \\
&\geq \Bigg[\tilde{a}_q \exp\left(\frac{c(\mathbf{N})}{2^{h-1}\hat{\Gamma}_q}\right)(\mathcal{P}_q(t))^{c(\mathbf{N})} + \sum_{\ell\in L^{\uparrow}} \tilde{a}_\ell \exp\left(\frac{c(\mathbf{N})}{2^{h-1}\hat{\Gamma}_\ell}\right)(\mathcal{P}_\ell(t))^{c(\mathbf{N})} \\
&\qquad + (\mathcal{P}_q(t))^{c(\mathbf{N})} \sum_{\ell\in L^{\downarrow}} \tilde{a}_\ell \left(\frac{\mathcal{P}_\ell(t+1)}{\mathcal{P}_q(t)}\right)^{c(\mathbf{N})}\Bigg]^{1/c(\mathbf{N})} \qquad \text{(by the inductive hypothesis)} \\
&= \Bigg[(\mathcal{P}_q(t))^{c(\mathbf{N})}\left(\tilde{a}_q \exp\left(\frac{c(\mathbf{N})}{2^{h-1}\hat{\Gamma}_q}\right) + \sum_{\ell\in L^{\downarrow}} \tilde{a}_\ell \left(\frac{\mathcal{P}_\ell(t+1)}{\mathcal{P}_q(t)}\right)^{c(\mathbf{N})}\right) \\
&\qquad + \sum_{\ell\in L^{\uparrow}} \tilde{a}_\ell \exp\left(\frac{c(\mathbf{N})}{2^{h-1}\hat{\Gamma}_\ell}\right)(\mathcal{P}_\ell(t))^{c(\mathbf{N})}\Bigg]^{1/c(\mathbf{N})}.
\end{aligned}
$$

Recall that for all $\ell \in L^{\downarrow}$, $\lim_{\epsilon\searrow 0} \frac{\mathcal{P}_\ell(t+1)}{\mathcal{P}_q(t)} = +\infty$, so $\lim_{\epsilon\searrow 0} \left(\frac{\mathcal{P}_\ell(t+1)}{\mathcal{P}_q(t)}\right)^{c(\mathbf{N})} = 0$, i.e. there exists a sufficiently small ϵ_4 such that if $\epsilon \leq \epsilon_4$, then

$$
\tilde{a}_q \exp\left(\frac{c(\mathbf{N})}{2^{h-1}\hat{\Gamma}_q}\right) + \sum_{\ell\in L^{\downarrow}} \tilde{a}_\ell \left(\frac{\mathcal{P}_\ell(t+1)}{\mathcal{P}_q(t)}\right) \leq \tilde{a}_q \exp\left(\frac{c(\mathbf{N})}{2^{h}\hat{\Gamma}_q}\right).
$$

and hence

$$
\mathcal{P}_{\mathbf{N}}(t+1) \geq \left[\tilde{a}_q \exp\left(\frac{c(\mathbf{N})}{2^{h}\hat{\Gamma}_q}\right)(\mathcal{P}_q(t))^{c(\mathbf{N})} + \sum_{\ell\in L^{\uparrow}} \tilde{a}_\ell \exp\left(\frac{c(\mathbf{N})}{2^{h-1}\hat{\Gamma}_\ell}\right)(\mathcal{P}_\ell(t))^{c(\mathbf{N})}\right]^{1/c(\mathbf{N})}.
$$

On the other hand, since $c(\mathbf{N}) < 0$,

$$\mathcal{P}_{\mathbf{N}}(t) = \left(\sum_{\ell=1}^{k} \tilde{a}_\ell (\mathcal{P}_\ell(t))^{c(\mathbf{N})}\right)^{1/c(\mathbf{N})} \leq \left(\tilde{a}_q (\mathcal{P}_q(t))^{c(\mathbf{N})} + \sum_{\ell \in L^\uparrow} \tilde{a}_\ell (\mathcal{P}_\ell(t))^{c(\mathbf{N})}\right)^{1/c(\mathbf{N})}.$$

So

$$\begin{aligned}\frac{\mathcal{P}_{\mathbf{N}}(t+1)}{\mathcal{P}_{\mathbf{N}}(t)} &\geq \left(\frac{\tilde{a}_q \exp\left(\frac{c(\mathbf{N})}{2^h \hat{\Gamma}_q}\right)(\mathcal{P}_q(t))^{c(\mathbf{N})} + \sum_{\ell \in L^\uparrow} \tilde{a}_\ell \exp\left(\frac{c(\mathbf{N})}{2^{h-1}\hat{\Gamma}_\ell}\right)(\mathcal{P}_\ell(t))^{c(\mathbf{N})}}{\tilde{a}_q (\mathcal{P}_q(t))^{c(\mathbf{N})} + \sum_{\ell \in L^\uparrow} \tilde{a}_\ell (\mathcal{P}_\ell(t))^{c(\mathbf{N})}}\right)^{1/c(\mathbf{N})} \\ &\geq \left[\exp\left(\frac{c(\mathbf{N})}{2^h \hat{\Gamma}_{\mathbf{N}}}\right)\right]^{1/c(\mathbf{N})} = \exp\left(\frac{1}{2^h \hat{\Gamma}_{\mathbf{N}}}\right).\end{aligned}$$

To conclude, we have proved the following: when $\epsilon \leq \min\{\epsilon_2, \epsilon_3, \epsilon_4\}$, i.e. when $\mathcal{X}_{\mathbf{N}}(t) = \max\{1/\epsilon_2, 1/\epsilon_3, 1/\epsilon_4\}$, then $\frac{\mathcal{P}_{\mathbf{N}}(t+1)}{\mathcal{P}_{\mathbf{N}}(t)} \geq \exp\left(\frac{1}{2^h \hat{\Gamma}_{\mathbf{N}}}\right)$. So $\mathring{\mathcal{X}}_{\mathbf{N}} = \max\{1/\epsilon_2, 1/\epsilon_3, 1/\epsilon_4\}$. □

Lemma 10.7. *In any tatonnement process starting with positive prices, the consolidated spending of every node in a buyer's NCES utility tree is bounded away from zero.*

Proof: First, for every good g, let $\mathcal{M}_g$ denote $\max\{p_g^\circ, 2M\}$, which is an upper bound on the price of good g in the whole tatonnement process. For every internal node I in a NCES utility tree, let $\mathcal{M}_I$ denote the consolidated price of the node when the prices of every good g in the tree rooted at I are set to $\mathcal{M}_g$.

The proof proceeds in top-down manner. At the root of the utility tree, the consolidated spending is the budget of the buyer, which is a fixed positive constant. So the lemma is trivially true at the root. In the following two cases, we prove that when the consolidated spending of a node I is bounded away from zero, i.e. $\mathcal{S}_I(t) \geq b > 0$ for some positive constant b and for all t, then the consolidated spendings of all its children are bounded away from zero.

Case 1: I is a substitute node.

By Lemma 10.6, $\mathcal{P}_I(t) \geq \min\left\{\mathcal{P}_I(0), \frac{b}{2\mathring{\mathcal{X}}_I}\right\} > 0$. Then (10.3) provides a positive lower bound on the consolidated prices of all the children of I. Let J denote a child of I.

By the Substitute Effect and the Law of Consolidated Demand, $\mathcal{S}_J$ is minimized when $\mathcal{P}_J = \mathcal{M}_J$ and the consolidated prices of other children of I are at the lower bounds given in (10.3). Since $\mathcal{S}_I$ is bounded away from zero, and the consolidated prices of all the children of I are bounded away from zero and not infinitely large, by the definition of $\mathcal{S}_J$, $\mathcal{S}_J$ is bounded away from zero.

Case 2: I is a complement node.

Let J denote a child of I. For any fixed $\mathcal{P}_J$, by the Complement Effect, $\mathcal{X}_J$ is minimized when for any other child J' of I, $\mathcal{P}_{J'} = \mathcal{M}_{J'}$. Even in this extremal case, by the Law of Consolidated Demand, when $\mathcal{P}_J$ is reduced to be sufficiently small, $\mathcal{X}_J \geq \mathring{\mathcal{X}}_J$. Hence, $\mathcal{P}_J$ has a positive lower bound in the whole tatonnement process.

By the Complement Effect, $\mathcal{S}_J$ is minimized when $\mathcal{P}_J$ is at the lower bound, and for all other children J' of I, $\mathcal{P}_{J'} = \mathcal{M}_{J'}$. Since $\mathcal{S}_I$ is bounded away from zero, and the consolidated prices of all the children of I are bounded away from zero and not infinitely large, by the definition of $\mathcal{S}_J$, $\mathcal{S}_J$ is bounded away from zero. $\square$

Corollary 10.8. *The price of each good is bounded away from zero.*

Proof: By Lemma 10.7, the consolidated spending of every leaf node is bounded away from zero, i.e. there is a positive lower bound on the spending on every good. Finally, when there is excess demand for a good its price will only increase, thus the price of each good cannot drop below half the lower bound on the spending on that good. □

10.3 Upper Sandwiching Bounds and Convergence of Tatonnement

By Lemmas 6.6 and 10.4, the potential function for an NCES Fisher market is

$$\phi(p) = \sum_j p_j - \sum_i e_i \log \mathcal{P}_i(p), \tag{10.5}$$

in which i runs over all buyers. It is easy to check that this potential function is strictly convex, so the market equilibrium is unique.

As in Chapter 9.1.1, we aim to show an upper sandwiching bound of the form

$$\phi(p+\Delta p) - \ell_\phi(p+\Delta p; p) \leq \sum_\ell \gamma_\ell \cdot d_h(p_\ell + \Delta p_\ell, p_\ell),$$

where $p + \Delta p$ is restricted to be in the range of one price update started at p. Again, γ_ℓ need not be constant but could be functions of p_ℓ and x_ℓ. In what follows, we will show that if $|\Delta p_\ell / p_\ell|$ is sufficiently small, $\gamma_\ell = \Theta(x_\ell)$ suffice.

We state the following lemmas about convex and concave functions, which will be useful later. Let $\alpha_1, \alpha_2, \cdots, \alpha_k$ be positive numbers such that $\sum_{q=1}^k \alpha_q = 1$. Given $x_1, x_2, \cdots, x_k$, we let $\bar{x}$ denote the weighted average of the x_q with weights α_q, i.e. $\bar{x} := \sum_{q=1}^{k} \alpha_q x_q$. Recall from calculus that if f is twice differentiable and $f'' \geq 0$, f is convex. If f is a convex function, $f(\bar{x}) \leq \sum_{q=1}^{k} \alpha_q f(x_q)$, i.e. for a convex function, "function of average" is always less than or equal to "average of function". With stronger restrictions on f'', we have a "reversed" version and a stronger version of the above inequality:

Lemma 10.9. *If $f'' \leq M$, $f(\bar{x}) \geq \sum_{q=1}^{k} \alpha_q f(x_q) - \frac{M}{2} \sum_{q=1}^{k} \alpha_q (x_q - \bar{x})^2$.*

Lemma 10.10. *If $f'' \geq m \geq 0$, $f(\bar{x}) \leq \sum_{q=1}^{k} \alpha_q f(x_q) - \frac{m}{2} \sum_{q=1}^{k} \alpha_q (x_q - \bar{x})^2$.*

Let $\mathbf{N}$ denote an internal node of a NCES utility tree, and we let ℓ run over all its children. When the prices are changed from p to $p + \Delta p$, let $\mathcal{R}_{\mathbf{N}} = \mathcal{P}_{\mathbf{N}}(p + \Delta p)/\mathcal{P}_{\mathbf{N}}(p)$.

Lemma 10.11. *If $c(\mathbf{N}) > 0$, then $-\mathcal{S}_{\mathbf{N}} \log \mathcal{R}_{\mathbf{N}} \leq -\sum_{\ell} \mathcal{S}_{\ell} \log \mathcal{R}_{\ell}$.*

Proof: First, note that

$$\begin{aligned}
-\mathcal{S}_{\mathbf{N}} \log \mathcal{R}_{\mathbf{N}} &= -\mathcal{S}_{\mathbf{N}} \log \frac{\left(\sum_{\ell} \tilde{a}_{\ell} (\mathcal{P}_{\ell}(p + \Delta p))^{c(\mathbf{N})}\right)^{1/c(\mathbf{N})}}{\left(\sum_{\ell} \tilde{a}_{\ell} (\mathcal{P}_{\ell}(p))^{c(\mathbf{N})}\right)^{1/c(\mathbf{N})}} \\
&= -\frac{\mathcal{S}_{\mathbf{N}}}{c(\mathbf{N})} \log \left(\sum_{\ell} \left(\frac{\tilde{a}_{\ell} (\mathcal{P}_{\ell}(p))^{c(\mathbf{N})}}{\sum_q \tilde{a}_q (\mathcal{P}_q(p))^{c(\mathbf{N})}} \right) (\mathcal{R}_{\ell})^{c(\mathbf{N})} \right).
\end{aligned}$$

Let $\alpha_{\ell} := \frac{\tilde{a}_{\ell} (\mathcal{P}_{\ell}(p))^{c(\mathbf{N})}}{\sum_q \tilde{a}_q (\mathcal{P}_q(p))^{c(\mathbf{N})}}$. Note that $\sum_{\ell} \alpha_{\ell} = 1$. By Definition 10.3, $\mathcal{S}_{\ell} =$

$\alpha_\ell \mathcal{S}_{\mathbf{N}}$. Since the function $-\frac{\mathcal{S}_{\mathbf{N}}}{c(\mathbf{N})}\log(x)$ is convex in x,

$$\begin{aligned} -\mathcal{S}_{\mathbf{N}} \log \mathcal{R}_{\mathbf{N}} &= -\frac{\mathcal{S}_{\mathbf{N}}}{c(\mathbf{N})} \log\left(\sum_\ell \alpha_\ell (\mathcal{R}_\ell)^{c(\mathbf{N})}\right) \\ &\le -\frac{\mathcal{S}_{\mathbf{N}}}{c(\mathbf{N})} \sum_\ell \alpha_\ell \log(\mathcal{R}_\ell)^{c(\mathbf{N})} \\ &= -\sum_\ell \mathcal{S}_\ell \log \mathcal{R}_\ell. \end{aligned}$$

□

Lemma 10.12. *If* $c(\mathbf{N}) < 0$ *and for all* ℓ, $1 + \frac{1}{6(c(\mathbf{N})-1)} \le \mathcal{R}_\ell \le 1 - \frac{1}{6(c(\mathbf{N})-1)}$, *then* $-\mathcal{S}_{\mathbf{N}} \log \mathcal{R}_{\mathbf{N}} \le -\sum_\ell \mathcal{S}_\ell \log \mathcal{R}_\ell - \frac{21}{25} c(\mathbf{N}) \sum_\ell \mathcal{S}_\ell (\mathcal{R}_\ell - 1)^2$.

Proof: As in the proof of Lemma 10.11, let $\alpha_\ell := \frac{\tilde{a}_\ell \mathcal{P}_\ell(p)^{c(\mathbf{N})}}{\sum_q \tilde{a}_q \mathcal{P}_q(p)^{c(\mathbf{N})}}$. Let $\overline{\mathcal{R}} := \sum_\ell \alpha_\ell (\mathcal{R}_\ell)^{c(\mathbf{N})}$.

Since $\mathcal{R}_\ell \le 1 - \frac{1}{c(\mathbf{N})-1}$, $(\mathcal{R}_\ell)^{c(\mathbf{N})} \ge \left(1 - \frac{1}{c(\mathbf{N})-1}\right)^{c(\mathbf{N})} \ge \exp(-1/6)$. In Lemma 10.9, take $f(q) = -\log q$, then $f''((\mathcal{R}_\ell)^{c(\mathbf{N})}) = ((\mathcal{R}_\ell)^{c(\mathbf{N})})^{-2} \le \exp(1/3) < \frac{7}{5}$. Hence

$$\begin{aligned} -\mathcal{S}_{\mathbf{N}} \log \mathcal{R}_{\mathbf{N}} &= -\frac{\mathcal{S}_{\mathbf{N}}}{c(\mathbf{N})} \log\left(\sum_\ell \alpha_\ell (\mathcal{R}_\ell)^{c(\mathbf{N})}\right) \qquad \text{(Proved in Lemma 10.11)} \\ &\le -\frac{\mathcal{S}_{\mathbf{N}}}{c(\mathbf{N})} \left[\sum_\ell \alpha_\ell \log(\mathcal{R}_\ell)^{c(\mathbf{N})} + \frac{7}{10} \sum_\ell \alpha_\ell \left((\mathcal{R}_\ell)^{c(\mathbf{N})} - \overline{\mathcal{R}}\right)^2\right]. \\ &= -\sum_\ell \mathcal{S}_\ell \log \mathcal{R}_\ell - \frac{7}{10} \frac{\mathcal{S}_{\mathbf{N}}}{c(\mathbf{N})} \sum_\ell \alpha_\ell \left((\mathcal{R}_\ell)^{c(\mathbf{N})} - \overline{\mathcal{R}}\right)^2. \qquad \text{(Definition 10.3)} \end{aligned}$$

Consider the function $Q(R') = \sum_\ell \alpha_\ell \left((\mathcal{R}_\ell)^{c(\mathbf{N})} - R'\right)^2$. By simple calculus, one can show that $Q(R')$ is minimized at $R' = \overline{\mathcal{R}}$. In particular, $Q(\overline{\mathcal{R}}) \le Q(1)$, which yields

$$-\frac{\mathcal{S}_{\mathbf{N}}}{c(\mathbf{N})}\log\mathcal{R}_{\mathbf{N}}\sum_{\ell}\alpha_{\ell}\left((\mathcal{R}_{\ell})^{c(\mathbf{N})}-\overline{\mathcal{R}}\right)^{2}\leq-\frac{\mathcal{S}_{\mathbf{N}}}{c(\mathbf{N})}\sum_{\ell}\alpha_{\ell}\left((\mathcal{R}_{\ell})^{c(\mathbf{N})}-1\right)^{2}.$$

By simple calculus, one can show that if $1+\frac{1}{6(c(\mathbf{N})-1)}\leq\mathcal{R}_{\ell}\leq 1-\frac{1}{6(c(\mathbf{N})-1)}$, then

$$\left((\mathcal{R}_{\ell})^{c(\mathbf{N})}-1\right)^{2}\leq\frac{6}{5}(c(\mathbf{N}))^{2}(\mathcal{R}_{\ell}-1)^{2}.$$

Combining all the above inequalities,

$$\begin{aligned}-\mathcal{S}_{\mathbf{N}}\log\mathcal{R}_{\mathbf{N}} &\leq -\sum_{\ell}\mathcal{S}_{\ell}\log\mathcal{R}_{\ell}-\frac{7}{10}\frac{\mathcal{S}_{\mathbf{N}}}{c(\mathbf{N})}\sum_{\ell}\alpha_{\ell}\left((\mathcal{R}_{\ell})^{c(\mathbf{N})}-\overline{\mathcal{R}}\right)^{2}\\ &\leq -\sum_{\ell}\mathcal{S}_{\ell}\log\mathcal{R}_{\ell}-\frac{7}{10}\frac{\mathcal{S}_{\mathbf{N}}}{c(\mathbf{N})}\sum_{\ell}\alpha_{\ell}\left((\mathcal{R}_{\ell})^{c(\mathbf{N})}-1\right)^{2}\\ &\leq -\sum_{\ell}\mathcal{S}_{\ell}\log\mathcal{R}_{\ell}-\frac{21}{25}c(\mathbf{N})\sum_{\ell}\mathcal{S}_{\ell}\left(\mathcal{R}_{\ell}-1\right)^{2}. \qquad \text{(Definition 10.3)}\end{aligned}$$

□

Lemma 10.13. *If $\mathcal{R}_{\ell}\geq 1/2$ for all ℓ, then $\mathcal{S}_{\mathbf{N}}(\mathcal{R}_{\mathbf{N}}-1)^{2}\leq\sum_{\ell}\mathcal{S}_{\ell}(\mathcal{R}_{\ell}-1)^{2}$.*

Proof: By recalling that $\mathcal{S}_{\ell}=\mathcal{S}_{\mathbf{N}}\alpha_{\ell}$, we may rewrite the inequality in this lemma as $(\mathcal{R}_{\mathbf{N}} - 1)^2 \leq \sum_\ell \alpha_\ell(\mathcal{R}_\ell - 1)^2$. Then, note that

$$\begin{aligned}
\mathcal{R}_{\mathbf{N}} &= \frac{\mathcal{P}_{\mathbf{N}}(p+\Delta p)}{\mathcal{P}_{\mathbf{N}}(p)} \\
&= \frac{\left(\sum_\ell \tilde{a}_\ell(\mathcal{P}_\ell(p+\Delta p))^{c(\mathbf{N})}\right)^{1/c(\mathbf{N})}}{\left(\sum_\ell \tilde{a}_\ell(\mathcal{P}_\ell(p))^{c(\mathbf{N})}\right)^{1/c(\mathbf{N})}} \\
&= \left(\sum_\ell \left(\frac{\tilde{a}_\ell(\mathcal{P}_\ell(p))^{c(\mathbf{N})}}{\sum_q \tilde{a}_q(\mathcal{P}_q(p))^{c(\mathbf{N})}}\right)(\mathcal{R}_\ell)^{c(\mathbf{N})}\right)^{1/c(\mathbf{N})} \\
&= \left(\sum_\ell \alpha_\ell(\mathcal{R}_\ell)^{c(\mathbf{N})}\right)^{1/c(\mathbf{N})}.
\end{aligned}$$

So we may rewrite the inequality in this lemma as

$$\left(\left(\sum_\ell \alpha_\ell(\mathcal{R}_\ell)^{c(\mathbf{N})}\right)^{1/c(\mathbf{N})} - 1\right)^2 \leq \sum_\ell \alpha_\ell(\mathcal{R}_\ell - 1)^2.$$

We prove this inequality by considering the following constrained optimization problem: suppose the value of $\sum_\ell \alpha_\ell(\mathcal{R}_\ell)^{c(\mathbf{N})}$ is fixed, find the minimum value of $\sum_\ell \alpha_\ell(\mathcal{R}_\ell - 1)^2$. This optimization problem is readily solved by using Lagrange Multipliers as follows: at any stationary point, the ratio between the two quantities $\frac{\partial}{\partial \mathcal{R}_j}\sum_\ell \alpha_\ell(\mathcal{R}_\ell - 1)^2$ and $\frac{\partial}{\partial \mathcal{R}_j}\sum_\ell \alpha_\ell(\mathcal{R}_\ell)^{c(\mathbf{N})}$ is the same for all j. The ratio is equal to $\frac{2\alpha_j(\mathcal{R}_j - 1)}{c(\mathbf{N})\alpha_j(\mathcal{R}_j)^{c(\mathbf{N})-1}} = \frac{2}{c(\mathbf{N})}\mathcal{R}_\ell^{1-c(\mathbf{N})}(\mathcal{R}_\ell - 1)$. As the function $x^{1-c(\mathbf{N})}(x-1)$ is strictly increasing when $x \geq 1/2$, at any stationary point $\mathcal{R}_\ell$ must be the same for all ℓ. Since $\sum_\ell \alpha_\ell = 1$, $\mathcal{R}_{\mathbf{N}} = \left(\sum_\ell \alpha_\ell(\mathcal{R}_\ell)^{c(\mathbf{N})}\right)^{1/c(\mathbf{N})} = \mathcal{R}_\ell$ for all ℓ, i.e. there is a unique stationary point. It is easy to check that the stationary point is the global minimum. At this minimum point, $\sum_\ell \alpha_\ell(\mathcal{R}_\ell - 1)^2$ equals to $\left(\left(\sum_\ell \alpha_\ell(\mathcal{R}_\ell)^{c(\mathbf{N})}\right)^{1/c(\mathbf{N})} - 1\right)^2$. $\square$

Corollary 10.14. *Let $L_{\mathbf{N}}$ denote the set of leaves in the utility tree rooted at $\mathbf{N}$. If $\mathcal{R}_j \geq 1/2$ for all $j \in L_{\mathbf{N}}$, then $\mathcal{S}_{\mathbf{N}}(\mathcal{R}_{\mathbf{N}} - 1)^2 \leq \sum_{j \in L_{\mathbf{N}}} \mathcal{S}_j(\mathcal{R}_j - 1)^2$.*

Proof: It is easy to check that if $\mathcal{R}_j \geq 1/2$ for all $j \in L_{\mathbf{N}}$, then for any node I in the tree, $\mathcal{R}_I \geq 1/2$. So we can bootstrap Lemma 10.13 to prove this corollary. $\square$

We will use the inequality

$$\epsilon - \frac{2}{5}\epsilon^2 \geq \log(1+\epsilon) \geq \epsilon - \frac{2}{3}\epsilon^2, \tag{10.6}$$

which holds when $|\epsilon| \leq 1/4$.

For any good j, let $\bar{c}_j := \min_v\{c_v, 0\}$, in which v runs over all ancestors of good j in every buyer's NCES utility tree.

Lemma 10.15. *If for any good j, $\max\{3/4, 1 + \frac{1}{\bar{c}_j - 1}\} \leq \frac{p_j + \Delta p_j}{p_j} \leq \min\{5/4, 1 - \frac{1}{\bar{c}_j - 1}\}$, then*

$$\phi(p+\Delta p) - \ell_\phi(p+\Delta p; p) \leq \sum_j \left[4 - \frac{126}{25} \cdot \min_i \left(\sum_{c'_{ij}<0} c'_{ij}\right)\right] x_j \cdot d_h(p_j + \Delta p_j, p_j),$$

where c'_{ij} runs over all negative c-values of all ancestors of good j in buyer i's utility tree.

Proof: First, note that

$$
\begin{aligned}
\phi(p+\Delta p) - \ell_\phi(p+\Delta p; p) &= \phi(p+\Delta p) - \phi(p) + \sum_j z_j \Delta p_j \\
&= \sum_j \Delta p_j + \sum_j z_j \Delta p_j - \sum_i e_i \log \frac{\mathcal{P}_{i,root}(p+\Delta p)}{\mathcal{P}_{i,root}(p)} \\
&= \sum_j x_j \Delta p_j - \sum_i e_i \log \mathcal{R}_{i,root}.
\end{aligned}
$$

Next, we derive an upper bound on $-e_i \log \mathcal{R}_{i,root}$. From the root downwards, apply Lemmas 10.11 and 10.12 repeatedly depending on the sign of the c-values. Each node with a negative c-value provides an extra term to the final sum. By Corollary 10.14, each of these extra terms will contribute an overall term $-\frac{21}{25} c \mathcal{S}_j (\mathcal{R}_j - 1)^2$ to each of its leaf good j. These yield

$$
-\sum_i e_i \log \mathcal{R}_{i,root} \leq \sum_i \sum_j \left(-\mathcal{S}_{ij} \log \mathcal{R}_j - \frac{21}{25} \left(\sum_{c'_{ij} < 0} c'_{ij} \right) \mathcal{S}_{ij} (\mathcal{R}_j - 1)^2 \right).
$$

By (10.6),

$$
-\mathcal{S}_{ij} \log \mathcal{R}_j \leq -\mathcal{S}_{ij} \left(\frac{\Delta p_j}{p_j} - \frac{2}{3} \frac{(\Delta p_j)^2}{(p_j)^2} \right) = -x_{ij} \Delta p_j + \frac{2}{3} x_{ij} \frac{(\Delta p_j)^2}{p_j}.
$$

Hence,

$$
\begin{aligned}
&\phi(p+\Delta p) - \ell_\phi(p+\Delta p;p) \\
&\le \sum_j x_j \Delta p_j + \sum_{i,j}\left(-x_{ij}\Delta p_j + \frac{2}{3}x_{ij}\frac{(\Delta p_j)^2}{p_j}\right) - \frac{21}{25}\sum_{i,j}\left(\sum_{c'_{ij}<0} c'_{ij}\right)\mathcal{S}_{ij}\frac{(\Delta p_j)^2}{(p_j)^2} \\
&\le \frac{2}{3}\sum_j x_j\frac{(\Delta p_j)^2}{p_j} - \frac{21}{25}\sum_j \min_i\left(\sum_{c'_{ij}<0} c'_{ij}\right)\sum_i x_{ij}\frac{(\Delta p_j)^2}{p_j} \\
&= \sum_j\left[\frac{2}{3} - \frac{21}{25}\min_i\left(\sum_{c'_{ij}<0} c'_{ij}\right)\right] x_j\frac{(\Delta p_j)^2}{p_j} \\
&\le \sum_j\left[4 - \frac{126}{25}\cdot\min_i\left(\sum_{c'_{ij}<0} c'_{ij}\right)\right] x_j \cdot d_h(p_j+\Delta p_j, p_j).
\end{aligned}
$$

$\square$

10.3.1 Bounding Γ_j

In Lemma 10.15, we show that $\gamma_j^t \ge \left[4 - \frac{126}{25}\cdot\min_i\left(\sum_{c'_{ij}<0} c'_{ij}\right)\right] x_j^t$ suffice to ensure the upper sandwiching bound. Since

$$\left[4 - \frac{126}{25}\cdot\min_i\left(\sum_{c'_{ij}<0} c'_{ij}\right)\right] x_j^t \le \left[8 - \frac{252}{25}\cdot\min_i\left(\sum_{c'_{ij}<0} c'_{ij}\right)\right]\cdot\max\left\{1, z_j^t\right\}$$

for all z_j^t and $x_j^t = z_j^t + 1$, so $\Gamma_j \ge \left[8 - \frac{252}{25}\cdot\min_i\left(\sum_{c'_{ij}<0} c'_{ij}\right)\right]$ suffice. The tatonnement rule (10.1) becomes

$$p_j^{t+1} = p_j^t \cdot \exp\left(\frac{\min\{z(t),1\}}{\Gamma_j}\right). \tag{10.7}$$

10.3.2 Convergence

Theorem 10.16. *Suppose all prices in p^0 are positive. For all NCES Fisher markets, the sequence of prices p^t defined by the update rule* (10.7) *converges to a market equilibrium.*

Proof: Following the proof of Lemma 9.2 (with Lemma 10.15 replacing Lemma 9.6), it is easy to show that

$$\phi(p^t) - \phi(p^{t+1}) \geq \Theta\left(\sum_j \frac{p_j(z_j)^2}{\Gamma_j}\right).$$

By Corollary 10.8, the sequence p^t is guaranteed to stay in the compact region $\mathfrak{C} = \times_j[\kappa, \mathcal{M}_j]$ for some positive number κ. Since all prices in $\mathfrak{C}$ are bounded away from zero and upper bounded by $\mathcal{M}_j$, ϕ, as given in (10.5), has finite upper and lower bounds in $\mathfrak{C}$, and hence $\phi(p^0) - \phi(p^t)$ is finitely bounded for all t. Since $\mathfrak{C}$ is compact, the sequence p^t has at least one limit point; let $\bar{p}$ denote one of the limit points.

Suppose that $\bar{p}$ is not a market equilibrium. By Corollary 10.8, all prices are bounded away from zero throughout the tatonnement process, so all prices in $\bar{p}$ are positive. Since $\bar{p}$ is not a market equilibrium, $z_k(\bar{p}) \neq 0$ for some good k. Further, since all prices in $\bar{p}$ are positive, $z_k(\cdot)$ is continuous at $\bar{p}$. Hence, there exists a sufficiently small neighbourhood of $\bar{p}$ such that for all p' in the neighbourhood, $\sum_j \frac{p'_j(z_j(p'))^2}{\Gamma_j} \geq \frac{p'_k(z_k(p'))^2}{\Gamma_k} \geq \delta$ for some $\delta > 0$.

Since $\bar{p}$ is a limit point of the sequence p^t, the tatonnement enters the neighbourhood infinitely often, and thus $\phi(p^t)$ drops by $\Omega(\delta)$ for infinitely many times.

But this is impossible, since $\phi(p^t)$ is a decreasing sequence and $\phi(p^0) - \phi(p^t)$ is finitely bounded for all t. So a contradiction occurs, which forces that $\bar{p}$ must be a market equilibrium.

By the arguments above, all limit points of the sequence p^t are market equilibria. But the potential function ϕ for NCES Fisher market is strictly convex in $\mathfrak{C}$, so there is a unique market equilibrium in $\mathfrak{C}$. Hence, $\bar{p}$ is the unique limit point of the sequence p^t, i.e. $\lim_{t\to\infty} p^t = \bar{p}$. $\square$

Appendix A

Missing Proofs

Proof of Lemma 3.6:

(a) Note that $\left(1-\frac{\lambda}{2}\right)(1+\lambda) = 1 + \frac{\lambda-\lambda^2}{2}$. Given $0 \le \lambda \le 1$, $\lambda - \lambda^2 \ge 0$. Hence $\left(1-\frac{\lambda}{2}\right)(1+\lambda) \ge 1$. We are done.

(b) For any λ satisfying $0 \le \lambda \le 1$, consider the function

$$h_\lambda(x) := 1 - \lambda\left(1 - \frac{1}{x}\right) - x^{-\lambda/2}.$$

Note that $\frac{dh_\lambda(x)}{dx} = \frac{\lambda}{x^2}\left(\frac{x^{1-\lambda/2}}{2} - 1\right)$. When $1 \le x \le 2$, $\frac{x^{1-\lambda/2}}{2} \le 1$ and hence $\frac{dh_\lambda(x)}{dx} \le 0$. Then for any $1 \le x \le 2$, $h_\lambda(x) \le h_\lambda(1) = 0$ and we are done.

(c) For any λ satisfying $0 \le \lambda \le 1$, consider the function

$$g_\lambda(x) := x^{-\lambda}\left[1 + \lambda(x-1)\right].$$

Note that $\frac{dg_\lambda(x)}{dx} = \lambda(1-\lambda)x^{-\lambda}\left(1 - x^{-\lambda}\right)$. When $1 \le x \le 2$, $x^{-\lambda} \le 1$ and hence

$\frac{dg_\lambda(x)}{dx} \geq 0$. Then for any $1 \leq x \leq 2$, $g_\lambda(x) \geq g_\lambda(1) = 1$ and we are done. $\square$

Proof of Lemma 3.13:

(a) For any $0 \leq x \leq 1$, consider the function

$$h_x^a(\epsilon) := (1+\epsilon)^x - 1 - \epsilon x.$$

Note that for any $0 \leq \epsilon \leq 1$, $\frac{dh_x^a(\epsilon)}{d\epsilon} = x[(1+\epsilon)^{x-1} - 1] \leq 0$, and hence $h_x^a(\epsilon) \leq h_x^a(0) = 0$. We are done.

(b) First, we need the following inequality: when $0 \leq \epsilon < 1$, $-\ln(1-\epsilon) \leq \epsilon + \frac{\epsilon^2}{2(1-\epsilon)}$. To prove this inequality, consider the function $y(\epsilon) = \epsilon + \frac{\epsilon^2}{2(1-\epsilon)} + \ln(1-\epsilon)$. Then $\frac{dy(\epsilon)}{d\epsilon} = 1 + \frac{\epsilon^2}{2(1-\epsilon)^2} + \frac{\epsilon}{1-\epsilon} - \frac{1}{1-\epsilon} = \frac{\epsilon^2}{2(1-\epsilon)^2} \geq 0$. Hence $y(\epsilon) \geq y(0) = 0$.

Next, consider the function

$$h_\epsilon^b(x) := 1 - (1-\epsilon)^x - \left(1 + \frac{\epsilon}{2(1-\epsilon)}\right)\epsilon x.$$

Then $\frac{dh_\epsilon^b(x)}{dx} = -(1-\epsilon)^x \ln(1-\epsilon) - \left(1 + \frac{\epsilon}{2(1-\epsilon)}\right)\epsilon \leq (1-\epsilon)^x \left(1 + \frac{\epsilon}{2(1-\epsilon)}\right)\epsilon - \left(1 + \frac{\epsilon}{2(1-\epsilon)}\right)\epsilon = \left(1 + \frac{\epsilon}{2(1-\epsilon)}\right)\epsilon\left((1-\epsilon)^x - 1\right) \leq 0$. Hence $h_\epsilon^b(x) \leq h_\epsilon^b(0) = 0$.

(c) Expanding $(1-\epsilon)^{1-E} - 1$ by Newton's binomial formula gives

$$\begin{aligned}(1-\epsilon)^{1-E} - 1 &= \sum_{i=1}^{\infty} \frac{(E-1)E(E+1)\cdots(E-2+i)}{i!}\epsilon^i \\ &= (E-1)\epsilon\left(1 + \sum_{i=1}^{\infty} \frac{E(E+1)\cdots(E-1+i)}{(i+1)!}\epsilon^i\right)\end{aligned}$$

Note that every term in the summation is positive, and the ratio between the

$(i+1)$-st term and the i-th term is $\frac{\frac{E(E+1)\cdots(E+i)}{(i+2)!}\epsilon^{i+1}}{\frac{E(E+1)\cdots(E-1+i)}{(i+1)!}\epsilon^{i}} = \frac{E+i}{2+i}\epsilon$.

When $E \leq 2$, the ratio is at most ϵ, so the summation is at most $\frac{\epsilon}{1-\epsilon}$, i.e. $(1-\epsilon)^{1-E} - 1 \leq (E-1)\epsilon\left(1+\frac{\epsilon}{1-\epsilon}\right) = \frac{E-1}{1-\epsilon}\epsilon$.

When $E \geq 2$, the ratio is at most $E\epsilon/2$, which is less than 1 by assumption. Then the summation is at most $\frac{E\epsilon/2}{1-E\epsilon/2}$, i.e. $(1-\epsilon)^{1-E} - 1 \leq (E-1)\lambda\left(1+\frac{E\epsilon/2}{1-E\epsilon/2}\right) = \frac{E-1}{1-E\epsilon/2}\epsilon$.

(d) For any $E \geq 1$, consider the function

$$h_E^d(\epsilon) := 1 - (1+\epsilon)^{1-E} - (E-1)\epsilon.$$

Note that for any $0 \leq \epsilon \leq 1$, $\frac{dh_E^d(\epsilon)}{d\epsilon} = (E-1)(1+\epsilon)^{-E} - (E-1) \leq 0$, and hence $h_E^d(\epsilon) \leq h_E^d(0) = 0$. We are done.

(e) This is a corollary of (c). In (c), taking $E = x+1$ yields $(1-\epsilon)^{-x} \leq 1 + \frac{x}{1-\frac{(x+1)\epsilon}{2}}\epsilon$. As $x \geq 1$, $\frac{x+1}{2} \leq x$ and hence $\frac{x}{1-\frac{(x+1)\epsilon}{2}} \leq \frac{x}{1-x\epsilon}$. We are done. $\square$

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