In this report we investigate the solution of boundary value problems on polygonal domains for elliptic partial differential equations.

On the Solution of Elliptic Partial Differential Equations on Regions with Corners II: Detailed Analysis

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1 Introduction

Classical potential theory reduces elliptic partial differential equations to second kind boundary integral equations, by representing the solutions by single-layer or double-layer potentials on the boundaries of the regions.

While there are many regimes in which such boundary integral equations have been studied, this report deals only with polygonal boundaries. In this case, both the kernels and the solutions of the integral equations of potential theory are singular at the corners. The existence and uniqueness of the solutions in the L^2-sense have long been known (see [3], [14]), and the properties of the solutions to both the differential and integral equations on polygons have been the subject of extensive research (see, for example, [9], [16], [15]; reviews of the literature can be found in, for example, [7], [10]). While much is known about the spaces to which to solutions belong, and the leading singular terms of the solutions are well-known, it was observed in [12] that the solutions admit an explicit representation which appears to have been overlooked.

In [12], it is shown that when the second kind boundary integral equations of potential theory are solved on polygonal domains, the solutions near corners are explicitly representable by linear combinations of certain non-integer powers, except at finitely many angles of the corners. In this report, we extend the results of [12] to all angles of the corners. In particular, we show that the solutions to the boundary integral equations near corners are, for any angle, representable by certain linear combinations of non-integer powers and non-integer powers multiplied by logarithms. We prove this result by constructing a mapping from the coefficients of these singular terms to the coefficients of the Taylor series representing the boundary data, and show that this mapping is invertible for any angle of the corner.

The structure of the report is as follows. Section 2 provides an overview of the principal results of the report. In Section 3, we introduce the necessary mathematical preliminaries. Section 4 contains the analytical apparatus. Sections 5 and 6 analyze the Neumann and Dirichlet cases respectively.

2 Overview

The following two subsections 2.1 and 2.2 summarize the Neumann and Dirichlet cases respectively. The principal results of this report are theorems 2.1 and 2.2.

Suppose that $\gamma\colon [-1,1] \to \mathbb{R}^2$ is a wedge in $\mathbb{R}^2$ with a corner at $\gamma(0)$, and with interior angle $\pi\alpha$. Suppose further that γ is parameterized by arc length, and let $\nu(t)$ denote the inward-facing unit normal to the curve γ at t. Let Γ denote the set $\gamma([-1,1])$. By extending the sides of the wedge to infinity, we divide $\mathbb{R}^2$ into two open sets Ω_1 and Ω_2 (see Figure 1).

2.1 The Neumann Case

Let $\phi\colon \mathbb{R}^2 \setminus \Gamma \to \mathbb{R}$ denote the potential induced by a charge distribution on γ with density $\rho\colon [-1,1] \to \mathbb{R}$. In other words, let ϕ be defined by the formula

$$\phi(x) = -\frac{1}{2\pi}\int_{-1}^{1} \log(\|\gamma(t) - x\|)\rho(t)\,dt, \tag{1}$$

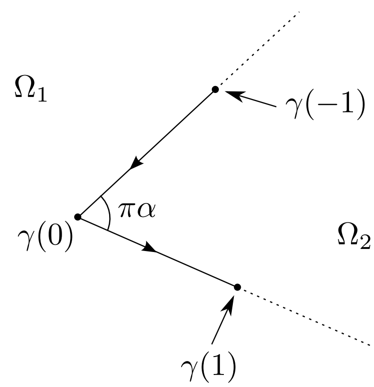

Figure 1: A wedge in $\mathbb{R}^2$

for all $x \in \mathbb{R}^2 \setminus \Gamma$, where $\|\cdot\|$ denotes the Euclidean norm. Suppose that $g\colon [-1,1] \to \mathbb{R}$ is defined by the formula

$$g(t) = \lim_{\substack{x \to \gamma(t) \\ x \in \Omega_1}} \frac{\partial \phi(x)}{\partial \nu(t)}, \tag{2}$$

for all $-1 \leq t \leq 1$, i.e. g is the limit of the normal derivative of integral (1) when x approaches the point $\gamma(t)$ from outside. It is well known that

$$g(s) = \frac{1}{2}\rho(s) + \frac{1}{2\pi}\int_{-1}^{1} K(s,t)\rho(t)\,dt, \tag{3}$$

for all $-1 \leq s \leq 1$, where

$$K(s,t) = \frac{\langle \gamma(t) - \gamma(s), \nu(s) \rangle}{\|\gamma(t) - \gamma(s)\|^2}, \tag{4}$$

for all $-1 \leq s, t \leq 1$, where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^2$.

In this report, we prove the following theorem, which holds for any angle $0 < \pi\alpha < 2\pi$, and which is the first of the two principal results of this report.

Theorem 2.1. *Suppose that $0 < \alpha < 2$ and that N is a positive integer. Letting $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ denote the floor and ceiling functions respectively, suppose that*

$$\overline{L} = \left\lceil \frac{\alpha N}{2} \right\rceil, \tag{5}$$

$$\underline{L} = \left\lfloor \frac{\alpha N}{2} \right\rfloor, \tag{6}$$

and

$$\underline{M} = \left\lfloor \frac{(2-\alpha)N}{2} \right\rfloor, \tag{7}$$

$$\overline{M} = \left\lceil \frac{(2-\alpha)N}{2} \right\rceil, \tag{8}$$

and observe that $\overline{L}+\underline{M}=N$ *and* $\overline{M}+\underline{L}=N$*. Suppose further that* ρ *is defined by the formula*

$$\rho(t)=\sum_{i=1}^{\overline{L}} b_i|t|^{\frac{2i-1}{\alpha}-1}+\sum_{\substack{i=1\\ i\notin S}}^{\underline{M}} b_{\overline{L}+i}|t|^{\frac{2i}{2-\alpha}-1}+\sum_{\substack{i=1\\ i\in S}}^{\underline{M}} b_{\overline{L}+i}|t|^{\frac{2i}{2-\alpha}-1}\log(|t|)$$
$$+\sum_{i=1}^{\overline{M}} c_i\operatorname{sgn}(t)|t|^{\frac{2i-1}{2-\alpha}-1}+\sum_{\substack{i=1\\ i\notin T}}^{\underline{L}} c_{\overline{M}+i}\operatorname{sgn}(t)|t|^{\frac{2i}{\alpha}-1}+\sum_{\substack{i=1\\ i\in T}}^{\underline{L}} c_{\overline{M}+i}\operatorname{sgn}(t)|t|^{\frac{2i}{\alpha}-1}\log(|t|), \quad (9)$$

for all $-1\le t\le 1$*, where* $b_1,b_2,\dots,b_N$ *and* $c_1,c_2,\dots,c_N$ *are arbitrary real numbers,*

$$S=\Big\{i\in\mathbb{Z},1\le i\le \underline{M}:\frac{2i}{2-\alpha}=\frac{2j-1}{\alpha}\quad \textit{for some integer } 1\le j\le \overline{L}\Big\}, \quad (10)$$

and

$$T=\Big\{i\in\mathbb{Z},1\le i\le \underline{L}:\frac{2i}{\alpha}=\frac{2j-1}{2-\alpha}\quad \textit{for some integer } 1\le j\le \overline{M}\Big\}. \quad (11)$$

Suppose finally that g *is defined by (3). Then there exist sequences of real numbers* $\beta_0,\beta_1,\dots$ *and* $\gamma_0,\gamma_1,\dots$ *such that*

$$g(t)=\sum_{n=0}^{\infty}\beta_n|t|^n+\sum_{n=0}^{\infty}\gamma_n\operatorname{sgn}(t)|t|^n, \quad (12)$$

for all $-1\le t\le 1$*. Conversely, suppose that* g *has the form (12). Suppose further that* N *is an arbitrary positive integer. Then,* for all angles $\pi\alpha$*, there exist unique real numbers* $b_1,b_2,\dots,b_N$ *and* $c_1,c_2,\dots,c_N$ *such that* ρ*, defined by (9), solves equation (3) to within an error* $O(t^N)$*.*

In other words, if ρ has the form (9), then g is smooth on each of the intervals $[-1,0]$ and $[0,1]$. Conversely, if g is smooth on each of the intervals $[-1,0]$ and $[0,1]$, then for each positive integer N, and for each angle $\pi\alpha$, there exists a unique solution ρ of the form (9) that solves equation (3) to within an error $O(t^N)$.

2.2 The Dirichlet Case

Let $\phi\colon\mathbb{R}^2\setminus\Gamma\to\mathbb{R}$ denote the potential induced by a dipole distribution on γ with density $\rho\colon[-1,1]\to\mathbb{R}$. In other words, let ϕ be defined by the formula

$$\phi(x)=\frac{1}{2\pi}\int_{-1}^{1}\frac{\langle x-\gamma(t),\nu(t)\rangle}{\|x-\gamma(t)\|^2}\rho(t)\,dt, \quad (13)$$

for all $x\in\mathbb{R}^2\setminus\Gamma$, where $\|\cdot\|$ denotes the Euclidean norm and $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathbb{R}^2$. Suppose that $g\colon[-1,1]\to\mathbb{R}$ is defined by the formula

$$g(t)=\lim_{\substack{x\to\gamma(t)\\ x\in\Omega_2}}\phi(x), \quad (14)$$

for all $-1 \le t \le 1$, i.e. g is the limit of integral (13) when x approaches the point $\gamma(t)$ from inside. It is well known that

$$g(s) = \frac{1}{2}\rho(s) + \frac{1}{2\pi}\int_{-1}^{1} K(t,s)\rho(t)\,dt, \tag{15}$$

for all $-1 \le s \le 1$, where

$$K(t,s) = \frac{\langle \gamma(s) - \gamma(t), \nu(t)\rangle}{\|\gamma(s) - \gamma(t)\|^2}, \tag{16}$$

for all $-1 \le s, t \le 1$, where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^2$.

In this report, we prove the following theorem, which holds for any angle $0 < \pi\alpha < 2\pi$, and which is the second of the two principal results of this report.

Theorem 2.2. *Suppose that $0 < \alpha < 2$ and that N is a positive integer. Letting $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ denote the floor and ceiling functions respectively, suppose that*

$$\overline{L} = \left\lceil \frac{\alpha N}{2} \right\rceil, \tag{17}$$

$$\underline{L} = \left\lfloor \frac{\alpha N}{2} \right\rfloor, \tag{18}$$

and

$$\underline{M} = \left\lfloor \frac{(2-\alpha)N}{2} \right\rfloor, \tag{19}$$

$$\overline{M} = \left\lceil \frac{(2-\alpha)N}{2} \right\rceil, \tag{20}$$

and observe that $\overline{L} + \underline{M} = N$ and $\overline{M} + \underline{L} = N$. Suppose further that ρ is defined by the formula

$$\begin{aligned} \rho(t) = b_0 &+ \sum_{i=1}^{\overline{L}} b_i |t|^{\frac{2i-1}{\alpha}} + \sum_{\substack{i=1 \\ i \notin S}}^{\underline{M}} b_{\overline{L}+i} |t|^{\frac{2i}{2-\alpha}} + \sum_{\substack{i=1 \\ i \in S}}^{\underline{M}} b_{\overline{L}+i} |t|^{\frac{2i}{2-\alpha}} \log(|t|) + c_0 \cdot \operatorname{sgn}(t) \\ &+ \sum_{i=1}^{\overline{M}} c_i \operatorname{sgn}(t) |t|^{\frac{2i-1}{2-\alpha}} + \sum_{\substack{i=1 \\ i \notin T}}^{\underline{L}} c_{\overline{M}+i} \operatorname{sgn}(t) |t|^{\frac{2i}{\alpha}} + \sum_{\substack{i=1 \\ i \in T}}^{\underline{L}} c_{\overline{M}+i} \operatorname{sgn}(t) |t|^{\frac{2i}{\alpha}} \log(|t|), \end{aligned} \tag{21}$$

for all $-1 \le t \le 1$, where $b_0, b_1, \dots, b_N$ and $c_0, c_1, \dots, c_N$ are arbitrary real numbers,

$$S = \left\{ i \in \mathbb{Z}, 1 \le i \le \underline{M} : \frac{2i}{2-\alpha} = \frac{2j-1}{\alpha} \quad \textit{for some integer } 1 \le j \le \overline{L} \right\}, \tag{22}$$

and

$$T = \left\{ i \in \mathbb{Z}, 1 \le i \le \underline{L} : \frac{2i}{\alpha} = \frac{2j-1}{2-\alpha} \quad \textit{for some integer } 1 \le j \le \overline{M} \right\}. \tag{23}$$

Suppose finally that g is defined by (15). Then there exist sequences of real numbers $\beta_0, \beta_1, \dots$ and $\gamma_0, \gamma_1, \dots$ such that

$$g(t) = \sum_{n=0}^{\infty} \beta_n |t|^n + \sum_{n=0}^{\infty} \gamma_n \operatorname{sgn}(t) |t|^n, \tag{24}$$

for all $-1 \leq t \leq 1$*. Conversely, suppose that* g *has the form (24). Suppose further that* N *is an arbitrary positive integer. Then,* for all angles $\pi\alpha$*, there exist unique real numbers* $b_0, b_1, \ldots, b_N$ *and* $c_0, c_1, \ldots, c_N$ *such that* ρ*, defined by (21), solves equation (15) to within an error* $O(t^{N+1})$*.*

In other words, if ρ has the form (21), then g is smooth on each of the intervals $[-1, 0]$ and $[0, 1]$. Conversely, if g is smooth on each of the intervals $[-1, 0]$ and $[0, 1]$, then for each positive integer N, and for each angle $\pi\alpha$, there exists a unique solution ρ of the form (21) that solves equation (15) to within an error $O(t^{N+1})$.

3 Mathematical Preliminaries

3.1 Boundary Value Problems

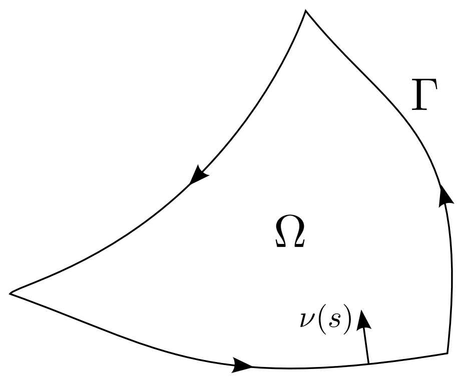

Figure 2: A curve in $\mathbb{R}^2$

Suppose that $\gamma\colon [0, L] \to \mathbb{R}^2$ is a simple closed curve of length L with a finite number of corners. Suppose further that γ is analytic except at the corners. We denote the interior of γ by Ω and the boundary of Ω by Γ, and let $\nu(t)$ denote the normalized internal normal to γ at $t \in [0, L]$. Supposing that g is some function $g\colon [0, L] \to \mathbb{R}$, we will solve the following four problems.

1) *Interior Neumann problem* (INP): Find a function $\phi\colon \Omega \to \mathbb{R}$ such that

$$\nabla^2 \phi(x) = 0 \qquad \text{for } x \in \Omega, \tag{25}$$

$$\lim_{\substack{x \to \gamma(t) \\ x \in \Omega}} \frac{\partial \phi(x)}{\partial \nu(t)} = g(t) \qquad \text{for } t \in [0, L]. \tag{26}$$

2) *Exterior Neumann problem* (ENP): Find a function $\phi\colon \mathbb{R}^2 \setminus \overline{\Omega} \to \mathbb{R}$ such that

$$\nabla^2 \phi(x) = 0 \qquad \text{for } x \in \mathbb{R}^2 \setminus \overline{\Omega}, \tag{27}$$

$$\lim_{\substack{x \to \gamma(t) \\ x \in \mathbb{R}^2 \setminus \overline{\Omega}}} \frac{\partial \phi(x)}{\partial \nu(t)} = g(t) \qquad \text{for } t \in [0, L]. \tag{28}$$

3) *Interior Dirichlet problem* (IDP): Find a function $\phi\colon \Omega \to \mathbb{R}$ such that

$$\nabla^2 \phi(x) = 0 \qquad \text{for } x \in \Omega, \tag{29}$$

$$\lim_{\substack{x \to \gamma(t) \\ x \in \Omega}} \phi(x) = g(t) \qquad \text{for } t \in [0, L]. \tag{30}$$

4) *Exterior Dirichlet problem* (EDP): Find a function $\phi\colon \mathbb{R}^2 \setminus \overline{\Omega} \to \mathbb{R}$ such that

$$\nabla^2 \phi(x) = 0 \qquad \text{for } x \in \mathbb{R}^2 \setminus \overline{\Omega}, \tag{31}$$

$$\lim_{\substack{x \to \gamma(t) \\ x \in \mathbb{R}^2 \setminus \overline{\Omega}}} \phi(x) = g(t) \qquad \text{for } t \in [0, L]. \tag{32}$$

Suppose that $g \in L^2([0, L])$. Then the interior and exterior Dirichlet problems have unique solutions. If g also satisfies the condition

$$\int_0^L g(t)\, dt = 0, \tag{33}$$

then the interior and exterior Neumann problems have unique solutions up to an additive constant (see, for example, [8], [5]).

3.2 Integral Equations of Potential Theory

In classical potential theory, boundary value problems are solved by representing the function ϕ by integrals of potentials over the boundary. The potential of a *unit charge* located at $x_0 \in \mathbb{R}^2$ is the function $\psi^0_{x_0}\colon \mathbb{R}^2 \setminus x_0 \to \mathbb{R}$, defined via the formula

$$\psi^0_{x_0}(x) = \log(\|x - x_0\|), \tag{34}$$

for all $x \in \mathbb{R}^2 \setminus x_0$, where $\|\cdot\|$ denotes the Euclidean norm. The potential of a *unit dipole* located at $x_0 \in \mathbb{R}^2$ and oriented in direction $h \in \mathbb{R}^2$, $\|h\| = 1$, is the function $\psi^1_{x_0,h}\colon \mathbb{R}^2 \setminus x_0 \to \mathbb{R}$, defined via the formula

$$\psi^1_{x_0,h}(x) = \frac{\langle h, x_0 - x\rangle}{\|x_0 - x\|^2}, \tag{35}$$

for all $x \in \mathbb{R}^2 \setminus x_0$, where $\langle \cdot, \cdot \rangle$ denotes the inner product.

Charge and dipole distributions with density $\rho\colon [0, L] \to \mathbb{R}$ on Γ produce potentials given by the formulas

$$\phi(x) = \int_0^L \psi^0_{\gamma(t)}(x) \rho(t)\, dt, \tag{36}$$

and

$$\phi(x) = \int_0^L \psi^1_{\gamma(t),\nu(t)}(x) \rho(t)\, dt, \tag{37}$$

respectively, for any $x \in \mathbb{R}^2 \setminus \Gamma$.

Reduction of Boundary Value Problems to Integral Equations

The following four theorems reduce the boundary value problems of Section 3.1 to boundary integral equations. They are found in, for example, [11], [14].

Theorem 3.1 (Exterior Neumann problem)**.** *Suppose that $\rho \in L^2([0,L])$. Suppose further that $g\colon [0,L] \to \mathbb{R}$ is defined by the formula*

$$g(s) = -\pi\rho(s) + \int_0^L \psi^1_{\gamma(s),\nu(s)}(\gamma(t))\rho(t)\,dt, \tag{38}$$

for any $s \in [0,L]$. Then g is in $L^2([0,L])$, and a solution ϕ to the exterior Neumann problem with right hand side g is obtained by substituting ρ into (36). Moreover, for any $g \in L^2([0,L])$, equation (38) has a unique solution $\rho \in L^2([0,L])$.

Theorem 3.2 (Interior Dirichlet problem)**.** *Suppose that $\rho \in L^2([0,L])$. Suppose further that $g\colon [0,L] \to \mathbb{R}$ is defined by the formula*

$$g(s) = -\pi\rho(s) + \int_0^L \psi^1_{\gamma(t),\nu(t)}(\gamma(s))\rho(t)\,dt, \tag{39}$$

for any $s \in [0,L]$. Then g is in $L^2([0,L])$, and the solution ϕ to the interior Dirichlet problem with right hand side g is obtained by substituting ρ into (37). Moreover, for any $g \in L^2([0,L])$, equation (39) has a unique solution $\rho \in L^2([0,L])$.

The following two theorems make use of the function $\overline{\omega}\colon [0,L] \to \mathbb{R}$, defined as the solution to the equation

$$\int_0^L \overline{\omega}(t)\log(\|x-\gamma(t)\|)\,dt = 1, \tag{40}$$

for all $x \in \overline{\Omega}$. In other words, we define the function $\overline{\omega}$ as the density of the charge distribution on Γ when $\overline{\Omega}$ is a conductor.

Theorem 3.3 (Interior Neumann problem)**.** *Suppose that $\rho \in L^2([0,L])$. Suppose further that $g\colon [0,L] \to \mathbb{R}$ is defined by the formula*

$$g(s) = \pi\rho(s) + \int_0^L \psi^1_{\gamma(s),\nu(s)}(\gamma(t))\rho(t)\,dt, \tag{41}$$

for any $s \in [0,L]$. Then g is in $L^2([0,L])$, and a solution ϕ to the exterior Neumann problem with right hand side g is obtained by substituting ρ into (36).

Now suppose that g is an arbitrary function in $L^2([0,L])$ such that

$$\int_0^L g(t)\,dt = 0. \tag{42}$$

Then equation (41) has a solution $\rho \in L^2([0,L])$. Moreover, if ρ_1 and ρ_2 are both solutions to equation (41), then there exists a real number C such that

$$\rho_1(t) - \rho_2(t) = C\overline{\omega}(t), \tag{43}$$

for $t \in [0,L]$, where $\overline{\omega}$ is the solution to (40).

Theorem 3.4 (Exterior Dirichlet problem)**.** *Suppose that $\rho \in L^2([0,L])$. Suppose further that $g\colon [0,L] \to \mathbb{R}$ is defined by the formula*

$$g(s) = \pi\rho(s) + \int_0^L \psi^1_{\gamma(t),\nu(t)}(\gamma(s))\rho(t)\,dt, \tag{44}$$

for any $s \in [0,L]$. Then g is in $L^2([0,L])$, and the solution ϕ to the interior Dirichlet problem with right hand side g is obtained by substituting ρ into (37).

Now suppose that g is an arbitrary function in $L^2([0,L])$ such that

$$\int_0^L g(t)\overline{\omega}(t)\,dt = 0, \tag{45}$$

where $\overline{\omega}$ is the solution to (40). Then equation (44) has a solution $\rho \in L^2([0,L])$. Moreover, if ρ_1 and ρ_2 are both solutions to equation (44), then there exists a real number C such that

$$\rho_1(t) - \rho_2(t) = C, \tag{46}$$

for $t \in [0,L]$.

Observation 3.1. Equation (38) is the adjoint of equation (39), and equation (41) is the adjoint of equation (44).

Observation 3.2. Suppose that the curve $\gamma\colon [0,L] \to \mathbb{R}^2$ is *not* closed. We observe that if $\rho \in L^2([0,L])$, and g is defined by either (38), (39), (41), or (44), then $g \in L^2([0,L])$. Moreover, if $g \in L^2([0,L])$, then equations (38), (39), (41), and (44) have unique solutions $\rho \in L^2([0,L])$.

Properties of the Kernels of Equations (38), (39), (41), and (44)

The following theorem shows that if a curve γ is has k continuous derivatives, where $k \geq 2$, then the kernels of equations (38), (39), (41), and (44) have $k-2$ continuous derivatives. It is found in, for example, [2].

Theorem 3.5. *Suppose that $\gamma\colon [0,L] \to \mathbb{R}^2$ is a curve in $\mathbb{R}^2$ that is parameterized by arc length. Suppose further that $k \geq 2$ is an integer. If γ is C^k on a neighborhood of a point s, where $0 < s < L$, then*

$$\psi^1_{\gamma(s),\nu(s)}(\gamma(t)), \tag{47}$$

$$\psi^1_{\gamma(t),\nu(t)}(\gamma(s)), \tag{48}$$

are C^{k-2} functions of t on a neighborhood of s and

$$\lim_{t\to s} \psi^1_{\gamma(s),\nu(s)}(\gamma(t)) = \lim_{t\to s} \psi^1_{\gamma(t),\nu(t)}(\gamma(s)) = -\frac{1}{2}k(s), \tag{49}$$

where $k\colon [0,L] \to \mathbb{R}$ is the signed *curvature of γ. Furthermore, if γ is analytic on a neighborhood of a point s, where $0 < s < L$, then (47) and (48) are analytic functions of t on a neighborhood of s.*

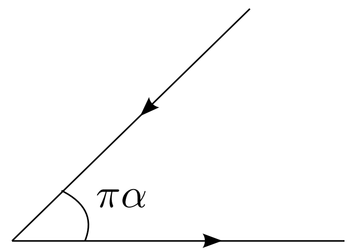

Figure 3: A wedge in $\mathbb{R}^2$

When the curve γ is a wedge, the kernels of equations (38), (39), (41), and (44) have a particularly simple form, which is given by the following lemma. It is proved in [13].

Lemma 3.6. *Suppose $\gamma\colon [-1,1] \to \mathbb{R}^2$ is defined by the formula*

$$\gamma(t) = \begin{cases} -t \cdot (\cos(\pi\alpha), \sin(\pi\alpha)) & \text{if } -1 \le t < 0, \\ (t, 0) & \text{if } 0 \le t \le 1, \end{cases} \tag{50}$$

shown in Figure 3. Then, for all $0 < s \le 1$,

$$\psi^1_{\gamma(s),\nu(s)}(\gamma(t)) = \begin{cases} \dfrac{t\sin(\pi\alpha)}{s^2 + t^2 + 2st\cos(\pi\alpha)} & \text{if } -1 \le t < 0, \\ 0 & \text{if } 0 \le t \le 1, \end{cases} \tag{51}$$

and, for all $-1 \le s < 0$,

$$\psi^1_{\gamma(s),\nu(s)}(\gamma(t)) = \begin{cases} 0 & \text{if } -1 \le t < 0, \\ \dfrac{-t\sin(\pi\alpha)}{s^2 + t^2 + 2st\cos(\pi\alpha)} & \text{if } 0 \le t \le 1. \end{cases} \tag{52}$$

Corollary 3.7. *Identities (51) and (52) remain valid after any rotation or translation of the curve γ in $\mathbb{R}^2$.*

Corollary 3.8. *When the curve γ is a straight line, $\psi^1_{\gamma(s),\nu(s)}(\gamma(t)) = 0$ for all $-1 \le s, t \le 1$.*

3.3 Boundary Integral Equations on a Wedge

Suppose that the curve $\gamma\colon [-1,1] \to \mathbb{R}^2$ is a wedge defined by (50) with interior angle $\pi\alpha$, where $0 < \alpha < 2$ (see Figure 3), and let $\nu(t)$ denote the inward-facing unit normal to the curve γ at the point $\gamma(t)$.

The Neumann Case

Let g be a function in $L^2([-1,1])$, and suppose that $\rho \in L^2([-1,1])$ solves the equation

$$g(s) = -\pi\rho(s) + \int_{-1}^{1} \psi^1_{\gamma(s),\nu(s)}(\gamma(t))\rho(t)\, dt, \tag{53}$$

for all $s \in [-1,1]$, where $\psi^1_{\gamma(s),\nu(s)}$ is defined by (35).

The following lemma combines equation (53) with identities (51) and (52).

Lemma 3.9. *Let $g \in L^2([-1,1])$, and suppose that $\rho \in L^2([-1,1])$ solves equation (53). Then*

$$g(s) = -\pi\rho(s) - \int_0^1 \frac{t\sin(\pi\alpha)}{s^2+t^2+2st\cos(\pi\alpha)}\rho(t)\,dt, \tag{54}$$

for all $-1 \le s < 0$, and

$$g(s) = -\pi\rho(s) + \int_{-1}^0 \frac{t\sin(\pi\alpha)}{s^2+t^2+2st\cos(\pi\alpha)}\rho(t)\,dt, \tag{55}$$

for all $0 < s \le 1$.

The following theorem uses a symmetry argument to reduce (54), (55) from two coupled integral equations on the interval $[-1,1]$ to two independent integral equations on the interval $[0,1]$. It is proved in [12].

Theorem 3.10. *Suppose that g is a function in $L^2([-1,1])$, and $\rho \in L^2([-1,1])$ solves equation (53). Suppose further that even functions $g_{\text{even}}, \rho_{\text{even}} \in L^2([-1,1])$ are defined via the formulas*

$$g_{\text{even}}(s) = \frac{1}{2}(g(s)+g(-s)), \tag{56}$$

$$\rho_{\text{even}}(s) = \frac{1}{2}(\rho(s)+\rho(-s)). \tag{57}$$

Then

$$g_{\text{even}}(s) = -\pi\rho_{\text{even}}(s) - \int_0^1 \frac{t\sin(\pi\alpha)}{s^2+t^2-2st\cos(\pi\alpha)}\rho_{\text{even}}(t)\,dt, \tag{58}$$

for all $0 < s \le 1$.

Likewise, suppose that odd functions $g_{\text{odd}}, \rho_{\text{odd}} \in L^2([-1,1])$ are defined via the formulas

$$g_{\text{odd}}(s) = \frac{1}{2}(g(s)-g(-s)), \tag{59}$$

$$\rho_{\text{odd}}(s) = \frac{1}{2}(\rho(s)-\rho(-s)). \tag{60}$$

Then

$$g_{\text{odd}}(s) = -\pi\rho_{\text{odd}}(s) + \int_0^1 \frac{t\sin(\pi\alpha)}{s^2+t^2-2st\cos(\pi\alpha)}\rho_{\text{odd}}(t)\,dt, \tag{61}$$

for all $0 < s \le 1$.

The Dirichlet Case

Let g be a function in $L^2([-1,1])$, and suppose that $\rho \in L^2([-1,1])$ solves the equation

$$g(s) = -\pi\rho(s) + \int_{-1}^1 \psi^1_{\gamma(t),\nu(t)}(\gamma(s))\rho(t)\,dt, \tag{62}$$

for all $s \in [-1,1]$, where $\psi^1_{\gamma(t),\nu(t)}$ is defined by (35).

The following lemma combines equation (62) with identities (51) and (52).

Lemma 3.11. *Let $g \in L^2([-1,1])$, and suppose that $\rho \in L^2([-1,1])$ solves equation (62). Then*

$$g(s) = -\pi\rho(s) - \int_0^1 \frac{s\sin(\pi\alpha)}{s^2+t^2+2st\cos(\pi\alpha)}\rho(t)\,dt, \tag{63}$$

for all $-1 \le s < 0$, and

$$g(s) = -\pi\rho(s) + \int_{-1}^0 \frac{s\sin(\pi\alpha)}{s^2+t^2+2st\cos(\pi\alpha)}\rho(t)\,dt, \tag{64}$$

for all $0 < s \le 1$.

The following theorem uses a symmetry argument to reduce (63), (64) from two coupled integral equations on the interval $[-1,1]$ to two independent integral equations on the interval $[0,1]$. It is proved in [12].

Theorem 3.12. *Suppose that g is a function in $L^2([-1,1])$, and $\rho \in L^2([-1,1])$ solves equation (62). Suppose further that even functions $g_{\text{even}}, \rho_{\text{even}} \in L^2([-1,1])$ are defined via the formulas*

$$g_{\text{even}}(s) = \frac{1}{2}(g(s)+g(-s)), \tag{65}$$
$$\rho_{\text{even}}(s) = \frac{1}{2}(\rho(s)+\rho(-s)). \tag{66}$$

Then

$$g_{\text{even}}(s) = -\pi\rho_{\text{even}}(s) - \int_0^1 \frac{s\sin(\pi\alpha)}{s^2+t^2-2st\cos(\pi\alpha)}\rho_{\text{even}}(t)\,dt, \tag{67}$$

for all $0 < s \le 1$.

Likewise, suppose that odd functions $g_{\text{odd}}, \rho_{\text{odd}} \in L^2([-1,1])$ are defined via the formulas

$$g_{\text{odd}}(s) = \frac{1}{2}(g(s)-g(-s)), \tag{68}$$
$$\rho_{\text{odd}}(s) = \frac{1}{2}(\rho(s)-\rho(-s)). \tag{69}$$

Then

$$g_{\text{odd}}(s) = -\pi\rho_{\text{odd}}(s) + \int_0^1 \frac{s\sin(\pi\alpha)}{s^2+t^2-2st\cos(\pi\alpha)}\rho_{\text{odd}}(t)\,dt, \tag{70}$$

for all $0 < s \le 1$.

3.4 Chebyshev Systems

The following defines the concept of a Chebyshev system.

Definition 3.1 (Chebyshev system)**.** *Let $a < b$ be real numbers and let n be a positive integer. A set of n functions $\varphi_1, \varphi_2, \dots, \varphi_n \colon [a,b] \to \mathbb{R}$ will be referred to as a* Chebyshev system *if each function is continuous on $[a,b]$ and the determinants*

$$\det \begin{pmatrix} \varphi_1(x_1) & \varphi_2(x_1) & \dots & \varphi_n(x_1) \\ \varphi_1(x_2) & \varphi_2(x_2) & \dots & \varphi_n(x_2) \\ \vdots & \vdots & & \vdots \\ \varphi_1(x_n) & \varphi_2(x_n) & \dots & \varphi_n(x_n) \end{pmatrix} \tag{71}$$

are nonzero for any set of n distinct points $x_1, x_2, \dots, x_n \in [a,b]$.

The following theorem states that the Lagrange interpolating polynomials form a Chebyshev system. For completeness, a proof is provided in Appendix A.

Theorem 3.13. *Suppose that K and L are nonnegative integers such that $K \le L$. Suppose further that $\mu_1, \mu_2, \dots, \mu_L$ are distinct real numbers. Suppose finally that $p \colon \mathbb{R} \to \mathbb{R}$ is defined by*

$$p(x) = \prod_{\ell=1}^{L} (x - \mu_\ell) \prod_{k=1}^{K} (x - \mu_k), \tag{72}$$

for all $x \in \mathbb{R}$. Then the set of $L + K$ functions

$$\left\{ \frac{p(x)}{x - \mu_\ell} \right\}_{\ell=1}^{L} \cup \left\{ \frac{p(x)}{(x - \mu_k)^2} \right\}_{k=1}^{K}, \tag{73}$$

where $x \in \mathbb{R}$, is a Chebyshev system.

3.5 Miscellaneous Analytical Facts

This section contains a number of miscellaneous elementary technical lemmas.

Theorem 3.14. *Suppose that $0 < \alpha < 2$. Then*

$$\frac{\sin(\pi\mu(1-\alpha))}{\sin(\pi\mu)} = -1, \tag{74}$$

if and only if

$$\mu = \frac{2n-1}{\alpha}, \tag{75}$$

or

$$\mu = \frac{2n}{2-\alpha}, \tag{76}$$

for some integer n. Likewise,

$$\frac{\sin(\pi\mu(1-\alpha))}{\sin(\pi\mu)} = 1, \tag{77}$$

if and only if

$$\mu = \frac{2n}{\alpha}, \tag{78}$$

or

$$\mu = \frac{2n-1}{2-\alpha}, \tag{79}$$

for some integer n.

Theorem 3.15. *Suppose that $0 < \alpha < 2$ and that n and m are positive integers. Then*

$$\frac{2n}{2-\alpha} = \frac{2m-1}{\alpha}, \tag{80}$$

if and only if

$$\alpha = \frac{4m-2}{2n+2m-1}. \tag{81}$$

Proof. Suppose that

$$\frac{2n}{2-\alpha} = \frac{2m-1}{\alpha}. \tag{82}$$

Multiplying both sides of (82) by

$$\frac{2-\alpha}{2m-1}, \tag{83}$$

we have

$$\frac{2n}{2m-1} = \frac{2}{\alpha} - 1. \tag{84}$$

Thus,

$$\frac{2}{\alpha} = \frac{2n+2m-1}{2m-1}, \tag{85}$$

so

$$\alpha = \frac{4m-2}{2n+2m-1}. \tag{86}$$

■

Corollary 3.16. *Suppose that $0 < \alpha < 2$ and that n and m are positive integers. Then*

$$\frac{2n}{\alpha} = \frac{2m-1}{2-\alpha}, \tag{87}$$

if and only if

$$\alpha = \frac{4n}{2n+2m-1}. \tag{88}$$

Theorem 3.17. *Suppose that $0 < \alpha < 2$ and that n is a positive integer. Suppose further that*

$$\frac{2n}{2-\alpha} = \frac{2m-1}{\alpha}, \tag{89}$$

for some positive integer m. Then

$$\frac{2n}{2-\alpha} = n + m - \frac{1}{2}. \tag{90}$$

Proof. By Theorem 3.15,

$$\alpha = \frac{4m-2}{2n+2m-1}. \tag{91}$$

Therefore,

$$\frac{2m-1}{\alpha} = \frac{2n+2m-1}{2} = n + m - \frac{1}{2}. \tag{92}$$

■

Corollary 3.18. *Suppose that $0 < \alpha < 2$ and that n is a positive integer. Suppose further that*

$$\frac{2n}{\alpha} = \frac{2m-1}{2-\alpha}, \tag{93}$$

for some positive integer m. Then

$$\frac{2n}{\alpha} = n + m - \frac{1}{2}. \tag{94}$$

Theorem 3.19. *Suppose that $0 < \alpha < 2$. Suppose further that*

$$\mu = \frac{2n}{2-\alpha}, \tag{95}$$

where n is a positive integer. Then

$$(1-\alpha)\cos(\pi(1-\alpha)\mu) - \cot(\pi\mu)\sin(\pi(1-\alpha)\mu) = 0, \tag{96}$$

if and only if

$$\frac{2n}{2-\alpha} = \frac{2m-1}{\alpha}, \tag{97}$$

for some positive integer m.

Proof. Suppose that

$$\frac{2n}{2-\alpha} = \frac{2m-1}{\alpha}, \tag{98}$$

for some positive integer m. Then, by Theorem 3.17,

$$\mu = n + m - \frac{1}{2}, \tag{99}$$

so

$$\cos(\pi\mu) = \cos(\pi(n + m - \tfrac{1}{2})) = 0. \tag{100}$$

Furthermore, by Theorem 3.15,

$$\alpha = \frac{4m - 2}{2n + 2m - 1}. \tag{101}$$

Therefore,

$$1 - \alpha = \frac{2n - 2m + 1}{2n + 2m - 1}. \tag{102}$$

Combining (102) and (99),

$$(1 - \alpha)\mu = \frac{2n - 2m + 1}{2n + 2m - 1} \cdot \Big(n + m - \frac{1}{2}\Big) = n - m + \frac{1}{2}, \tag{103}$$

so

$$\cos(\pi(1 - \alpha)\mu) = \cos(\pi(n - m + \tfrac{1}{2})) = 0. \tag{104}$$

Thus, from (104) and (100), we have that

$$(1 - \alpha)\cos(\pi(1 - \alpha)\mu) - \cot(\pi\mu)\sin(\pi(1 - \alpha)\mu) = 0. \tag{105}$$

For the converse, suppose now that

$$(1 - \alpha)\cos(\pi(1 - \alpha)\mu) - \cot(\pi\mu)\sin(\pi(1 - \alpha)\mu) = 0. \tag{106}$$

By Theorem 3.14,

$$\frac{\sin(\pi(1 - \alpha)\mu)}{\sin(\pi\mu)} = -1. \tag{107}$$

Thus,

$$(1 - \alpha)\cos(\pi(1 - \alpha)\mu) + \cos(\pi\mu) = 0. \tag{108}$$

Combining (108) and (95), we have that

$$(1 - \alpha)\cos\big(\pi(1 - \alpha)\tfrac{2n}{2-\alpha}\big) + \cos\big(\pi\tfrac{2n}{2-\alpha}\big) = 0. \tag{109}$$

Since

$$\cos\big(\pi(1 - \alpha) \cdot \tfrac{2n}{2-\alpha}\big) = \cos\big(-\pi\tfrac{2n}{2-\alpha} + 2n\pi\big) = \cos\big(\pi\tfrac{2n}{2-\alpha}\big), \tag{110}$$

it follows that

$$(1 - \alpha)\cos\big(\pi\tfrac{2n}{2-\alpha}\big) + \cos\big(\pi\tfrac{2n}{2-\alpha}\big) = 0, \tag{111}$$

so

$$(2-\alpha)\cdot\cos\left(\pi\tfrac{2n}{2-\alpha}\right)=0. \tag{112}$$

Thus,

$$\cos\left(\pi\tfrac{2n}{2-\alpha}\right)=0. \tag{113}$$

Clearly, (113) implies that

$$\frac{2n}{2-\alpha}=k+\frac{1}{2}, \tag{114}$$

for some integer $k>n-1$. Therefore

$$\frac{4n}{2-\alpha}=2k+1, \tag{115}$$

and

$$2-\alpha=\frac{4n}{2k+1}, \tag{116}$$

so

$$\alpha=\frac{4k-4n+2}{2k+1}. \tag{117}$$

Letting $m=k-n+1$, we have

$$\alpha=\frac{4m-2}{2n+2m-1} \tag{118}$$

and $m>0$, so, by Theorem 3.15,

$$\frac{2n}{2-\alpha}=\frac{2m-1}{\alpha}. \tag{119}$$

■

Corollary 3.20. *Suppose that* $0<\alpha<2$. *Suppose further that*

$$\mu=\frac{2n}{\alpha}, \tag{120}$$

where n *is a positive integer. Then*

$$(1-\alpha)\cos(\pi(1-\alpha)\mu)-\cot(\pi\mu)\sin(\pi(1-\alpha)\mu)=0, \tag{121}$$

if and only if

$$\frac{2n}{\alpha}=\frac{2m-1}{2-\alpha}, \tag{122}$$

for some positive integer m.

Theorem 3.21. *Suppose* $0 < \alpha < 2$ *and that* i *and* n *are integers. If*

$$i = \frac{2n-1}{\alpha}, \tag{123}$$

then

$$\lim_{\nu \to \alpha} \frac{\sin(i\pi\nu)}{i - \frac{2n-1}{\nu}} = -\pi\alpha. \tag{124}$$

Likewise, if

$$i = \frac{2n}{2-\alpha}, \tag{125}$$

then

$$\lim_{\nu \to \alpha} \frac{\sin(i\pi\nu)}{i - \frac{2n}{2-\nu}} = -\pi(2-\alpha). \tag{126}$$

Proof. Suppose that

$$i = \frac{2n-1}{\alpha}. \tag{127}$$

Then, by L'Hôpital's rule,

$$\lim_{\nu \to \alpha} \frac{\sin(i\pi\nu)}{i - \frac{2n-1}{\nu}} = -\pi \cdot \lim_{\nu \to \alpha} \nu \cdot \frac{\sin(i\pi\nu - (2n-1)\pi)}{i\pi\nu - (2n-1)\pi} = -\pi\alpha. \tag{128}$$

Likewise, suppose that

$$i = \frac{2n}{2-\alpha}. \tag{129}$$

Then,

$$\lim_{\nu \to \alpha} \frac{\sin(i\pi\nu)}{i - \frac{2n}{2-\nu}} = -\pi \cdot \lim_{\nu \to \alpha} (2-\nu) \cdot \frac{\sin(i\pi(2-\nu) - 2n\pi)}{i\pi(2-\nu) - 2n\pi} = -\pi(2-\alpha). \tag{130}$$

■

Theorem 3.22. *Suppose* $0 < \alpha < 2$ *and that* i *and* n *are integers. If*

$$i = \frac{2n-1}{2-\alpha}, \tag{131}$$

then

$$\lim_{\nu \to \alpha} \frac{\sin(i\pi\nu)}{i - \frac{2n-1}{2-\nu}} = \pi(2-\alpha). \tag{132}$$

Likewise, if

$$i = \frac{2n}{\alpha}, \tag{133}$$

then

$$\lim_{\nu \to \alpha} \frac{\sin(i\pi\nu)}{i - \frac{2n}{\nu}} = \pi\alpha. \tag{134}$$

Theorem 3.23. *Suppose that $0 < \alpha < 2$, and that L and M are arbitrary nonnegative integers. Suppose further that $p\colon (0,2) \times \mathbb{Z} \to \mathbb{R}$ is defined by*

$$p(\nu, i) = \prod_{\ell=1}^{L} \Big(i - \frac{2\ell - 1}{\nu} \Big) \prod_{m=1}^{M} \Big(i - \frac{2m}{2-\nu} \Big), \tag{135}$$

for all $0 < \nu < 2$ and integers i. Then, for each integer i, the limit

$$\lim_{\nu \to \alpha} \frac{\sin(i\pi\nu)}{p(\nu, i)} \tag{136}$$

exists and is finite.

Proof. Suppose that i is an integer and that

$$i = \frac{2\ell - 1}{\alpha}, \tag{137}$$

for some integer $1 \le \ell \le L$. By Theorem 3.17,

$$\frac{2\ell - 1}{\alpha} \neq \frac{2m}{2 - \alpha}, \tag{138}$$

for all positive integers m, so, by Theorem 3.21,

$$\lim_{\nu \to \alpha} \frac{\sin(i\pi\nu)}{p(\nu, i)} \tag{139}$$

exists and is finite.

Likewise, suppose that i is an integer and that

$$i = \frac{2m}{2 - \alpha}, \tag{140}$$

for some integer $1 \le m \le M$. By Theorem 3.17,

$$\frac{2m}{2 - \alpha} \neq \frac{2\ell - 1}{\alpha}, \tag{141}$$

for all positive integers ℓ. Thus, by Theorem 3.21,

$$\lim_{\nu \to \alpha} \frac{\sin(i\pi\nu)}{p(\nu, i)} \tag{142}$$

exists and is finite.

■

Corollary 3.24. *Suppose that $0 < \alpha < 2$, and that L and M are arbitrary nonnegative integers. Suppose further that $p\colon (0,2) \times \mathbb{Z} \to \mathbb{R}$ is defined by*

$$p(\nu, i) = \prod_{\ell=1}^{L} \Big(i - \frac{2\ell - 1}{2-\nu} \Big) \prod_{m=1}^{M} \Big(i - \frac{2m}{\nu} \Big), \tag{143}$$

for all $0 < \nu < 2$ and integers i. Then, for each integer i, the limit

$$\lim_{\nu \to \alpha} \frac{\sin(i\pi\nu)}{p(\nu, i)} \tag{144}$$

exists and is finite.

Theorem 3.25. *Suppose that x is a real number, and let $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ denote the floor and ceiling functions respectively. Then*

$$\lfloor x + \tfrac{1}{2} \rfloor \le \lceil x \rceil. \tag{145}$$

Proof. Let $y = x - \lfloor x \rfloor$. Then $0 \le y < 1$, and

$$\lfloor x + \tfrac{1}{2} \rfloor = \lfloor \lfloor x \rfloor + y + \tfrac{1}{2} \rfloor = \lfloor x \rfloor + \lfloor y + \tfrac{1}{2} \rfloor. \tag{146}$$

If $0 \le y < \frac{1}{2}$, then $\lfloor y + \frac{1}{2} \rfloor = \lfloor y \rfloor$. On the other hand, if $\frac{1}{2} \le y < 1$, then $\lfloor y + \frac{1}{2} \rfloor = \lceil y \rceil$. Thus,

$$\lfloor y + \tfrac{1}{2} \rfloor \le \lceil y \rceil. \tag{147}$$

Combining (147) and (146),

$$\lfloor x + \tfrac{1}{2} \rfloor \le \lfloor x \rfloor + \lceil y \rceil = \lceil \lfloor x \rfloor + y \rceil = \lceil x \rceil. \tag{148}$$

■

4 Analytical Apparatus

4.1 Integral Identities

The following theorem is proved in [12], and is an identity involving the integral of a ratio of a polynomial and a non-integer power.

Theorem 4.1. *Suppose that $0 < \alpha < 2$, and that $\operatorname{Re}\mu > -1$ and $\mu \ne 1, 2, 3, \ldots$. Then*

$$\int_0^1 \frac{x^\mu \sin(\pi\alpha)}{a^2 - 2ax\cos(\pi\alpha) + x^2}\, dx = \pi a^{\mu-1} \frac{\sin\bigl(\mu\pi(1-\alpha)\bigr)}{\sin(\mu\pi)} + \sum_{k=0}^{\infty} \frac{\sin\bigl((k+1)\pi\alpha\bigr)}{\mu - k - 1} a^k, \tag{149}$$

for all $0 < a < 1$.

The following theorem computes the integral in (149) when the integrand is multiplied by $\log(x)$. It is proved by differentiating (149) with respect to μ.

Theorem 4.2. *Suppose that $0 < \alpha < 2$, and that $\operatorname{Re}(\mu) > -1$ and $\mu \ne 1, 2, 3, \ldots$. Then*

$$\begin{aligned}\int_0^1 \frac{x^\mu \log(x) \sin(\pi\alpha)}{a^2 - 2ax\cos(\pi\alpha) + x^2}\, dx &= \pi a^{\mu-1} \log(a) \frac{\sin(\mu\pi(1-\alpha))}{\sin(\mu\pi)} \\ &+ \frac{\pi^2 a^{\mu-1}}{\sin(\pi\mu)} \bigl((1-\alpha)\cos(\pi(1-\alpha)\mu) - \cot(\pi\mu)\sin(\pi(1-\alpha)\mu)\bigr) \\ &- \sum_{k=0}^{\infty} \frac{\sin((k+1)\pi\alpha)}{(\mu-k-1)^2} a^k,\end{aligned} \tag{150}$$

for all $0 < a < 1$.

4.2 The Invertibility of a Certain Linear Mapping

In this section, we show that a certain matrix is invertible. The principal result of this section is Theorem 4.4.

The following is a technical lemma involving the limits of a certain ratio of a sine function and a rational function.

Lemma 4.3. *Suppose that N is a positive integer and that $0 < \alpha < 2$. Letting $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ denote the floor and ceiling functions respectively, suppose further that*

$$\bar{L} = \left\lceil \frac{\alpha N}{2} \right\rceil, \tag{151}$$

and

$$\underline{M} = \left\lfloor \frac{(2-\alpha)N}{2} \right\rfloor, \tag{152}$$

and observe that $\bar{L} + \underline{M} = N$. Suppose finally that $p\colon (0,2)\times\mathbb{Z} \to \mathbb{R}$ is defined by

$$p(\nu, i) = \prod_{j=1}^{\bar{L}} \Big(i - \frac{2j-1}{\nu}\Big) \prod_{k=1}^{\underline{M}} \Big(i - \frac{2k}{2-\nu}\Big), \tag{153}$$

for all $0 < \nu < 2$ and integers i. Then, for each integer $1 \le i \le N$,

$$\lim_{\nu\to\alpha} \frac{\sin(i\pi\nu)}{p(\nu, i)} \tag{154}$$

exists and is finite, and

$$\lim_{\nu\to\alpha} \frac{\sin(i\pi\nu)}{p(\nu, i)} \neq 0. \tag{155}$$

Proof. Suppose that $1 \le i \le N$ is an integer. Then, by Theorem 3.23, the limit

$$\lim_{\nu\to\alpha} \frac{\sin(i\pi\nu)}{p(\nu, i)} \tag{156}$$

exists and is finite. Suppose further that αi is an integer. Then

$$\sin(i\pi\alpha) = 0, \tag{157}$$

and, furthermore,

$$\lim_{\nu\to\alpha} \frac{\sin(i\pi\nu)}{p(\nu, i)} \neq 0 \tag{158}$$

if and only if either

$$i = \frac{2j-1}{\alpha}, \tag{159}$$

for some integer j, where

$$1 \le j \le \left\lceil \frac{N\alpha}{2} \right\rceil, \tag{160}$$

or

$$i = \frac{2k}{2-\alpha}, \tag{161}$$

for some integer k, where

$$1 \leq k \leq \left\lfloor \frac{(2-\alpha)N}{2} \right\rfloor. \tag{162}$$

Suppose that $i\alpha$ is an odd integer. Then

$$i\alpha = 2j - 1, \tag{163}$$

for some integer j. Therefore,

$$i = \frac{2j-1}{\alpha}, \tag{164}$$

for some integer j. Observing that

$$j = \frac{\alpha i}{2} + \frac{1}{2}, \tag{165}$$

and combining (165) with $1 \leq i \leq N$, we have

$$\frac{\alpha}{2} + \frac{1}{2} \leq j \leq \frac{N\alpha}{2} + \frac{1}{2}, \tag{166}$$

so

$$1 \leq j \leq \left\lfloor \frac{N\alpha}{2} + \frac{1}{2} \right\rfloor. \tag{167}$$

Thus, by Theorem 3.25,

$$1 \leq j \leq \left\lceil \frac{N\alpha}{2} \right\rceil. \tag{168}$$

Suppose now that $i\alpha$ is an even integer. Then

$$i(2-\alpha) = 2k, \tag{169}$$

for some integer k. Therefore,

$$i = \frac{2k}{2-\alpha}, \tag{170}$$

for some integer k. Observing that

$$k = \frac{(2-\alpha)i}{2}, \tag{171}$$

and combining (171) with $1 \leq i \leq N$, we have

$$\frac{2-\alpha}{2} \leq k \leq \frac{(2-\alpha)N}{2}. \tag{172}$$

Thus,

$$1 \leq k \leq \left\lfloor \frac{(2-\alpha)N}{2} \right\rfloor. \tag{173}$$

■

The following theorem is the principal result of this section.

Theorem 4.4. *Suppose that $0 < \alpha < 2$. Suppose further that N is a positive integer, and, letting $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ denote the floor and ceiling functions respectively, suppose that*

$$\overline{L} = \left\lceil \frac{\alpha N}{2} \right\rceil, \tag{174}$$

and

$$\underline{M} = \left\lfloor \frac{(2-\alpha)N}{2} \right\rfloor, \tag{175}$$

and observe that $\overline{L} + \underline{M} = N$. Suppose finally that A is the $N \times N$ matrix defined by the formula

$$A_{i,j} = \lim_{\nu \to \alpha} \frac{\sin(i\pi\nu)}{i - \frac{2j-1}{\nu}}, \tag{176}$$

$$A_{i,\overline{L}+k} = \begin{cases} \lim_{\nu \to \alpha} \dfrac{\sin(i\pi\nu)}{i - \frac{2k}{2-\nu}} & \text{if } \frac{2k}{2-\alpha} \neq \frac{2\ell-1}{\alpha} \text{ for all integers } 1 \le \ell \le \overline{L}, \\ \dfrac{\sin(i\pi\alpha)}{(i - \frac{2k}{2-\alpha})^2} & \text{if } \frac{2k}{2-\alpha} = \frac{2\ell-1}{\alpha} \text{ for some integer } 1 \le \ell \le \overline{L}, \end{cases} \tag{177}$$

for all integers $1 \le i \le N$, $1 \le j \le \overline{L}$ and $1 \le k \le \underline{M}$ (Theorems 3.21 and 3.17 show that matrix A is well-defined). Then matrix A is invertible.

Proof. Suppose that $p\colon (0,2) \times \mathbb{Z} \to \mathbb{R}$ is defined by

$$p(\nu, i) = \prod_{j=1}^{\overline{L}} \Big(i - \frac{2j-1}{\nu}\Big) \prod_{k=1}^{\underline{M}} \Big(i - \frac{2k}{2-\nu}\Big), \tag{178}$$

for all $0 < \nu < 2$ and integers i. Suppose further that D is the diagonal $N \times N$ matrix defined by

$$D_{i,j} = \begin{cases} \lim_{\nu \to \alpha} \dfrac{\sin(i\pi\nu)}{p(\nu, i)} & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases} \tag{179}$$

for all integers $1 \le i, j \le N$. By Lemma 4.3, the matrix D is well-defined and invertible. Therefore,

$$(D^{-1}A)_{i,j} = \lim_{\nu \to \alpha} \frac{p(\nu, i)}{i - \frac{2j-1}{\nu}}, \tag{180}$$

$$(D^{-1}A)_{i,\overline{L}+k} = \begin{cases} \lim_{\nu \to \alpha} \dfrac{p(\nu, i)}{i - \frac{2k}{2-\nu}} & \text{if } \frac{2k}{2-\alpha} \neq \frac{2\ell-1}{\alpha} \text{ for all integers } 1 \le \ell \le \overline{L}, \\ \dfrac{p(\alpha, i)}{(i - \frac{2k}{2-\alpha})^2} & \text{if } \frac{2k}{2-\alpha} = \frac{2\ell-1}{\alpha} \text{ for some integer } 1 \le \ell \le \overline{L}, \end{cases} \tag{181}$$

for all integers $1 \le i \le N$, $1 \le j \le \overline{L}$ and $1 \le k \le \underline{M}$. By Theorem 3.13, the matrix $D^{-1}A$ is invertible and, since D is invertible, so is the matrix A.

■

5 Analysis of the Integral Equation: the Neumann Case

Suppose that the curve $\gamma\colon [-1,1] \to \mathbb{R}^2$ is a wedge defined by (50) with interior angle $\pi\alpha$, where $0 < \alpha < 2$ (see Figure 3), and let $\nu(t)$ denote the inward-facing unit normal to the curve γ at the point $\gamma(t)$. Let g be a function in $L^2([-1,1])$, and suppose that $\rho \in L^2([-1,1])$ solves the equation

$$-\pi\rho(s) + \int_{-1}^{1} \psi^1_{\gamma(s),\nu(s)}(\gamma(t))\rho(t)\,dt = g(s), \tag{182}$$

for all $s \in [-1,1]$, where $\psi^1_{\gamma(s),\nu(s)}$ is defined by (35).

In this section, we analyze this boundary integral equation, which is well-posed even though the curve γ is open (see Observation 3.2). We investigate the behavior of (182) for functions $\rho \in L^2([-1,1])$ of the forms

$$\rho(t) = |t|^{\mu-1}, \tag{183}$$
$$\rho(t) = |t|^{\mu-1}\log(|t|), \tag{184}$$
$$\rho(t) = \mathrm{sgn}(t)|t|^{\mu-1}, \tag{185}$$
$$\rho(t) = \mathrm{sgn}(t)|t|^{\mu-1}\log(|t|), \tag{186}$$

where $\mu > \frac{1}{2}$ is a real number. If ρ has the forms (183), (184), (185), (186), then for most values of μ the resulting g is singular. In Section 5.1, we observe that for certain μ, the function g is smooth. In Section 5.2, we fix g and view (182) as an integral equation in ρ. We then observe that if g is smooth, then the solution ρ is representable by a series of functions of the forms (183), (184), (185), (186).

5.1 The Singularities in the Solution of Equation (182)

In this section we observe that for certain functions ρ, the function g is representable by convergent Taylor series on the intervals $[-1,0]$ and $[0,1]$.

The Even Case

Suppose that $\rho \in L^2([-1,1])$ is an even function, and suppose that $g \in L^2([-1,1])$ is defined by (182). By Theorem 3.10, g is also even and

$$g(s) = -\pi\rho(s) - \int_0^1 \frac{t\sin(\pi\alpha)}{s^2+t^2-2st\cos(\pi\alpha)}\rho(t)\,dt, \tag{187}$$

for all $0 < s \le 1$.

Suppose that $\rho(t) = t^{\mu-1}$ for all $0 \le t \le 1$. The following theorem shows that for certain values of μ, the function g in (187) is representable by a convergent Taylor series on the interval $[0,1]$. This theorem is proved in [12]; a proof is provided here for completeness.

Theorem 5.1. *Suppose that $0 < \alpha < 2$ is a real number and n is a positive integer. Then*

$$\pi s^{\frac{2n-1}{\alpha}-1} + \int_0^1 \frac{t\sin(\pi\alpha)}{s^2+t^2-2st\cos(\pi\alpha)} t^{\frac{2n-1}{\alpha}-1}\,dt = -\sum_{m=1}^{\infty} \Big(\lim_{\nu\to\alpha} \frac{\sin(m\pi\nu)}{m-\frac{2n-1}{\nu}}\Big) s^{m-1}, \tag{188}$$

$$\pi s^{\frac{2n}{2-\alpha}-1} + \int_0^1 \frac{t\sin(\pi\alpha)}{s^2+t^2-2st\cos(\pi\alpha)} t^{\frac{2n}{2-\alpha}-1}\,dt = -\sum_{m=1}^{\infty} \Big(\lim_{\nu\to\alpha} \frac{\sin(m\pi\nu)}{m-\frac{2n}{2-\nu}}\Big) s^{m-1}, \tag{189}$$

for all $0 < s \le 1$.

Proof. Substituting

$$\mu = \frac{2n-1}{\alpha} \tag{190}$$

into identity (149) and applying Theorem 3.14, we have

$$\pi s^{\frac{2n-1}{\alpha}-1} + \int_0^1 \frac{t\sin(\pi\alpha)}{s^2+t^2-2st\cos(\pi\alpha)} t^{\frac{2n-1}{\alpha}-1}\,dt = -\sum_{m=1}^{\infty} \frac{\sin(m\pi\alpha)}{m-\frac{2n-1}{\alpha}} s^{m-1}, \tag{191}$$

for all $0 < \alpha < 2$ such that

$$\frac{2n-1}{\alpha} \neq 1, 2, 3, \dots . \tag{192}$$

Viewing both sides of (191) as functions of α, we observe that Theorem 3.21 extends identity (191) to all $0 < \alpha < 2$. The proof of identity (189) is essentially identical.

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Suppose now that $\rho(t) = t^{\mu-1}\log(t)$ for all $0 \le t \le 1$. The following theorem shows that, for certain values of μ, the function g in (187) is representable by a convergent Taylor series on the interval $[0, 1]$.

Theorem 5.2. *Suppose that $0 < \alpha < 2$ is a real number and n is a positive integer. Suppose further that*

$$\frac{2n}{2-\alpha} = \frac{2k-1}{\alpha}, \tag{193}$$

for some positive integer k. Then

$$\pi s^{\frac{2n}{2-\alpha}-1}\log(s) + \int_0^1 \frac{t\sin(\pi\alpha)}{s^2+t^2-2st\cos(\pi\alpha)} t^{\frac{2n}{2-\alpha}-1}\log(t)\,dt = -\sum_{m=1}^{\infty} \frac{\sin(m\pi\alpha)}{(m-\frac{2n}{2-\alpha})^2} s^{m-1}, \tag{194}$$

for all $0 < s \le 1$.

Proof. Identity (194) follows from combining

$$\mu = \frac{2n}{2-\alpha} \tag{195}$$

and (193) with (150) and theorems 3.14 and 3.19. By Theorem 3.17 and (193),

$$\frac{2n}{2-\alpha} = n + k - \frac{1}{2}, \tag{196}$$

so the right hand side of (194) is well-defined.

■

The Odd Case

Suppose that $\rho \in L^2([-1,1])$ is an odd function, and suppose that $g \in L^2([-1,1])$ is defined by (182). By Theorem 3.10, g is also odd and

$$g(s) = -\pi\rho(s) + \int_0^1 \frac{t\sin(\pi\alpha)}{s^2+t^2-2st\cos(\pi\alpha)}\rho(t)\,dt, \tag{197}$$

for all $0 < s \le 1$.

Suppose that $\rho(t) = t^{\mu-1}$ for all $0 \le t \le 1$. The following theorem shows that for certain values of μ, the function g in (197) is representable by a convergent Taylor series on the interval $[0,1]$.

Theorem 5.3. *Suppose that $0 < \alpha < 2$ is a real number and n is a positive integer. Then*

$$-\pi s^{\frac{2n-1}{2-\alpha}-1} + \int_0^1 \frac{t\sin(\pi\alpha)}{s^2+t^2-2st\cos(\pi\alpha)} t^{\frac{2n-1}{2-\alpha}-1}\,dt = -\sum_{m=1}^{\infty}\Big(\lim_{\nu\to\alpha}\frac{\sin(m\pi\nu)}{m-\frac{2n-1}{2-\nu}}\Big)s^{m-1}, \tag{198}$$

$$-\pi s^{\frac{2n}{\alpha}-1} + \int_0^1 \frac{t\sin(\pi\alpha)}{s^2+t^2-2st\cos(\pi\alpha)} t^{\frac{2n}{\alpha}-1}\,dt = -\sum_{m=1}^{\infty}\Big(\lim_{\nu\to\alpha}\frac{\sin(m\pi\nu)}{m-\frac{2n}{\nu}}\Big)s^{m-1}, \tag{199}$$

for all $0 < s \le 1$.

Suppose now that $\rho(t) = t^{\mu-1}\log(t)$ for all $0 \le t \le 1$. The following theorem shows that, for certain values of μ, the function g in (197) is representable by a convergent Taylor series on the interval $[0,1]$.

Theorem 5.4. *Suppose that $0 < \alpha < 2$ is a real number and n is a positive integer. Suppose further that*

$$\frac{2n}{\alpha} = \frac{2k-1}{2-\alpha}, \tag{200}$$

for some positive integer k. Then

$$-\pi s^{\frac{2n}{\alpha}-1}\log(s) + \int_0^1 \frac{t\sin(\pi\alpha)}{s^2+t^2-2st\cos(\pi\alpha)} t^{\frac{2n}{\alpha}-1}\log(t)\,dt = -\sum_{m=1}^{\infty}\frac{\sin(m\pi\alpha)}{(m-\frac{2n}{\alpha})^2}s^{m-1}, \tag{201}$$

for all $0 < s \le 1$.

5.2 Series Representation of the Solution of Equation (182)

Suppose that g is representable by convergent Taylor series on the intervals $[-1,0]$ and $[0,1]$. Suppose further that ρ solves equation (182). In this section we observe that ρ is representable by a linear combination of certain non-integer powers and non-integer powers multiplied by logarithms, on the intervals $[-1,0]$ and $[0,1]$.

The Even Case

Suppose that $g \in L^2([-1,1])$ is an even function, and suppose that $\rho \in L^2([-1,1])$ satisfies equation (182). By Theorem 3.10, ρ is also even and

$$-\pi\rho(s) - \int_0^1 \frac{t\sin(\pi\alpha)}{s^2+t^2-2st\cos(\pi\alpha)}\rho(t)\,dt = g(s), \tag{202}$$

for all $0 < s \leq 1$, where $0 < \alpha < 2$.

Suppose that N is a positive integer. Letting $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ denote the floor and ceiling functions respectively, suppose that

$$\overline{L} = \left\lceil \frac{\alpha N}{2} \right\rceil, \tag{203}$$

$$\tag{204}$$

and

$$\underline{M} = \left\lfloor \frac{(2-\alpha)N}{2} \right\rfloor, \tag{205}$$

$$\tag{206}$$

and observe that $\overline{L} + \underline{\mathrm{M}} = N$. The following theorem shows that if g is representable by a convergent Taylor series on $[0,1]$, then there exist real numbers numbers $b_1, b_2, \ldots, b_N$ such that the function

$$\rho(t) = \sum_{i=1}^{\overline{L}} b_i t^{\frac{2i-1}{\alpha}-1} + \sum_{\substack{i=1\\ i\notin S}}^{\underline{M}} b_{\overline{L}+i} t^{\frac{2i}{2-\alpha}-1} + \sum_{\substack{i=1\\ i\in S}}^{\underline{M}} b_{\overline{L}+i} t^{\frac{2i}{2-\alpha}-1}\log(t), \tag{207}$$

where $0 < t \leq 1$ and

$$S = \Big\{ i \in \mathbb{Z}, 1 \leq i \leq \underline{M} : \frac{2i}{2-\alpha} = \frac{2j-1}{\alpha} \quad \text{for some integer } 1 \leq j \leq \overline{L} \Big\}, \tag{208}$$

solves equation (202) to within an error $O(t^N)$.

Theorem 5.5. *Suppose that $0 < \alpha < 2$ and that N is a positive integer. Letting $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ denote the floor and ceiling functions respectively, suppose that*

$$\overline{L} = \left\lceil \frac{\alpha N}{2} \right\rceil, \tag{209}$$

and

$$\underline{M} = \left\lfloor \frac{(2-\alpha)N}{2} \right\rfloor, \tag{210}$$

and observe that $\overline{L} + \underline{M} = N$. Suppose that $g\colon [0,1] \to \mathbb{R}$ is defined by the formula

$$g(t) = \sum_{i=0}^{\infty} c_{i+1} t^i, \tag{211}$$

for all $0 \le t \le 1$*, where* $c_1, c_2, \dots$ *are real numbers. Then,* for any $0 < \alpha < 2$*, there exist unique real numbers* $b_1, b_2, \dots, b_N$ *such that if*

$$\rho(t) = \sum_{i=1}^{\overline{L}} b_i t^{\frac{2i-1}{\alpha}-1} + \sum_{\substack{i=1 \\ i \notin S}}^{\underline{M}} b_{\overline{L}+i} t^{\frac{2i}{2-\alpha}-1} + \sum_{\substack{i=1 \\ i \in S}}^{\underline{M}} b_{\overline{L}+i} t^{\frac{2i}{2-\alpha}-1} \log(t), \tag{212}$$

where $0 < t \le 1$ *and*

$$S = \Big\{ i \in \mathbb{Z}, 1 \le i \le \underline{M} : \frac{2i}{2-\alpha} = \frac{2j-1}{\alpha} \quad \text{for some integer } 1 \le j \le \overline{L} \Big\}, \tag{213}$$

then

$$-\pi\rho(s) - \int_0^1 \frac{t \sin(\pi\alpha)}{s^2 + t^2 - 2st\cos(\pi\alpha)} \rho(t)\, dt = g(s) + \sum_{i=N}^{\infty} d_i s^i, \tag{214}$$

for all $0 < s \le 1$*, for some real numbers* $d_N, d_{N+1}, \dots$ *.*

Proof. By Theorem 4.4, the $N \times N$ matrix A, defined by (176) and (177), is invertible for any $0 < \alpha < 2$. Therefore, there exist unique real numbers $b_1, b_2, \dots, b_N$ such that

$$\sum_{j=1}^{n} A_{i,j} b_j = c_i, \tag{215}$$

for each $i = 1, 2, \dots, N$. Suppose that $\rho\colon [0,1] \to \mathbb{R}$ is defined by (212). By Theorems 5.1 and 5.2,

$$\begin{aligned}
&-\pi\rho(s) - \int_0^1 \frac{t \sin(\pi\alpha)}{s^2 + t^2 - 2st\cos(\pi\alpha)} \rho(t)\, dt \\
&= \sum_{j=1}^{\overline{L}} b_j \sum_{i=1}^{\infty} \Big(\lim_{\nu \to \alpha} \frac{\sin(i\pi\nu)}{i - \frac{2j-1}{\nu}} \Big) s^{i-1} + \sum_{\substack{j=1 \\ j \notin S}}^{\underline{M}} b_{\overline{L}+j} \sum_{i=1}^{\infty} \Big(\lim_{\nu \to \alpha} \frac{\sin(i\pi\nu)}{i - \frac{2j}{2-\nu}} \Big) s^{i-1} \\
&\quad + \sum_{\substack{j=1 \\ j \in S}}^{\underline{M}} b_{\overline{L}+j} \sum_{i=1}^{\infty} \frac{\sin(i\pi\alpha)}{(i - \frac{2j}{2-\alpha})^2} s^{i-1} \\
&= \sum_{j=1}^{\overline{L}} b_j \sum_{i=1}^{N} \Big(\lim_{\nu \to \alpha} \frac{\sin(i\pi\nu)}{i - \frac{2j-1}{\nu}} \Big) s^{i-1} + \sum_{\substack{j=1 \\ j \notin S}}^{\underline{M}} b_{\overline{L}+j} \sum_{i=1}^{N} \Big(\lim_{\nu \to \alpha} \frac{\sin(i\pi\nu)}{i - \frac{2j}{2-\nu}} \Big) s^{i-1} \\
&\quad + \sum_{\substack{j=1 \\ j \in S}}^{\underline{M}} b_{\overline{L}+j} \sum_{i=1}^{N} \frac{\sin(i\pi\alpha)}{(i - \frac{2j}{2-\alpha})^2} s^{i-1} + \sum_{i=N}^{\infty} d_i s^i,
\end{aligned} \tag{216}$$

where $0 < s \le 1$ and S is defined by (213), for some real numbers $d_N, d_{N+1}, \dots$.

Therefore,

$$
\begin{aligned}
&-\pi\rho(s) - \int_0^1 \frac{t\sin(\pi\alpha)}{s^2+t^2-2st\cos(\pi\alpha)}\rho(t)\,dt \\
&= \sum_{i=1}^{N}\sum_{j=1}^{N} A_{i,j} b_j s^{i-1} + \sum_{i=N}^{\infty} d_i s^i \\
&= \sum_{i=1}^{N} c_i s^{i-1} + \sum_{i=N}^{\infty} d_i s^i \\
&= g(s) + \sum_{i=N}^{\infty} (d_i - c_{i+1}) s^i, \qquad (217)
\end{aligned}
$$

for all $0 < s \leq 1$.

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The Odd Case

Suppose that $g \in L^2([-1,1])$ is an odd function, and suppose that $\rho \in L^2([-1,1])$ satisfies equation (182). By Theorem 3.10, ρ is also odd and

$$
-\pi\rho(s) + \int_0^1 \frac{t\sin(\pi\alpha)}{s^2+t^2-2st\cos(\pi\alpha)}\rho(t)\,dt = g(s), \qquad (218)
$$

for all $0 < s \leq 1$, where $0 < \alpha < 2$.

Suppose that N is a positive integer. Letting $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ denote the floor and ceiling functions respectively, suppose that

$$
\overline{M} = \left\lceil \frac{(2-\alpha)N}{2} \right\rceil, \qquad (219)
$$

$$
\qquad (220)
$$

and

$$
\underline{L} = \left\lfloor \frac{\alpha N}{2} \right\rfloor, \qquad (221)
$$

$$
\qquad (222)
$$

and observe that $\overline{M} + \underline{L} = N$. The following theorem shows that if g is representable by a convergent Taylor series on $[0,1]$, then there exist real numbers numbers $b_1, b_2, \ldots, b_N$ such that the function

$$
\rho(t) = \sum_{i=1}^{\overline{M}} b_i t^{\frac{2i-1}{2-\alpha}-1} + \sum_{\substack{i=1 \\ i\notin T}}^{\underline{L}} b_{\overline{M}+i} t^{\frac{2i}{\alpha}-1} + \sum_{\substack{i=1 \\ i\in T}}^{\underline{L}} b_{\overline{M}+i} t^{\frac{2i}{\alpha}-1}\log(t), \qquad (223)
$$

where $0 < t \leq 1$ and

$$
T = \Big\{ i \in \mathbb{Z}, 1 \leq i \leq \underline{L} : \frac{2i}{\alpha} = \frac{2j-1}{2-\alpha} \quad \text{for some integer } 1 \leq j \leq \overline{M} \Big\}, \qquad (224)
$$

solves equation (218) to within an error $O(t^N)$.

Theorem 5.6. *Suppose that $0 < \alpha < 2$ and that N is a positive integer. Letting $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ denote the floor and ceiling functions respectively, suppose that*

$$\overline{M} = \left\lceil \frac{(2-\alpha)N}{2} \right\rceil, \tag{225}$$

and

$$\underline{L} = \left\lfloor \frac{\alpha N}{2} \right\rfloor, \tag{226}$$

and observe that $\overline{M} + \underline{L} = N$. Suppose that $g\colon [0,1] \to \mathbb{R}$ is defined by the formula

$$g(t) = \sum_{i=0}^{\infty} c_{i+1} t^i, \tag{227}$$

for all $0 \le t \le 1$, where $c_1, c_2, \dots$ are real numbers. Then, for any $0 < \alpha < 2$, *there exist unique real numbers $b_1, b_2, \dots, b_N$ such that if*

$$\rho(t) = \sum_{i=1}^{\overline{M}} b_i t^{\frac{2i-1}{2-\alpha}-1} + \sum_{\substack{i=1\\ i\notin T}}^{\underline{L}} b_{\overline{M}+i} t^{\frac{2i}{\alpha}-1} + \sum_{\substack{i=1\\ i\in T}}^{\underline{L}} b_{\overline{M}+i} t^{\frac{2i}{\alpha}-1}\log(t), \tag{228}$$

where $0 < t \le 1$ and

$$T = \Big\{ i \in \mathbb{Z}, 1 \le i \le \underline{L} : \frac{2i}{\alpha} = \frac{2j-1}{2-\alpha} \quad \text{for some integer } 1 \le j \le \overline{M} \Big\}, \tag{229}$$

then

$$-\pi\rho(s) + \int_0^1 \frac{t\sin(\pi\alpha)}{s^2+t^2-2st\cos(\pi\alpha)}\rho(t)\,dt = g(s) + \sum_{i=N}^{\infty} d_i s^i, \tag{230}$$

for all $0 < s \le 1$, for some real numbers $d_N, d_{N+1}, \dots$.

6 Analysis of the Integral Equation: the Dirichlet Case

Suppose that the curve $\gamma\colon [-1,1] \to \mathbb{R}^2$ is a wedge defined by (50) with interior angle $\pi\alpha$, where $0 < \alpha < 2$ (see Figure 3), and let $\nu(t)$ denote the inward-facing unit normal to the curve γ at the point $\gamma(t)$. Let g be a function in $L^2([-1,1])$, and suppose that $\rho \in L^2([-1,1])$ solves the equation

$$-\pi\rho(s) + \int_{-1}^{1} \psi^1_{\gamma(t),\nu(t)}(\gamma(s))\rho(t)\,dt = g(s), \tag{231}$$

for all $s \in [-1,1]$, where $\psi^1_{\gamma(t),\nu(t)}$ is defined by (35).

In this section, we analyze this boundary integral equation, which is well-posed even though the curve γ is open (see Observation 3.2). We investigate the behavior of (231) for functions $\rho \in L^2([-1,1])$ of the forms

$$\rho(t) = |t|^\mu, \tag{232}$$

$$\rho(t) = |t|^\mu \log(|t|), \tag{233}$$

$$\rho(t) = \operatorname{sgn}(t)|t|^\mu, \tag{234}$$

$$\rho(t) = \operatorname{sgn}(t)|t|^\mu \log(|t|), \tag{235}$$

where $\mu > \frac{1}{2}$ is a real number. If ρ has the forms (232), (233), (234), (235), then for most values of μ the resulting g is singular. In Section 6.1, we observe that for certain μ, the function g is smooth. In Section 6.2, we fix g and view (231) as an integral equation in ρ. We then observe that if g is smooth, then the solution ρ is representable by a series of functions of the forms (232), (233), (234), (235).

6.1 The Singularities in the Solution of Equation (231)

In this section we observe that for certain functions ρ, the function g is representable by convergent Taylor series on the intervals $[-1, 0]$ and $[0, 1]$.

The Even Case

Suppose that $\rho \in L^2([-1, 1])$ is an even function, and suppose that $g \in L^2([-1, 1])$ is defined by (231). By Theorem 3.12, g is also even and

$$g(s) = -\pi\rho(s) - \int_0^1 \frac{s\sin(\pi\alpha)}{s^2 + t^2 - 2st\cos(\pi\alpha)}\rho(t)\,dt, \tag{236}$$

for all $0 < s \le 1$.

Suppose that $\rho(t) = t^\mu$ for all $0 \le t \le 1$. The following theorem shows that for certain values of μ, the function g in (236) is representable by a convergent Taylor series on the interval $[0, 1]$.

Theorem 6.1. *Suppose that $0 < \alpha < 2$ is a real number and n is a positive integer. Then*

$$\pi + \int_0^1 \frac{s\sin(\pi\alpha)}{s^2 + t^2 - 2st\cos(\pi\alpha)}\,dt = (2-\alpha)\pi - \sum_{m=1}^{\infty} \frac{\sin(m\pi\alpha)}{m} s^m, \tag{237}$$

$$\pi s^{\frac{2n-1}{\alpha}} + \int_0^1 \frac{s\sin(\pi\alpha)}{s^2 + t^2 - 2st\cos(\pi\alpha)} t^{\frac{2n-1}{\alpha}}\,dt = -\sum_{m=1}^{\infty} \Big(\lim_{\nu\to\alpha} \frac{\sin(m\pi\nu)}{m - \frac{2n-1}{\nu}}\Big) s^m, \tag{238}$$

$$\pi s^{\frac{2n}{2-\alpha}} + \int_0^1 \frac{s\sin(\pi\alpha)}{s^2 + t^2 - 2st\cos(\pi\alpha)} t^{\frac{2n}{2-\alpha}}\,dt = -\sum_{m=1}^{\infty} \Big(\lim_{\nu\to\alpha} \frac{\sin(m\pi\nu)}{m - \frac{2n}{2-\nu}}\Big) s^m, \tag{239}$$

for all $0 < s \le 1$.

Suppose now that $\rho(t) = t^\mu \log(t)$ for all $0 \le t \le 1$. The following theorem shows that, for certain values of μ, the function g in (236) is representable by a convergent Taylor series on the interval $[0, 1]$.

Theorem 6.2. *Suppose that $0 < \alpha < 2$ is a real number and n is a positive integer. Suppose further that*

$$\frac{2n}{2-\alpha} = \frac{2k-1}{\alpha}, \tag{240}$$

for some positive integer k. Then

$$\pi s^{\frac{2n}{2-\alpha}} \log(s) + \int_0^1 \frac{s\sin(\pi\alpha)}{s^2 + t^2 - 2st\cos(\pi\alpha)} t^{\frac{2n}{2-\alpha}} \log(t)\,dt = -\sum_{m=1}^{\infty} \frac{\sin(m\pi\alpha)}{(m - \frac{2n}{2-\alpha})^2} s^m, \tag{241}$$

for all $0 < s \le 1$.

The Odd Case

Suppose that $\rho \in L^2([-1,1])$ is an odd function, and suppose that $g \in L^2([-1,1])$ is defined by (231). By Theorem 3.12, g is also odd and

$$g(s) = -\pi\rho(s) + \int_0^1 \frac{s\sin(\pi\alpha)}{s^2+t^2-2st\cos(\pi\alpha)}\rho(t)\,dt, \tag{242}$$

for all $0 < s \le 1$.

Suppose that $\rho(t) = t^\mu$ for all $0 \le t \le 1$. The following theorem shows that for certain values of μ, the function g in (242) is representable by a convergent Taylor series on the interval $[0,1]$.

Theorem 6.3. *Suppose that $0 < \alpha < 2$ is a real number and n is a positive integer. Then*

$$-\pi + \int_0^1 \frac{s\sin(\pi\alpha)}{s^2+t^2-2st\cos(\pi\alpha)}\,dt = -\alpha\pi - \sum_{m=1}^{\infty}\frac{\sin(m\pi\alpha)}{m}s^m, \tag{243}$$

$$-\pi s^{\frac{2n-1}{2-\alpha}} + \int_0^1 \frac{s\sin(\pi\alpha)}{s^2+t^2-2st\cos(\pi\alpha)}t^{\frac{2n-1}{2-\alpha}}\,dt = -\sum_{m=1}^{\infty}\Big(\lim_{\nu\to\alpha}\frac{\sin(m\pi\nu)}{m-\frac{2n-1}{2-\nu}}\Big)s^m, \tag{244}$$

$$-\pi s^{\frac{2n}{\alpha}} + \int_0^1 \frac{s\sin(\pi\alpha)}{s^2+t^2-2st\cos(\pi\alpha)}t^{\frac{2n}{\alpha}}\,dt = -\sum_{m=1}^{\infty}\Big(\lim_{\nu\to\alpha}\frac{\sin(m\pi\nu)}{m-\frac{2n}{\nu}}\Big)s^m, \tag{245}$$

for all $0 < s \le 1$.

Suppose now that $\rho(t) = t^\mu\log(t)$ for all $0 \le t \le 1$. The following theorem shows that, for certain values of μ, the function g in (242) is representable by a convergent Taylor series on the interval $[0,1]$.

Theorem 6.4. *Suppose that $0 < \alpha < 2$ is a real number and n is a positive integer. Suppose further that*

$$\frac{2n}{\alpha} = \frac{2k-1}{2-\alpha}, \tag{246}$$

for some positive integer k. Then

$$-\pi s^{\frac{2n}{\alpha}}\log(s) + \int_0^1 \frac{s\sin(\pi\alpha)}{s^2+t^2-2st\cos(\pi\alpha)}t^{\frac{2n}{\alpha}}\log(t)\,dt = -\sum_{m=1}^{\infty}\frac{\sin(m\pi\alpha)}{(m-\frac{2n}{\alpha})^2}s^m, \tag{247}$$

for all $0 < s \le 1$.

6.2 Series Representation of the Solution of Equation (231)

Suppose that g is representable by convergent Taylor series on the intervals $[-1,0]$ and $[0,1]$. Suppose further that ρ solves equation (231). In this section we observe that ρ is representable by a linear combination of certain non-integer powers and non-integer powers multiplied by logarithms, on the intervals $[-1,0]$ and $[0,1]$.

The Even Case

Suppose that $g \in L^2([-1,1])$ is an even function, and suppose that $\rho \in L^2([-1,1])$ satisfies equation (231). By Theorem 3.12, ρ is also even and

$$-\pi\rho(s) - \int_0^1 \frac{s\sin(\pi\alpha)}{s^2+t^2-2st\cos(\pi\alpha)}\rho(t)\,dt = g(s), \tag{248}$$

for all $0 < s \leq 1$, where $0 < \alpha < 2$.

Suppose that N is a positive integer. Letting $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ denote the floor and ceiling functions respectively, suppose that

$$\overline{L} = \left\lceil \frac{\alpha N}{2} \right\rceil, \tag{249}$$

$$\tag{250}$$

and

$$\underline{M} = \left\lfloor \frac{(2-\alpha)N}{2} \right\rfloor, \tag{251}$$

$$\tag{252}$$

and observe that $\overline{L} + \underline{\mathrm{M}} = N$. The following theorem shows that if g is representable by a convergent Taylor series on $[0,1]$, then there exist real numbers numbers $b_0, b_1, \ldots, b_N$ such that the function

$$\rho(t) = b_0 + \sum_{i=1}^{\overline{L}} b_i t^{\frac{2i-1}{\alpha}} + \sum_{\substack{i=1 \\ i \notin S}}^{\underline{M}} b_{\overline{L}+i} t^{\frac{2i}{2-\alpha}} + \sum_{\substack{i=1 \\ i \in S}}^{\underline{M}} b_{\overline{L}+i} t^{\frac{2i}{2-\alpha}} \log(t), \tag{253}$$

where $0 < t \leq 1$ and

$$S = \Big\{ i \in \mathbb{Z}, 1 \leq i \leq \underline{M} : \frac{2i}{2-\alpha} = \frac{2j-1}{\alpha} \quad \text{for some integer } 1 \leq j \leq \overline{L} \Big\}, \tag{254}$$

solves equation (248) to within an error $O(t^{N+1})$.

Theorem 6.5. *Suppose that $0 < \alpha < 2$ and that N is a positive integer. Letting $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ denote the floor and ceiling functions respectively, suppose that*

$$\overline{L} = \left\lceil \frac{\alpha N}{2} \right\rceil, \tag{255}$$

and

$$\underline{M} = \left\lfloor \frac{(2-\alpha)N}{2} \right\rfloor, \tag{256}$$

and observe that $\overline{L} + \underline{M} = N$. Suppose that $g \colon [0,1] \to \mathbb{R}$ is defined by the formula

$$g(t) = \sum_{i=0}^{\infty} c_i\, t^i, \tag{257}$$

for all $0 \le t \le 1$*, where* $c_0, c_1, \ldots$ *are real numbers. Then,* for any $0 < \alpha < 2$*, there exist unique real numbers* $b_0, b_1, \ldots, b_N$ *such that if*

$$\rho(t) = b_0 + \sum_{i=1}^{\overline{L}} b_i t^{\frac{2i-1}{\alpha}} + \sum_{\substack{i=1 \\ i \notin S}}^{\underline{M}} b_{\overline{L}+i} t^{\frac{2i}{2-\alpha}} + \sum_{\substack{i=1 \\ i \in S}}^{\underline{M}} b_{\overline{L}+i} t^{\frac{2i}{2-\alpha}} \log(t), \tag{258}$$

where $0 < t \le 1$ *and*

$$S = \Big\{ i \in \mathbb{Z}, 1 \le i \le \underline{M} : \frac{2i}{2-\alpha} = \frac{2j-1}{\alpha} \quad \text{for some integer } 1 \le j \le \overline{L} \Big\}, \tag{259}$$

then

$$-\pi\rho(s) - \int_0^1 \frac{s\sin(\pi\alpha)}{s^2+t^2-2st\cos(\pi\alpha)} \rho(t)\,dt = g(s) + \sum_{i=N+1}^{\infty} d_i s^i, \tag{260}$$

for all $0 < s \le 1$*, for some real numbers* $d_{N+1}, d_{N+2}, \ldots$ *.*

The Odd Case

Suppose that $g \in L^2([-1,1])$ is an odd function, and suppose that $\rho \in L^2([-1,1])$ satisfies equation (231). By Theorem 3.12, ρ is also odd and

$$-\pi\rho(s) + \int_0^1 \frac{s\sin(\pi\alpha)}{s^2+t^2-2st\cos(\pi\alpha)} \rho(t)\,dt = g(s), \tag{261}$$

for all $0 < s \le 1$, where $0 < \alpha < 2$.

Suppose that N is a positive integer. Letting $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ denote the floor and ceiling functions respectively, suppose that

$$\overline{M} = \Big\lceil \frac{(2-\alpha)N}{2} \Big\rceil, \tag{262}$$

$$\tag{263}$$

and

$$\underline{L} = \Big\lfloor \frac{\alpha N}{2} \Big\rfloor, \tag{264}$$

$$\tag{265}$$

and observe that $\overline{M} + \underline{L} = N$. The following theorem shows that if g is representable by a convergent Taylor series on $[0,1]$, then there exist real numbers numbers $b_0, b_1, \ldots, b_N$ such that the function

$$\rho(t) = b_0 + \sum_{i=1}^{\overline{M}} b_i t^{\frac{2i-1}{2-\alpha}} + \sum_{\substack{i=1 \\ i \notin T}}^{\underline{L}} b_{\overline{M}+i} t^{\frac{2i}{\alpha}} + \sum_{\substack{i=1 \\ i \in T}}^{\underline{L}} b_{\overline{M}+i} t^{\frac{2i}{\alpha}} \log(t), \tag{266}$$

where $0 < t \le 1$ and

$$T = \Big\{ i \in \mathbb{Z}, 1 \le i \le \underline{L} : \frac{2i}{\alpha} = \frac{2j-1}{2-\alpha} \quad \text{for some integer } 1 \le j \le \overline{M} \Big\}, \tag{267}$$

solves equation (261) to within an error $O(t^{N+1})$.

Theorem 6.6. *Suppose that $0 < \alpha < 2$ and that N is a positive integer. Letting $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ denote the floor and ceiling functions respectively, suppose that*

$$\overline{M} = \left\lceil \frac{(2-\alpha)N}{2} \right\rceil, \tag{268}$$

and

$$\underline{L} = \left\lfloor \frac{\alpha N}{2} \right\rfloor, \tag{269}$$

and observe that $\overline{M} + \underline{L} = N$. Suppose that $g\colon [0,1] \to \mathbb{R}$ is defined by the formula

$$g(t) = \sum_{i=0}^{\infty} c_i t^i, \tag{270}$$

for all $0 \le t \le 1$, where $c_0, c_1, \dots$ are real numbers. Then, for any *$0 < \alpha < 2$, there exist unique real numbers $b_0, b_1, \dots, b_N$ such that if*

$$\rho(t) = b_0 + \sum_{i=1}^{\overline{M}} b_i t^{\frac{2i-1}{2-\alpha}} + \sum_{\substack{i=1 \\ i \notin T}}^{\underline{L}} b_{\overline{M}+i} t^{\frac{2i}{\alpha}} + \sum_{\substack{i=1 \\ i \in T}}^{\underline{L}} b_{\overline{M}+i} t^{\frac{2i}{\alpha}} \log(t), \tag{271}$$

where $0 < t \le 1$ and

$$T = \left\{ i \in \mathbb{Z}, 1 \le i \le \underline{L} : \frac{2i}{\alpha} = \frac{2j-1}{2-\alpha} \quad \text{for some integer } 1 \le j \le \overline{M} \right\}, \tag{272}$$

then

$$-\pi\rho(s) + \int_0^1 \frac{s\sin(\pi\alpha)}{s^2 + t^2 - 2st\cos(\pi\alpha)} \rho(t)\, dt = g(s) + \sum_{i=N+1}^{\infty} d_i s^i, \tag{273}$$

for all $0 < s \le 1$, for some real numbers $d_{N+1}, d_{N+2}, \dots$.

7 Appendix A

In this section we provide a proof of Theorem 3.13, which is restated here as Theorem 7.1.

The following theorem states that the Lagrange interpolating polynomials form a Chebyshev system (see Definition 3.1).

Theorem 7.1. *Suppose that K and L are nonnegative integers such that $K \le L$. Suppose further that $\mu_1, \mu_2, \dots, \mu_L$ are distinct real numbers. Suppose finally that $p\colon \mathbb{R} \to \mathbb{R}$ is defined by*

$$p(x) = \prod_{\ell=1}^{L} (x - \mu_\ell) \prod_{k=1}^{K} (x - \mu_k), \tag{274}$$

for all $x \in \mathbb{R}$. Then the set of $L + K$ functions

$$\left\{ \frac{p(x)}{x - \mu_\ell} \right\}_{\ell=1}^{L} \cup \left\{ \frac{p(x)}{(x - \mu_k)^2} \right\}_{k=1}^{K}, \tag{275}$$

where $x \in \mathbb{R}$, is a Chebyshev system.

Proof. Let $\varphi_1, \varphi_2, \ldots, \varphi_L \colon \mathbb{R} \to \mathbb{R}$ be defined by

$$\varphi_i(x) = \begin{cases} \dfrac{p(x)}{(x-\mu_i)^2} & \text{if } 1 \le i \le K, \\ \dfrac{p(x)}{x-\mu_i} & \text{if } K+1 \le i \le L, \end{cases} \tag{276}$$

where $x \in \mathbb{R}$ and p is defined by (274), for all integers $1 \le i \le L$. Furthermore, let $\psi_1, \psi_2, \ldots, \psi_K \colon \mathbb{R} \to \mathbb{R}$ be defined by

$$\psi_i(x) = \frac{p(x)}{x-\mu_i}, \tag{277}$$

where $x \in \mathbb{R}$ and p is defined by (274), for all integers $1 \le i \le K$. Suppose that

$$C_i = \prod_{\substack{\ell=1 \\ \ell \ne i}}^{L} (\mu_i - \mu_\ell) \prod_{\substack{k=1 \\ k \ne i}}^{K} (\mu_i - \mu_k), \tag{278}$$

for all integers $1 \le i \le L$. Then it is straightforward to show that

$$\frac{\varphi_i(\mu_j)}{C_i} = \begin{cases} 1 & j = i, \\ 0 & j \ne i, \end{cases} \tag{279}$$

for all integers $1 \le i, j \le L$, and

$$\psi_i(\mu_j) = 0, \tag{280}$$

for all integers $1 \le i \le K$, $1 \le j \le L$, and

$$\frac{\psi_i'(\mu_j)}{C_i} = \begin{cases} 1 & j = i, \\ 0 & j \ne i, \end{cases} \tag{281}$$

for all integers $1 \le i, j \le K$.

Suppose that $f \colon \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \sum_{i=1}^{L} b_i \frac{\varphi_i(x)}{C_i} + \sum_{i=1}^{K} c_i \frac{\psi_i(x)}{C_i}, \tag{282}$$

where $x \in \mathbb{R}$ and C_i is defined by (278), and where $b_1, b_2, \ldots, b_L$ and $c_1, c_2, \ldots, c_K$ are arbitrary real numbers. Suppose further that there exists a set of $L+K$ distinct points $y_1, y_2, \ldots, y_{L+K}$ such that

$$f(y_i) = 0, \tag{283}$$

for each integer $1 \le i \le L+K$. Since f is a polynomial of order at most $L+K-1$ with $L+K$ distinct roots, it must be uniformly zero. We will show that this implies $b_i = 0$ for all $1 \le i \le L$ and $c_i = 0$ for all $1 \le i \le K$.

First, suppose that $b_i \ne 0$ for some integer $1 \le i \le L$. Then, by (282), (279), and (280),

$$f(\mu_i) = b_i, \tag{284}$$

so f is not uniformly zero.

Next, suppose that $c_i \neq 0$ for some integer $1 \leq i \leq K$. If $b_\ell \neq 0$ for some $1 \leq \ell \leq L$, then f is not uniformly zero, so suppose further that $b_\ell = 0$ for all $1 \leq \ell \leq L$. Combining (282) with (281), we have

$$f'(\mu_i) = c_i, \tag{285}$$

so f is not uniformly zero.

Therefore, the only vector $(b_1, b_2, \ldots, b_L, c_1, c_2, \ldots, c_K)^T$ for which

$$\begin{pmatrix} \varphi_1(y_1) & \ldots & \varphi_L(y_1) & \psi_1(y_1) & \ldots & \psi_K(y_1) \\ \varphi_1(y_2) & \ldots & \varphi_L(y_2) & \psi_1(y_2) & \ldots & \psi_K(y_2) \\ \vdots & & \vdots & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & & \vdots \\ \varphi_1(y_{L+K}) & \ldots & \varphi_L(y_{L+K}) & \psi_1(y_{L+K}) & \ldots & \psi_K(y_{L+K}) \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_L \\ c_1 \\ \vdots \\ c_K \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{pmatrix} \tag{286}$$

is true is the zero vector. Thus

$$\det \begin{pmatrix} \varphi_1(y_1) & \ldots & \varphi_L(y_1) & \psi_1(y_1) & \ldots & \psi_K(y_1) \\ \varphi_1(y_2) & \ldots & \varphi_L(y_2) & \psi_1(y_2) & \ldots & \psi_K(y_2) \\ \vdots & & \vdots & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & & \vdots \\ \varphi_1(y_{L+K}) & \ldots & \varphi_L(y_{L+K}) & \psi_1(y_{L+K}) & \ldots & \psi_K(y_{L+K}) \end{pmatrix} \neq 0, \tag{287}$$

for any set of $L + K$ distinct points $y_1, y_2, \ldots, y_{L+K}$.

■

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