Domain Decomposition Methods for Raviart-Thomas Vector Fields

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by

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Professor Olof B. Widlund

To my family

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Abstract

Raviart-Thomas finite elements are very useful for problems posed in $H(\text{div})$ since they are $H(\text{div})$-conforming. We introduce two domain decomposition methods for solving vector field problems posed in $H(\text{div})$ discretized by Raviart-Thomas finite elements.

A two-level overlapping Schwarz method is developed. The coarse part of the preconditioner is based on energy-minimizing extensions and the local parts consist of traditional solvers on overlapping subdomains. We prove that our method is scalable and that the condition number grows linearly with the logarithm of the number of degrees of freedom in the individual subdomains and linearly with the relative overlap between the overlapping subdomains. The condition number of the method is also independent of the values and jumps of the coefficients across the interface between subdomains. We provide numerical results to support our theory.

We also consider a balancing domain decomposition by constraints (BDDC) method. The BDDC preconditioner consists of a coarse part involving primal constraints across the interface between subdomains and local parts related to the Schur complements corresponding to the local subdomain problems. We provide bounds of the condition number of the preconditioned linear system and suggest that the condition number has a polylogarithmic bound in terms of the number of degrees of freedom in the individual subdomains from our numerical experiments for arbitrary jumps of the coefficients across the subdomain interfaces.

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Chapter 1

Introduction

1.1 An Overview

In order to obtain the approximate solution of a certain partial differential equation (PDE) numerically, we can use finite elements, finite differences, or other schemes. After discretization, we often face a large, ill-conditioned, linear system of algebraic equations. It is often hard to solve such a linear system by using traditional direct methods due to the limitation of computing resources. Even though we can apply iterative methods such as Krylov type methods, we will need many iterations because of the large condition number. To avoid this, we will introduce preconditioners which make the condition number and the iteration count of the preconditioned linear system much smaller than that of the original linear system.

The main role of domain decomposition methods is providing good preconditioners. They typically involve solving one global coarse problem and many small local subproblems. Both the coarse global problem and the local subproblems are small compared to the original problem so that each problem may be handled by exact solvers. We divide the original domain into often many subdomains, to obtain local subproblems related to each subdomain, and use a coarse grid for the coarse problem. We can also solve the local subproblems independently and domain decomposition algorithms can therefore be implemented effectively on parallel machines.

There are two major families of domain decomposition methods: overlapping Schwarz methods with overlapping subdomains and iterative substructuring methods with nonoverlapping subdomains. In overlapping Schwarz methods, we can begin with a set of nonoverlapping subdomains. We then enlarge each subdomain by adding layers of elements. The local subproblems are defined on the extended subdomains and the global problem is associated with the coarse meshes defined by the subdomains.

The original overlapping Schwarz algorithms were originally developed for scalar elliptic problems; see [61,67] and references therein. Later these methods have been widely extended to various problems.

In iterative substructuring methods, we first reduce the original linear system to a Schur complement system by implicitly eliminating the interior unknowns of each subdomains. We then consider appropriate preconditioners for the Schur complement system. Two main classes of iterative substructuring methods are the balancing Neumann Neumann (BNN) type and the finite element tearing and interconnecting (FETI) type algorithms; see [26–28, 37, 42].

There are many variants of the iterative substructuring methods. Among them, balancing domain decomposition by constraints (BDDC) and dual-primal finite element tearing and interconnecting (FETI-DP) are currently the most important.

In this dissertation, we will mainly focus on BDDC methods. The BDDC methods, introduced by Dohrmann in [18], are modified BNN methods with a global component obtained by using primal continuity constraints. For a pioneering analysis, see [43, 44].

In this dissertation, we will consider two-level overlapping Schwarz methods and BDDC methods for solving vector field problems. Overlapping Schwarz methods for vector field problems with constant coefficients were previously introduced in [34, 64]. Later nonoverlapping domain decomposition methods were considered in [71] and BNN, FETI, and FETI-DP methods were developed in [63,65,66]. Other methods, such as multigrid methods, have also been considered; see [3, 32, 70]. While many iterative substructuring methods have been studied for discontinuous coefficients cases, there has been little supporting theory for the overlapping Schwarz methods until recently. For the purpose of handling the discontinuity of the coefficients, we borrow the advanced coarse space techniques of [19, 20] developed for almost incompressible elasticity. We also consider the BDDC methods with various primal constraints.

1.2 Krylov Subspace Methods

The discretization of elliptic PDEs usually yields symmetric, sparse, and positive definite linear systems. As we noted earlier, direct methods often require too much work and a huge memory, if the number of degrees of freedom is very large, especially for 3D problems. Hence, we use iterative methods for solving such linear systems. Jacobi, Gauss-Seidel, and successive over-relaxation (SOR) methods were introduced early as iterative methods. Later, techniques based on projection processes were developed. Among them, Krylov subspace methods are the most important and popular currently. We will consider the Krylov methods in this section.

1.2.1 Overview of Krylov Subspace Methods

We consider the following linear system:

$$Au = f. \tag{1.1}$$

The Krylov methods are based on a projection into a lower-dimensional Krylov subspace which is given by

$$\mathcal{K}_m(A, r_0) = \text{span}\{r_0, Ar_0, A^2r_0, \cdots, A^{m-1}r_0\}, \tag{1.2}$$

where $r_0 = f - Au_0$ and u_0 is an initial guess. In other words, the approximate solution obtained from a Krylov subspace method has the following form:

$$x_m = x_0 + p_{m-1}(A)r_0, \tag{1.3}$$

where p_{m-1} is a polynomial of degree $m-1$. There are various types of Krylov subspace methods, e.g., conjugate gradient methods, Lanczos methods, and generalized minimal residuals. The methods chosen depends on the type of problems. We will mainly consider the preconditioned conjugate gradient methods for solving symmetric, positive definite linear systems in this dissertation.

1.2.2 The Conjugate Gradient Method

We consider the case where A is sparse, symmetric, and positive definite. The conjugate gradient algorithm for solving (1.1) is given in Figure 1.1.

Initialize: $r_0 := f - Au_0$, $p_0 := r_0$

Iterate $k = 0, 1, \cdots$ until convergence

$$\alpha_k := r_k^T r_k / p_k^T A p_k$$

$$u_{k+1} := u_k + \alpha_k p_k$$

$$r_{k+1} := r_k - \alpha_k A p_k$$

$$\beta_k := r_{k+1}^T r_{k+1} / r_k^T r_k$$

$$p_{k+1} := r_{k+1} + \beta_k p_k$$

Figure 1.1: Conjugate Gradient Algorithm

We define the A-norm and the error vector for the analysis of the convergence rate of the conjugate gradient algorithms. The A-norm is defined as follows:

$$\|u\|_A^2 := u^T A u$$

and the error vector e_n is given by $e_n := u_n - u_*$, where u_n is from the algorithm in Figure 1.1 and u_* is the solution of (1.1). We then have the following error estimate.

Lemma 1.2.1. *Let A be a symmetric positive matrix. Then, the A-norm of the errors for the conjugate gradient methods satisfy the following bound:*

$$\frac{\|e_n\|_A}{\|e_0\|_A} \leqslant 2 \left(\frac{\sqrt{\kappa(A)} - 1}{\sqrt{\kappa(A)} + 1} \right)^n, \tag{1.4}$$

where $\kappa(A)$ is the condition number of A in the 2-norm.

By Lemma 1.2.1, if the condition number of A is very large, the convergence rate may be unsatisfactory. Hence, we introduce a preconditioner P, a symmetric positive definite, and the following modified linear systems:

$$P^{-1}Au = P^{-1}f \tag{1.5}$$

or

$$P^{-1/2}AP^{-1/2}v = P^{-1/2}f, \ v = P^{1/2}u. \tag{1.6}$$

The preconditioned conjugate gradient algorithm is given in Figure 1.2 for solving (1.5) or (1.6) instead of (1.1). For the preconditioned conjugate gradient

Initialize: $r_0 := f - Au_0$, $z_0 := P^{-1}r_0$, $p_0 := z_0$

Iterate $k = 0, 1, \cdots$ until convergence

$$\alpha_k := r_k^T z_k / p_k^T A p_k$$

$$u_{k+1} := u_k + \alpha_k p_k$$

$$r_{k+1} := r_k - \alpha_k A p_k$$

$$z_{k+1} := P^{-1} r_{k+1}$$

$$\beta_k := z_{k+1}^T r_{k+1} / z_k^T r_k$$

$$p_{k+1} := z_{k+1} + \beta_k p_k$$

Figure 1.2: Preconditioned Conjugate Gradient Algorithm

algorithm, we have the following estimate.

Lemma 1.2.2. *Let A and P be a symmetric positive matrix. Then, the A-norm of the errors for the conjugate gradient methods satisfy the following bound:*

$$\frac{\|e_n\|_A}{\|e_0\|_A} \leqslant 2\left(\frac{\sqrt{\kappa(P^{-1}A)}-1}{\sqrt{\kappa(P^{-1}A)}+1}\right)^n, \tag{1.7}$$

where $\kappa(P^{-1}A)$ is the condition number of $P^{-1}A$ in the 2-norm.

If we find a preconditioner which makes the condition number of $P^{-1}A$ much smaller than that of A, we will have a better convergence rate according to Lemma 1.2.2. There are then extra costs associated with a matrix-vector product with P^{-1} and we have to make sure that the computational cost for this matrix-vector multiplication should be cheap enough to save computing time.

It is usually very hard to obtain exact spectral information on a matrix. However, if we use a tridiagonal matrix related to the coefficients of the conjugate gradient algorithm, we can obtain approximate eigenvalues. This information can be used to estimate the condition number. We can construct the tridiagonal matrix as follows:

$$J^{(m)} = \begin{bmatrix} \frac{1}{\alpha_0} & \frac{\sqrt{\beta_0}}{\alpha_0} & & & \\ \frac{\sqrt{\beta_0}}{\alpha_0} & \frac{1}{\alpha_1}+\frac{\beta_0}{\alpha_0} & \frac{\sqrt{\beta_1}}{\alpha_1} & & \\ & \frac{\sqrt{\beta_1}}{\alpha_1} & \ddots & \ddots & \\ & & \ddots & \ddots & \frac{\sqrt{\beta_{m-2}}}{\alpha_{m-2}} \\ & & & \frac{\sqrt{\beta_{m-2}}}{\alpha_{m-2}} & \frac{1}{\alpha_{m-1}}+\frac{\beta_{m-2}}{\alpha_{m-2}}. \end{bmatrix}. \tag{1.8}$$

By considering the spectral information of $J^{(m)}$, we can estimate the eigenvalues of A. We note that extreme eigenvalues of $J^{(m)}$ converge rapidly after a few iterations. For more detail, see [57, Chapter 6.7].

1.3 Mixed Finite Element Methods

We consider mixed finite element methods in this section. Let V and Π be two Hilbert spaces and $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ and $b(\cdot,\cdot) : V \times \Pi \to \mathbb{R}$ be two continuous bilinear forms:

$$\begin{aligned} a(u,v) &\leqslant C\|u\|_V\|v\|_V, \forall u,v \in V \\ b(v,p) &\leqslant C\|v\|_V\|p\|_\Pi, \forall v \in V, p \in \Pi. \end{aligned}$$

We consider the following variational problem to find $v \in V$ and $p \in \Pi$:

$$\begin{aligned} a(u,v) + b(v,p) &= F(v), \forall v \in V \\ b(u,q) &= G(q), \forall q \in \Pi, \end{aligned} \tag{1.9}$$

where $F \in V'$ and $G \in \Pi'$.

Before we consider the existence and uniqueness of (1.9), we introduce a coercitity and an inf-sup condition.

Definition 1.3.1. *If there exists a positive constant α such that for all $v \in V$, $\alpha\|v\|_V^2 \leqslant a(v,v)$, we call $a(\cdot,\cdot)$ coercive.*

Definition 1.3.2. *$b(\cdot,\cdot)$ has an inf-sup condition if $b(\cdot,\cdot)$ satisfies the following condition: there exists a positive constant β such that*

$$\beta\|p\|_\Pi \leqslant \sup_{w \in V} \frac{b(w,p)}{\|w\|_V}, \forall p \in \Pi.$$

We note that the inf-sup condition is also called the Ladyzhenskaya-Babuška-Brezzi condition.

We consider a closed subspace of V:

$$Z := \{v \in V \mid b(v,q) = 0, \forall q \in \Pi\}. \tag{1.10}$$

The following theorem determines the well-posedness of problem (1.9).

Theorem 1.3.1. *If $a(\cdot,\cdot)$ is coercive for all $v \in Z$ and $b(\cdot,\cdot)$ satisfies the inf-sup condition, then there exists a unique solution pair to (1.9).*

Proof. See [7, Theorem 4.3]. □

In some cases, we need the following additional bilinear form:

$$c(\cdot,\cdot) : \Pi \times \Pi \to \mathbb{R}, c(p,p) \geqslant 0, \forall p \in \Pi$$

to consider a saddle point problem. The problem has the following form:

$$\begin{aligned} a(u,v) + b(v,p) &= F(v), \forall v \in V \\ b(u,q) - c(p,q) &= G(q), \forall q \in \Pi. \end{aligned}$$

For more details, see [7, Chapter III.4], [12, Chapter 12], and [14].

1.4 Organization of the Dissertation

The remaining parts of this dissertation are organized as follows. In Chapter 2, we introduce various function spaces, finite element spaces, and their properties. We present our model problem in Chapter 3. We next review domain decomposition algorithms and theories in Chapter 4. We finally provide an overlapping

Schwarz method and a BDDC method for solving vector field problems in Chapter 5 and Chapter 6, respectively.

Chapter 2

Function and Finite Element Spaces

2.1 Continuous Spaces

2.1.1 Sobolev Spaces

We will use Sobolev spaces and corresponding norms and seminorms for bounded open Lipschitz domains Ω. Let us consider the L^2-space first. It is the space of square integrable functions and its norm is given by

$$\|u\|_{0;\Omega}^2 := \int_\Omega |u|^2 \, dx.$$

We next consider the following scaled H^1-norm:

$$\|u\|_{1;\Omega}^2 := |u|_{1;\Omega}^2 + \frac{1}{H^2}\|u\|_{0;\Omega}^2,$$

where H is the diameter of Ω and the seminorm $|\cdot|_{1;\Omega}$ is defined by

$$|u|_{1;\Omega}^2 := \int_\Omega |\nabla u|^2 \, dx.$$

$H^1(\Omega) \subset L^2(\Omega)$ is the space of functions with finite scaled H^1-norms.

We next introduce $H^{\frac{1}{2}}(\partial\Omega)$, the trace space of $H^1(\Omega)$. The scaled norm of $H^{\frac{1}{2}}(\partial\Omega)$ is given by

$$\|u\|_{\frac{1}{2};\partial\Omega}^2 := |u|_{\frac{1}{2};\partial\Omega}^2 + \frac{1}{H}\|u\|_{0;\partial\Omega}^2,$$

where the seminorm is defined by

$$|u|_{\frac{1}{2};\partial\Omega}^2 := \int_{\partial\Omega}\int_{\partial\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^d} \, dxdy$$

and d is the dimension of Ω. Moreover, the dual space of $H^{\frac{1}{2}}(\partial\Omega)$ is denoted by $H^{-\frac{1}{2}}(\partial\Omega)$.

We have the following two lemmas related to trace spaces.

Lemma 2.1.1. (Trace theorem) *Let Ω be a Lipschitz domain. Then, there is a bounded linear operator γ_0, which maps a smooth function into its restriction on the boundary, that can be extended continuously to an operator $\gamma_0 : H^1(\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$.*

Lemma 2.1.2. (Extension theorem) *Let Ω be a Lipschitz domain. There exists a continuous lifting operator $\mathcal{R}_0 : H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega)$, such that for all $u \in H^{\frac{1}{2}}(\partial\Omega)$, $\gamma_0(\mathcal{R}_0 u) = u$.*

We next introduce Poincaré's and Friedrichs' inequalities, which are very useful tools for the analysis of domain decomposition methods.

Lemma 2.1.3. (Poincaré's inequality) *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz region with diameter H_Ω and $u \in H^1(\Omega)$. Then, there exist constants C_1 and C_2 such* *that*

$$\|u\|_{0;\Omega}^2 \leqslant C_1 H_\Omega^2 |u|_{1;\Omega}^2 + C_2 \frac{1}{H_\Omega^d} \left(\int_\Omega u \, dx \right)^2,$$

where the constants depend only on the shape of Ω.

Lemma 2.1.4. (Friedrichs' inequality) *Let* $\Omega \subset \mathbb{R}^d$ *be a bounded Lipschitz region with diameter* H_Ω *and* $\Gamma \subseteq \partial\Omega$ *have nonvanishing* $(d-1)$*-dimensional measure. Then, there exist constants* C_1 *and* C_2*, depending only on the shape of* Ω *and* Γ*, such that*

$$\|u\|_{0;\Omega}^2 \leqslant C_1 H_\Omega^2 |u|_{1;\Omega}^2 + C_2 H_\Omega \|u\|_{0;\Gamma}^2,$$

for all $u \in H^1(\Omega)$*. Moreover, if* u *vanishes on* Γ*,*

$$\|u\|_{0;\Omega}^2 \leqslant C_1 H_\Omega^2 |u|_{1;\Omega}^2$$

and thus the H^1*-norm and* H^1*-seminorm are then equivalent.*

2.1.2 $H(\mathrm{div})$ and $H(\mathbf{curl})$ Space

2.1.2.1 The space $H(\mathrm{div})$

For a given vector function $\mathbf{u} \in \mathbb{R}^2$ or $\mathbb{R}^3$, we define the divergence operator as

$$\mathrm{div}\, \mathbf{u} := \sum_{i=1}^{d} \frac{\partial u_i}{\partial x_i},$$

where d is 2 or 3 and where u_i is the i-th component of $\mathbf{u}$. The space $H(\mathrm{div};\Omega)$ is defined by

$$H(\mathrm{div};\Omega) := \{\mathbf{u} \in (L^2(\Omega))^d \,|\, \mathrm{div}\, \mathbf{u} \in L^2(\Omega)\}$$

with the following scaled graph norm:

$$\|\mathbf{u}\|_{\mathrm{div};\Omega}^2 := \|\mathbf{u}\|_{0;\Omega}^2 + H^2\|\mathrm{div}\,\mathbf{u}\|_{0;\Omega}^2,$$

where d is the dimension of Ω and H is the diameter of Ω. We note that the scaling is different from that of $\|u\|_{1;\Omega}^2$. It is known that the normal component of $\mathbf{u} \in H(\mathrm{div};\Omega)$ is in $H^{-\frac{1}{2}}(\partial\Omega)$; see [14, 50]. The norm for the space $H^{-\frac{1}{2}}(\partial\Omega)$ is given by

$$\|\mathbf{u}\cdot\mathbf{n}\|_{-\frac{1}{2};\partial\Omega} := \sup_{\phi\in H^{\frac{1}{2}}(\partial\Omega),\phi\neq 0} \frac{< \mathbf{u}\cdot\mathbf{n}, \phi >}{\|\phi\|_{\frac{1}{2};\partial\Omega}}.$$

The angle brackets stand for the duality product of $H^{-\frac{1}{2}}(\partial\Omega)$ and $H^{\frac{1}{2}}(\partial\Omega)$. We denote by $H_0(\mathrm{div};\Omega)$ the subspace of $H(\mathrm{div};\Omega)$ with a vanishing normal component on $\partial\Omega$.

Lemma 2.1.5. *There exists a constant C, which is independent of the diameter of Ω, such that, for all $\mathbf{u} \in H(\mathrm{div};\Omega)$,*

$$\|\mathbf{u}\cdot\mathbf{n}\|_{-\frac{1}{2};\partial\Omega}^2 \leqslant C(\|\mathbf{u}\|_{0;\Omega}^2 + H^2\|\mathrm{div}\mathbf{u}\|_{0;\Omega}^2),$$

where H is the diameter of Ω.

Proof. This follows directly from Green's identity on a domain with a diameter one and by applying a dilation; see [71, Lemma 2.1]. $\square$

2.1.2.2 The space $H(\mathbf{curl})$

The **curl** operator for a given vector function $\mathbf{u} \in \mathbb{R}^3$ is defined by

$$\mathbf{curl}\,\mathbf{u} := \begin{bmatrix} \dfrac{\partial u_3}{\partial x_2} - \dfrac{\partial u_2}{\partial x_3} \\ \dfrac{\partial u_1}{\partial x_3} - \dfrac{\partial u_3}{\partial x_1} \\ \dfrac{\partial u_2}{\partial x_1} - \dfrac{\partial u_1}{\partial x_2} \end{bmatrix}.$$

We note that the curl operator for 2D is just a simple rotation of the divergence operator. For 2D cases, we therefore can use the results for $H(\mathrm{div};\Omega)$ to obtain results for $H(\mathrm{curl};\Omega)$.

$H(\mathbf{curl};\Omega)$ is defined by

$$H(\mathbf{curl};\Omega) := \{\mathbf{u} \in (L^2(\Omega))^3 \,|\, \mathbf{curl}\,\mathbf{u} \in (L^2(\Omega))^3\}$$

with the following scaled graph norm:

$$\|\mathbf{u}\|^2_{\mathbf{curl};\Omega} := \|\mathbf{u}\|^2_{0;\Omega} + H^2\|\mathbf{curl}\,\mathbf{u}\|^2_{0;\Omega},$$

where H is the diameter of Ω.

We can now define the tangential component of $\mathbf{u}$ on the boundary as follows:

$$\mathbf{u}_t := \mathbf{u} - (\mathbf{u}\cdot\mathbf{n})\mathbf{n} = (\mathbf{n}\times\mathbf{u})\times\mathbf{n}.$$

We note that this tangential component is in $H^{-\frac{1}{2}}(\partial\Omega)^3$; see [50]. Let $H_0(\mathbf{curl};\Omega)$ denote the subspace of $H(\mathbf{curl};\Omega)$ with a vanishing tangential component.

2.1.3 Helmholtz Decompositions

We introduce the following Helmholtz decompositions.

Lemma 2.1.6. (Helmholtz decompositions) *Let Ω be convex and let $H_0(\mathbf{curl}\,;\Omega)$ be the subset of $H(\mathbf{curl}\,;\Omega)$ with a vanishing tangential component on $\partial\Omega$ and $H_0(\mathrm{div}\,;\Omega)$ the subset of $H(\mathrm{div}\,;\Omega)$ with a vanishing normal component on $\partial\Omega$. Then, $H_0(\mathbf{curl}\,;\Omega)$, $H_0(\mathrm{div}\,;\Omega)$, $H(\mathbf{curl}\,;\Omega)$, and $H(\mathrm{div}\,;\Omega)$ have the following generalized orthogonal Helmholtz decompositions:*

$$
\begin{aligned}
H_0(\mathbf{curl}\,;\Omega) &= \mathbf{grad}\, H_0^1(\Omega) \oplus H_0^{\perp}(\mathbf{curl}\,;\Omega),\\
H(\mathbf{curl}\,;\Omega) &= \mathbf{grad}\, H^1(\Omega) \oplus H^{\perp}(\mathbf{curl}\,;\Omega)
\end{aligned}
$$

and

$$
\begin{aligned}
H_0(\mathrm{div}\,;\Omega) &= \mathbf{curl}\, H_0(\mathbf{curl}\,;\Omega) \oplus H_0^{\perp}(\mathrm{div}\,;\Omega)\\
&= \mathbf{curl}\, H_0^{\perp}(\mathbf{curl}\,;\Omega) \oplus H_0^{\perp}(\mathrm{div}\,;\Omega),\\
H(\mathrm{div}\,;\Omega) &= \mathbf{curl}\, H(\mathbf{curl}\,;\Omega) \oplus H^{\perp}(\mathrm{div}\,;\Omega)\\
&= \mathbf{curl}\, H^{\perp}(\mathbf{curl}\,;\Omega) \oplus H^{\perp}(\mathrm{div}\,;\Omega),
\end{aligned}
$$

where

$$
\begin{aligned}
H_0^{\perp}(\mathbf{curl}\,;\Omega) &:= H_0(\mathbf{curl}\,;\,\Omega) \cap H(\mathrm{div}_0\,;\Omega),\\
H^{\perp}(\mathbf{curl}\,;\Omega) &:= H(\mathbf{curl}\,;\,\Omega) \cap H_0(\mathrm{div}_0\,;\Omega)
\end{aligned}
$$

and

$$
\begin{aligned}
H_0^{\perp}(\operatorname{div};\Omega) &:= H_0(\operatorname{div};\Omega)\cap H(\mathbf{curl}_0;\Omega),\\
H^{\perp}(\operatorname{div};\Omega) &:= H(\operatorname{div};\Omega)\cap H_0(\mathbf{curl}_0;\Omega).
\end{aligned}
$$

Here, $H(\mathbf{curl}_0;\Omega)$, $H(\operatorname{div}_0;\Omega)$, $H_0(\mathbf{curl}_0;\Omega)$, and $H_0(\operatorname{div}_0;\Omega)$ are defined as follows:

$$
\begin{aligned}
H(\mathbf{curl}_0;\Omega) &:= \{\mathbf{u}\in H(\mathbf{curl};\Omega), \mathbf{curl}\,\mathbf{u}=0\},\\
H_0(\mathbf{curl}_0;\Omega) &:= \{\mathbf{u}\in H_0(\mathbf{curl};\Omega), \mathbf{curl}\,\mathbf{u}=0\}
\end{aligned}
$$

and

$$
\begin{aligned}
H(\operatorname{div}_0;\Omega) &:= \{\mathbf{u}\in H(\operatorname{div};\Omega), \operatorname{div}\mathbf{u}=0\},\\
H_0(\operatorname{div}_0;\Omega) &:= \{\mathbf{u}\in H_0(\operatorname{div};\Omega), \operatorname{div}\mathbf{u}=0\}.
\end{aligned}
$$

Proof. See [17, Proposition 1, p.215]. □

Remark 2.1.1. *There is another stable decomposition which holds for more general region; see [35, Lemma 3.10] for details.*

We summarize the properties of the decomposition.

Lemma 2.1.7. *For $\mathbf{u}\in H^{\perp}(\mathbf{curl};\Omega)$ and $\mathbf{v}\in H^{\perp}(\operatorname{div};\Omega)$, we have the following estimates:*

$$
\|\mathbf{u}\|_{0;\Omega} \leqslant CH\|\mathbf{curl}\,\mathbf{u}\|_{0;\Omega}, \forall \mathbf{u}\in H^{\perp}(\mathbf{curl};\Omega)
$$

and

$$\|\mathbf{v}\|_{0;\Omega} \leqslant CH\|\operatorname{div}\mathbf{v}\|_{0;\Omega}, \forall \mathbf{v} \in H^{\perp}(\operatorname{div}; \Omega),$$

where H is the diameter of Ω.

Proof. See [1, Prop. 4.6]. □

Lemma 2.1.8. *If Ω is convex, the spaces $H^{\perp}(\mathbf{curl}; \Omega)$ and $H^{\perp}(\operatorname{div}; \Omega)$ are continuously embedded in $(H^1(\Omega))^d$, where d is the dimension of Ω.*

Proof. See [1, Theorem 2.17] and [34, Lemma 4.1]. □

2.2 Finite Element Spaces

2.2.1 Raviart-Thomas and Nédélec Elements

Let $\mathcal{T}_h$ be a given triangulation and K be the elements of $\mathcal{T}_h$. We assume that $\mathcal{T}_h$ is shape-regular.

We first consider the Raviart-Thomas elements. The lowest order Raviart-Thomas element space is defined by

$$X_h := \{\mathbf{u} \mid \mathbf{u}_{|K} \in \mathcal{RT}(K), K \in \mathcal{T}_h \text{ and } \mathbf{u} \in H(\mathrm{div}; \Omega)\},$$

where $\mathcal{RT}(K)$ is given by

$$\mathcal{RT}(K) := \mathbf{a} + b\mathbf{x},$$

for triangular or tetrahedral elements and by

$$\mathcal{RT}(K) := \mathbf{a} + \mathbf{b} \cdot \mathbf{x},$$

for quadrilateral or hexahedral elements, where $\mathbf{a}$ and $\mathbf{b}$ are vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ and b is a scalar.

The degrees of freedom are defined by the average values of the normal components over the edges and the faces of $\mathcal{T}_h$ for two and three dimensions, respectively, i.e., by

$$\lambda(\mathbf{u}) := \frac{1}{|F|}\int_F \mathbf{u}\cdot\mathbf{n}\, ds,\;\; F \subset \partial K.$$

The l^2-norm of the vector of these coefficients can be related to the L^2-norm of $\mathbf{u}$. We have the following lemma.

Lemma 2.2.1. *Let $K \in \mathcal{T}_h$. Then, for all $\mathbf{u} \in X_h$, there exist constants depending only on the aspect ratio of K, such that*

$$c\sum_{f\subset\partial K} h_f^d \lambda(\mathbf{u})^2 \leqslant \|\mathbf{u}\|_{0;K}^2 \leqslant C\sum_{f\subset\partial K} h_f^d \lambda(\mathbf{u})^2, \tag{2.1}$$

where h_f is the diameter of f and d is the dimension of the region.

Proof. The proof of this lemma is just a simple modification of [56, Proposition 6.3.1]. □

The basis functions of the lowest order Raviart-Thomas element space are supported in two elements of $\mathcal{T}_h$ and their normal component equals 1 on a specified edge (2D) or face (3D) and 0 on the other edges (2D) or faces (3D).

We also define $X_{0;h}$ which is the subspace of X_h with a vanishing normal component on the boundary of the domain Ω, i.e.,

$$X_{0;h}(\Omega) := X_h(\Omega) \cap H_0(\mathrm{div};\Omega).$$

We need to define trace spaces. Let $F_h(\partial\Omega)$ be the space of functions which are constant on each edge (2D) or face (3D) of the edges or faces of the elements of $\mathcal{T}_h$ which are contained in $\partial\Omega$. We also define $F_{0;h}(\partial\Omega)$ as the subspace of $F_h(\partial\Omega)$ with mean value zero over $\partial\Omega$.

We next introduce the Nédélec elements. The lowest order Nédélec element space is defined by

$$N_h := \{\mathbf{u} \mid \mathbf{u}_{|K} \in \mathcal{ND}(K), K \in \mathcal{T}_h \text{ and } \mathbf{u} \in H(\mathbf{curl};\Omega)\},$$

where

$$\mathcal{ND}(K) := \mathbf{a} + \mathbf{x} \times \mathbf{b},$$

for triangular or tetrahedral elements, by

$$\mathcal{ND}(K) := Q_{0,1} \times Q_{1,0},$$

with Q_{k_1,k_2}, the space of polynomial of degree k_i in the i-th variable, for quadrilateral elements, and by

$$\mathcal{ND}(K) := Q_{0,1,1} \times Q_{1,0,1} \times Q_{1,1,0},$$

with Q_{k_1,k_2,k_3}, the space of polynomial of degree k_i in the i-th variable, for hexahedral elements.

The degrees of freedom are defined by the average value of the tangential component over the edges e

$$\lambda_e(\mathbf{u}) = \frac{1}{|e|} \int_e \mathbf{u} \cdot \mathbf{t}_e \, ds, \;\; e \subset \partial K,$$

for each $K \in \mathcal{T}_h$. We have the following lemma similar to Lemma 2.2.1.

Lemma 2.2.2. *Let $K \in \mathcal{T}_h$. Then, for all $\mathbf{u} \in N_h$, there exist constants depending only on the aspect ratio of K, such that*

$$c \sum_{e \subset \partial K} h_e^d \lambda_e(\mathbf{u})^2 \leqslant \|\mathbf{u}\|_{0;K}^2 \leqslant C \sum_{e \subset \partial K} h_e^d \lambda_e(\mathbf{u})^2, \tag{2.2}$$

where h_e is the length of e and d is the dimension of the region.

Proof. See Lemma 2.2.1 and [56, Proposition 6.3.1] $\square$

The basis functions, associated with individual edges, of the lowest order Nédélec element space are supported in the union of the elements of $\mathcal{T}_h$ that have the edge in common and their tangential components are 1 on the specified edge and 0 on all other edges.

We also define $N_{0;h}$, which is the subspace of N_h with a vanishing tangential component on the boundary of the domain Ω, i.e.,

$$N_{0;h}(\Omega) := N_h(\Omega) \cap H_0(\mathbf{curl}; \Omega).$$

2.2.2 Commuting Property

Let S_h be the continuous P_1 space and $S_{0;h}$ be the subspace of S_h with zero boundary values. We also denote by Q_h the P_0 space of constant functions on each element. Figure 2.1 and Figure 2.2 show the symbolic notation for local degrees of freedom of each element for 2D and 3D, respectively. We note that Lagrange P_1 elements, Raviart-Thomas elements, and Nédélec elements are conforming elements in H^1, $H(\mathrm{div})$, and $H(\mathbf{curl})$, respectively.

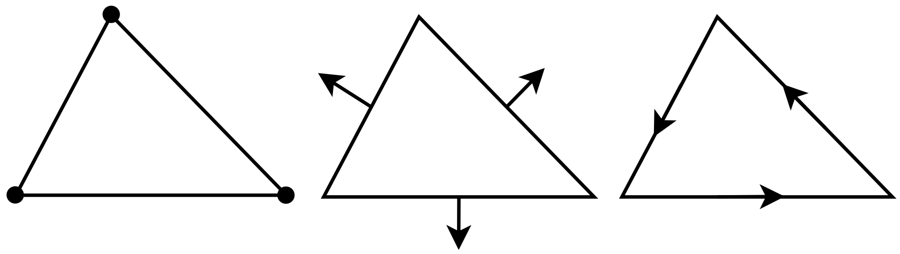

Figure 2.1: P_1, Raviart-Thomas, and Nédélec element (2D).

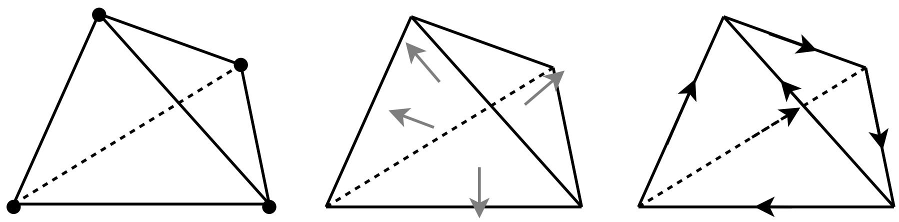

Figure 2.2: P_1, Raviart-Thomas, and Nédélec element (3D).

We next define three interpolation operators I_h, Π_h^{ND}, Π_h^{RT}, and Π_h onto S_h, N_h, X_h, and Q_h, respectively. We then have the following commuting diagram.

$$\begin{tikzcd} H^1 \arrow[r, "\mathbf{grad}"] \arrow[d, "I_h"] & H(\mathbf{curl}) \arrow[r, "\mathbf{curl}"] \arrow[d, "\Pi_h^{ND}"] & H(\mathrm{div}) \arrow[r, "\mathrm{div}"] \arrow[d, "\Pi_h^{RT}"] & L^2 \arrow[d, "\Pi_h"] \\ S_h \arrow[r, "\mathbf{grad}"] & N_h \arrow[r, "\mathbf{curl}"] & X_h \arrow[r, "\mathrm{div}"] & Q_h \end{tikzcd} \tag{2.3}$$

2.2.3 Discrete Helmholtz Decompositions

We have the following decompositions for finite element space.

Lemma 2.2.3. (Discrete Helmholtz decompositions) *If Ω is convex, then we have decompositions for the finite element spaces similar to those in the continuous cases. Thus,*

$$\begin{aligned} N_{0;h}(\Omega) &= \mathbf{grad}\, S_{0;h}(\Omega) \oplus N_{0;h}^{\perp}(\Omega), \\ N_h(\Omega) &= \mathbf{grad}\, S_h(\Omega) \oplus N_h^{\perp}(\Omega) \end{aligned}$$

and

$$\begin{aligned} X_{0;h}(\Omega) &= \mathbf{curl}\, N_{0;h}(\Omega) \oplus X_{0;h}^{\perp}(\Omega) \\ &= \mathbf{curl}\, N_{0;h}^{\perp}(\Omega) \oplus X_{0;h}^{\perp}(\Omega), \\ X_h(\Omega) &= \mathbf{curl}\, N_h(\Omega) \oplus X_h^{\perp}(\Omega) \\ &= \mathbf{curl}\, N_h^{\perp}(\Omega) \oplus X_h^{\perp}(\Omega), \end{aligned}$$

where $X_{0;h}^{\perp}(\Omega)$, $N_{0;h}^{\perp}(\Omega)$, $X_h^{\perp}(\Omega)$, *and* $N_h^{\perp}(\Omega)$ *are orthogonal complements of the kernel of the* **curl** *or* div *operator.*

Proof. See [32, Theorem 2.36]. □

Remark 2.2.1. *There is a discrete version of the alternative decomposition of Remark 2.1.1 as well; see [35, Lemma 5.1]. This discrete decomposition has one additional term compared to the discrete Helmholtz decomposition. This third term is related to the error of a Scott-Zhang interpolation given in [60].*

We have the following lemma, similar to Lemma 2.1.7, for the discrete Helmholtz decomposition.

Lemma 2.2.4. *For* $\mathbf{u} \in N_h^{\perp}(\Omega)$ *and* $\mathbf{v} \in X_h^{\perp}(\Omega)$, *we have the following estimates:*

$$\|\mathbf{u}\|_{0;\Omega} \leqslant CH\|\mathbf{curl}\,\mathbf{u}\|_{0;\Omega}, \forall \mathbf{u} \in N_h^{\perp}(\Omega)$$

and

$$\|\mathbf{v}\|_{0;\Omega} \leqslant CH\|\mathrm{div}\,\mathbf{v}\|_{0;\Omega}, \forall \mathbf{v} \in X_h^{\perp}(\Omega),$$

where H *is the diameter of* Ω.

Chapter 3

A Model Problem

3.1 Introduction

Because of the physical relevance of $H(\mathrm{div})$ and $H(\mathbf{curl})$, there are many applications posed in these spaces. For example, the space $H(\mathbf{curl})$ is suitable for electromagnetism and some formulations of Navier-Stokes equations; see [29, 30, 32]. For $H(\mathrm{div})$, the space usually occurs in mixed formulations of second order elliptic equations; see [47, 48, 68]. Moreover, for the incompressible Navier-Stokes equations, we need solutions of problems posed in $H(\mathrm{div})$ for the sequential regularization method; see [41]. We will consider a standard $H(\mathrm{div})$ problem in this dissertation.

3.2 A Model Problem

We consider the following boundary value problem:

$$\begin{aligned} L\mathbf{u} := -\mathbf{grad}\,(\alpha\,\mathrm{div}\,\mathbf{u}) + \beta\,\mathbf{u} &= \mathbf{f} \text{ in } \Omega, \\ \mathbf{u}\cdot\mathbf{n} &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{3.1}$$

Here Ω is a bounded polygon in $\mathbb{R}^2$ or a polyhedron in $\mathbb{R}^3$ and $\mathbf{n}$ is the outward normal vector of its boundary. We assume that $\mathbf{f}$ is in $(L^2(\Omega))^2$ or $(L^2(\Omega))^3$ and that α and β are positive $L^\infty(\Omega)$ functions.

Let us also consider the following elliptic equation:

$$\begin{aligned} -\mathrm{div}\left(\beta^{-1}\nabla w\right) + \alpha^{-1} w &= g \text{ in } \Omega, \\ \beta^{-1}\nabla w\cdot\mathbf{n} &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{3.2}$$

If we set $\mathbf{f} = \nabla(\alpha g)$, then a mixed formulation of problem (3.2) is equivalent to the original vector field problem (3.1). We note that $\mathbf{u} = \beta^{-1}\nabla w$. Therefore, if we solve the vector field problem (3.1), we can obtain the gradient of the solution of (3.2).

In order to see this, we introduce an intermediate variable $\mathbf{q} := \beta^{-1}\nabla w$ as an additional unknown. We then have the following mixed variational problem:

$$\begin{aligned} \int_\Omega \beta\,\mathbf{q}\cdot\mathbf{p}\,dx + \int_\Omega w\,\mathrm{div}\,\mathbf{p}\,dx &= 0,\ \mathbf{p}\in H_0(\mathrm{div}\,;\Omega), \\ \int_\Omega \mathrm{div}\,\mathbf{q}\,v\,dx - \int_\Omega \alpha^{-1} w v\,dx &= -\int_\Omega g v\,dx, v\in L^2(\Omega). \end{aligned}$$

This gives us $\mathbf{q} = \mathbf{u}$. We can show that this problem is well-posed; see [14, Section

II.1.2] and Section 1.3.

Another application is given by a least-squares formulation. Consider the following least-squares functional:

$$G(\mathbf{u}, w; g) := \|\beta^{-1}\nabla w - \mathbf{u}\|_{0;\Omega}^2 + \| - \operatorname{div} \mathbf{u} + \alpha^{-1} w - g\|_{0;\Omega}^2. \tag{3.3}$$

The problem (3.1) arises from (3.3). We can solve (3.2) directly and obtain ∇w by simply taking the gradient of w. However, we will lose accuracy. In order to preserve the accuracy, we can use a mixed or least-squares formulation. For more detail, see [15, 71].

3.3 Variational and Discretized Formulas

We will consider a variational formulation of the original problem:

$$\mathbf{a}(\mathbf{u}, \mathbf{v}) := \int_\Omega \alpha \operatorname{div} \mathbf{u} \operatorname{div} \mathbf{v} \, dx + \beta \, \mathbf{u} \cdot \mathbf{v} \, dx = \int_\Omega \mathbf{f} \cdot \mathbf{v} \, dx, \; \mathbf{v} \in H_0(\operatorname{div}; \Omega). \tag{3.4}$$

Restricting the bilinear form $\mathbf{a}(\mathbf{u}, \mathbf{v})$ to the finite element space of the lowest order Raviart-Thomas elements, we obtain the stiffness matrix A.

We decompose the domain Ω into N nonoverlapping subdomains Ω_i of diameter H_i and then consider triangulations of all the subdomains. Thus, we introduce two triangulations $\mathcal{T}_H$ and $\mathcal{T}_h$. $\mathcal{T}_H$ is a shape-regular coarse triangulation and $\mathcal{T}_h$ is a refinement of $\mathcal{T}_H$ which provides a shape-regular and quasi-uniform triangulation of the individual coarse mesh elements. We assume that each subdomain Ω_i is a union of coarse elements of $\mathcal{T}_H$ and that the number of such elements forming each subdomain is uniformly bounded. Moreover, we denote by h_i the minimum diameter of the triangulation of Ω_i.

We next consider extended subregions Ω_i' obtained from Ω_i by adding layers of elements. Thus, $\partial\Omega_i'$ does not cut through any elements. We also define the interface Γ:

$$\Gamma = \left(\bigcup_{i=0}^{N} \partial\Omega_i \right) \backslash \partial\Omega.$$

Moreover, let Γ_h be the set of interface nodes.

From now on, we will assume that the coefficients α and β are constant in each subdomain and that they thus only can have jumps across the interface Γ. We can then write the problem (3.4) in the following way:

$$\mathbf{a}(\mathbf{u}, \mathbf{v}) := \sum_{i=1}^{N} \alpha_i \int_{\Omega_i} \operatorname{div} \mathbf{u} \operatorname{div} \mathbf{v} \, dx + \beta_i \int_{\Omega_i} \mathbf{u} \cdot \mathbf{v} \, dx, \ \mathbf{u}, \mathbf{v} \in H_0(\operatorname{div}; \Omega).$$

We can also define local energy bilinear forms:

$$\mathbf{a}_i(\mathbf{u}, \mathbf{u}) := \alpha_i \int_{\Omega_i} \operatorname{div} \mathbf{u} \operatorname{div} \mathbf{u} \, dx + \beta_i \int_{\Omega_i} \mathbf{u} \cdot \mathbf{u} \, dx \tag{3.5}$$

and

$$\begin{aligned} \widetilde{\mathbf{a}}_i(\mathbf{u}, \mathbf{u}) &:= \int_{\Omega_i'} \alpha \operatorname{div} \mathbf{u} \operatorname{div} \mathbf{u} \, dx + \int_{\Omega_i'} \beta \, \mathbf{u} \cdot \mathbf{u} \, dx \\ &= \alpha_i \int_{\Omega_i} \operatorname{div} \mathbf{u} \operatorname{div} \mathbf{u} \, dx + \beta_i \int_{\Omega_i} \mathbf{u} \cdot \mathbf{u} \, dx \\ &+ \sum_{i \neq j, \Omega_i' \cap \Omega_j \neq \phi} \alpha_j \int_{\Omega_i' \cap \Omega_j} \operatorname{div} \mathbf{u} \operatorname{div} \mathbf{u} \, dx + \beta_j \int_{\Omega_i' \cap \Omega_j} \mathbf{u} \cdot \mathbf{u} \, dx. \end{aligned} \tag{3.6}$$

We note that for overlapping Schwarz methods, we can obtain the local subproblems from the global stiffness matrix A. However, it is sometimes impossible to obtain the corresponding subproblems and suitable boundary conditions directly from A for other type of domain decomposition algorithm such as those of the Neumann-Neumann and FETI families. We need a special care to consider those type of methods.

Chapter 4

Domain Decomposition Methods

4.1 Introduction

The very first domain decomposition method was considered by Schwarz in 1870. The method was designed to solve elliptic boundary value problems on the union of two subdomains. Even though it is not a numerical scheme, the classical alternating Schwarz method gives us good insight.

One-level additive Schwarz methods were originally introduced in [49, 51]. For elliptic PDEs, the solution on the entire domain depends on the right hand side and the boundary values. The information of the right hand side at a point can be transmitted to all points in the domain only by passing through neighboring subdomains. Hence, the number of subdomains will effect the efficiency of any one-level methods. For more detail, see [61, 67].

In the domain decomposition theory, methods with more than one level can often provide good scalability, i.e., have a convergence rate which depends only on the size of the subdomain problems and not on any other parameters, e.g., the number of subdomains; scalability can be obtained by introducing a coarse global problem. For more detail for two-level methods, see [22, 24, 25].

The other type of domain decomposition methods, iterative substructuring methods, have also been considered. The traditional substructuring methods construct the Schur complement system and solve it by a direct method. Iterative substructuring methods, also known as nonoverlapping methods, are iterative methods for solving the Schur complement system. These methods were first developed by Bramble, Pasciak, and Schatz; see [8–11]. In [23], Dryja, Smith, and Widlund analyzed iterative substructuring methods by using an abstract Schwarz framework and many variants of iterative substructuring methods were developed. Among them, Neumann-Neumann type and FETI type are the most popular and have been widely used for many problems. Neumann-Neumann type algorithms were first introduced in [31] and extended to BNN methods with an additional level; see [26,37]. One-level FETI methods, which in fact implicitly include a coarse component, were introduced in [28] and analyzed in [45]. Later, FETI-DP methods were introduced by Farhat, Lesoinne, Le Tallec, Pierson, and Rixen with coarse-level primal constraints; see [27]. A theoretical result was first provided by Mandel and Tezaur in [46]. As we mentioned earlier, the BDDC methods were introduced by Dohrmann and analyzed by Dohrmann and Mandel; see [18, 43].

In this chapter, we introduce two-level overlapping Schwarz methods and BDDC methods. We also consider the abstract Schwarz analysis, which is very helpful in developing the domain decomposition theory.

4.2 Overlapping Schwarz Methods

We consider a two-level overlapping Schwarz algorithm to solve the linear system $Au = f$. The additive overlapping Schwarz preconditioner usually has the following form:

$$P^{-1} = R_0^T A_0^{-1} R_0 + \sum_{i=1}^{N} R_i^T A_i^{-1} R_i, \tag{4.1}$$

where A_0 is the matrix of the global coarse problem, the A_i's are obtained from local subproblems related to the subdomains, and R_0 and R_i's are restriction operators to the coarse and local spaces, respectively; see [61, 67] for more details.

We introduce certain assumptions for our overlapping Schwarz method.

Assumption 4.2.1. *For $i = 1, \cdots, N$, there exists $\delta_i > 0$, such that, if x belongs to Ω_i', then*

$$\mathrm{dist}(x, \partial\Omega_j' \backslash \partial\Omega) \geqslant \delta_i,$$

for a suitable $j = j(x)$, possibly equal to i, with $x \in \Omega_j'$.

Assumption 4.2.2. (Finite Covering) *The partition $\{\Omega_i'\}$ can be colored using a finite number of N^c colors, in such a way that subregions with the same color are disjoint.*

4.3 Abstract Schwarz Analysis

We introduce the abstract Schwarz framework in this section. It is frequently used for analyzing domain decomposition methods.

Let V be a finite dimensional space. We consider the following symmetric, positive definite bilinear form:

$$\mathbf{a}(\cdot,\cdot) : V \times V \to \mathbb{R} \tag{4.2}$$

and the following variational problem:

Find $\mathbf{u} \in V$ such that

$$\mathbf{a}(\mathbf{u},\mathbf{v}) = \mathbf{f}(\mathbf{v}),\ \forall \mathbf{v} \in V, \tag{4.3}$$

where $\mathbf{f}$ is a bounded linear functional on V. We next consider auxiliary spaces $V_0, V_1, \cdots, V_N$ and extension operators

$$R_i^T : V_i \to V,\ i = 0, \cdots, N. \tag{4.4}$$

We can then consider the following decomposition:

$$V = R_0^T V_0 + \sum_{i=0}^{N} R_i^T V_i. \tag{4.5}$$

The space V_0 is associated with a coarse problem and each V_i is related to an extended overlapping subdomain Ω_i'. We note that the decomposition (4.5) is not necessarily a direct sum of the spaces. We define the Schwarz operators P_i by

$$P_i := R_i^T A_i^{-1} R_i A,\ i = 0, \cdots, N \tag{4.6}$$

and the additive Schwarz operator by

$$P_{ad} := \sum_{i=0}^{N} P_i = P^{-1} A. \tag{4.7}$$

We then have the following lemmas for the additive Schwarz operator.

Lemma 4.3.1. *If for all* $\mathbf{u} \in V$ *a representation,* $\mathbf{u} = \sum_{i=0}^{N} \mathbf{u}_i$*, can be found, such that*

$$\sum_{i=0}^{N} \mathbf{a}(\mathbf{u}_i, \mathbf{u}_i) \leqslant C_0^2 \mathbf{a}(\mathbf{u}, \mathbf{u}), \tag{4.8}$$

then

$$\mathbf{a}(P_{ad}\mathbf{u}, \mathbf{u}) \geqslant C_0^{-2} \mathbf{a}(\mathbf{u}, \mathbf{u}), \tag{4.9}$$

for all $\mathbf{u} \in V$

Lemma 4.3.2. *The largest eigenvalue of the additive operator* P_{ad} *is bounded from above by* $(N_c + 1)$*, where* N_c *is defined in Assumption 4.2.2.*

By Lemma 4.3.1 and 4.3.2, the condition number of the additive Schwarz operator is bounded by $C_0^2(N_c + 1)$. For more detail, see [67, Chapter 2].

4.4 Some Useful Operators

Let $W^{(i)}$ be the space of the lowest order Raviart-Thomas finite elements on Ω_i with a zero normal component on $\partial\Omega \cap \partial\Omega_i$. We decompose $W^{(i)}$ into two parts, the interior part and the interface part and denote the corresponding spaces by $W_I^{(i)}$ and $W_\Gamma^{(i)}$, respectively. Moreover, the interface part $W_\Gamma^{(i)}$ is decomposed into a primal space $W_\Pi^{(i)}$ and a dual space $W_\Delta^{(i)}$. Hence, we obtain the following decomposition:

$$W^{(i)} = W_I^{(i)} \oplus W_\Gamma^{(i)} = W_I^{(i)} \oplus W_\Delta^{(i)} \oplus W_\Pi^{(i)}.$$

Furthermore, we use the following product spaces:

$$W := \prod_{i=1}^{N} W^{(i)}, \ W_I := \prod_{i=1}^{N} W_I^{(i)},$$

$$W_\Gamma := \prod_{i=1}^{N} W_\Gamma^{(i)}, \ W_\Delta := \prod_{i=1}^{N} W_\Delta^{(i)},$$

and

$$W_\Pi := \prod_{i=1}^{N} W_\Pi^{(i)}.$$

We then have

$$W = W_I \oplus W_\Gamma = W_I \oplus W_\Delta \oplus W_\Pi.$$

In general, the functions in W_Γ have discontinuous normal components across the interface while those of the finite element solutions are continuous. We denote the space with continuous normal components by $\widehat{W}_\Gamma$. We next consider a space $\widetilde{W}_\Gamma$ as well. The functions in $\widetilde{W}_\Gamma$ satisfy the primal constraints. By using the above definitions, we can decompose $\widehat{W}_\Gamma$ and $\widetilde{W}_\Gamma$ into $\widehat{W}_\Delta \oplus \widehat{W}_\Pi$ and $W_\Delta \oplus \widehat{W}_\Pi$, respectively, where $\widehat{W}_\Delta$ is the continuous dual variable space and $\widehat{W}_\Pi$ is the continuous primal variable space. We obtain the local stiffness matrix $A^{(i)}$ by restricting to the finite element space $W^{(i)}$:

$$A^{(i)} \begin{bmatrix} u_I^{(i)} \\ u_\Delta^{(i)} \\ u_\Pi^{(i)} \end{bmatrix} = \begin{bmatrix} A_{II}^{(i)} & A_{I\Delta}^{(i)} & A_{I\Pi}^{(i)} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} & A_{\Delta\Pi}^{(i)} \\ A_{\Pi I}^{(i)} & A_{\Pi\Delta}^{(i)} & A_{\Pi\Pi}^{(i)} \end{bmatrix} \begin{bmatrix} u_I^{(i)} \\ u_\Delta^{(i)} \\ u_\Pi^{(i)} \end{bmatrix} = \begin{bmatrix} f_I^{(i)} \\ f_\Delta^{(i)} \\ f_\Pi^{(i)} \end{bmatrix}. \tag{4.10}$$

We obtain the global problem by assembling the local subdomain problems:

$$A \begin{bmatrix} u_I \\ u_\Delta \\ u_\Pi \end{bmatrix} = \begin{bmatrix} A_{II} & A_{I\Delta} & A_{I\Pi} \\ A_{\Delta I} & A_{\Delta\Delta} & A_{\Delta\Pi} \\ A_{\Pi I} & A_{\Pi\Delta} & A_{\Pi\Pi} \end{bmatrix} \begin{bmatrix} u_I \\ u_\Delta \\ u_\Pi \end{bmatrix} = \begin{bmatrix} f_I \\ f_\Delta \\ f_\Pi \end{bmatrix},$$

where $u_I \in W_I$, $u_\Delta \in \widehat{W}_\Delta$, and $u_\Pi \in \widehat{W}_\Pi$.

We now define several operators which perform restrictions, extensions, scalings, and averages between different spaces. We first consider the restriction operators. $R_\Gamma^{(i)}$ maps the space $\widehat{W}_\Gamma$ to the subdomain subspace $W_\Gamma^{(i)}$. Similarly, we can define $\overline{R}_\Gamma^{(i)} : \widetilde{W}_\Gamma \to W_\Gamma^{(i)}$. Moreover, $R_\Delta^{(i)} : W_\Delta \to W_\Delta^{(i)}$ and $R_\Pi^{(i)} : \widehat{W}_\Pi \to W_\Pi^{(i)}$ map global interface vectors on Γ to their components on $\Gamma_i := \partial\Omega_i \cap \Gamma$. $R_{\Gamma\Delta}$ and $R_{\Gamma\Pi}$ are the restriction operators from the intermediate space $\widetilde{W}_\Gamma$ to W_Δ and $\widehat{W}_\Pi$, respectively. R_Γ and $\overline{R}_\Gamma$ are the direct sums of the $R_\Gamma^{(i)}$ and $\overline{R}_\Gamma^{(i)}$, respectively. Furthermore, $\widetilde{R}_\Gamma : \widehat{W}_\Gamma \to \widetilde{W}_\Gamma$ is the direct sum of $R_{\Gamma\Pi}$ and $R_\Delta^{(i)} R_{\Gamma\Delta}$

We now define the scaling operators. A scaling factor $\delta_i^\dagger(x)$ is defined by

$$\delta_i^\dagger(x) := \frac{\chi_i^\gamma(x)}{\sum_{j\in\mathcal{N}_x} \chi_j^\gamma(x)},\ x \in \Gamma_h \cap \partial\Omega_{i,h}, \tag{4.11}$$

for some $\gamma \in [\frac{1}{2}, \infty)$, where $\mathcal{N}_x$ is the set of indices j of the subdomains such that $x \in \partial\Omega_{j,h}$ and $\chi_j(x)$ is a function of the coefficients $\alpha(x)$ and $\beta(x)$ of (3.4) at $x \in \Omega_j$. We can easily check that $\sum \delta_i^\dagger \equiv 1$.

We note that there is only one non-zero element in each row of $R_\Gamma^{(i)}$ and $R_\Delta^{(i)}$ associated with a coarse edge or face. We then define the scaling operators $R_{D,\Gamma}^{(i)}$ and $R_{D,\Delta}^{(i)}$ by multiplying each row of $R_\Gamma^{(i)}$ and $R_\Delta^{(i)}$ by the scaling factor $\delta_i^\dagger(x)$. We define similarly $R_{D,\Gamma}$, $R_{D,\Delta}$, and $\widetilde{R}_{D,\Gamma}$. $R_{D,\Gamma}$ and $R_{D,\Delta}$ that are the direct sums of $R^{(i)}_{D,\Gamma}$ and $R^{(i)}_{D,\Delta}$, respectively. $\widetilde{R}_{D,\Gamma}$ is the direct sum of $R_{\Gamma\Pi}$ and $R_{D,\Gamma}R_{\Gamma\Delta}$. We note that

$$R^T_\Gamma R_{D,\Gamma} = R^T_{D,\Gamma} R_\Gamma = I \tag{4.12}$$

and

$$\widetilde{R}^T_\Gamma \widetilde{R}_{D,\Gamma} = \widetilde{R}^T_{D,\Gamma} \widetilde{R}_\Gamma = I. \tag{4.13}$$

We consider the matrix B_Δ, which is defined in terms of subdomain operators $B^{(i)}_\Delta$:

$$B_\Delta = \left[B^{(1)}_\Delta, B^{(2)}_\Delta, \cdots, B^{(N)}_\Delta \right]. \tag{4.14}$$

Each $B^{(i)}_\Delta$ is constructed from $\{-1, 0, 1\}$ and B_Δ expresses the following continuity constraints across the interface:

$$B_\Delta u_\Delta = \sum_{i=1}^{N} B^{(i)}_\Delta u^{(i)}_\Delta = 0. \tag{4.15}$$

We define $B^{(i)}_{D,\Delta}$ in the following way: each row of B_Δ with a non-zero entry corresponds to a point on $\Gamma_i \cap \Gamma_j$. We obtain $B^{(i)}_{D,\Delta}$ by multiplying such a row of B_Δ by the scaling factor $\delta^\dagger_j(x)$ in (4.11). Furthermore, B_Γ is defined by $B_\Gamma := B_\Delta R_{\Gamma\Delta}$ and let $B_{D,\Delta} R_{\Gamma\Delta}$ be denoted by $B_{D,\Gamma}$.

Finally, we introduce an average operator $E_D : \widetilde{W}_\Gamma \to \widehat{W}_\Gamma$ defined by

$$E_D := \widetilde{R}_\Gamma \widetilde{R}^T_{D,\Gamma}. \tag{4.16}$$

This operator provides a weighted average across the interface Γ. Moreover, we define a jump operator as follows:

$$P_D := B_{D,\Gamma}^T B_\Gamma. \tag{4.17}$$

We have the following properties of the average and jump operators.

Lemma 4.4.1. *We have the following identities:*

$$E_D + P_D = I, E_D^2 = E_D, P_D^2 = P_D, \text{ and } E_D P_D = P_D E_D = 0.$$

Proof. See [40, Lemma 1]. ☐

Lemma 4.4.2. *For all $u_\Gamma \in \widehat{W}_\Gamma$,*

$$E_D u_\Gamma = u_\Gamma. \tag{4.18}$$

4.5 Schur Complements and Discrete Harmonic Extensions

4.5.1 Reduced Interface Problem

We consider the following local stiffness matrix:

$$A^{(i)} = \begin{bmatrix} A_{II}^{(i)} & A_{I\Gamma}^{(i)} \\ A_{\Gamma I}^{(i)} & A_{\Gamma\Gamma}^{(i)} \end{bmatrix}, \ i = 1, \cdots, N. \tag{4.19}$$

We first eliminate all interior unknowns locally by using direct solvers. After this step, we obtain the local Schur complement:

$$S^{(i)} = A^{(i)}_{\Gamma\Gamma} - A^{(i)}_{\Gamma I} A^{(i)-1}_{II} A^{(i)}_{I\Gamma}. \tag{4.20}$$

Moreover, the global Schur complement S is given by the direct sum of $S^{(i)}$, i.e.,

$$S := \begin{bmatrix} S^{(1)} & & & \\ & S^{(2)} & & \\ & & \ddots & \\ & & & S^{(N)} \end{bmatrix}. \tag{4.21}$$

By using the local Schur complements, we can build a reduced global interface problem given by

$$\widehat{S}_{\Gamma} u_{\Gamma} = g_{\Gamma}, \tag{4.22}$$

where

$$\widehat{S}_{\Gamma} = \sum_{i=1}^{N} R^{(i)T}_{\Gamma} S^{(i)} R^{(i)}_{\Gamma}$$

and

$$g_{\Gamma} = \sum_{i=1}^{N} R^{(i)T}_{\Gamma} \left\{ \begin{bmatrix} f_{\Delta} \\ f_{\Pi} \end{bmatrix} - \begin{bmatrix} A^{(i)}_{\Delta I} \\ A^{(i)}_{\Pi I} \end{bmatrix} A^{(i)-1}_{II} f^{(i)}_{I} \right\}.$$

We note that once $u^{(i)}_{\Gamma}$ is available, we can compute the interior values $u^{(i)}_{I}$ by solving the following local equation:

$$A^{(i)}_{II} u^{(i)}_{I} = f^{(i)}_{I} - A^{(i)}_{I\Gamma} u^{(i)}_{\Gamma}.$$

We will consider a preconditioner to solve the interface problem (4.22).

4.5.2 Discrete Harmonic Extensions

The space of discrete harmonic extensions, which are directly related to the Schur complements, is an essential subspace for domain decomposition methods, especially for iterative substructuring methods. Let us consider the following local linear system:

$$A_{II}^{(i)} u_I^{(i)} + A_{I\Gamma}^{(i)} u_\Gamma^{(i)} = 0. \tag{4.23}$$

If $u_\Gamma^{(i)}$ is given, $u_I^{(i)}$ is completely determined by $u_\Gamma^{(i)}$. We call $u^{(i)}$ the discrete harmonic function on Ω_i with the given interface value $u_\Gamma^{(i)}$ and write $u^{(i)}$ as $u^{(i)} := \mathcal{H}_i(u_\Gamma^{(i)})$. We also denote the piecewise discrete harmonic extension of u_Γ by $\mathcal{H}(u_\Gamma)$.

We have the following minimal property of discrete harmonic functions.

Lemma 4.5.1. *Let $u_\Gamma^{(i)}$ be the restriction of a finite element function to $\partial\Omega_i \cap \Gamma$. Then, the discrete harmonic extension $u^{(i)} = \mathcal{H}_i(u_\Gamma^{(i)})$ of $u_\Gamma^{(i)}$ into Ω_i satisfies*

$$u^{(i)^T} A^{(i)} u^{(i)} = \min_{v^{(i)}|_{\partial\Omega_i \cap \Gamma} = u_\Gamma^{(i)}} v^{(i)^T} A^{(i)} v^{(i)} \tag{4.24}$$

and

$$u_\Gamma^{(i)^T} S^{(i)} u_\Gamma^{(i)} = u^{(i)^T} A^{(i)} u^{(i)}. \tag{4.25}$$

Similarly, if u_Γ is the restriction of a finite element function to Γ, the piecewise discrete harmonic extension $u = \mathcal{H}(u_\Gamma)$ of u_Γ into the interior of the subdomains satisfies

$$u^T A u = \min_{v|_\Gamma = u_\Gamma} v^T A v \tag{4.26}$$

and

$$u_\Gamma^T S u_\Gamma = u^T A u. \tag{4.27}$$

We note that we can work with the discrete harmonic extensions instead of functions defined on the interface Γ.

4.6 BDDC Methods

4.6.1 The Algorithm

We follow the description of the algorithm as introduced in [40, Section 4]. We first present a change of variables to express the primal constraints. After this process, we will have common edge or face averages across the interface, i.e., these averages will serve as primal variables.

Consider the unknowns corresponding to the degrees of freedom on an edge or a face F_{ij} and denote them by $u_{F_{ij}}^1, u_{F_{ij}}^2, \cdots, u_{F_{ij}}^m, \cdots, u_{F_{ij}}^l$. We note that $u^{(i)}$ is written in the following form:

$$u^{(i)} = \left[{u_I^{(i)}}^T, {u_{\overline{\Gamma}}^{(i)}}^T, u_{F_{ij}}^1, u_{F_{ij}}^2, \cdots, u_{F_{ij}}^m, \cdots, u_{F_{ij}}^l \right]^T$$

with $\overline{\Gamma} = (\Gamma \cap \partial\Omega_i) \setminus F_{ij}$. We then consider the local linear systems:

$$A^{(i)}u^{(i)} = \begin{bmatrix} A^{(i)}_{II} & A^{(i)}_{I\overline{\Gamma}} & A^{(i)}_{I1} & \cdots & A^{(i)}_{Im} & \cdots & A^{(i)}_{Il} \\ A^{(i)}_{\overline{\Gamma}I} & A^{(i)}_{\overline{\Gamma}\overline{\Gamma}} & A^{(i)}_{\overline{\Gamma}1} & \cdots & A^{(i)}_{\overline{\Gamma}m} & \cdots & A^{(i)}_{\overline{\Gamma}l} \\ A^{(i)}_{1I} & A^{(i)}_{1\overline{\Gamma}} & A^{(i)}_{11} & \cdots & A^{(i)}_{1m} & \cdots & A^{(i)}_{1l} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ A^{(i)}_{mI} & A^{(i)}_{m\overline{\Gamma}} & A^{(i)}_{m1} & \cdots & A^{(i)}_{mm} & \cdots & A^{(i)}_{ml} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ A^{(i)}_{lI} & A^{(i)}_{l\overline{\Gamma}} & A^{(i)}_{l1} & \cdots & A^{(i)}_{lm} & \cdots & A^{(i)}_{ll} \end{bmatrix} \begin{bmatrix} u^{(i)}_I \\ u^{(i)}_{\overline{\Gamma}} \\ u^1_{F_{ij}} \\ \vdots \\ u^m_{F_{ij}} \\ \vdots \\ u^l_{F_{ij}} \end{bmatrix} = \begin{bmatrix} f^{(i)}_I \\ f^{(i)}_{\overline{\Gamma}} \\ f^1_{F_{ij}} \\ \vdots \\ f^m_{F_{ij}} \\ \vdots \\ f^l_{F_{ij}} \end{bmatrix}.$$

We define $T^{(i)}_{F_{ij}}$ by:

$$T^{(i)}_{F_{ij}} := \begin{bmatrix} 1 & & 1 & & \\ & \ddots & \vdots & & \\ -1 & \cdots & 1 & \cdots & -1 \\ & & \vdots & \ddots & \\ & & 1 & & 1 \end{bmatrix}.$$

We now perform a change of variables:

$$
\begin{bmatrix} u^1_{F_{ij}} \\ \vdots \\ u^m_{F_{ij}} \\ \vdots \\ u^l_{F_{ij}} \end{bmatrix} = T^{(i)}_{F_{ij}} \begin{bmatrix} \hat{u}^1_{F_{ij}} \\ \vdots \\ \hat{u}^m_{F_{ij}} \\ \vdots \\ \hat{u}^l_{F_{ij}} \end{bmatrix} = \begin{bmatrix} 1 & & 1 & & \\ & \ddots & \vdots & & \\ -1 & \cdots & 1 & \cdots & -1 \\ & & \vdots & \ddots & \\ & & 1 & & 1 \end{bmatrix} \begin{bmatrix} \hat{u}^1_{F_{ij}} \\ \vdots \\ \hat{u}^m_{F_{ij}} \\ \vdots \\ \hat{u}^l_{F_{ij}} \end{bmatrix}
$$

$$
= \begin{bmatrix} 1 \\ \vdots \\ 1 \\ \vdots \\ 1 \end{bmatrix} \hat{u}^m_{F_{ij}} + \begin{bmatrix} \hat{u}^1_{F_{ij}} \\ \vdots \\ -\hat{u}^1_{F_{ij}} - \cdots - \hat{u}^{m-1}_{F_{ij}} - \hat{u}^{m+1}_{F_{ij}} - \cdots - \hat{u}^l_{F_{ij}} \\ \vdots \\ \hat{u}^l_{F_{ij}} \end{bmatrix}.
$$

The transformed stiffness matrix is given by

$$
T^{(i)^T} \begin{bmatrix} A^{(i)}_{II} & A^{(i)}_{I\Gamma} \\ A^{(i)}_{\Gamma I} & A^{(i)}_{\Gamma\Gamma} \end{bmatrix} T^{(i)}, \tag{4.28}
$$

where $T^{(i)}$ is a block diagonal matrix of the following form:

$$
T^{(i)} = \begin{bmatrix} I & \\ & T^{(i)}_{\Gamma} \end{bmatrix}.
$$

We note that $T^{(i)}_{\Gamma}$ is a direct sum of matrices and each block matrix consists of the matrix $T^{(i)}_{F_{ij}}$ associated with the edge or face F_{ij}.

We also note that this change of variables is a local procedure which can be performed edge by edge or face by face. From now on, we assume that the variables have been changed. We consider the partially assembled stiffness matrix $\widetilde{A}$ given by

$$\widetilde{A} = \begin{bmatrix} A_{II}^{(1)} & A_{I\Delta}^{(1)} & & & & A_{I\Pi}^{(1)} R_{\Pi}^{(1)} \\ A_{\Delta I}^{(1)} & A_{\Delta\Delta}^{(1)} & & & & A_{\Delta\Pi}^{(1)} R_{\Pi}^{(1)} \\ & & \ddots & & & \vdots \\ & & & A_{II}^{(N)} & A_{I\Delta}^{(N)} & A_{I\Pi}^{(N)} R_{\Pi}^{(N)} \\ & & & A_{\Delta I}^{(N)} & A_{\Delta\Delta}^{(N)} & A_{\Delta\Pi}^{(1)} R_{\Pi}^{(1)} \\ R_{\Pi}^{(1)T} A_{\Pi I}^{(1)} & R_{\Pi}^{(1)T} A_{\Pi\Delta}^{(1)} & \cdots & R_{\Pi}^{(N)T} A_{\Pi I}^{(N)} & R_{\Pi}^{(N)T} A_{\Pi\Delta}^{(N)} & \sum_{i=1}^{N} R_{\Pi}^{(i)T} A_{\Pi\Pi}^{(i)} R_{\Pi}^{(i)} \end{bmatrix}.$$

We can now define a different Schur complement $\widetilde{S}_{\Gamma}$. By eliminating the interior residuals, we obtain the following linear system which defines $\widetilde{S}_{\Gamma}$:

$$\widetilde{A} \begin{bmatrix} u_I^{(1)} \\ u_{\Delta}^{(1)} \\ \vdots \\ u_I^{(N)} \\ u_{\Delta}^{(N)} \\ u_{\Pi} \end{bmatrix} = \begin{bmatrix} 0 \\ R_{\Delta}^{(1)} R_{\Gamma\Delta} \widetilde{S}_{\Gamma} u_{\Gamma} \\ \vdots \\ 0 \\ R_{\Delta}^{(N)} R_{\Gamma\Delta} \widetilde{S}_{\Gamma} u_{\Gamma} \\ R_{\Gamma\Pi} \widetilde{S}_{\Gamma} u_{\Gamma} \end{bmatrix}.$$

We note that $\widetilde{S}_{\Gamma}$ is a partially assembled Schur complement. Hence, we need to further assemble it to obtain the fully assembled Schur complement $\widehat{S}_{\Gamma}$. By using restriction and extension operators, we find that $\widehat{S}_{\Gamma} = \widetilde{R}_{\Gamma}^T \widetilde{S}_{\Gamma} \widetilde{R}_{\Gamma}$. We can then rewrite the interface problem (4.22) as follows:

$$\widetilde{R}_{\Gamma}^T \widetilde{S}_{\Gamma} \widetilde{R}_{\Gamma} u_{\Gamma} = g_{\Gamma}.$$

The BDDC preconditioner has the following form:

$$M^{-1} = \widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma},$$

where

$$\widetilde{S}_\Gamma^{-1} := R_{\Gamma\Delta}^T \left(\sum_{i=1}^N \begin{bmatrix} 0 & R_\Delta^{(i)T} \end{bmatrix} \begin{bmatrix} A_{II}^{(i)} & A_{I\Delta}^{(i)} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ R_\Delta^{(i)} \end{bmatrix} \right) R_{\Gamma\Delta} + \Phi S_{\Pi\Pi}^{-1} \Phi^T \tag{4.29}$$

with

$$\Phi := R_{\Gamma\Pi}^T - R_{\Gamma\Delta}^T \sum_{i=1}^N \begin{bmatrix} 0 & R_\Delta^{(i)T} \end{bmatrix} \begin{bmatrix} A_{II}^{(i)} & A_{I\Delta}^{(i)} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{bmatrix}^{-1} \begin{bmatrix} A_{\Pi I}^{(i)T} \\ A_{\Pi\Delta}^{(i)T} \end{bmatrix} R_\Pi^{(i)}$$

and

$$S_{\Pi\Pi} := \sum_{i=1}^N R_\Pi^{(i)T} \left(A_{\Pi\Pi}^{(i)} - \begin{bmatrix} A_{\Pi I}^{(i)} & A_{\Pi\Delta}^{(i)} \end{bmatrix} \begin{bmatrix} A_{II}^{(i)} & A_{I\Delta}^{(i)} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{bmatrix}^{-1} \begin{bmatrix} A_{\Pi I}^{(i)T} \\ A_{\Pi\Delta}^{(i)T} \end{bmatrix} \right) R_\Pi^{(i)}.$$

The first term of (4.29) is related to the local problems and the second to the coarse-level problem related to the primal constraints Π. We obtain the following preconditioned linear system:

$$M^{-1}\widehat{S}_\Gamma u_\Gamma = \widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} \widetilde{R}_\Gamma^T \widetilde{S}_\Gamma \widetilde{R}_\Gamma u_\Gamma = \widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} g_\Gamma = M^{-1} g_\Gamma \tag{4.30}$$

We use the preconditioned conjugate gradient method to solve the linear system (4.30).

4.6.2 Convergence Analysis

We now derive an upper and a lower bound of the eigenvalues of the preconditioned linear system (4.30)

Lemma 4.6.1. *Let* $M^{-1} = \widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma}$. *Then,* $\forall u_\Gamma \in \widehat{W}_\Gamma$, *we have the following estimate:*

$$u_\Gamma^T M u_\Gamma \leqslant u_\Gamma^T \widehat{S}_\Gamma u_\Gamma$$

Proof. We will follow [44, Theorem 25]; see also [39, Theorem 1] and [69, Theorem 6.1]. Let $w_\Gamma := M u_\Gamma = (\widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma})^{-1} u_\Gamma$. We note that $w_\Gamma \in \widehat{W}_\Gamma$. We use the property (4.13) and a generalized Cauchy-Schwarz inequality to obtain

$$\begin{aligned}
u_\Gamma^T M u_\Gamma &= u_\Gamma^T (\widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma})^{-1} u_\Gamma = u_\Gamma^T w_\Gamma && (4.31)\\
&= u_\Gamma^T \widetilde{R}_\Gamma^T \widetilde{S}_\Gamma \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} w_\Gamma = \left\langle \widetilde{R}_\Gamma u_\Gamma, \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} w_\Gamma \right\rangle_{\widetilde{S}_\Gamma} && (4.32)\\
&\leqslant \left\langle \widetilde{R}_\Gamma u_\Gamma, \widetilde{R}_\Gamma u_\Gamma \right\rangle_{\widetilde{S}_\Gamma}^{\frac{1}{2}} \left\langle \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} w_\Gamma, \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} w_\Gamma \right\rangle_{\widetilde{S}_\Gamma}^{\frac{1}{2}} && (4.33)\\
&= \left(u_\Gamma^T \widetilde{R}_\Gamma^T \widetilde{S}_\Gamma \widetilde{R}_\Gamma u_\Gamma \right)^{\frac{1}{2}} \left(w_\Gamma^T \widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{S}_\Gamma \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} w_\Gamma \right)^{\frac{1}{2}}\\
&= \left(u_\Gamma^T \widehat{S}_\Gamma u_\Gamma \right)^{\frac{1}{2}} \left(u_\Gamma^T M u_\Gamma \right)^{\frac{1}{2}}.
\end{aligned}$$

Hence, we obtain

$$u_\Gamma^T M u_\Gamma \leqslant u_\Gamma^T \widehat{S}_\Gamma u_\Gamma.$$

□

We now obtain an upper bound of the eigenvalues of the generalized eigenvalue problem $\widehat{S} u_\Gamma = \lambda M u_\Gamma$.

Lemma 4.6.2. *Let* $M^{-1} = \widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma}$. *We also assume that* $\|E_D u_\Gamma\|_{\widetilde{S}_\Gamma}^2 \leqslant$

$\omega^2\|u_\Gamma\|^2_{\widetilde{S}_\Gamma}$, for all $u_\Gamma \in \widehat{W}_\Gamma$. Then, we have the following estimate:

$$u_\Gamma^T \widehat{S}_\Gamma u_\Gamma \leqslant \omega^2 u_\Gamma^T M u_\Gamma.$$

Proof. We will also follow the idea in [44, Theorem 25]; see also [39, Theorem 1] and [69, Theorem 6.1]. Let $w_\Gamma := M u_\Gamma = (\widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma})^{-1} u_\Gamma$. We note that $w_\Gamma \in \widehat{W}_\Gamma$. By the property (4.13) and a generalized Cauchy-Schwarz inequality, we have

$$\begin{aligned}
u_\Gamma^T \widehat{S}_\Gamma u_\Gamma &= u_\Gamma^T \widetilde{R}_\Gamma^T \widetilde{S}_\Gamma \widetilde{R}_\Gamma u_\Gamma = u_\Gamma^T \widetilde{R}_\Gamma^T \widetilde{S}_\Gamma \widetilde{R}_\Gamma \widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} w_\Gamma \\
&= \left\langle \widetilde{R}_\Gamma u_\Gamma, E_D \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} w_\Gamma \right\rangle_{\widetilde{S}_\Gamma} \\
&\leqslant \left\langle \widetilde{R}_\Gamma u_\Gamma, \widetilde{R}_\Gamma u_\Gamma \right\rangle_{\widetilde{S}_\Gamma}^{\frac{1}{2}} \left\langle E_D \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} w_\Gamma, E_D \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} w_\Gamma \right\rangle_{\widetilde{S}_\Gamma}^{\frac{1}{2}} \\
&\leqslant C \left\langle \widetilde{R}_\Gamma u_\Gamma, \widetilde{R}_\Gamma u_\Gamma \right\rangle_{\widetilde{S}_\Gamma}^{\frac{1}{2}} \omega \left\langle \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} w_\Gamma, \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} w_\Gamma \right\rangle_{\widetilde{S}_\Gamma}^{\frac{1}{2}} \\
&= \omega \left(u_\Gamma^T \widetilde{R}_\Gamma^T \widetilde{S}_\Gamma \widetilde{R}_\Gamma u_\Gamma \right)^{\frac{1}{2}} \left(w_\Gamma^T \widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{S}_\Gamma \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} w_\Gamma \right)^{\frac{1}{2}} \\
&= \omega \left(u_\Gamma^T \widehat{S}_\Gamma u_\Gamma \right)^{\frac{1}{2}} \left(u_\Gamma^T M u_\Gamma \right)^{\frac{1}{2}}
\end{aligned}$$

Therefore, we obtain

$$u_\Gamma^T \widehat{S}_\Gamma u_\Gamma \leqslant \omega^2 u_\Gamma^T M u_\Gamma.$$

$\square$

Chapter 5

An Overlapping Schwarz Algorithm for Raviart-Thomas Vector Fields

5.1 Introduction

The purpose of this chapter is to develop an overlapping Schwarz method for the model problem (3.1) discretized by the Raviart-Thomas elements. As we mentioned earlier in this thesis, many iterative substructuring methods have been suggested for discontinuous coefficient cases. However, there has been little supporting theory for the overlapping Schwarz methods in case of coefficients which have jumps. In order to deal with this difficulty, we use the coarse space techniques of [19,20] developed for almost incompressible elasticity. Similar alternative coarse space methods have also been developed for Reissner-Mindlin plates problems; see [38].

The rest of this chapter is organized as follows. We describe the algorithm in Section 5.2. In Section 5.3, we introduce some useful technical tools and present our main result. Finally, Section 5.4 contains supporting numerical experiments.

5.2 Overlapping Schwarz Algorithm

5.2.1 The Traditional Coarse Component

We focus on the restriction operator R_0 onto the coarse space. The restriction operator is obtained by the interpolation from the subspaces defining the coarse component to the global space. More precisely, R_0 are exactly the coefficients obtained by interpolating the traditional coarse basis functions onto the fine mesh.

5.2.2 An Alternative Coarse Component

Instead of the conventional coarse basis, we will use energy-minimal, discrete harmonic extensions to define the new coarse basis functions. The coarse part of the preconditioner is of the form $R_0^T A_0^{-1} R_0$ and we need to redefine R_0 and A_0. For each face (or edge) F_{ij}, a subset of the interface Γ, we can define a submatrix of the stiffness matrix A. It corresponds to the two subdomains which have F_{ij} in common:

$$\begin{bmatrix} A_{II}^{(i)} & 0 & A_{IF_{ij}}^{(i)} \\ 0 & A_{II}^{(j)} & A_{IF_{ij}}^{(j)} \\ A_{F_{ij}I}^{(i)} & A_{F_{ij}I}^{(j)} & A_{F_{ij}F_{ij}} \end{bmatrix}.$$

Let $\widetilde{u}_{ij} := [u_I^{(i)T} \; u_I^{(j)T} \; u_{F_{ij}}^T]^T$, where $\widetilde{u}_{ij}$ is the discrete harmonic extension, i.e., $A_{II}^{(i)} u_I^{(i)} + A_{IF_{ij}}^{(i)} u_{F_{ij}} = 0$ and $A_{II}^{(j)} u_I^{(j)} + A_{IF_{ij}}^{(j)} u_{F_{ij}} = 0$; cf. [67, Chapter 4.4]. We can write $\widetilde{u}_{ij}$ as $\widetilde{u}_{ij} = [(E_i u_{F_{ij}})^T \ (E_j u_{F_{ij}})^T \ u_{F_{ij}}^T]^T$ where $E_i := -{A_{II}^{(i)}}^{-1} A_{IF_{ij}}^{(i)}$ and $E_j := -{A_{II}^{(j)}}^{-1} A_{IF_{ij}}^{(j)}$. Also, let u_{ij} be the extension of $\widetilde{u}_{ij}$ to a global space obtained by an extension by zero. We note that the vector u_{ij} is completely determined by $u_{F_{ij}}$.

We choose $u_{F_{ij}}^T = [1, 1, \cdots, 1]$ to define the vector u_{ij} corresponding to a coarse basis function for the face (or edge) F_{ij}. We can now define A_0 and R_0, after introducing a suitable global indexing, by

$$(A_0)_{mn} := u_{ij}^T A u_{kl},$$

where F_{ij} and F_{kl} are the m-th and n-th face of Γ, respectively. Furthermore, let

$$R_0 := \begin{bmatrix} & \vdots & \\ - & u_{ij}^T & - \\ & \vdots & \end{bmatrix}.$$

5.2.3 Local Components

For the local components, each R_i is a rectangular matrix with elements equal to 0 or 1. Each R_i just provides the indices relevant to an individual extended subdomain Ω_i'. This means that each R_i extracts the degrees of freedom of Ω_i', the extended subregion obtained from Ω_i by adding layers of elements. We can then define a submatrix of the original stiffness matrix A by the following formula:

$$A_i = R_i A R_i^T.$$

Thus, A_i is just the principal minor of the original stiffness matrix A defined by R_i. By using these matrices, we can build the local part $\sum_{i=1}^{N} R_i^T A_i^{-1} R_i$ of the preconditioner.

5.2.4 The Additive Schwarz Operator

We now construct our preconditioner. Let $P_i = R_i^T A_i^{-1} R_i A$. The preconditioned linear operator has the following form:

$$P_{ad} = \sum_{i=0}^{N} P_i = \sum_{i=0}^{N} R_i^T A_i^{-1} R_i A.$$

When we apply the operator P_{ad} to a vector, the action of A_0^{-1} and A_i^{-1} can be performed by solving a global coarse problem and a local subproblem, respectively. By using a suitable indexing, we can perform most work of the preconditioned conjugate gradient method locally and in parallel except for the work of the coarse part and the communication between subdomains; see [61], [67, Chapter 3].

5.2.5 Remarks on the Implementation

We rewrite the preconditioned conjugate gradient method algorithm in Figure 5.1. We use (4.1) as a preconditioner. We note that all the local subproblem can be solved in parallel.

We remark that R_i does not appear in practical implementation. We can perform the computation R_i times a vector by using suitable indexing.

Initialize:

$r_0 := f - Au_0$

$z_0 := \left(R_0^T A_0^{-1} R_0 + \sum_{i=1}^N R_i^T A_i^{-1} R_i\right) r_0$

$p_0 := z_0$

Iterate $k = 0, 1, \cdots$ until convergence

$\alpha_k := r_k^T z_k / p_k^T A p_k$

$x_{k+1} := x_k + \alpha_k p_k$

$r_{k+1} := r_k - \alpha_k A p_k$

$z_{k+1} := \left(R_0^T A_0^{-1} R_0 + \sum_{i=1}^N R_i^T A_i^{-1} R_i\right) r_{k+1}$

$\beta_k := z_{k+1}^T r_{k+1} / z_k^T r_k$

$p_{k+1} := z_{k+1} + \beta_k p_k$

Figure 5.1: Implementation of the two-level overlapping Schwarz method as a preconditioned conjugate gradient method.

5.3 Technical Tools and the Main Result

5.3.1 Technical Tools

We will now consider the 3D case only; the arguments are quite similar for 2D. The condition $\mu \in F_{0;h}(\partial\Omega_i)$, which means that $\int_{\partial\Omega_i} \mu ds = 0$, is very important; cf. [19, 47, 68]. This means that it is important to find a suitable $\mathbf{v}$ which makes the flux of $\mathbf{u} - \mathbf{v}$ zero across $\partial\Omega_i$. To make this possible let us consider the coarse interpolation operator Π_H^{RT} onto the Raviart-Thomas space of the coarse mesh. For a given F, a coarse face contained in the interface Γ, we define the degree of freedom by

$$\lambda_F(\Pi_H^{RT}\mathbf{u}) := \frac{1}{|F|}\int_F \mathbf{u} \cdot \mathbf{n} ds.$$

Trivially,

$$\int_F (\mathbf{u} - \Pi_H^{RT}\mathbf{u}) \cdot \mathbf{n}\, ds = 0.$$

We will need some estimates for Π_H^{RT}.

Lemma 5.3.1. (Stability estimate for the coarse interpolation) *For all* $\mathbf{u} \in X_h$*, we have the following estimates:*

$$\|\mathrm{div}\,(\Pi_H^{RT}\mathbf{u})\|_{0;\Omega_i}^2 \leqslant \|\mathrm{div}\,\mathbf{u}\|_{0;\Omega_i}^2 \tag{5.1}$$

and

$$\|\Pi_H^{RT}\mathbf{u}\|_{0;\Omega_i}^2 \leqslant C((1+\log\frac{H_i}{h_i})\|\mathbf{u}\|_{0;\Omega_i}^2 + H_i^2\|\mathrm{div}\,\mathbf{u}\|_{0;\Omega_i}^2). \tag{5.2}$$

The constant C *depends only on the aspect ratios of the elements of* $\mathcal{T}_H$ *and the elements of* $\mathcal{T}_h$*.*

Proof. The first estimate (5.1) follows by the commuting property (2.3):

$$\mathrm{div}\,(\Pi_H^{RT}\mathbf{u}) = \Pi_H(\mathrm{div}\,\mathbf{u}),$$

where Π_H is the L^2-projection onto the space of piecewise constant on the coarse mesh; see Section 2.2.2 and [7, p.150 5.3].

For the second estimate (5.2), we use Green's identity and the face basis function; see [23], [67, Lemma 4.25]. We also use the fact that the L^2-norms of functions in the Raviart-Thomas finite element space can be bounded from above and below by a weighted l_2-norm of their degrees of freedom; see Lemma 2.2.1. For details, see [70, Lemma 2.4] and [71, Lemma 4.1]. □

Lemma 5.3.2. *Let* $\mathbf{u} \in N_h$*,* $\mathbf{v} \in X_h$*, and* θ_i *be a continuous, piecewise linear* *scalar function supported in* Ω_i*. Then,*

$$\|\Pi_h^{ND}(\theta_i \mathbf{u})\|_{0;\Omega_i}^2 \leqslant C\|\theta_i \mathbf{u}\|_{0;\Omega_i}^2,$$

$$\|\mathbf{curl}\,(\Pi_h^{ND}(\theta_i \mathbf{u}))\|_{0;\Omega_i}^2 \leqslant C\|\mathbf{curl}\,(\theta_i \mathbf{u})\|_{0;\Omega_i}^2,$$

$$\|\Pi_h^{RT}(\theta_i \mathbf{v})\|_{0;\Omega_i}^2 \leqslant C\|\theta_i \mathbf{v}\|_{0;\Omega_i}^2,$$

and

$$\|\mathrm{div}\,(\Pi_h^{RT}(\theta_i \mathbf{v}))\|_{0;\Omega_i}^2 \leqslant C\|\mathrm{div}\,(\theta_i \mathbf{v})\|_{0;\Omega_i}^2,$$

where Π_h^{ND} *and* Π_h^{RT} *are the interpolation operators onto the lowest order Nédélec finite element space and the lowest order Raviart-Thomas finite element space, respectively.*

Proof. We use error estimates of the operators Π_h^{ND} and Π_h^{RT} and inverse inequalities; see [7, Lemma 5.5]. For more details, see [64, Lemma 4.3] and [67, Lemma 10.8 and Lemma 10.13]. $\square$

Definition 5.3.1. (Projection Operators) *Let* $\Theta_{\mathbf{curl}}^{\perp}$ *and* $\Theta_{\mathrm{div}}^{\perp}$ *be the orthogonal projections from* $H(\mathbf{curl}\,;\Omega)$ *onto* $H^{\perp}(\mathbf{curl}\,;\Omega)$ *and from* $H(\mathrm{div}\,;\Omega)$ *onto* $H^{\perp}(\mathrm{div}\,;\Omega)$*, respectively. We next define a projection* P_h^{ND} *from* $H(\mathbf{curl}\,;\Omega)$ *onto* V_{ND}^{+} *and* P_h^{RT} *from* $H(\mathrm{div}\,;\Omega)$ *onto* V_{RT}^{+}*, with* $V_{ND}^{+} = \Theta_{\mathbf{curl}}^{\perp}(N_h^{\perp})$ *and* $V_{RT}^{+} = \Theta_{\mathrm{div}}^{\perp}(X_h^{\perp})$*.*

Remark 5.3.1. *We can easily check that* $\mathbf{curl}\,(P_h^{ND}\mathbf{u}^{\perp}) = \mathbf{curl}\,(\Theta_{\mathbf{curl}}^{\perp}\mathbf{u}^{\perp}) = \mathbf{curl}\,\mathbf{u}^{\perp}$ *and* $\mathrm{div}\,(P_h^{RT}\mathbf{v}^{\perp}) = \mathrm{div}\,(\Theta_{\mathrm{div}}^{\perp}\mathbf{v}^{\perp}) = \mathrm{div}\,\mathbf{v}^{\perp}$ *whenever* $\mathbf{u}^{\perp} \in N_h^{\perp}$ *and* $\mathbf{v}^{\perp} \in X_h^{\perp}$*.*

Lemma 5.3.3. *Let* Ω_i *be convex. Then, we have the following error estimates:*

$$\|\mathbf{u}_h^{\perp} - P_h^{ND}\mathbf{u}_h^{\perp}\|_{0;\Omega_i} \leqslant Ch_i\|\mathbf{curl}\,\mathbf{u}_h^{\perp}\|_{0;\Omega_i}, \forall \mathbf{u}_h^{\perp} \in N_h^{\perp}(\Omega_i)$$

and

$$\|\mathbf{v}_h^\perp - P_h^{RT}\mathbf{v}_h^\perp\|_{0;\Omega_i} \leqslant Ch_i\|\mathrm{div}\,\mathbf{v}_h^\perp\|_{0;\Omega_i},\ \forall\mathbf{v}_h^\perp \in X_h^\perp(\Omega_i),$$

with C independent of h_i, $\mathbf{u}_h^\perp$, and $\mathbf{v}_h^\perp$.

Proof. We can use almost the same idea as in [64, Lemma 3.3] and [34, Lemma 4.2, 4.3 and 4.4]. □

We recall that all subdomains are convex.

Lemma 5.3.4. *Let $\Omega_{i,\delta_i} \subset \Omega_i$ be the set of all points which are within a distance δ_i of the boundary of Ω_i. Then, there exists a constant C such that $\forall\mathbf{u}^\perp \in N_h^\perp$ and $\forall\mathbf{v}^\perp \in X_h^\perp$,*

$$\frac{1}{\delta_i^2}\|\mathbf{u}^\perp\|^2_{0;\Omega_i\cap\Omega_{i,\delta_i}} \leqslant C(1+\frac{H_i}{\delta_i})\|\mathbf{curl}\,\mathbf{u}^\perp\|^2_{0;\Omega_i}$$

and

$$\frac{1}{\delta_i^2}\|\mathbf{v}^\perp\|^2_{0;\Omega_i\cap\Omega_{i,\delta_i}} \leqslant C(1+\frac{H_i}{\delta_i})\|\mathrm{div}\,\mathbf{v}^\perp\|^2_{0;\Omega_i}.$$

Similarly, for a subdomain Ω_j which has a face in common with Ω_i, we have

$$\frac{1}{\delta_i^2}\|\mathbf{u}^\perp\|^2_{0;\Omega_i'\cap\Omega_j} \leqslant C(1+\frac{H_i}{\delta_i})\|\mathbf{curl}\,\mathbf{u}^\perp\|^2_{0;\Omega_j}$$

and

$$\frac{1}{\delta_i^2}\|\mathbf{v}^\perp\|^2_{0;\Omega_i'\cap\Omega_j} \leqslant C(1+\frac{H_i}{\delta_i})\|\mathrm{div}\,\mathbf{v}^\perp\|^2_{0;\Omega_j}.$$

Moreover, for $\forall m \in I_{jl}$, where I_{jl} is the set of indices of the subdomains which have an edge E_{jl} common with Ω_i and $\Psi_{jl} := \bigcap_{m\in I_{jl}}\Omega_m'$, we have

$$\frac{1}{\delta_i^2}\|\mathbf{u}^\perp\|^2_{0;\Psi_{jl}\cap\Omega_m} \leqslant C(1+\frac{H_i}{\delta_i})\|\mathbf{curl}\,\mathbf{u}^\perp\|^2_{0;\Omega_m},$$

and

$$\frac{1}{\delta_i^2}\|\mathbf{v}^\perp\|_{0;\Psi_{jl}\cap\Omega_m}^2 \leqslant C(1+\frac{H_i}{\delta_i})\|\mathrm{div}\,\mathbf{v}^\perp\|_{0;\Omega_m}^2.$$

Proof. By the triangle inequality,

$$\|\mathbf{u}^\perp\|_{0;\Omega_i\cap\Omega_{i,\delta_i}}^2 \leqslant 2(\|\mathbf{u}^\perp - P_h^{ND}\mathbf{u}^\perp\|_{0;\Omega_i\cap\Omega_{i,\delta_i}}^2 + \|P_h^{ND}\mathbf{u}^\perp\|_{0;\Omega_i\cap\Omega_{i,\delta_i}}^2). \tag{5.3}$$

Consider the first term. By Lemma 5.3.3,

$$\frac{1}{\delta_i^2}\|\mathbf{u}^\perp - P_h^{ND}\mathbf{u}^\perp\|_{0;\Omega_i\cap\Omega_{i,\delta_i}}^2 \leqslant \frac{1}{\delta_i^2}\|\mathbf{u}^\perp - P_h^{ND}\mathbf{u}^\perp\|_{0;\Omega_i}^2 \leqslant \frac{h_i^2}{\delta_i^2}\|\mathbf{curl}\,\mathbf{u}^\perp\|_{0;\Omega_i}^2.$$

By the fact that $\dfrac{h_i}{\delta_i}$ is bounded by 1, the first term of (5.3) is bounded by $\|\mathbf{curl}\,\mathbf{u}^\perp\|_{0;\Omega_i}^2$. For the second term, we will use an argument similar to that of [67, Lemma 3.10]. By a Friedrichs inequality, Lemma 2.1.8, and Remark 5.3.1, we have

$$\begin{aligned}
\frac{1}{\delta_i^2}\|P_h^{ND}\mathbf{u}^\perp\|_{0;\Omega_i\cap\Omega_{i,\delta_i}}^2 &\leqslant C(|P_h^{ND}\mathbf{u}^\perp|_{1;\Omega_i\cap\Omega_{i,\delta_i}}^2 + \frac{1}{\delta_i}\|P_h^{ND}\mathbf{u}^\perp\|_{0;\partial\Omega_i}^2) \\
&\leqslant C(|P_h^{ND}\mathbf{u}^\perp|_{1;\Omega_i}^2 + \frac{1}{\delta_i}\|P_h^{ND}\mathbf{u}^\perp\|_{0;\partial\Omega_i}^2) \\
&\leqslant C(\|\mathbf{curl}\,P_h^{ND}\mathbf{u}^\perp\|_{0;\Omega_i}^2 + \frac{1}{\delta_i}\|P_h^{ND}\mathbf{u}^\perp\|_{0;\partial\Omega_i}^2) \\
&= C(\|\mathbf{curl}\,\mathbf{u}^\perp\|_{0;\Omega_i}^2 + \frac{1}{\delta_i}\|P_h^{ND}\mathbf{u}^\perp\|_{0;\partial\Omega_i}^2).
\end{aligned}$$

By a trace estimate and by combining [67, Lemma A.6], the embedding $L^2(\partial\Omega_i) \subset$

$H^{\frac{1}{2}}(\partial\Omega_i)$ with scaling, Lemma 2.1.7, Lemma 2.1.8, and Remark 5.3.1, we find that

$$\begin{aligned}
\frac{1}{\delta_i}\|P_h^{ND}\mathbf{u}^\perp\|_{0;\partial\Omega_i}^2 &\leqslant C(\frac{H_i}{\delta_i}|P_h^{ND}\mathbf{u}^\perp|_{1;\Omega_i}^2 + \frac{1}{\delta_i H_i}\|P_h^{ND}\mathbf{u}^\perp\|_{0;\Omega_i}^2) \\
&\leqslant C(\frac{H_i}{\delta_i}\|\mathbf{curl}\, P_h^{ND}\mathbf{u}^\perp\|_{0;\Omega_i}^2 + \frac{1}{\delta_i H_i}\|P_h^{ND}\mathbf{u}^\perp\|_{0;\Omega_i}^2) \\
&\leqslant C(\frac{H_i}{\delta_i}\|\mathbf{curl}\, P_h^{ND}\mathbf{u}^\perp\|_{0;\Omega_i}^2 + \frac{1}{\delta_i H_i}H_i^2\|\mathbf{curl}\, P_h^{ND}\mathbf{u}^\perp\|_{0;\Omega_i}^2) \\
&\leqslant C\frac{H_i}{\delta_i}\|\mathbf{curl}\, P_h^{ND}\mathbf{u}^\perp\|_{0;\Omega_i}^2 = C\frac{H_i}{\delta_i}\|\mathbf{curl}\,\mathbf{u}^\perp\|_{0;\Omega_i}^2.
\end{aligned}$$

Therefore,

$$\frac{1}{\delta_i^2}\|\mathbf{u}^\perp\|_{0;\Omega_i\cap\Omega_{i,\delta_i}}^2 \leqslant C(1+\frac{H_i}{\delta_i})\|\mathbf{curl}\,\mathbf{u}^\perp\|_{0;\Omega_i}^2.$$

We can use exactly the same idea for all the other estimates. ☐

5.3.2 Stability Estimates

We consider the coarse part first.

Lemma 5.3.5. (Coarse Space Estimate) *Let $\mathbf{u}_0$ be the discrete harmonic extension of the given interface values of $\Pi_H^{RT}\mathbf{u}$. Then,*

$$\mathbf{a}(\mathbf{u}_0,\mathbf{u}_0) \leqslant C\left(\max_{1\leqslant i\leqslant N}(1+\log\frac{H_i}{h_i})\right)\mathbf{a}(\mathbf{u},\mathbf{u}), \tag{5.4}$$

where C is independent of α_i, β_i, H_i, h_i, and the jumps of the coefficients.

Proof. First, let us assume that $H_i^2\beta_i \leqslant \alpha_i$. Let $\mathbf{u}_H := \Pi_H^{RT}\mathbf{u}$. We note that $\mathbf{u}_0$ is the discrete harmonic extension with the same interface value as $\mathbf{u}_H$ on $\partial\Omega_i$. By the minimal-energy property of the discrete harmonic extension and Lemma 5.3.1, we find that

$$\begin{aligned}
\mathbf{a}_i(\mathbf{u}_0,\mathbf{u}_0) &\leqslant \mathbf{a}_i(\mathbf{u}_H,\mathbf{u}_H) \\
&\leqslant C(1+\log\frac{H_i}{h_i})(\beta_i\|\mathbf{u}\|^2_{0;\Omega_i}+(\alpha_i+H_i^2\beta_i)\|\operatorname{div}\mathbf{u}\|^2_{0;\Omega_i}) \\
&\leqslant C(1+\log\frac{H_i}{h_i})(\alpha_i\|\operatorname{div}\mathbf{u}\|^2_{0;\Omega_i}+\beta_i\|\mathbf{u}\|^2_{0;\Omega_i}).
\end{aligned}$$

Hence, we obtain

$$\mathbf{a}_i(\mathbf{u}_0,\mathbf{u}_0)\leqslant C(1+\log\frac{H_i}{h_i})\mathbf{a}_i(\mathbf{u},\mathbf{u}). \tag{5.5}$$

We now assume that $H_i^2\beta_i\geqslant\alpha_i$. We use a method similar to that of [55, Lemma 4.1]. We introduce piecewise linear scalar cut-off functions χ_1 and χ_2. The two functions satisfy the following conditions: χ_1 is equal to 1 on all interior small faces of F_{ij} and $\chi_1|_{\partial\Omega_i\setminus F_{ij}}=0$. The extension of χ_1 takes values between 0 and 1; c.f. [67, Section 4.6.3] and [21, Section 4]. χ_2 has the value 1 on $\partial\Omega_i$. Also, $\chi_k|_{\Omega_i\setminus\Omega_{i,d_k}}=0$ for some $h_i\leqslant d_k\leqslant H_i$. Moreover, $\|\nabla\chi_k\|_\infty\leqslant\dfrac{C}{d_k}$ for $k=1$ and 2. Then, the following estimates hold; cf. [19, Section 4 and 5], [20, Section 4], and [67, Lemma 4.25]:

$$\|\chi_1\|^2_{0;\Omega_i}\leqslant CH_i^2d_1,$$

$$|\chi_1|^2_{1;\Omega_i}\leqslant C(1+\log\frac{H_i}{h_i})\frac{H_i^2}{d_1}.$$

Let ϕ_{ij} be the coarse basis function corresponding to the face F_{ij}. This means that the normal component of ϕ_{ij} is 1 on F_{ij} and 0 on the other faces of $\Omega_i\cup\Omega_j$ and the interior values of ϕ_{ij} are obtained by the discrete harmonic extension. We also consider the standard basis function $\widetilde{\phi}_{ij}$ obtained from the coarse mesh. We note that ϕ_{ij} and $\widetilde{\phi}_{ij}$ have the same normal component on each coarse face. We can easily show that $\|\widetilde{\phi}_{ij}\|^2_{0;\Omega_i}\leqslant CH_i^3$ and $\|\operatorname{div}\widetilde{\phi}_{ij}\|^2_{0;\Omega_i}\leqslant CH_i$. The function $\mathbf{u}_0$ can be expressed as follows:

$$\mathbf{u}_0 = \sum_{F_{ij}\subset\Gamma} \lambda_{F_{ij}} \phi_{ij}.$$

Hence, it is enough to consider these terms one by one. We provide bounds of the coefficient and the energy of the basis functions separately.

We first consider the coefficients. We modify the proof of [70, Lemma 2.4]. Let f_k be the small faces which contain edges of ∂F_{ij}. We note that on f_k, χ_1 has values between 0 and 1. Also we know that the number of such faces, n_F, is bounded by $C(H/h)$; for details, see [70, Lemma 2.4]. We find that

$$\begin{aligned} |F_{ij}|\lambda_{F_{ij}}(\mathbf{u}) &= \int_{F_{ij}} \mathbf{u}\cdot\mathbf{n}\,ds = \int_{F_{ij}} \chi_1\mathbf{u}\cdot\mathbf{n}\,ds + \sum_{k=1}^{n_F} c_k|f_k|(\mathbf{u}\cdot\mathbf{n}_{f_k}) \\ &= \int_{\Omega_i} \chi_1 \operatorname{div}\mathbf{u} + \nabla\chi_1\cdot\mathbf{u}\,dx + \sum_{k=1}^{n_F} c_k|f_k|(\mathbf{u}\cdot\mathbf{n}_{f_k}), \end{aligned}$$

where $|c_k| < 1$. We note that $(\sum_{k=1}^{n_F} c_k|f_k|(\mathbf{u}\cdot\mathbf{n}_{f_k}))^2$ is bounded by $CH_i\|\mathbf{u}\|^2_{0;\Omega_i}$; see [70, (2.16)]. Hence,

$$\begin{aligned} |\lambda_{F_{ij}}(\mathbf{u})|^2 &\leqslant C\frac{1}{H_i^4}((\int_{\Omega_i}\chi_1\operatorname{div}\mathbf{u}\,dx)^2 + (\int_{\Omega_i}\nabla\chi_1\cdot\mathbf{u}\,dx)^2 + H_i\|\mathbf{u}\|^2_{0;\Omega_i}) \\ &\leqslant C\frac{1}{H_i^4}(\|\chi_1\|^2_{0;\Omega_i}\|\operatorname{div}\mathbf{u}\|^2_{0;\Omega_i} + \|\nabla\chi_1\|^2_{0;\Omega_i}\|\mathbf{u}\|^2_{0;\Omega_i} + H_i\|\mathbf{u}\|^2_{0;\Omega_i}) \\ &\leqslant C\frac{1}{H_i^4}(H_i^2 d_1\|\operatorname{div}\mathbf{u}\|^2_{0;\Omega_i} + (1+\log\frac{H_i}{h_i})\frac{H_i^2}{d_1}\|\mathbf{u}\|^2_{0;\Omega_i} + H_i\|\mathbf{u}\|^2_{0;\Omega_i}) \quad (5.6) \\ &\leqslant C\frac{1}{H_i^4}(H_i^2 d_1\|\operatorname{div}\mathbf{u}\|^2_{0;\Omega_i} + (1+\log\frac{H_i}{h_i})\frac{H_i^2}{d_1}\|\mathbf{u}\|^2_{0;\Omega_i}) \quad (5.7) \\ &\leqslant C\frac{1}{H_i^2}(d_1\|\operatorname{div}\mathbf{u}\|^2_{0;\Omega_i} + (1+\log\frac{H_i}{h_i})\frac{1}{d_1}\|\mathbf{u}\|^2_{0;\Omega_i}) \\ &\leqslant C(1+\log\frac{H_i}{h_i})\frac{1}{H_i^2}(d_1\|\operatorname{div}\mathbf{u}\|^2_{0;\Omega_i} + \frac{1}{d_1}\|\mathbf{u}\|^2_{0;\Omega_i}). \quad (5.8) \end{aligned}$$

We note that due to the fact that $\dfrac{H_i^2}{d_1} \geqslant H_i$, the last term of (5.6) can be absorbed into the L^2-term of (5.7). By the fact that ϕ_{ij} and $\widetilde{\phi}_{ij}$ have the same normal component on each face of $\partial\Omega_i$ and ϕ_{ij} is the energy minimal extension of $\widetilde{\phi}_{ij}$, we obtain

$$\begin{aligned} & \alpha_i \|\mathrm{div}\, \phi_{ij}\|_{0;\Omega_i}^2 + \beta_i \|\phi_{ij}\|_{0;\Omega_i}^2 \\ \leqslant\ & \alpha_i \|\mathrm{div}\,(\Pi_h^{RT}(\chi_2 \widetilde{\phi}_{ij}))\|_{0;\Omega_i}^2 + \beta_i \|\Pi_h^{RT}(\chi_2 \widetilde{\phi}_{ij})\|_{0;\Omega_i}^2. \end{aligned}$$

By Lemma 5.3.2 and estimates for the basis functions, we have

$$\begin{aligned} \|\mathrm{div}\,(\Pi_h^{RT}(\chi_2 \widetilde{\phi}_{ij}))\|_{0;\Omega_i}^2 &\leqslant C\|\mathrm{div}\,(\chi_2 \widetilde{\phi}_{ij})\|_{0;\Omega_i}^2 \\ &\leqslant C\|\chi_2\|_\infty^2 \|\mathrm{div}\, \widetilde{\phi}_{ij}\|_{0;\Omega_{i,d2}}^2 + C\|\nabla \chi_2\|_\infty^2 \|\widetilde{\phi}_{ij}\|_{0;\Omega_{i,d2}}^2 \\ &\leqslant C(d_2 + \frac{1}{d_2^2} H_i^2 d_2) \leqslant C(d_2 + \frac{H_i^2}{d_2}) \end{aligned} \tag{5.9}$$

and

$$\begin{aligned} \|\Pi_h^{RT}(\chi_2 \widetilde{\phi}_{ij})\|_{0;\Omega_i}^2 &\leqslant C\|(\chi_2 \widetilde{\phi}_{ij})\|_{0;\Omega_i}^2 \\ &\leqslant C\|\chi_2\|_\infty^2 \|\widetilde{\phi}_{ij}\|_{0;\Omega_{i,d2}}^2 \leqslant C H_i^2 d_2. \end{aligned} \tag{5.10}$$

By (5.8), (5.9), and (5.10), we find that

$$
\begin{aligned}
& \alpha_i \|\lambda_{F_{ij}}(\mathbf{u}) \left(\operatorname{div}(\Pi_h^{RT}(\chi_2 \widetilde{\phi}_{ij}))\right) \|^2_{0;\Omega_i} + \beta_i \|\lambda_{F_{ij}}(\mathbf{u}) \left(\Pi_h^{RT}(\chi_2 \widetilde{\phi}_{ij})\right) \|^2_{0;\Omega_i} \\
\leqslant\ & C(\alpha_i |\lambda_{F_{ij}}(\mathbf{u})|^2 (d_2 + \frac{H_i^2}{d_2}) + \beta_i |\lambda_{F_{ij}}(\mathbf{u})|^2 H_i^2 d_2) \\
\leqslant\ & C(1 + \log \frac{H_i}{h_i})((\alpha_i(\frac{d_1}{d_2} + \frac{d_1 d_2}{H_i^2}) + \beta_i d_1 d_2) \|\operatorname{div} \mathbf{u}\|^2_{0;\Omega_i} \\
& + (\alpha_i(\frac{1}{d_1 d_2} + \frac{d_2}{d_1 H_i^2}) + \beta_i \frac{d_2}{d_1}) \|\mathbf{u}\|^2_{0;\Omega_i}). \qquad (5.11)
\end{aligned}
$$

Let $d_1 = \sqrt{\frac{\alpha_i}{\beta_i}}$ and $d_2 = H_i \sqrt{\frac{1}{1+\frac{\beta_i H_i^2}{\alpha_i}}}$. We note that $h_i \leqslant d_1, d_2 \leqslant H_i$.
We then obtain

$$
\begin{aligned}
& \mathbf{a}_i(\mathbf{u}_0, \mathbf{u}_0) \\
\leqslant\ & \sum \alpha_i \|\lambda_{F_{ij}}(\mathbf{u}) \left(\operatorname{div} \phi_{ij}\right) \|^2_{0;\Omega_i} + \beta_i \|\lambda_{F_{ij}}(\mathbf{u}) \left(\phi_{ij}\right) \|^2_{0;\Omega_i} \\
\leqslant\ & \sum \alpha_i \|\lambda_{F_{ij}}(\mathbf{u}) \left(\operatorname{div}(\Pi_h^{RT}(\chi_2 \widetilde{\phi}_{ij}))\right) \|^2_{0;\Omega_i} + \beta_i \|\lambda_{F_{ij}}(\mathbf{u}) \left(\Pi_h^{RT}(\chi_2 \widetilde{\phi}_{ij})\right) \|^2_{0;\Omega_i} \\
\leqslant\ & C(1 + \log \frac{H_i}{h_i}) \sqrt{1 + \frac{\alpha_i}{\beta_i H_i^2}} \mathbf{a}_i(\mathbf{u}, \mathbf{u}).
\end{aligned}
$$

Since $H_i^2 \beta_i \geqslant \alpha_i$, $\sqrt{1 + \frac{\alpha_i}{\beta_i H_i^2}}$ is bounded by a constant. Hence,

$$
\mathbf{a}_i(\mathbf{u}_0, \mathbf{u}_0) \leqslant C(1 + \log \frac{H_i}{h_i}) \mathbf{a}_i(\mathbf{u}, \mathbf{u}). \qquad (5.12)
$$

In all cases, we obtain the same result (5.5) and (5.12). We can conclude that (5.4) holds by summing over all subdomains. □

Remark 5.3.2. *In [70, Chapter 2.2], the constant depends on* $\max_i(1 + \frac{\beta_i H_i^2}{\alpha_i})$. *As we see from the numerical experiments in [70, 71] and this paper, the results do not depend on* α_i*'s and* β_i*'s at all. We have therefore improved the previous results* *in [70, 71] by removing this dependence of the coefficients.*

We now consider the local components. Let $\mathbf{v} = \mathbf{u} - \mathbf{u}_0$. We know that $\mathbf{v} \in X_h$. By Lemma 2.2.3, we can find $\mathbf{w}^\perp \in N_h^\perp$ and $\mathbf{v}^\perp \in X_h^\perp$ such that $\mathbf{v} = \mathbf{curl}\, \mathbf{w}^\perp + \mathbf{v}^\perp$.

Let θ_i be a piecewise linear function associated with the subdomain Ω_i. Each θ_i is constructed in a similar way as in [67, Lemma 3.4]. We construct $\tilde{\theta}_i(x)$ which satisfies the following conditions:

$$\tilde{\theta}_i(x) = \begin{cases} 1, & \text{dist}\,(x, \partial\Omega_i) \geqslant \delta_i, \\ 0, & x \in \partial\Omega_i, \end{cases}$$

and if $\text{dist}\,(x, \partial\Omega_i) \leqslant \delta_i$, $\|\nabla\tilde{\theta}_i(x)\| \leqslant \dfrac{C}{\delta_i}$.

We set

$$\theta_i = I^h(\tilde{\theta}_i).$$

Lemma 5.3.6. *Let* $\mathbf{v}_i = \Pi_h^{RT}(\theta_i \mathbf{v}^\perp)$ *and* $\mathbf{w}_i = \Pi_h^{ND}(\theta_i \mathbf{w}^\perp)$. *Then,*

$$\sum_{i=1}^N \widetilde{\mathbf{a}}_i(\mathbf{v}_i, \mathbf{v}_i) \;\; \leqslant \;\; C\left(\max_{1\leqslant i\leqslant N}(1 + \frac{H_i}{\delta_i})\right)\mathbf{a}(\mathbf{v}^\perp, \mathbf{v}^\perp) \tag{5.13}$$

and

$$\sum_{i=1}^N \widetilde{\mathbf{a}}_i(\mathbf{curl}\,\mathbf{w}_i, \mathbf{curl}\,\mathbf{w}_i) \leqslant \quad C\left(\max_{1\leqslant i\leqslant N}(1 + \frac{H_i}{\delta_i})\right)\mathbf{a}(\mathbf{curl}\,\mathbf{w}^\perp, \mathbf{curl}\,\mathbf{w}^\perp), \tag{5.14}$$

with C independent of α_i, β_i, H_i, h_i, δ_i, and the jumps of the coefficients.

Proof. We note that θ_i is supported in $\overline{\Omega}_i$. By Lemma 5.3.2,

$$
\begin{aligned}
\widetilde{\mathbf{a}}_i(\mathbf{v}_i,\mathbf{v}_i) &= \mathbf{a}_i(\mathbf{v}_i,\mathbf{v}_i) = \alpha_i\|\mathrm{div}\,\mathbf{v}_i\|^2_{0;\Omega_i} + \beta_i\|\mathbf{v}_i\|^2_{0;\Omega_i} \\
&= \alpha_i\|\mathrm{div}\,(\Pi_h^{RT}(\theta_i\mathbf{v}^\perp))\|^2_{0;\Omega_i} + \beta_i\|\Pi_h^{RT}(\theta_i\mathbf{v}^\perp)\|^2_{0;\Omega_i} \\
&\leqslant C(\alpha_i\|\mathrm{div}\,(\theta_i\mathbf{v}^\perp)\|^2_{0;\Omega_i} + \beta_i\|\theta_i\mathbf{v}^\perp\|^2_{0;\Omega_i}).
\end{aligned}
$$

Consider the L^2-term:

$$
\|\theta_i\mathbf{v}^\perp\|^2_{0;\Omega_i} \leqslant \|\theta_i\|^2_\infty\|\mathbf{v}^\perp\|^2_{0;\Omega_i} \leqslant \|\mathbf{v}^\perp\|^2_{0;\Omega_i}.
$$

We now consider the divergence term. By Lemma 5.3.4,

$$
\begin{aligned}
\|\mathrm{div}\,(\theta_i\mathbf{v}^\perp)\|^2_{0;\Omega_i} &\leqslant C(\|\nabla\theta_i\cdot\mathbf{v}^\perp\|^2_{0;\Omega_i} + \|\theta_i\mathrm{div}\,\mathbf{v}^\perp\|^2_{0;\Omega_i}) \\
&\leqslant C(\|\nabla\theta_i\|^2_\infty\|\mathbf{v}^\perp\|^2_{0;\Omega_i} + \|\theta_i\|^2_\infty\|\mathrm{div}\,\mathbf{v}^\perp\|^2_{0;\Omega_i}) \\
&\leqslant C(\frac{1}{\delta_i^2}\|\mathbf{v}^\perp\|^2_{0;\Omega_i} + \|\mathrm{div}\,\mathbf{v}^\perp\|^2_{0;\Omega_i}) \qquad (5.15) \\
&\leqslant C(1+\frac{H_i}{\delta_i})\|\mathrm{div}\,\mathbf{v}^\perp\|^2_{0;\Omega_i}. \qquad (5.16)
\end{aligned}
$$

Therefore,

$$
\widetilde{\mathbf{a}}_i(\mathbf{v}_i,\mathbf{v}_i) \leqslant C(1+\frac{H_i}{\delta_i})\mathbf{a}_i(\mathbf{v}^\perp,\mathbf{v}^\perp).
$$

We now consider (5.14). We note that $\widetilde{\mathbf{a}}_i(\mathbf{curl}\,\mathbf{w}_i,\mathbf{curl}\,\mathbf{w}_i) = \mathbf{a}_i(\mathbf{curl}\,\mathbf{w}_i,\mathbf{curl}\,\mathbf{w}_i)$.

By Lemma 5.3.2,

$$
\begin{aligned}
\mathbf{a}_i(\mathbf{curl}\,\mathbf{w}_i, \mathbf{curl}\,\mathbf{w}_i) &= \beta_i \|\mathbf{curl}\,\mathbf{w}_i\|^2_{0;\Omega_i} \\
&= \beta_i \|\mathbf{curl}\,(\Pi_h^{ND}(\theta_i \mathbf{w}^\perp))\|^2_{0;\Omega_i} \\
&\leqslant C\beta_i \|\mathbf{curl}\,(\theta_i \mathbf{w}^\perp)\|^2_{0;\Omega_i} \\
&\leqslant C\beta_i (\|\nabla\theta_i \times \mathbf{w}^\perp\|^2_{0;\Omega_i} + \|\theta_i \mathbf{curl}\,\mathbf{w}^\perp\|^2_{0;\Omega_i}) \\
&\leqslant C\beta_i (\|\nabla\theta_i\|^2_\infty \|\mathbf{w}^\perp\|^2_{0;\Omega_i} + \|\theta_i\|^2_\infty \|\mathbf{curl}\,\mathbf{w}^\perp\|^2_{0;\Omega_i}) \\
&\leqslant C\beta_i (\frac{1}{\delta_i^2} \|\mathbf{w}^\perp\|^2_{0;\Omega_i} + \|\mathbf{curl}\,\mathbf{w}^\perp\|^2_{0;\Omega_i}).
\end{aligned}
$$

By Lemma 5.3.4, the following inequality holds:

$$
\begin{aligned}
\widetilde{\mathbf{a}}_i(\mathbf{curl}\,\mathbf{w}_i, \mathbf{curl}\,\mathbf{w}_i) &\leqslant C\beta_i (1 + \frac{H_i}{\delta_i}) \|\mathbf{curl}\,\mathbf{w}^\perp\|^2_{0;\Omega_i} \\
&= C(1 + \frac{H_i}{\delta_i}) \mathbf{a}_i(\mathbf{curl}\,\mathbf{w}^\perp, \mathbf{curl}\,\mathbf{w}^\perp).
\end{aligned}
$$

We obtain (5.14) by summing over the subdomains. $\square$

We next build another cut-off function $\theta_{F_{ij}}$, which is supported in the set

$$\Xi_{ij} := (\Omega_i' \cap \Omega_j) \cup (\Omega_i \cap \Omega_j') \cup (F_{ij});$$

cf. [19, Section 4 and 5]. $\theta_{F_{ij}}$ satisfies the following conditions:

$$0 \leqslant \theta_{F_{ij}} \leqslant 1,$$

$$\theta_{F_{ij}}|_{\partial\Xi_{ij}} = 0,$$

and

$$\|\nabla\theta_{F_{ij}}\|_\infty \leqslant \frac{C}{\delta_i};$$

see [67, Lemma 3.4] for details.

Lemma 5.3.7. *Let* $\mathbf{v}_{ij} = \Pi_h^{RT}(\frac{1}{2}\theta_{F_{ij}}\mathbf{v}^\perp)$ *and* $\mathbf{w}_{ij} = \Pi_h^{ND}(\frac{1}{2}\theta_{F_{ij}}\mathbf{w}^\perp)$*. Then, we have*

$$\sum_{i=1}^{N}\sum_{F_{ij}\subset\partial\Omega_i} \widetilde{\mathbf{a}}_i(\mathbf{v}_{ij},\mathbf{v}_{ij}) \leqslant C\left(\max_{1\leqslant i\leqslant N}(1+\frac{H_i}{\delta_i})\right)\mathbf{a}(\mathbf{v}^\perp,\mathbf{v}^\perp) \tag{5.17}$$

$$\begin{aligned}&\sum_{i=1}^{N}\sum_{F_{ij}\subset\partial\Omega_i} \widetilde{\mathbf{a}}_i(\mathbf{curl}\,\mathbf{w}_{ij},\mathbf{curl}\,\mathbf{w}_{ij})\\ \leqslant\ & C\left(\max_{1\leqslant i\leqslant N}(1+\frac{H_i}{\delta_i})\right)\mathbf{a}(\mathbf{curl}\,\mathbf{w}^\perp,\mathbf{curl}\,\mathbf{w}^\perp),\end{aligned} \tag{5.18}$$

with C *independent of* α_i, β_i, H_i, h_i, δ_i, *and the jumps of the coefficients.*

Proof. Because $\theta_{F_{ij}}$ is supported in $\overline{\Xi}_{ij}$, we have

$$\begin{aligned}\widetilde{\mathbf{a}}_i(\mathbf{v}_{ij},\mathbf{v}_{ij}) &= \int_{\Xi_{ij}}\alpha \mathrm{div}\,\mathbf{v}_{ij}\mathrm{div}\,\mathbf{v}_{ij}\,dx + \int_{\Xi_{ij}}\beta\mathbf{v}_{ij}\cdot\mathbf{v}_{ij}\,dx\\ &= \int_{\Omega_i\cap\Omega_j'}\alpha_i \mathrm{div}\,\mathbf{v}_{ij}\mathrm{div}\,\mathbf{v}_{ij}\,dx + \int_{\Omega_i\cap\Omega_j'}\beta_i\mathbf{v}_{ij}\cdot\mathbf{v}_{ij}\,dx\\ &\quad + \int_{\Omega_i'\cap\Omega_j}\alpha_j \mathrm{div}\,\mathbf{v}_{ij}\mathrm{div}\,\mathbf{v}_{ij}\,dx + \int_{\Omega_i'\cap\Omega_j}\beta_j\mathbf{v}_{ij}\cdot\mathbf{v}_{ij}\,dx.\end{aligned}$$

By Lemma 5.3.2,

$$\alpha_j\|\mathrm{div}\,\mathbf{v}_{ij}\|^2_{0;\Omega_i'\cap\Omega_j} \leqslant C\alpha_j\|\mathrm{div}\,(\theta_{F_{ij}}\mathbf{v}^\perp)\|^2_{0;\Omega_i'\cap\Omega_j}$$

and

$$\beta_j \|\mathbf{v}_{ij}\|^2_{0;\Omega_i'\cap\Omega_j} \leqslant C\beta_j \|\theta_{F_{ij}}\mathbf{v}^\perp\|^2_{0;\Omega_i'\cap\Omega_j}.$$

Moreover,

$$\begin{aligned}
\|\mathrm{div}\,(\theta_{F_{ij}}\mathbf{v}^\perp)\|^2_{0;\Omega_i'\cap\Omega_j} &\leqslant C(\|\nabla\theta_{F_{ij}}\cdot\mathbf{v}^\perp\|^2_{0;\Omega_i'\cap\Omega_j} + \|\theta_{F_{ij}}\mathrm{div}\,\mathbf{v}^\perp\|^2_{0;\Omega_i'\cap\Omega_j}) \\
&\leqslant C(\|\nabla\theta_{F_{ij}}\|^2_\infty\|\mathbf{v}^\perp\|^2_{0;\Omega_i'\cap\Omega_j} + \|\theta_{F_{ij}}\|^2_\infty\|\mathrm{div}\,\mathbf{v}^\perp\|^2_{0;\Omega_i'\cap\Omega_j}) \\
&\leqslant C(\frac{1}{\delta_i^2}\|\mathbf{v}^\perp\|^2_{0;\Omega_i'\cap\Omega_j} + \|\mathrm{div}\,\mathbf{v}^\perp\|^2_{0;\Omega_i'\cap\Omega_j}) \qquad (5.19) \\
&\leqslant C(1+\frac{H_i}{\delta_i})\|\mathrm{div}\,\mathbf{v}^\perp\|^2_{0;\Omega_j} \qquad (5.20)
\end{aligned}$$

and

$$\|\theta_{F_{ij}}\mathbf{v}^\perp\|^2_{0;\Omega_i'\cap\Omega_j} \leqslant \|\mathbf{v}^\perp\|^2_{0;\Omega_i'\cap\Omega_j} \leqslant \|\mathbf{v}^\perp\|^2_{0;\Omega_j}.$$

We obtain (5.20) from (5.19) by using Lemma 5.3.4. Hence,

$$\alpha_j\|\mathrm{div}\,\mathbf{v}_{ij}\|^2_{0;\Omega_i'\cap\Omega_j} \leqslant C\alpha_j(1+\frac{H_i}{\delta_i})\|\mathrm{div}\,\mathbf{v}^\perp\|^2_{0;\Omega_j}$$

and

$$\beta_j\|\mathbf{v}_{ij}\|^2_{0;\Omega_i'\cap\Omega_j} \leqslant C\beta_j\|\mathbf{v}^\perp\|^2_{0;\Omega_j}.$$

Similarly,

$$\alpha_i\|\mathrm{div}\,\mathbf{v}_{ij}\|^2_{0;\Omega_i\cap\Omega_j'} \leqslant C\alpha_i(1+\frac{H_i}{\delta_i})\|\mathrm{div}\,\mathbf{v}^\perp\|^2_{0;\Omega_i}$$

and

$$\beta_i\|\mathbf{v}_{ij}\|^2_{0;\Omega_i\cap\Omega_j'} \leqslant C\beta_i\|\mathbf{v}^\perp\|^2_{0;\Omega_i}.$$

Therefore, we can obtain (5.17) by a coloring argument and summing over all the partitions Ξ_{ij}.

We now consider (5.18):

$$\begin{aligned}\widetilde{\mathbf{a}}_i(\mathbf{curl}\,\mathbf{w}_{ij}, \mathbf{curl}\,\mathbf{w}_{ij}) &= \int_{\Xi_{ij}} \beta \mathbf{curl}\, w_{ij} \mathbf{curl}\, w_{ij}\, dx \\ &= \beta_i \|\mathbf{curl}\,\mathbf{w}_{ij}\|^2_{0;\Omega_i\cap\Omega'_j} + \beta_j \|\mathbf{curl}\,\mathbf{w}_{ij}\|^2_{0;\Omega'_i\cap\Omega_j}.\end{aligned}$$

By Lemma 5.3.2,

$$\beta_i \|\mathbf{curl}\,\mathbf{w}_{ij}\|^2_{0;\Omega_i\cap\Omega'_j} \leqslant C\beta_i \|\mathbf{curl}\,(\theta_{F_{ij}}\mathbf{w}^\perp)\|^2_{0;\Omega_i\cap\Omega'_j}$$

and

$$\beta_j \|\mathbf{curl}\,\mathbf{w}_{ij}\|^2_{0;\Omega'_i\cap\Omega_j} \leqslant C\beta_j \|\mathbf{curl}\,(\theta_{F_{ij}}\mathbf{w}^\perp)\|^2_{0;\Omega'_i\cap\Omega_j}.$$

Therefore,

$$\begin{aligned}\beta_j \|\mathbf{curl}\,(\theta_{F_{ij}}\mathbf{w}^\perp)\|^2_{0;\Omega'_i\cap\Omega_j} &\leqslant C\beta_j (\|\nabla\theta_{F_{ij}} \times \mathbf{w}^\perp\|^2_{0;\Omega'_i\cap\Omega_j} + \|\theta_{F_{ij}} \mathbf{curl}\,\mathbf{w}^\perp\|^2_{0;\Omega'_i\cap\Omega_j}) \\ &\leqslant C\beta_j (\|\nabla\theta_{F_{ij}}\|^2_\infty \|\mathbf{w}^\perp\|^2_{0;\Omega'_i\cap\Omega_j} + \|\theta_{F_{ij}}\|^2_\infty \|\mathbf{curl}\,\mathbf{w}^\perp\|^2_{0;\Omega'_i\cap\Omega_j}) \\ &\leqslant C\beta_j (\frac{1}{\delta_i^2} \|\mathbf{w}^\perp\|^2_{0;\Omega'_i\cap\Omega_j} + \|\mathbf{curl}\,\mathbf{w}^\perp\|^2_{0;\Omega'_i\cap\Omega_j}).\end{aligned}$$

By Lemma 5.3.4,

$$\beta_j \|\mathbf{curl}\,(\theta_{F_{ij}}\mathbf{w}^\perp)\|^2_{0;\Omega'_i\cap\Omega_j} \leqslant C\beta_j (1 + \frac{H_i}{\delta_i}) \|\mathbf{curl}\,\mathbf{w}^\perp\|^2_{0;\Omega_j}.$$

Similarly,

$$\beta_i \|\mathbf{curl}\,(\theta_{F_{ij}}\mathbf{w}^\perp)\|^2_{0;\Omega_i\cap\Omega'_j} \leqslant C\beta_i (1 + \frac{H_i}{\delta_i}) \|\mathbf{curl}\,\mathbf{w}^\perp\|^2_{0;\Omega_i}.$$

Therefore,

$$
\begin{aligned}
&\widetilde{\mathbf{a}}_i(\mathbf{curl}\,\mathbf{w}_{ij}, \mathbf{curl}\,\mathbf{w}_{ij}) \\
\leqslant\;& C\beta_i(1+\frac{H_i}{\delta_i})\|\mathbf{curl}\,\mathbf{w}^{\perp}\|^2_{0;\Omega_i} + C\beta_j(1+\frac{H_i}{\delta_i})\|\mathbf{curl}\,\mathbf{w}^{\perp}\|^2_{0;\Omega_j}.
\end{aligned}
$$

Finally, (5.18) holds by a coloring argument and summing over all the partitions Ξ_{ij}. □

We finally construct the remaining parts of the partition of unity. For each edge $E_{jl} \subset \partial\Omega_i$, which equals $\overline{F}_{ij} \cap \overline{F}_{il}$, consider a cut-off function $\theta_{E_{jl}}$ which is supported in the set

$$\Psi_{jl} =: \bigcap_{m\in I_{jl}} \Omega'_m,$$

where I_{jl} is the set of indices of the subdomains which have the edge E_{jl} in common with Ω_i; cf. [19, Section 4 and 5]. $\theta_{E_{jl}}$ satisfies following conditions:

$$0 \leqslant \theta_{E_{jl}} \leqslant 1,$$

$$\|\nabla\theta_{E_{jl}}\|_\infty \leqslant \frac{C}{\delta_i},$$

and

$$\sum_{i=1}^{N}(\theta_i + \sum_{F_{ij}\subset\partial\Omega_i}\theta_{F_{ij}} + \sum_{E_{jl}\subset\partial\Omega_i}\theta_{E_{jl}}) = 1.$$

Lemma 5.3.8. *Let* $\mathbf{v}_{E_{jl}} = \Pi_h^{RT}(\frac{1}{|I_{jl}|}\theta_{E_{jl}}\mathbf{v}^{\perp})$ *and* $\mathbf{w}_{E_{jl}} = \Pi_h^{ND}(\frac{1}{|I_{jl}|}\theta_{E_{jl}}\mathbf{w}^{\perp})$. *Then,*

$$\sum_{i=1}^{N}\sum_{E_{jl}\subset\partial\Omega_i}\widetilde{\mathbf{a}}_i(\mathbf{v}_{E_{jl}}, \mathbf{v}_{E_{jl}}) \;\leqslant\; C\left(\max_{1\leqslant i\leqslant N}(1+\frac{H_i}{\delta_i})\right)\mathbf{a}(\mathbf{v}^{\perp}, \mathbf{v}^{\perp}) \qquad (5.21)$$

and

$$
\begin{aligned}
& \sum_{i=1}^{N} \sum_{E_{jl} \subset \partial\Omega_i} \widetilde{\mathbf{a}}_i(\mathbf{curl}\, \mathbf{w}_{E_{jl}}, \mathbf{curl}\, \mathbf{w}_{E_{jl}}) \\
\leqslant \quad & C \left(\max_{1 \leqslant i \leqslant N} (1 + \frac{H_i}{\delta_i}) \right) \mathbf{a}(\mathbf{curl}\, \mathbf{w}^{\perp}, \mathbf{curl}\, \mathbf{w}^{\perp}),
\end{aligned} \tag{5.22}
$$

with C independent of α_i, β_i, H_i, h_i, δ_i, and the jumps of the coefficients.

Proof. Because $\theta_{E_{jl}}$ is supported in $\overline{\Psi}_{jl}$, we find that

$$
\widetilde{\mathbf{a}}_i(\mathbf{v}_{E_{jl}}, \mathbf{v}_{E_{jl}}) = \sum_{m \in I_{jl}} (\alpha_m \|\mathrm{div}\, \mathbf{v}_{E_{jl}}\|^2_{0;\Psi_{jl} \cap \Omega_m} + \beta_m \|\mathbf{v}_{E_{jl}}\|^2_{0;\Psi_{jl} \cap \Omega_m}).
$$

We can apply the same idea to each subset $\Psi_{jl} \cap \Omega_m$. It suffices to consider just one subset.

By Lemma 5.3.2,

$$
\|\mathrm{div}\, \mathbf{v}_{E_{jl}}\|^2_{0;\Psi_{jl} \cap \Omega_m} \leqslant C \|\mathrm{div}\, (\theta_{E_{jl}} \mathbf{v}^{\perp})\|^2_{0;\Psi_{jl} \cap \Omega_m}
$$

and

$$
\|\mathbf{v}_{E_{jl}}\|^2_{0;\Psi_{jl} \cap \Omega_m} \leqslant C \|\theta_{E_{jl}} \mathbf{v}^{\perp}\|^2_{0;\Psi_{jl} \cap \Omega_m} \leqslant C \|\mathbf{v}^{\perp}\|^2_{0;\Psi_{jl} \cap \Omega_m}.
$$

Therefore,

$$
\begin{aligned}
& C \|\mathrm{div}\, (\theta_{E_{jl}} \mathbf{v}^{\perp})\|^2_{0;\Psi_{jl} \cap \Omega_m} \\
\leqslant \quad & C (\|\nabla \theta_{E_{jl}}\|^2_{\infty} \|\mathbf{v}^{\perp}\|^2_{0;\Psi_{jl} \cap \Omega_m} + \|\theta_{E_{jl}}\|^2_{\infty} \|\mathrm{div}\, \mathbf{v}^{\perp}\|^2_{0;\Psi_{jl} \cap \Omega_m}) \\
\leqslant \quad & C (\frac{1}{\delta_i^2} \|\mathbf{v}^{\perp}\|^2_{0;\Psi_{jl} \cap \Omega_m} + \|\mathrm{div}\, \mathbf{v}^{\perp}\|^2_{0;\Psi_{jl} \cap \Omega_m}).
\end{aligned}
$$

By Lemma 5.3.4,

$$C\|\mathrm{div}\,(\theta_{E_{jl}}\mathbf{v}^{\perp})\|^2_{0;\Psi_{jl}\cap\Omega_m} \leqslant C(1+\frac{H_i}{\delta_i})\|\mathrm{div}\,\mathbf{v}^{\perp}\|^2_{0;\Omega_m}.$$

By a coloring argument and summing over all the partitions, we obtain

$$\sum_{i=1}^{N}\sum_{E_{jl}\subset\partial\Omega_i}\widetilde{\mathbf{a}}_i(\mathbf{v}_{E_{jl}},\mathbf{v}_{E_{jl}}) \leqslant C\left(\max_{1\leqslant i\leqslant N}(1+\frac{H_i}{\delta_i})\right)\mathbf{a}(\mathbf{v}^{\perp},\mathbf{v}^{\perp}).$$

Consider the second estimate (5.22):

$$\widetilde{\mathbf{a}}_i(\mathbf{curl}\,\mathbf{w}_{E_{jl}},\mathbf{curl}\,\mathbf{w}_{E_{jl}}) = \sum_{m\in I_{jl}}\beta_m\|\mathbf{curl}\,\mathbf{w}_{E_{jl}}\|^2_{0;\Psi_{jl}\cap\Omega_m}.$$

By Lemma 5.3.2,

$$\begin{aligned}
&\|\mathbf{curl}\,\mathbf{w}_{E_{jl}}\|^2_{0;\Psi_{jl}\cap\Omega_m}\\
\leqslant\;& \|\mathbf{curl}\,(\theta_{E_{jl}}\mathbf{w}^{\perp})\|^2_{0;\Psi_{jl}\cap\Omega_m}\\
\leqslant\;& C(\|\nabla\theta_{E_{jl}}\|^2_{\infty}\|\mathbf{w}^{\perp}\|^2_{0;\Psi_{jl}\cap\Omega_m}+\|\theta_{E_{jl}}\|^2_{\infty}\|\mathbf{curl}\,\mathbf{w}^{\perp}\|^2_{0;\Psi_{jl}\cap\Omega_m})\\
\leqslant\;& C(\frac{1}{\delta_i^2}\|\mathbf{w}^{\perp}\|^2_{0;\Psi_{jl}\cap\Omega_m}+\|\mathbf{curl}\,\mathbf{w}^{\perp}\|^2_{0;\Psi_{jl}\cap\Omega_m}).
\end{aligned}$$

By Lemma 5.3.4,

$$\|\mathbf{curl}\,(\theta_{E_{jl}}\mathbf{w}^{\perp})\|^2_{0;\Psi_{jl}\cap\Omega_m} \leqslant C(1+\frac{H_i}{\delta_i})\|\mathbf{curl}\,\mathbf{w}^{\perp}\|^2_{0;\Omega_m}.$$

Therefore, we obtain

$$\sum_{i=1}^{N} \sum_{E_{jl} \subset \partial\Omega_i} \widetilde{\mathbf{a}}_i(\mathbf{curl}\,\mathbf{w}_{E_{jl}}, \mathbf{curl}\,\mathbf{w}_{E_{jl}}) \leqslant C \left(\max_{1 \leqslant i \leqslant N} (1 + \frac{H_i}{\delta_i}) \right) \mathbf{a}(\mathbf{curl}\,\mathbf{w}^{\perp}, \mathbf{curl}\,\mathbf{w}^{\perp}),$$

by summing over all the partitions. $\square$

5.3.3 Main Result

We recall (4.1), (4.6), and (4.7). Let $P_i = R_i^T A_i^{-1} R_i A$ and $P_{ad} = \sum_{i=0}^{N} P_i$.

Theorem 5.3.9. (Condition number estimate) *The condition number of the preconditioned system satisfies*

$$\kappa(P_{ad}) \leqslant C \left(\max_{1 \leqslant i \leqslant N} (1 + \log \frac{H_i}{h_i}) \right) \left(\max_{1 \leqslant i \leqslant N} (1 + \frac{H_i}{\delta_i}) \right),$$

where C is a constant which does not depend on the number of subdomains, H_i, h_i, and δ_i. C is also independent of the coefficients α_i, β_i, and the jumps of the coefficients between subdomains.

Proof. We obtain this main result by using Lemmas 5.3.5, 5.3.6, 5.3.7, 5.3.8 and the triangle inequality. $\square$

Remark 5.3.3. *In the previous result in [34], there was a second factor of $(1+\frac{H}{\delta})$. We have improved the result by reducing the power of the $\frac{H}{\delta}$ term. Moreover, we deal with coefficients which have jumps.*

5.4 Numerical Experiments

5.4.1 The 2D Case

We apply the overlapping Schwarz method with the energy-minimizing coarse space to our model problem. We use $\Omega = [0,1]^2$ and the lowest order Raviart-Thomas elements. We decompose the domain into N^2 identical square subdomains. In each subdomain, we assume that the coefficients α and β are constant. We consider cases where the coefficients have jumps across the interface between the subdomains, in particular, the checkerboard distribution pattern of Fig. 5.2. We use a fixed β for the whole domain and different values of α for the black and white regions. We have $\alpha = 1$ for the black regions and another specified value for the white regions.

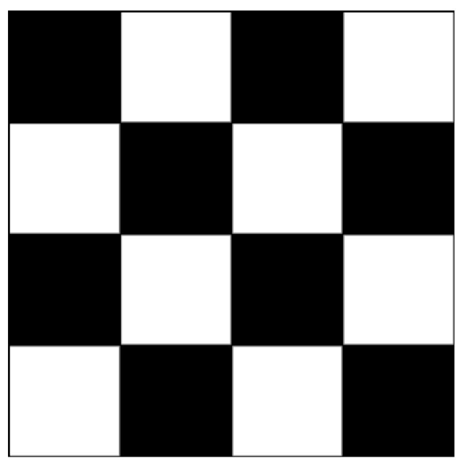

Figure 5.2: Checkerboard distribution of the coefficients

Each subdomain Ω_i has side length H and each mesh triangle has h as a minimum side length. We also introduce extended subdomains whose boundaries do not cut any mesh elements; recall Assumption 4.2.1. We use the preconditioned conjugate gradient method to solve the linear system of the finite element discretization.

In order to estimate the condition numbers, we use the method outlined in [53]. We stop the iteration when the residual l_2-norm has been reduced by a factor of 10^{-8}.

We perform two different kinds of experiments. We first fix the overlap $\frac{H}{\delta}$ and vary $\frac{H}{h}$. We next fix the size of $\frac{H}{h}$ and use various size of $\frac{H}{\delta}$. Table 5.1 and Table 5.2 show the first results and Table 5.3 and Table 5.4 show the second results.

In the first set of experiments, we see that the condition numbers and iteration counts do not depend on the size of $\frac{H}{h}$. In the second set, we can conclude that the condition numbers grow linearly with $\frac{H}{\delta}$. For both cases, the condition numbers and iteration counts are also quite independent of the jumps of coefficients between the subdomains. Fig. 5.4 shows that the estimated condition number depends linearly on $\frac{H}{\delta}$ and Fig. 5.3 shows that the estimated condition number is independent of $\frac{H}{h}$. Even though these results are independent of $\frac{H}{h}$, our numerical results are consistent with our main result.

Table 5.1: Condition numbers and iteration counts. $\alpha_i = 1$ or specified values as indicated in a checkerboard pattern, $\beta_i \equiv 1$ and $\frac{H}{\delta} = 8$ (2D case)

	$\alpha_i = 0.01$		$\alpha_i = 0.1$		$\alpha_i = 1$		$\alpha_i = 10$		$\alpha_i = 100$	
$\frac{H}{h}$	cond	iters	cond	iters	cond	iters	cond	iters	cond	iters
8	7.35	19	10.98	23	13.96	22	14.76	23	14.84	23
16	7.32	19	10.95	23	13.91	22	14.70	23	14.79	23
32	7.31	19	10.95	23	13.85	22	14.69	23	14.77	23
64	7.31	19	10.95	23	12.87	22	14.69	24	14.77	23

5.4.2 The 3D Case

For the 3D case, we use $\Omega = [0,1]^3$ and hexahedral instead of tetrahedral elements. In a way similar to the 2D case, we decompose the domain into N^3

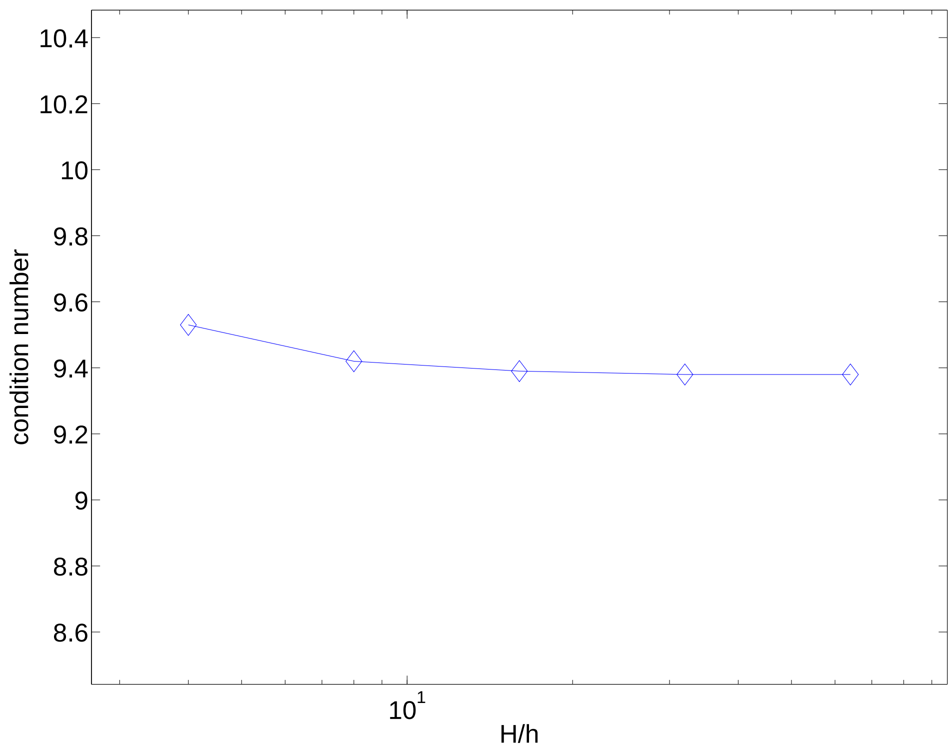

Figure 5.3: Estimated condition number, versus $\frac{H}{h}$; $\alpha_i = 1$ and $\alpha_i = 100$ in a checkerboard pattern, $\beta_i \equiv 1$ and $\frac{H}{\delta} = 4$ (2D case)

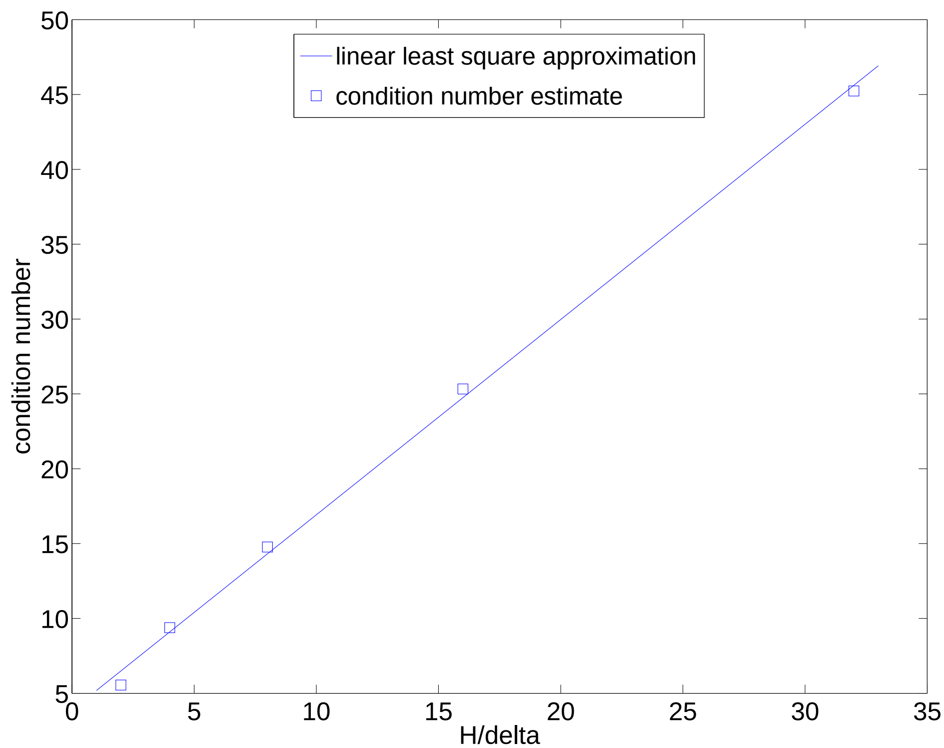

Figure 5.4: Estimated condition number and linear least square fitting, versus $\frac{H}{\delta}$; $\alpha_i = 1$ and $\alpha_i = 100$ in a checkerboard pattern, $\beta_i \equiv 1$ and $\frac{H}{h} = 32$ (2D case)

Table 5.2: Condition numbers and iteration counts. $\alpha_i = 1$ or specified values as indicated in a checkerboard pattern, $\beta_i \equiv 1$ and $\frac{H}{\delta} = 4$ (2D case)

	$\alpha_i = 0.01$		$\alpha_i = 0.1$		$\alpha_i = 1$		$\alpha_i = 10$		$\alpha_i = 100$	
$\frac{H}{h}$	cond	iters	cond	iters	cond	iters	cond	iters	cond	iters
4	5.44	17	7.46	20	9.17	19	9.50	21	9.53	20
8	5.38	17	7.41	20	9.07	19	9.38	21	9.42	20
16	5.36	17	7.39	20	9.01	19	9.36	21	9.39	20
32	5.35	17	7.38	20	8.45	19	9.35	21	9.38	20
64	5.35	17	7.38	20	6.34	17	9.35	21	9.38	20

Table 5.3: Condition numbers and iteration counts. $\alpha_i = 1$ or specified values as indicated in a checkerboard pattern, $\beta_i \equiv 1$ and $\frac{H}{h} = 16$ (2D case)

	$\alpha_i = 0.01$		$\alpha_i = 0.1$		$\alpha_i = 1$		$\alpha_i = 10$		$\alpha_i = 100$	
$\frac{H}{\delta}$	cond	iters	cond	iters	cond	iters	cond	iters	cond	iters
2	5.09	15	5.49	17	5.18	17	6.37	17	5.66	15
4	5.36	17	7.39	20	9.01	19	9.36	21	9.39	20
8	7.32	19	10.95	23	13.91	22	14.70	23	14.79	23
16	11.62	23	18.04	28	23.25	26	25.14	29	25.36	27

Table 5.4: Condition numbers and iteration counts. $\alpha_i = 1$ or specified values as indicated in a checkerboard pattern, $\beta_i \equiv 1$ and $\frac{H}{h} = 32$ (2D case)

	$\alpha_i = 0.01$		$\alpha_i = 0.1$		$\alpha_i = 1$		$\alpha_i = 10$		$\alpha_i = 100$	
$\frac{H}{\delta}$	cond	iters	cond	iters	cond	iters	cond	iters	cond	iters
2	5.05	15	5.48	17	5.18	16	6.32	17	5.55	15
4	5.36	17	7.39	20	8.45	19	9.35	21	9.38	20
8	7.31	19	10.95	23	13.85	22	14.69	23	14.77	23
16	11.61	23	18.03	28	23.22	27	25.11	29	25.33	27
32	19.97	29	31.30	36	38.91	34	44.50	38	45.24	33

fixed $\frac{H}{h}$, Table 5.6 and Table 5.8 show the results.

We find that the 3D case is very similar to the 2D case. This means that the condition numbers and iteration counts are independent of $\frac{H}{h}$ and the condition numbers depend linearly on the value of $\frac{H}{\delta}$. Moreover, they appear to be independent of the jumps of coefficients between subdomains. We see that the estimated condition numbers depend linearly on $\frac{H}{\delta}$ in Fig. 5.6 and Fig. 5.8. We also see that the estimated condition numbers do not depend on $\frac{H}{h}$ in Fig. 5.5 and Fig. 5.7. Our numerical results for the 3D case are consistent with our main result as well.

Table 5.5: Condition numbers and iteration counts. $\alpha_i = 1$ or specified values as indicated in a checkerboard pattern, $\beta_i \equiv 1$ and $\frac{H}{\delta} = 3$ (3D case)

	$\alpha_i = 0.01$		$\alpha_i = 0.1$		$\alpha_i = 1$		$\alpha_i = 10$		$\alpha_i = 100$	
$\frac{H}{h}$	cond	iters	cond	iters	cond	iters	cond	iters	cond	iters
3	8.37	19	8.70	19	9.47	20	9.68	20	9.71	20
6	8.44	19	8.70	20	9.51	20	9.73	21	9.76	23
12	8.46	20	8.67	21	9.52	21	9.74	22	9.73	23

Table 5.6: Condition numbers and iteration counts. $\alpha_i = 1$ or specified values as indicated in a checkerboard pattern, $\beta_i \equiv 1$ and $\frac{H}{h} = 12$ (3D case)

	$\alpha_i = 0.01$		$\alpha_i = 0.1$		$\alpha_i = 1$		$\alpha_i = 10$		$\alpha_i = 100$	
$\frac{H}{\delta}$	cond	iters	cond	iters	cond	iters	cond	iters	cond	iters
3	8.46	20	8.67	21	9.52	21	9.74	22	9.73	23
6	9.69	21	12.21	23	15.91	23	16.66	26	16.75	26
12	13.61	23	19.05	27	27.33	28	29.30	28	29.53	28

Table 5.7: Condition numbers and iteration counts. $\beta_i = 1$ or specified values as indicated in a checkerboard pattern, $\alpha_i \equiv 1$ and $\frac{H}{\delta} = 3$ (3D case)

	$\beta_i = 0.01$		$\beta_i = 0.1$		$\beta_i = 1$		$\beta_i = 10$		$\beta_i = 100$	
$\frac{H}{h}$	cond	iters	cond	iters	cond	iters	cond	iters	cond	iters
3	8.47	21	9.02	20	9.47	20	8.85	20	8.38	20
6	8.38	21	9.06	21	9.51	20	8.84	21	8.39	20
12	8.34	21	9.08	21	9.52	21	8.81	21	8.39	20

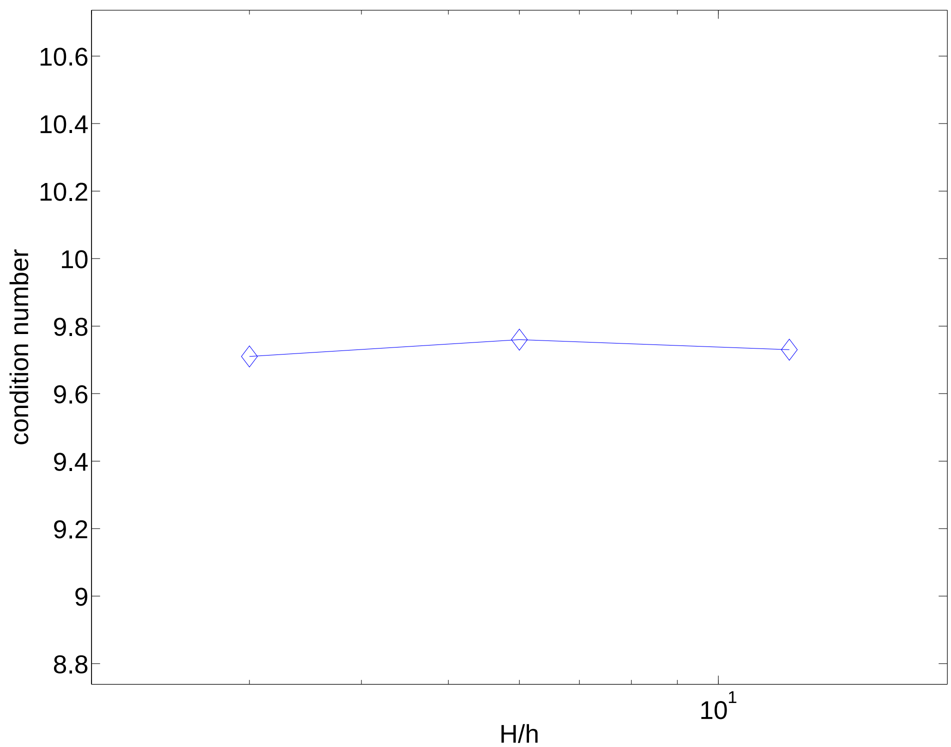

Figure 5.5: Estimated condition number, versus $\frac{H}{h}$; $\alpha_i = 1$ and $\alpha_i = 100$ in a checkerboard pattern, $\beta_i \equiv 1$ and $\frac{H}{\delta} = 3$ (3D case)

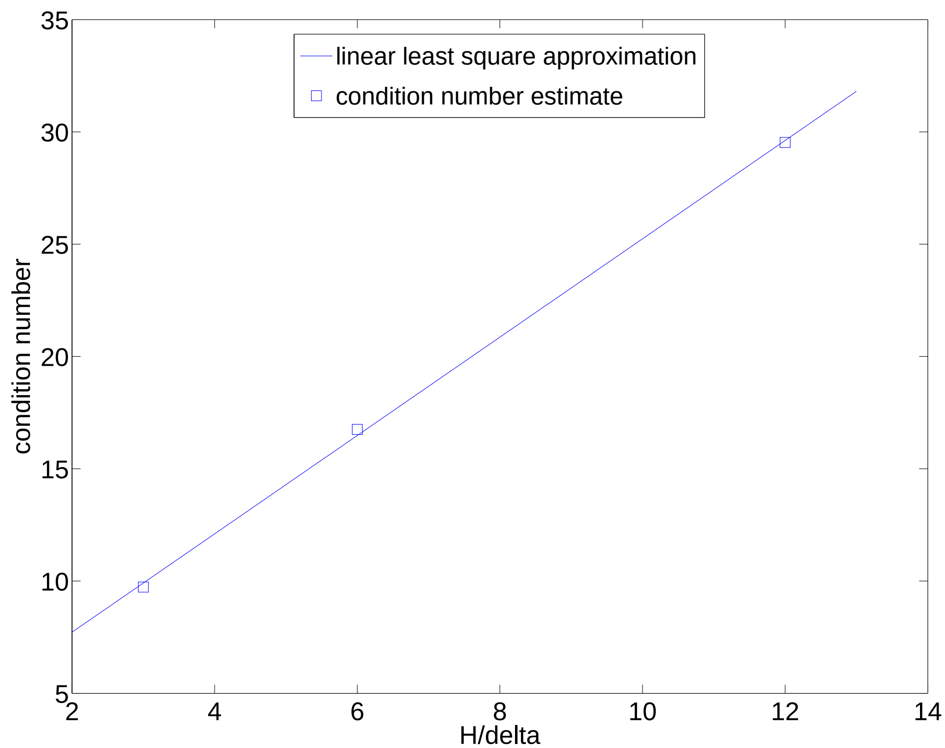

Figure 5.6: Estimated condition number and linear least square fitting, versus $\frac{H}{\delta}$; $\alpha_i = 1$ and $\alpha_i = 100$ in a checkerboard pattern, $\beta_i \equiv 1$ and $\frac{H}{h} = 12$ (3D case)

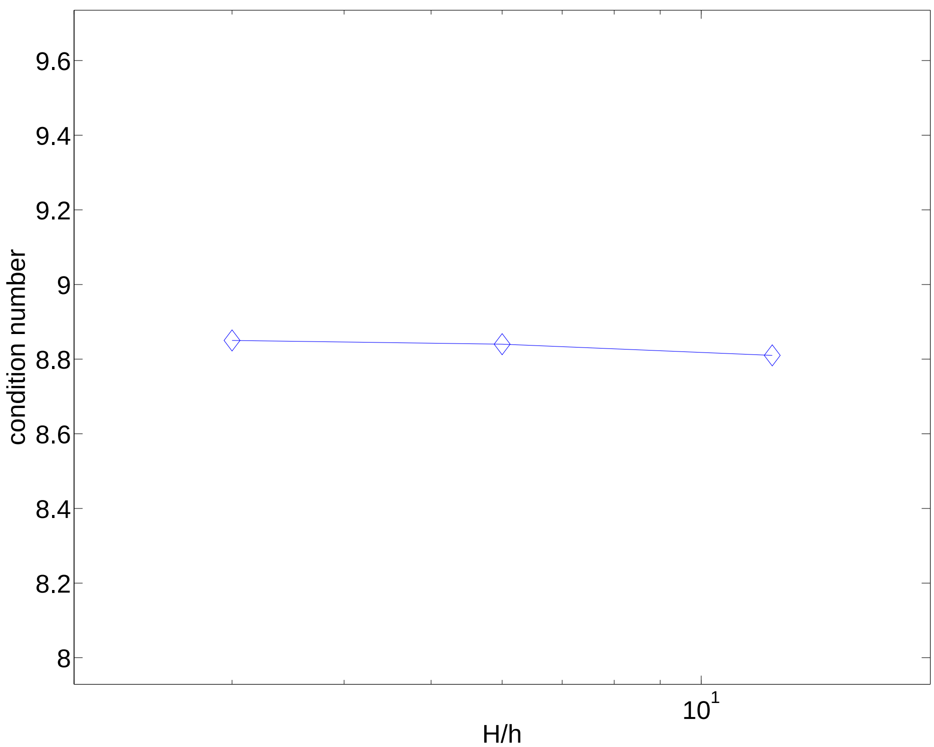

Figure 5.7: Estimated condition number, versus $\frac{H}{h}$; $\beta_i = 1$ and $\beta_i = 10$ in a checkerboard pattern, $\alpha_i \equiv 1$ and $\frac{H}{\delta} = 3$ (3D case)

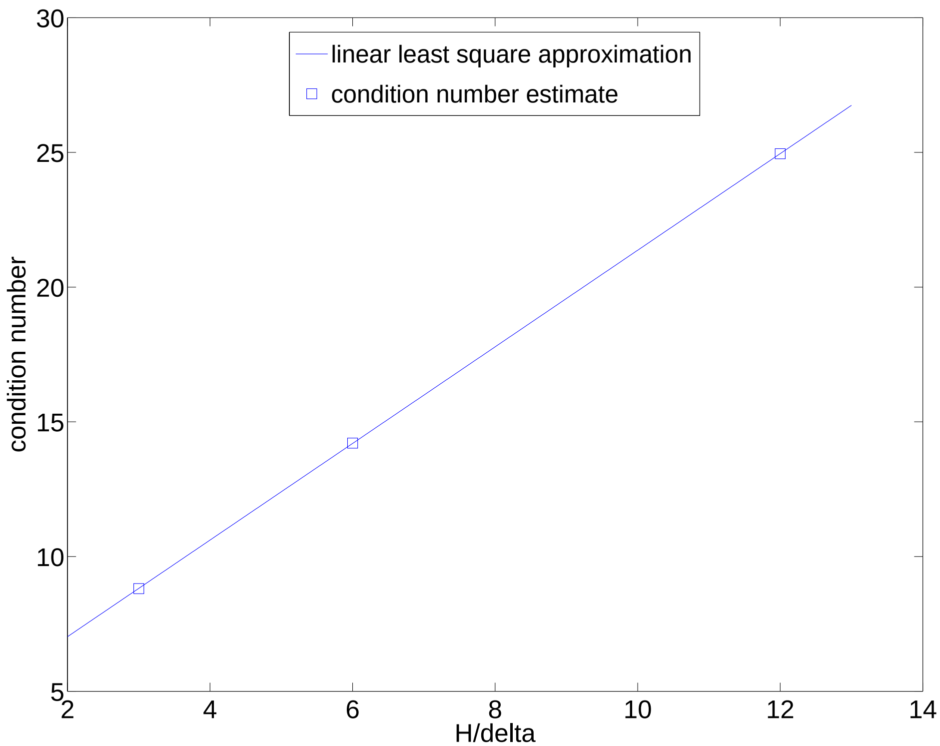

Figure 5.8: Estimated condition number and linear least square fitting, versus $\frac{H}{\delta}$; $\beta_i = 1$ and $\beta_i = 10$ in a checkerboard pattern, $\alpha_i \equiv 1$ and $\frac{H}{h} = 12$ (3D case)

Table 5.8: Condition numbers and iteration counts. $\beta_i = 1$ or specified values as indicated in a checkerboard pattern, $\alpha_i \equiv 1$ and $\frac{H}{h} = 12$ (3D case)

	$\beta_i = 0.01$		$\beta_i = 0.1$		$\beta_i = 1$		$\beta_i = 10$		$\beta_i = 100$	
$\frac{H}{\delta}$	cond	iters	cond	iters	cond	iters	cond	iters	cond	iters
3	8.34	21	9.08	21	9.52	21	8.81	21	8.39	20
6	10.14	23	15.17	23	15.91	23	14.21	22	9.65	21
12	15.31	23	27.22	27	27.33	28	24.95	26	14.14	23

5.4.3 Parallel Experiments

5.4.3.1 Portable Extensible Toolkit for Scientific Computation

We have developed parallel C codes for our algorithm using the Portable Extensible Toolkit for Scientific Computation (PETSc) library.

PETSc is being developed at the Mathematics and Computer Science Division (MCS) at Argonne National Laboratory (ANL); see [4–6]. It is a useful toolkit for writing large-scale codes on parallel (and serial) machines. PETSc provides a set of data structures and routines. The data structures involve both sequential and parallel index sets, vectors, and matrices and the set of routines contain linear and non-linear solver. Hence, it provides an easier way of implementing numerical methods defined by users. Furthermore, it supports numerous runtime options and debugging tools.

PETSc uses the Message Passing Interface (MPI) standard and is layered on top of MPI. It has intermediate tools to send or receive datatypes between processors. Hence, users do not need to know much MPI when using the system. Moreover, PETSc is transparent to the users. This means that same code works well on serial and parallel machines. It also supports interfaces for external software packages.

Domain decomposition methods are natural methods for effective parallel algorithms for distributed memory computers. Therefore, PETSc is a good choice to implement domain decomposition methods. Actually, it was originally designed by Barry Smith to provide a library for domain decomposition algorithms.

5.4.3.2 Parallel Experiments

We use 4×4 subdomains for the $2D$ problem. We stop the iteration after a reduction of 10^{-8} of the l^2-norm of the initial residual. We report the total times of all computation, including the assembling of the stiffness matrix, the local and coarse factorizations, and preconditioned conjugate gradient iterations, in seconds with various number of processors and degrees of freedom. Table 5.9 and Table 5.10 show the results. We use one layer of overlap for the first experiments and fixed $\frac{H}{\delta}$ for the second experiments.

We note that these codes were tested on *Crunchy* machine at the Courant Institute.

Table 5.9: The total times of computation with different number of processors and degrees of freedom. (4×4 subdomains, one layer of overlap)

	number of degrees of freedom				
# of proc	3136	12146	49408	197120	787456
1	0.383	0.617	1.736	8.225	53.814
2	0.245	0.391	1.080	4.829	29.914
4	0.118	0.205	0.636	2.882	16.924
8	0.100	0.165	0.505	2.244	13.539
16	0.116	0.181	0.502	1.953	11.041

Table 5.10: The total times of computation with different number of processors and degrees of freedom. (4×4 subdomains, $H/\delta = 8$)

	number of degrees of freedom				
# of proc	3136	12146	49408	197120	787456
1	0.383	0.581	1.704	8.563	58.612
2	0.245	0.369	1.019	4.848	31.916
4	0.118	0.195	0.597	2.789	17.178
8	0.100	0.151	0.459	1.984	12.580
16	0.116	0.165	0.427	1.678	9.576

Chapter 6

A BDDC Algorithm for Raviart-Thomas Vector Fields

6.1 Introduction

Two main families of iterative substructuring methods are the BNN type and the FETI type algorithms; see Section 4.6 and [26–28,37,42]. The BDDC methods, introduced by Dohrmann in [18], are modified BNN methods with a global component obtained by using primal continuity constraints. For a pioneering analysis, see also [43, 44]. The connection between the BNN and BDDC is quite similar to that of one level FETI and FETI-DP. An advantage of BDDC methods over the older BNN algorithms is that all matrices of BDDC methods are nonsingular.

In this chapter, we will consider BDDC algorithms for vector field problems formulated in $H(\mathrm{div})$. Nonoverlapping domain decomposition methods for vector field problems were first considered in [71]. We will use some auxiliary results from that paper to analyze our methods. Later BNN, FETI, and FETI-DP methods for these kinds of problems were developed in [63, 65, 66]. Other methods such as multigrid methods and overlapping Schwarz methods have also been introduced for vector field problems; see [2,3,32,34,35,52,64]. BDDC methods for other problems such as incompressible Stokes, almost incompressible elasticity, and flow in porous media have been proposed in [39, 54, 68, 69]. While many BDDC algorithms have been studied for one variable coefficient, we are faced with two sets of coefficients. We will try various weighted averages to deal with this difficulty.

Due to the fact that preconditioned linear systems from the BDDC and FETI-DP algorithms have the same spectrum except for possible eigenvalues at 0 and 1, we can also apply our result to FETI-DP methods with the same primal constraints; see [13, 40, 44].

The rest of this chapter is organized as follows. We describe the algorithm in Section 6.2. In Section 6.3, we introduce some useful technical tools and present our main result. Finally, Section 6.4 contains supporting numerical experiments.

6.2 The BDDC Method

6.2.1 Notations

We recall the notations in Section 4.4. $W^{(i)}$ is the space of lowest order Raviart-Thomas finite elements on Ω_i with a zero normal component on $\partial\Omega \cap \partial\Omega_i$. Each $W^{(i)}$ is decomposed into two parts: the interior part $W_I^{(i)}$ and the interface part $W_\Gamma^{(i)}$. Furthermore, we can decompose the interface part $W_\Gamma^{(i)}$ into a primal space and a dual space: $W_\Pi^{(i)}$ and $W_\Delta^{(i)}$, respectively.

We consider $W_\Gamma := \prod_{i=1}^{N} W_\Gamma^{(i)}$. We note that generally the functions in W_Γ have discontinuous normal components across the interface. We denote the space with continuous normal components by $\widehat{W}_\Gamma$. We next consider an intermediate space $\widetilde{W}_\Gamma$. The space $\widetilde{W}_\Gamma$ consists of functions which satisfy the primal constraints.

We also introduce some spaces related to $W_\Gamma^{(i)}$. We define $W_{0,\Gamma}^{(i)}$ as the subspace of $W_\Gamma^{(i)}$ with mean value zero over Γ_i, where $\Gamma_i := \Gamma \cap \partial\Omega_i$. Finally, let $W_{H,\Gamma}^{(i)}$ denote the space of constant functions on each subdomain edge or face.

6.2.2 The Algorithm

We just recall the algorithm in Section 4.6.1. We first consider the interface problem (4.22):

$$\widehat{S}_\Gamma u_\Gamma = g_\Gamma, \tag{6.1}$$

where

$$\widehat{S}_\Gamma = \sum_{i=1}^N R_\Gamma^{(i)T} S^{(i)} R_\Gamma^{(i)}$$

and

$$g_\Gamma = \sum_{i=1}^N R_\Gamma^{(i)T} \left\{ \begin{bmatrix} f_\Delta \\ f_\Pi \end{bmatrix} - \begin{bmatrix} A_{\Delta I}^{(i)} \\ A_{\Pi I}^{(i)} \end{bmatrix} A_{II}^{(i)-1} f_I^{(i)} \right\}.$$

We next consider the BDDC preconditioner:

$$M^{-1} = \widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma},$$

where

$$\widetilde{S}_\Gamma^{-1} := R_{\Gamma\Delta}^T \left(\sum_{i=1}^N \begin{bmatrix} 0 & R_\Delta^{(i)T} \end{bmatrix} \begin{bmatrix} A_{II}^{(i)} & A_{I\Delta}^{(i)} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ R_\Delta^{(i)} \end{bmatrix} \right) R_{\Gamma\Delta} + \Phi S_{\Pi\Pi}^{-1} \Phi^T$$

with

$$\Phi := R_{\Gamma\Pi}^T - R_{\Gamma\Delta}^T \sum_{i=1}^{N} \begin{bmatrix} 0 & R_{\Delta}^{(i)T} \end{bmatrix} \begin{bmatrix} A_{II}^{(i)} & A_{I\Delta}^{(i)} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{bmatrix}^{-1} \begin{bmatrix} A_{\Pi I}^{(i)T} \\ A_{\Pi\Delta}^{(i)T} \end{bmatrix} R_{\Pi}^{(i)}$$

and

$$S_{\Pi\Pi} := \sum_{i=1}^{N} R_{\Pi}^{(i)T} \left(A_{\Pi\Pi}^{(i)} - \begin{bmatrix} A_{\Pi I}^{(i)} & A_{\Pi\Delta}^{(i)} \end{bmatrix} \begin{bmatrix} A_{II}^{(i)} & A_{I\Delta}^{(i)} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{bmatrix}^{-1} \begin{bmatrix} A_{\Pi I}^{(i)T} \\ A_{\Pi\Delta}^{(i)T} \end{bmatrix} \right) R_{\Pi}^{(i)}.$$

6.2.3 Remarks on the Implementation

We use the preconditioned conjugate gradient method for the following preconditioned global interface problem:

$$M^{-1}\widehat{S}_{\Gamma}u_{\Gamma} = M^{-1}g_{\Gamma}.$$

The preconditioned conjugate gradient method algorithm is given in Figure 6.1.

We remark that we do not need $R_{\Gamma}^{(i)}, R_{\Pi}^{(i)}, \widetilde{R}_{D,\Gamma}, R_{\Gamma\Pi}$ and $R_{\Gamma\Delta}$ explicitly. By appropriate indexing, we can perform these procedures without constructing the restriction operators. We also note that we can compute $S^{(i)}$ times a vector by solving a local Dirichlet problem and some sparse matrix-vector products. This local work can be done efficiently in parallel.

Initialize:

$r_0 := g_\Gamma - \left(\sum_{i=1}^N R_\Gamma^{(i)T} S^{(i)} R_\Gamma^{(i)}\right) u_{\Gamma,0}$

$z_0 := \left(\widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma}\right) r_0$

$p_0 := z_0$

Iterate $k = 0, 1, \cdots$ until convergence

$$q_k := \left(\sum_{i=1}^N R_\Gamma^{(i)T} S^{(i)} R_\Gamma^{(i)}\right) p_k$$

$$\alpha_k := r_k^T z_k / p_k^T q_k$$

$$u_{\Gamma,k+1} := u_{\Gamma,k} + \alpha_k p_k$$

$$r_{k+1} := r_k - \alpha_k q_k$$

$$z_{k+1} := \left(\widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma}\right) r_{k+1}$$

$$\beta_k := z_{k+1}^T r_{k+1} / z_k^T r_k$$

$$p_{k+1} := z_{k+1} + \beta_k p_k$$

Figure 6.1: Implementation of the BDDC method as a preconditioned conjugate gradient method.

6.3 Technical Tools and the Main Result

6.3.1 Technical Tools

We will borrow some useful technical tools from [71].

Lemma 6.3.1. (Divergence free extension) *There exists an extension operator* $\widetilde{\mathcal{H}}_i : W_{0,\Gamma}^{(i)} \to W^{(i)}$, *which satisfies*

$$(\widetilde{\mathcal{H}}_i \mu) \cdot \mathbf{n} = \mu, \ \mathrm{div} \widetilde{\mathcal{H}}_i \mu = 0,$$

for all $\mu \in W_{0,\Gamma}^{(i)}$*, and which satisfies the following estimate:*

$$\|\widetilde{\mathcal{H}}_i\mu\|_{0;\Omega_i} \leqslant C\|\mu\|_{-\frac{1}{2};\partial\Omega_i}.$$

Proof. See [71, Lemma 4.3] and [70, Lemma 2.6]. □

Lemma 6.3.2. (Discrete harmonic extension) *There exists a discrete harmonic extension operator* $\mathcal{H}_i : W_{0,\Gamma}^{(i)} \to W^{(i)}$*, which satisfies*

$$(\mathcal{H}_i\mu) \cdot \mathbf{n} = \mu,$$

for all $\mu \in W_{0,\Gamma}^{(i)}$*, and which satisfies*

$$\alpha_i\|\operatorname{div}\mathcal{H}_i\mu\|_{0;\Omega_i}^2 + \beta_i\|\mathcal{H}_i\mu\|_{0;\Omega_i}^2 \leqslant C\beta_i\|\mu\|_{-\frac{1}{2};\partial\Omega_i}^2.$$

Proof. $\mathcal{H}_i$ is the minimal-energy extension for a given subdomain. Therefore, we obtain the following estimate:

$$\alpha_i\|\operatorname{div}\mathcal{H}_i\mu\|_{0;\Omega_i}^2 + \beta_i\|\mathcal{H}_i\mu\|_{0;\Omega_i}^2 \leqslant \alpha_i\|\operatorname{div}\widetilde{\mathcal{H}}_i\mu\|_{0;\Omega_i}^2 + \beta_i\|\widetilde{\mathcal{H}}_i\mu\|_{0;\Omega_i}^2.$$

But $\operatorname{div}\widetilde{\mathcal{H}}_i\mu = 0$ and by Lemma 6.3.1,

$$\alpha_i\|\operatorname{div}\mathcal{H}_i\mu\|_{0;\Omega_i}^2 + \beta_i\|\mathcal{H}_i\mu\|_{0;\Omega_i}^2 \leqslant C\beta_i\|\mu\|_{-\frac{1}{2};\partial\Omega_i}^2.$$

□

We next introduce partition of unity functions associated with faces of each subdomain Ω_i as defined in [67, Chapter 10.2.1]. Let $\zeta_{\mathcal{F}}$ be the characteristic function of $\mathcal{F}$ given by:

$$\zeta_{\mathcal{F}}(x) = \begin{cases} 1, & x \in \mathcal{F} \\ 0, & x \in \partial\Omega_i \backslash \mathcal{F}. \end{cases}$$

We then have

$$\sum_{\mathcal{F} \subset \partial\Omega_i} \zeta_{\mathcal{F}} \equiv 1, \text{a.e. on } \partial\Omega_i \backslash \Omega.$$

We have the following estimates for the face components:

Lemma 6.3.3. *Consider $W_{0,F}^{(i)}$, the subspace of $W_{0,\Gamma}^{(i)}$ with a vanishing normal component on $\partial\Omega_i \backslash F$, i.e., $\int_F \mu ds = 0$ if $\mu \in W_{0,F}^{(i)}$. Let $\mu_F \in W_{0,F}^{(i)}, F \subset \partial\Omega_i$ and $\mu := \sum_{F \subset \partial\Omega_i} \mu_F$. Then, for all $\mu_H \in W_{H,\Gamma}^{(i)}$,*

$$\|\mu_F\|^2_{-\frac{1}{2};\partial\Omega_i} \leqslant C(1 + \log\frac{H_i}{h_i})((1 + \log\frac{H_i}{h_i})\|\mu + \mu_H\|^2_{-\frac{1}{2};\partial\Omega_i} + \|\mu\|^2_{-\frac{1}{2};\partial\Omega_i}),$$

where C is independent not only of μ_H but also of h.

Proof. See [71, Lemma 4.4] and [70, Lemma 2.7]. □

Lemma 6.3.4. *Let Ω_i and Ω_j be two adjacent subdomains with a common face F. Let μ be a function in $H^{-\frac{1}{2}}(\partial\Omega_i)$, which vanishes outside of F. Then, there exists a constant C, such that*

$$\|\mu\|_{-\frac{1}{2};\partial\Omega_i} \leqslant C\|\mu\|_{-\frac{1}{2};\partial\Omega_j}.$$

Proof. See [62, Lemma 5.5.2]. □

Lemma 6.3.5. (Stable interpolation) *Let $\partial\Omega_i$ be Lipschitz. Then, there exist two operators $\Pi_h^S : H(\text{div}\,; \Omega_i) \to W^{(i)}$ and $P_h^S : L^2(\Omega_i) \to P_0(\Omega_i)$ which satisfy the* *following commutative property and invariance property:*

$$\operatorname{div}\left(\Pi_h^S \mathbf{u}\right) = P_h^S \left(\operatorname{div} \mathbf{u}\right), \tag{6.2}$$

$$\Pi_h^S \mathbf{u} = \mathbf{u}, \quad \forall \mathbf{u} \in W^{(i)}, \tag{6.3}$$

and

$$P_h^S v = v, \quad \forall v \in P_0(\Omega_i). \tag{6.4}$$

Moreover, the two operators are L^2-stable, i.e.,

$$\|\Pi_h^S \mathbf{u}\|_{0;\Omega_i} \leqslant C\|\mathbf{u}\|_{0;\Omega_i} \tag{6.5}$$

and

$$\|P_h^S v\|_{0;\Omega_i} \leqslant C\|v\|_{0;\Omega_i}. \tag{6.6}$$

Proof. See [16]. □

Remark 6.3.1. *The operators in [16] are designed for essential boundary conditions. For natural boundary conditions, see [58] and [59]. The references also introduce interpolations for H^1 and $H(\mathbf{curl})$.*

We introduce the following extension lemma from [33].

Lemma 6.3.6. *Let Ω be a bounded Lipschitz domain. Then, there exists an extension operator*

$$\mathcal{E} : H(\operatorname{div}; \Omega) \to H(\operatorname{div}; \mathbb{R}^3) \tag{6.7}$$

satisfying $\mathcal{E}\mathbf{u} = \mathbf{u}$ a.e. in Ω and the following continuity conditions:

$$\|\operatorname{div}\left(\mathcal{E}\mathbf{u}\right)\|_{0;\mathbb{R}^3} \leqslant C\|\operatorname{div} \mathbf{u}\|_{0;\Omega}, \tag{6.8}$$

and

$$\|\mathcal{E}\mathbf{u}\|_{0;\mathbb{R}^3} \leqslant C\|\mathbf{u}\|_{0;\Omega}. \tag{6.9}$$

Proof. This is just a special case of [33, Theorem 3.6]. □

We now consider an extension lemma for Raviart-Thomas elements.

Lemma 6.3.7. (Extension lemma) *Let Ω_i and Ω_j be two adjacent subdomains with a common face F_{ij}. We consider two subspaces $V^{(i)}$ and $V^{(j)}$, subspaces of $W^{(i)}$ and $W^{(j)}$, respectively:*

$$V^{(i)} := \left\{\mathbf{u}_h \in W^{(i)} \,|\, \mathbf{u}_h \cdot \mathbf{n} = 0 \text{ on } \partial\Omega_i \backslash F_{ij}\right\}$$

and

$$V^{(j)} := \left\{\mathbf{u}_h \in W^{(j)} \,|\, \mathbf{u}_h \cdot \mathbf{n} = 0 \text{ on } \partial\Omega_j \backslash F_{ij}\right\}.$$

Then, there exists an extension operator $E_{ji} : V^{(j)} \to V^{(i)}$ such that for all $\mathbf{u}_h \in V^{(j)}$,

$$(E_{ji}\mathbf{u}_h)\,|_{\Omega_j} = \mathbf{u}_h,$$

$$\|\mathrm{div}\,(E_{ji}\mathbf{u}_h)\,\|_{0;\Omega_i} \leqslant C\|\mathrm{div}\,\mathbf{u}_h\|_{0;\Omega_j}, \tag{6.10}$$

and

$$\|E_{ji}\mathbf{u}_h\|_{0;\Omega_i} \leqslant C\|\mathbf{u}_h\|_{0;\Omega_j}. \tag{6.11}$$

Proof. We use an idea similar to that of [36, Lemma 4.5]. Let $\mathcal{C}\Omega_i$ be the complement of Ω_i. By Lemma 6.3.6, there exists an extension operator $E_{\mathcal{C}\Omega_i} : H(\mathrm{div}\,;\mathcal{C}\Omega_i) \to H(\mathrm{div}\,;\mathbb{R}^3)$ such that

$$\|\mathrm{div}\,(E_{\mathcal{C}\Omega_i}\mathbf{u})\,\|_{0;\mathbb{R}^3} \leqslant C\|\mathrm{div}\,\mathbf{u}\|_{0;\mathcal{C}\Omega_i} \tag{6.12}$$

and

$$\|E_{\mathcal{C}\Omega_i}\mathbf{u}\|_{0;\mathbb{R}^3} \leqslant C\|\mathbf{u}\|_{0;\mathcal{C}\Omega_i} \tag{6.13}$$

with

$$\left(E_{\mathcal{C}\Omega_i}\mathbf{u}\right)|_{\mathcal{C}\Omega_i} = \mathbf{u} \tag{6.14}$$

for all $\mathbf{u} \in H(\mathrm{div}\,;\mathcal{C}\Omega_i)$.

For $\mathbf{u}_h \in V^{(j)}$, let $\widetilde{\mathbf{u}}$ be an extension of $\mathbf{u}_h$ by zero outside of $\Omega_i \cup F_{ij} \cup \Omega_j$. We note that $\|\widetilde{\mathbf{u}}\|_{0;\mathcal{C}\Omega_i} = \|\mathbf{u}_h\|_{0;\Omega_j}$ and $\|\mathrm{div}\,\widetilde{\mathbf{u}}\|_{0;\mathcal{C}\Omega_i} = \|\mathrm{div}\,\mathbf{u}_h\|_{0;\Omega_j}$. We now consider an extension operator E_{ji} defined by

$$E_{ji} := \begin{cases} \Pi_h^S\left(E_{\mathcal{C}\Omega_i}\widetilde{\mathbf{u}}\right) & \text{in} \quad \Omega_i \cup F_{ij} \cup \Omega_j, \\ 0 & \text{otherwise.} \end{cases}$$

By (6.2), (6.6), and (6.12), we have

$$\begin{aligned} \|\mathrm{div}\left(E_{ji}\mathbf{u}_h\right)\|_{0;\Omega_i} &= \|\mathrm{div}\left(\Pi_h^S\left(E_{\mathcal{C}\Omega_i}\widetilde{\mathbf{u}}\right)\right)\|_{0;\Omega_i} = \|P_h^S\left(\mathrm{div}\left(E_{\mathcal{C}\Omega_i}\widetilde{\mathbf{u}}\right)\right)\|_{0;\Omega_i} \\ &\leqslant C\|\mathrm{div}\left(E_{\mathcal{C}\Omega_i}\widetilde{\mathbf{u}}\right)\|_{0;\Omega_i} \leqslant C\|\mathrm{div}\left(E_{\mathcal{C}\Omega_i}\widetilde{\mathbf{u}}\right)\|_{0;\mathbb{R}^3} \\ &\leqslant C\|\mathrm{div}\,\widetilde{\mathbf{u}}\|_{0;\mathcal{C}\Omega_i} = C\|\mathrm{div}\,\mathbf{u}_h\|_{0;\Omega_j}. \end{aligned}$$

Moreover, by (6.5) and (6.13), we obtain

$$\begin{aligned} \|E_{ji}\mathbf{u}_h\|_{0;\Omega_i} &= \|\Pi_h^S\left(E_{\mathcal{C}\Omega_i}\widetilde{\mathbf{u}}\right)\|_{0;\Omega_i} \leqslant C\|E_{\mathcal{C}\Omega_i}\widetilde{\mathbf{u}}\|_{0;\Omega_i} \\ &\leqslant C\|E_{\mathcal{C}\Omega_i}\widetilde{\mathbf{u}}\|_{0;\mathbb{R}^3} \leqslant C\|\widetilde{\mathbf{u}}\|_{0;\mathcal{C}\Omega_i} = C\|\mathbf{u}_h\|_{0;\Omega_j}. \end{aligned}$$

Trivially, for all $\mathbf{u}_h \in V^{(j)}$, $\left(E_{ji}^k\mathbf{u}_h\right)|_{\Omega_j} = \mathbf{u}_h$. $\square$

6.3.2 Stability Estimates

Let $\|u_\Gamma\|^2_{\widetilde{S}_\Gamma} := u_\Gamma^T \widetilde{S}_\Gamma u_\Gamma$. The averaging operator E_D, defined in (4.16), satisfies the following estimate:

Lemma 6.3.8. *For all $u_\Gamma \in \widetilde{W}_\Gamma$, there is a constant C, such that*

$$\|E_D u_\Gamma\|^2_{\widetilde{S}_\Gamma} \leqslant C\eta \max_{1\leqslant i\leqslant N}(1+\log\frac{H_i}{h_i})^2 \|u_\Gamma\|^2_{\widetilde{S}_\Gamma}, \tag{6.15}$$

where $\eta = \max\limits_{1\leqslant i\leqslant N} \max\left\{\dfrac{\beta_i H_i^2}{\alpha_i}, 1\right\}$.

Proof. Let $u_i := \overline{R}_\Gamma^{(i)} u_\Gamma$. We also consider $u_\Gamma^0 := \sum_{i=1}^N \overline{R}_\Gamma^{(i)^T} \left(\left(\Pi_H^{RT}\left(\mathcal{H}_i u_i\right)\right)\cdot \mathbf{n}\right)$ and $v_\Gamma := u_\Gamma - u_\Gamma^0$. We note that $v_i := \overline{R}_\Gamma^{(i)} v_\Gamma \in W_{0,\Gamma}^{(i)}$. We have

$$\|E_D v_\Gamma\|^2_{\widetilde{S}_\Gamma} = \|\overline{R}_\Gamma\left(E_D v_\Gamma\right)\|^2_{S_\Gamma} = \sum_{i=1}^N \|\overline{R}_\Gamma^{(i)}\left(E_D v_\Gamma\right)\|^2_{S_\Gamma^{(i)}}.$$

We set $w_i := \overline{R}_\Gamma^{(i)}(E_D v_\Gamma) = \delta_i^\dagger v_i + \delta_j^\dagger v_j$ on F_{ij}.

We set $\chi_i = \beta_i$ and $\chi_j = \beta_j$ in (4.11).

We note that the following inequality holds for $\gamma \in [\frac{1}{2}, \infty)$:

$$\beta_i {\delta_j^\dagger}^2 \leqslant \min\{\beta_i, \beta_j\}. \tag{6.16}$$

By Lemma 6.3.2, we have

$$\|w_i\|^2_{S_\Gamma^{(i)}} \leqslant C\beta_i \|w_i\|^2_{-\frac{1}{2};\partial\Omega_i} \leqslant C\beta_i \sum_{F_{ij}\subset\partial\Omega_i} \|\zeta_{F_{ij}} w_i\|^2_{-\frac{1}{2};\partial\Omega_i}. \tag{6.17}$$

Moreover, by Lemma 6.3.4 and (6.16), we obtain

$$
\begin{aligned}
\beta_i \|\zeta_{F_{ij}} w_i\|^2_{-\frac{1}{2};\partial\Omega_i} &\leqslant 2\beta_i \left({\delta_i^\dagger}^2 \|\zeta_{F_{ij}} v_i\|^2_{-\frac{1}{2};\partial\Omega_i} + {\delta_j^\dagger}^2 \|\zeta_{F_{ij}} v_j\|^2_{-\frac{1}{2};\partial\Omega_i} \right) \\
&\leqslant C\beta_i \left({\delta_i^\dagger}^2 \|\zeta_{F_{ij}} v_i\|^2_{-\frac{1}{2};\partial\Omega_i} + {\delta_j^\dagger}^2 \|\zeta_{F_{ij}} v_j\|^2_{-\frac{1}{2};\partial\Omega_j} \right) \\
&\leqslant C \left(\beta_i \|\zeta_{F_{ij}} v_i\|^2_{-\frac{1}{2};\partial\Omega_i} + \beta_j \|\zeta_{F_{ij}} v_j\|^2_{-\frac{1}{2};\partial\Omega_j} \right). \qquad (6.18)
\end{aligned}
$$

Let $\mu_H := \left(\overline{R}_\Gamma^{(i)} u_\Gamma^0 \right) \cdot \mathbf{n}$. Then, for any $F_{ij} \subset \partial\Omega_i$,

$$
\int_{F_{ij}} (u_i - \mu_H)\, ds = 0.
$$

Hence, we obtain the following estimate by using Lemma 6.3.3:

$$
\begin{aligned}
& \beta_i \|\zeta_{F_{ij}} v_i\|^2_{-\frac{1}{2};\partial\Omega_i} \\
\leqslant\ & \beta_i \|\zeta_{F_{ij}} (u_i - \mu_H)\|^2_{-\frac{1}{2};\partial\Omega_i} \\
\leqslant\ & C\beta_i \left(1 + \log \frac{H_i}{h_i}\right) \left(\left(1 + \log \frac{H_i}{h_i}\right) \|u_i\|^2_{-\frac{1}{2};\partial\Omega_i} + \|u_i - \mu_H\|^2_{-\frac{1}{2};\partial\Omega_i} \right) \\
\leqslant\ & C\beta_i \left(\left(1 + \log \frac{H_i}{h_i}\right)^2 \|u_i\|^2_{-\frac{1}{2};\partial\Omega_i} + \left(1 + \log \frac{H_i}{h_i}\right) \|\mu_H\|^2_{-\frac{1}{2};\partial\Omega_i} \right). \qquad (6.19)
\end{aligned}
$$

Moreover, by Lemma 2.1.5, we have

$$
\begin{aligned}
\beta_i \|u_i\|^2_{-\frac{1}{2};\partial\Omega_i} &\leqslant C \left(\beta_i H_i^2 \|\mathrm{div}\,(\mathcal{H}_i u_i)\|^2_{0;\Omega_i} + \beta_i \|\mathcal{H}_i u_i\|^2_{0;\Omega_i} \right) \\
&\leqslant C\eta \left(\alpha_i \|\mathrm{div}\,(\mathcal{H}_i u_i)\|^2_{0;\Omega_i} + \beta_i \|\mathcal{H}_i u_i\|^2_{0;\Omega_i} \right) \\
&= C\eta \|u_i\|^2_{S_\Gamma^{(i)}}. \qquad (6.20)
\end{aligned}
$$

By Lemma 2.1.5 and Lemma 5.3.1, we obtain

$$\begin{aligned}
\beta_i\|\mu_H\|^2_{-\frac{1}{2};\partial\Omega_i} &\leqslant C\beta_i\left(H_i^2\|\mathrm{div}\left(\Pi_H^{RT}\left(\mathcal{H}_i u_i\right)\right)\|^2_{0;\Omega_i} + \|\Pi_H^{RT}\left(\mathcal{H}_i u_i\right)\|^2_{0;\Omega_i}\right) \\
&\leqslant C\beta_i H_i^2\|\mathrm{div}\left(\mathcal{H}_i u_i\right)\|^2_{0;\Omega_i} \\
&+ C\left(1+\log\frac{H_i}{h_i}\right)\left(\beta_i\|\mathcal{H}_i u_i\|^2_{0;\Omega_i} + \beta_i H_i^2\|\mathrm{div}\left(\mathcal{H}_i u_i\right)\|^2_{0;\Omega_i}\right) \\
&\leqslant C\left(1+\log\frac{H_i}{h_i}\right)\left(\beta_i\|\mathcal{H}_i u_i\|^2_{0;\Omega_i} + \beta_i H_i^2\|\mathrm{div}\left(\mathcal{H}_i u_i\right)\|^2_{0;\Omega_i}\right) \\
&\leqslant C\eta\left(1+\log\frac{H_i}{h_i}\right)\left(\beta_i\|\mathcal{H}_i u_i\|^2_{0;\Omega_i} + \alpha_i\|\mathrm{div}\left(\mathcal{H}_i u_i\right)\|^2_{0;\Omega_i}\right) \\
&= C\eta\left(1+\log\frac{H_i}{h_i}\right)\|u_i\|^2_{S_\Gamma^{(i)}}.
\end{aligned} \tag{6.21}$$

In a similar way, we can obtain a bound for $\beta_j\|\zeta_{F_{ij}}v_j\|^2_{-\frac{1}{2};\partial\Omega_j}$. Therefore, we finally obtain the following estimate by using (6.17), (6.18), (6.19), (6.20), and (6.21):

$$\|w_i\|^2_{S_\Gamma^{(i)}} \leqslant C\eta\left(1+\log\frac{H_i}{h_i}\right)^2\|u_i\|^2_{S_\Gamma^{(i)}} + C\eta\left(1+\log\frac{H_j}{h_j}\right)^2\|u_j\|^2_{S_\Gamma^{(j)}}. \tag{6.22}$$

□

Theorem 6.3.9. (Condition number estimate) *The condition number of the preconditioned linear system $M^{-1}\widehat{S}_\Gamma u_\Gamma = M^{-1}g_\Gamma$ satisfies*

$$\kappa(M^{-1}\widehat{S}_\Gamma) \leqslant C\eta\max_{1\leqslant i\leqslant N}(1+\log\frac{H_i}{h_i})^2,$$

where $\eta = \max_{1\leqslant i\leqslant N}\max\left\{\frac{\beta_i H_i^2}{\alpha_i}, 1\right\}$.

Proof. We can obtain this main result by using Lemmas 6.3.8, 4.6.1, and 4.6.2. □

We do not see the unfavorable factor η in the numerical experiments in Section 6.4. These results suggest the following two conjectures.

Conjecture 6.3.10. *For all $u_\Gamma \in \widetilde{W}_\Gamma$, there is a constant C, such that*

$$\|E_D u_\Gamma\|^2_{\widetilde{S}_\Gamma} \leqslant C \max_{1 \leqslant i \leqslant N} (1 + \log \frac{H_i}{h_i})^2 \|u_\Gamma\|^2_{\widetilde{S}_\Gamma}. \tag{6.23}$$

Conjecture 6.3.11. (Condition number estimate) *The condition number of the preconditioned linear system $M^{-1}\widehat{S}_\Gamma u_\Gamma = M^{-1} g_\Gamma$ satisfies*

$$\kappa(M^{-1}\widehat{S}_\Gamma) \leqslant C \max_{1 \leqslant i \leqslant N} (1 + \log \frac{H_i}{h_i})^2. \tag{6.24}$$

6.4 Numerical Results

6.4.1 The 2D One Variable Coefficient Case

We have applied the BDDC algorithm to our model problem (3.1). For algorithmic details, we follow the method introduced in [40] and Section 4.6.1. We use common edge averages across the interface as primal constraints. We set $\Omega = [0,1]^2$ and decompose the unit square into N^2 square subdomains. Each subdomain has a side length $H = \frac{1}{N}$. Moreover, we assume that the coefficients α and β have jumps across the interface between the subdomains with the checkerboard pattern of Fig. 6.2. We discretize the model problem (3.1) by using the lowest order Raviart-Thomas finite elements for triangles and use the preconditioned conjugate gradient method to solve the discretized problem. The iteration is stopped when the l_2-norm of the residual has been reduced by a factor of 10^{-8}.

We have two kinds of experimental sets. We first fix the value of β and vary α and use $\chi_i(x) = \alpha_i$ for the scaling factor in (4.11). Second, we fix the value of α and use various value of β and use $\chi_i(x) = \beta_i$ for the scaling factor in (4.11). Table

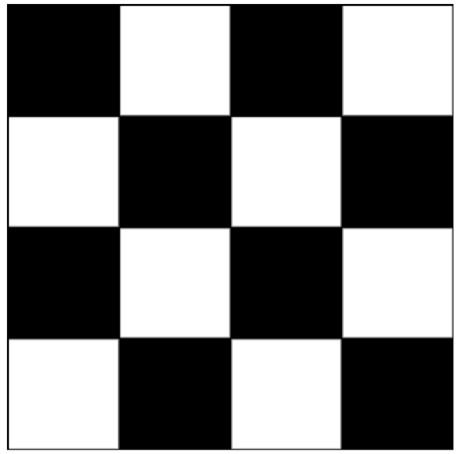

Figure 6.2: Checkerboard distribution of the coefficients (2D case)

6.1 and Table 6.2 show the results. In the graphs Fig 6.3 and Fig 6.4, we see that the condition numbers grow quadratically with the logarithm of $\frac{H}{h}$. Moreover, the condition number is insensitive to the jumps of coefficients.

Table 6.1: Condition numbers and iteration counts. $\alpha_i = 1$ or the specified value as indicated, in a checkerboard pattern and $\beta_i \equiv 1$ (2D case)

	$\alpha_i = 0.01$		$\alpha_i = 0.1$		$\alpha_i = 1$		$\alpha_i = 10$		$\alpha_i = 100$	
$\frac{H}{h}$	cond	iters	cond	iters	cond	iters	cond	iters	cond	iters
4	3.22	9	2.71	8	1.62	6	2.76	9	3.85	10
8	4.80	10	3.98	9	2.21	7	4.06	10	5.71	12
16	6.82	10	5.54	9	2.95	8	5.64	11	7.98	13
32	9.26	13	7.40	11	3.83	9	7.52	13	10.67	14
64	12.14	14	9.55	12	4.85	9	9.69	15	13.78	17
128	15.44	15	11.97	12	6.01	11	12.13	17	17.31	19

6.4.2 The 3D One Variable Coefficient Case

For the 3D case, we use the unit cube $[0,1]^3$ for Ω. In a way similar to the 2D case, we decompose the domain into N^3 subdomains with the side length $H = \frac{1}{N}$. We use the lowest order hexahedral Raviart-Thomas elements for this case and a

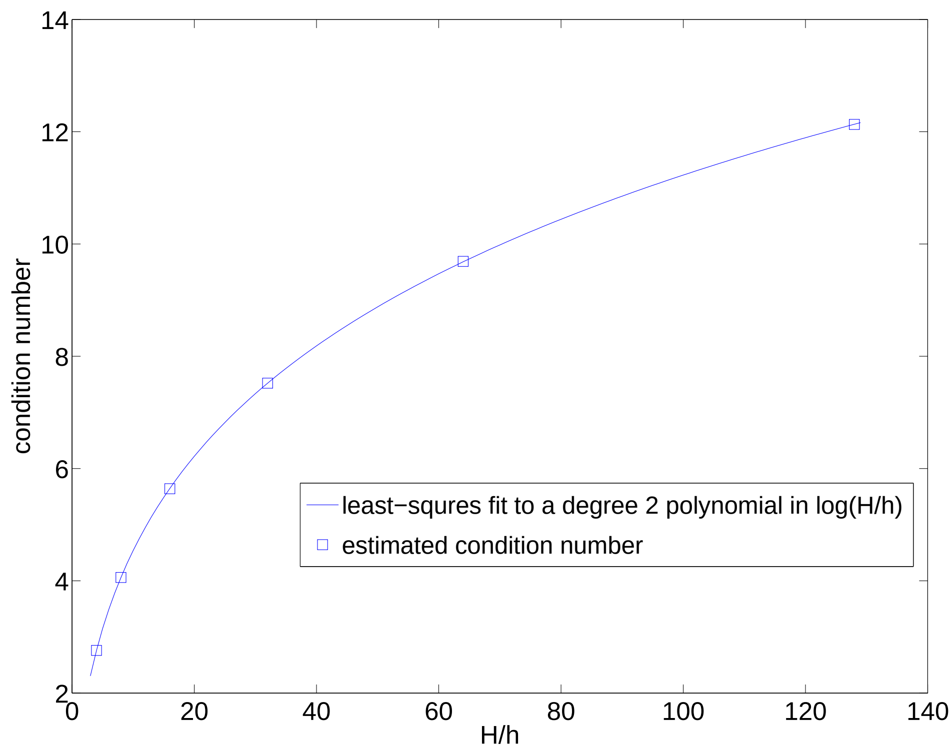

Figure 6.3: Estimated condition number and least-squares fit to a degree 2 polynomial in $\log \frac{H}{h}$, versus $\frac{H}{h}$; $\alpha_i = 1$ and $\alpha_i = 10$ in a checkerboard pattern and $\beta_i \equiv 1$ (2D case)

Table 6.2: Condition numbers and iteration counts. $\beta_i = 1$ or the specified value as indicated, in a checkerboard pattern and $\alpha_i \equiv 1$ (2D case)

	$\beta_i = 0.01$		$\beta_i = 0.1$		$\beta_i = 1$		$\beta_i = 10$		$\beta_i = 100$	
$\frac{H}{h}$	cond	iters	cond	iters	cond	iters	cond	iters	cond	iters
4	3.60	10	2.62	9	1.62	6	2.57	8	3.05	9
8	5.24	11	3.78	10	2.21	7	3.71	9	4.44	9
16	7.23	13	5.19	11	2.95	8	5.11	9	6.21	10
32	9.57	14	6.86	13	3.83	9	6.76	11	8.34	12
64	12.27	16	8.79	15	4.85	9	8.66	12	10.84	14
128	15.32	18	10.88	16	6.01	11	10.82	12	13.70	13

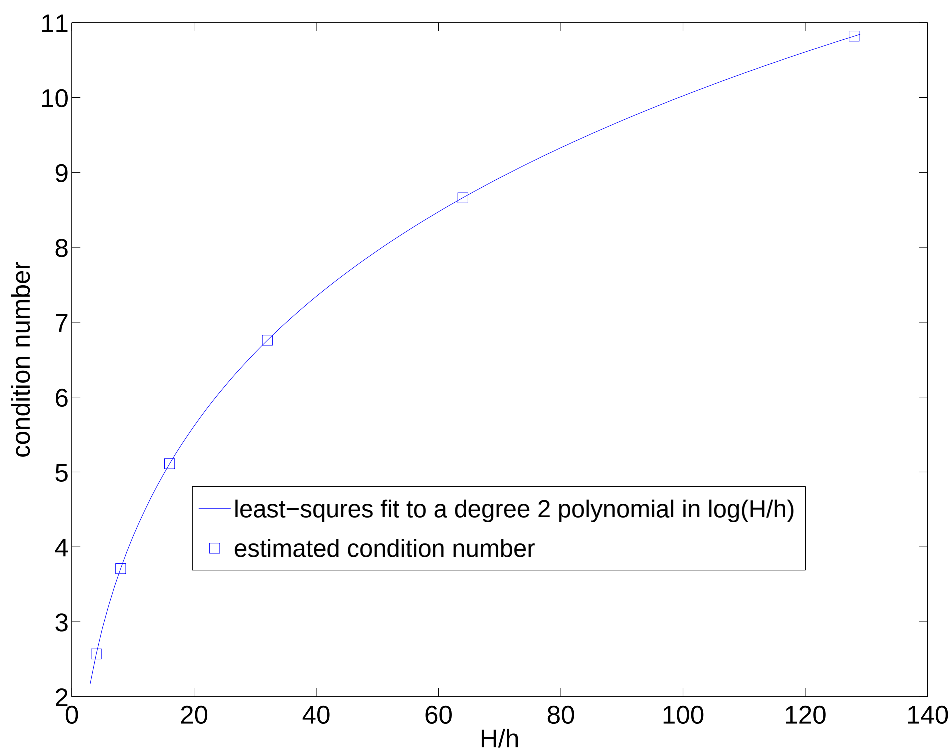

Figure 6.4: Estimated condition number and least-squares fit to a degree 2 polynomial in $\log \frac{H}{h}$, versus $\frac{H}{h}$; $\beta_i = 1$ and $\beta_i = 10$ in a checkerboard pattern and $\alpha_i \equiv 1$ (2D case)

similar checkerboard distribution of the coefficients as in the 2D case; see Fig. 6.5.

We use the stopping criteria of reducing the residual l_2-norm by a factor of 10^{-6}

for the preconditioned conjugate gradient method. We use common face averages across the interface as primal constraints. Other general settings are very similar to those of the 2D case.

We find that the results for the 3D case are quite similar to those of the 2D case; see Table 6.3, Table 6.4, Fig 6.6, and Fig 6.7. The condition numbers depend quadratically on the value of $\log \frac{H}{h}$ and are independent of the jumps of coefficients across the interface.

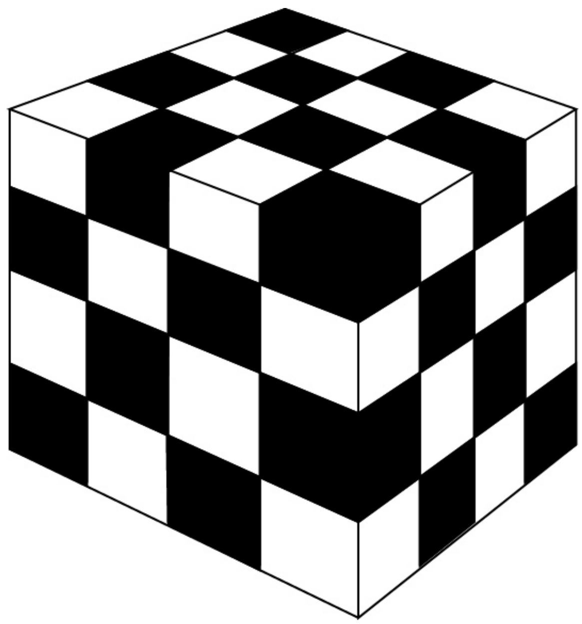

Figure 6.5: Checkerboard distribution of the coefficients (3D case)

Table 6.3: Condition numbers and iteration counts. $\alpha_i = 1$ or the specified value as indicated, in a checkerboard pattern and $\beta_i \equiv 1$ (3D case)

	$\alpha_i = 0.01$		$\alpha_i = 0.1$		$\alpha_i = 1$		$\alpha_i = 10$		$\alpha_i = 100$	
$\frac{H}{h}$	cond	iters	cond	iters	cond	iters	cond	iters	cond	iters
2	3.66	10	2.98	9	1.83	6	3.03	9	4.28	11
4	5.37	13	4.46	12	2.69	9	4.57	12	6.46	15
8	7.88	17	6.57	15	3.75	10	6.74	16	9.41	19
16	11.26	20	9.13	18	5.01	13	9.29	18	13.12	22

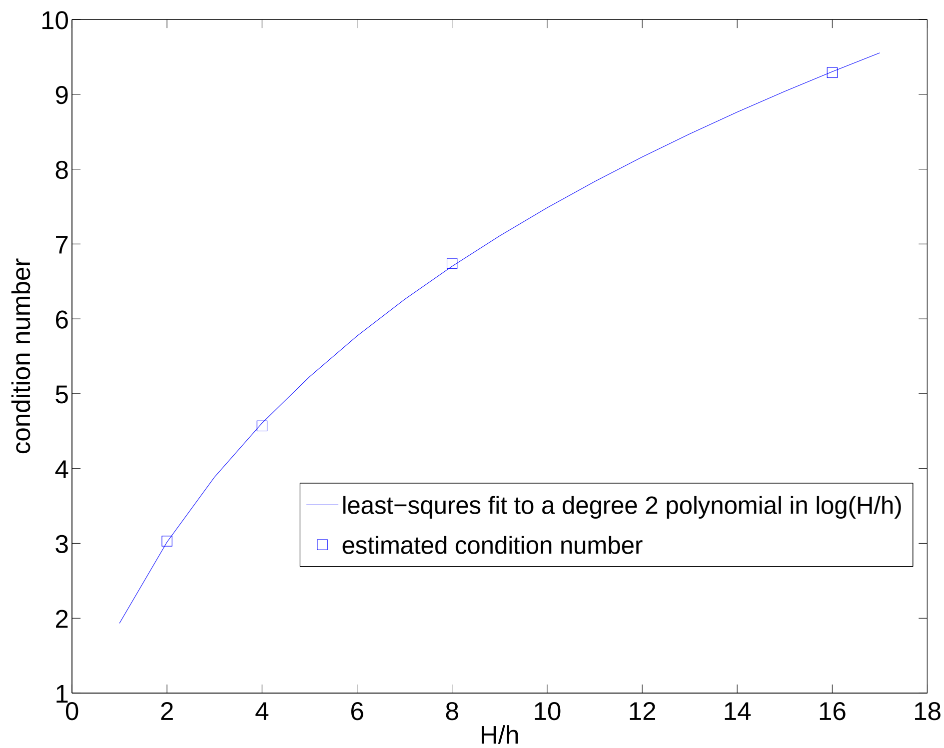

Figure 6.6: Estimated condition number and least-squares fit to a degree 2 polynomial in $\log \frac{H}{h}$, versus $\frac{H}{h}$; $\alpha_i = 1$ and $\alpha_i = 10$ in a checkerboard pattern and $\beta_i \equiv 1$ (3D case)

Table 6.4: Condition numbers and iteration counts. $\beta_i = 1$ or the specified value as indicated, in a checkerboard pattern and $\alpha_i \equiv 1$ (3D case)

	$\beta_i = 0.01$		$\beta_i = 0.1$		$\beta_i = 1$		$\beta_i = 10$		$\beta_i = 100$	
$\frac{H}{h}$	cond	iters	cond	iters	cond	iters	cond	iters	cond	iters
2	4.13	11	2.93	9	1.83	6	2.88	9	3.58	10
4	6.21	15	4.45	12	2.69	9	4.37	12	5.21	14
8	9.15	19	6.56	16	3.75	10	6.44	15	7.80	17
16	12.93	21	9.14	18	5.01	13	9.00	18	11.20	20

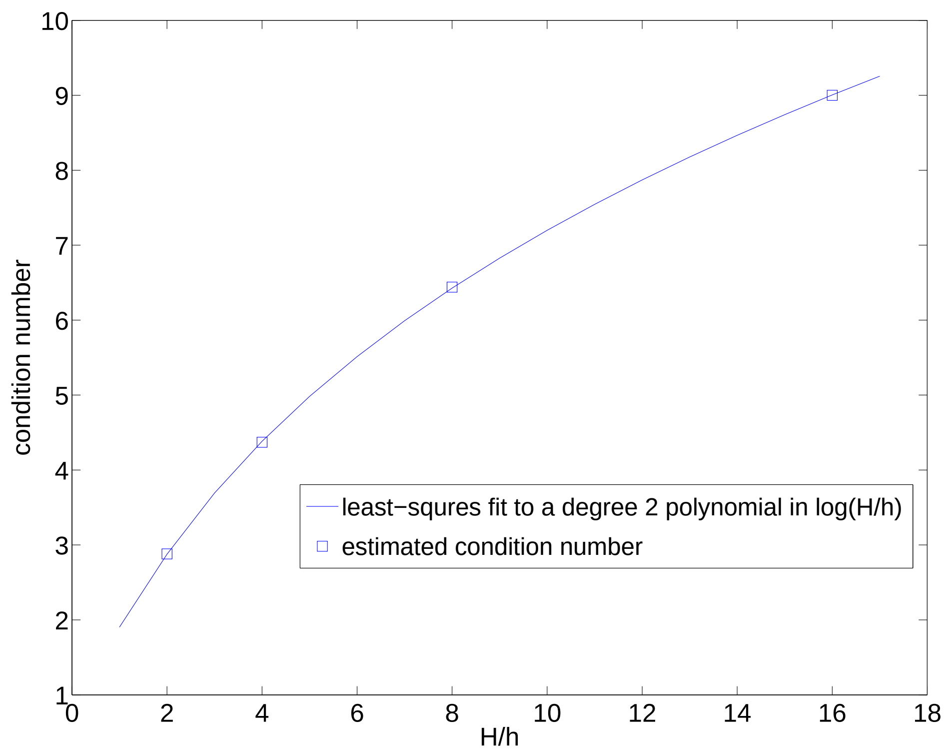

Figure 6.7: Estimated condition number and least-squares fit to a degree 2 polynomial in $\log \frac{H}{h}$, versus $\frac{H}{h}$; $\beta_i = 1$ and $\beta_i = 10$ in a checkerboard pattern and $\alpha_i \equiv 1$ (3D case)

6.4.3 A Random Coefficient Case

In this experiment, we use a different distribution of the coefficients than in Section 6.4.1 and Section 6.4.2. In each set of experiments, we first fix 4 constants, a_B,

a_W, b_B, and b_W. We next choose $N^2/2$ random numbers, $r_1, r_2, \cdots, r_{N^2/2}$, in $[0, 1]$ with a uniform distribution. We consider $N^2/2$ numbers $10^{a_B \cdot r_1}, 10^{a_B \cdot r_2}, \cdots 10^{a_B \cdot r_{N^2/2}}$. We then assign these numbers for the α values of the black subdomains. By the same process, we can distribute the coefficients for the other subdomains.

Table 6.5: Experimental sets

	upper bounds of coefficients
set 1	$a_B = 1$, $a_W = 1$, $b_B = 1$, $b_W = 1$
set 2	$a_B = 2$, $a_W = -2$, $b_B = 1$, $b_W = 1$
set 3	$a_B = 1$, $a_W = 1$, $b_B = 2$, $b_W = -2$
set 4	$a_B = 2$, $a_W = -2$, $b_B = 2$, $b_W = -2$
set 5	$a_B = 2$, $a_W = -2$, $b_B = -2$, $b_W = 2$

We perform 5 different sets of experiments. For the first set, we have gentle jumps across the interface Γ for both α and β. We next use large discontinuities for α and mild discontinuities for β and we then reverse the situation for set 3. We use large jumps of the coefficients across the interface for both α and β for the fourth and fifth sets. We provide balanced coefficients, which means that $\frac{\alpha_i}{\beta_i}$ are in certain range, for set 4 while we have both mass-dominant cases and divergence-dominant cases for set 5. For details, see Table 6.5. In each set, we try two different weight functions for the scaling factor in (4.11); we first use $\chi_i = \alpha_i$ and then $\chi_i = \beta_i$. Other general settings are quite similar to those of Section 6.4.1 and Section 6.4.2.

Table 6.6 and Table 6.7 show the results of our experiments for 2D. The results for 3D are given in Table 6.8 and Table 6.9. As we see from the tables, the algorithms work quite well if we choose the weight function $\chi_i = \beta_i$ in all sets of experiments. However, in the opposite case, where $\chi_i = \alpha_i$, the method is extremely vulnerable to jumps of β across the interface.

Table 6.6: Condition numbers and iteration counts with $\chi_i = \alpha_i$. Coefficients as indicated in a checkerboard pattern. (2D case)

	set 1		set 2		set 3		set 4		set 5	
$\frac{H}{h}$	cond	iters	cond	iters	cond	iters	cond	iters	cond	iters
8	5.46	15	5.34	15	3.67e3	219	6.71	16	2.52e3	184
16	7.19	16	7.29	16	4.71e3	267	8.75	17	3.31e3	225
32	9.26	18	9.61	18	6.30e3	296	11.15	19	4.31e3	263
64	11.67	20	12.31	19	8.13e3	322	13.92	20	5.49e3	294

Table 6.7: Condition numbers and iteration counts with $\chi_i = \beta_i$. Coefficients as indicated in a checkerboard pattern. (2D case)

	set 1		set 2		set 3		set 4		set 5	
$\frac{H}{h}$	cond	iters	cond	iters	cond	iters	cond	iters	cond	iters
8	2.55	10	2.67	11	5.45	15	4.39	14	4.04	14
16	3.44	12	3.59	13	7.41	17	6.01	16	5.61	16
32	4.49	13	4.68	14	9.72	19	7.82	17	7.59	18
64	5.70	14	5.94	15	12.47	21	9.84	19	9.94	20

Table 6.8: Condition numbers and iteration counts with $\chi_i = \alpha_i$. Coefficients as indicated in a checkerboard pattern. (3D case)

	set 1		set 2		set 3		set 4		set 5	
$\frac{H}{h}$	cond	iters	cond	iters	cond	iters	cond	iters	cond	iters
2	5.92	14	16.30	22	4.06e3	282	1.26e2	64	6.96e3	318
4	9.60	18	26.20	29	6.12e3	404	2.00e2	84	8.78e3	451
8	14.40	22	40.42	36	9.01e3	489	2.98e2	102	1.28e4	573

Table 6.9: Condition numbers and iteration counts with $\chi_i = \beta_i$. Coefficients as indicated in a checkerboard pattern. (3D case)

	set 1		set 2		set 3		set 4		set 5	
$\frac{H}{h}$	cond	iters	cond	iters	cond	iters	cond	iters	cond	iters
2	2.24	8	2.11	8	4.31	13	4.36	13	3.92	12
4	3.35	10	3.17	10	6.58	16	6.71	17	6.09	15
8	4.83	13	4.61	13	9.79	20	10.16	20	9.15	19

Bibliography

[1] C. Amrouche, C. Bernardi, M. Dauge, and V. Girault, *Vector potentials in three-dimensional non-smooth domains*, Math. Methods Appl. Sci., 21 (1998), pp. 823–864.

[2] D. N. Arnold, R. S. Falk, and R. Winther, *Preconditioning in* $H(\mathrm{div})$ *and applications*, Math. Comp., 66 (1997), pp. 957–984.

[3] D. N. Arnold, R. S. Falk, and R. Winther, *Multigrid in* $H(\mathrm{div})$ *and* $H(\mathrm{curl})$, Numer. Math., 85 (2000), pp. 197–217.

[4] S. Balay, J. Brown, , K. Buschelman, V. Eijkhout, W. D. Gropp, D. Kaushik, M. G. Knepley, L. C. McInnes, B. F. Smith, and H. Zhang, *PETSc users manual*, Tech. Report ANL-95/11 - Revision 3.1, Argonne National Laboratory, 2010.

[5] S. Balay, J. Brown, K. Buschelman, W. D. Gropp, D. Kaushik, M. G. Knepley, L. C. McInnes, B. F. Smith, and H. Zhang, *PETSc Web page*, 2011. http://www.mcs.anl.gov/petsc.

[6] S. Balay, W. D. Gropp, L. C. McInnes, and B. F. Smith, *Efficient management of parallelism in object oriented numerical software libraries*, in

Modern Software Tools in Scientific Computing, E. Arge, A. M. Bruaset, and H. P. Langtangen, eds., Birkhäuser Press, 1997, pp. 163–202.

[7] D. Braess, *Finite elements*, Cambridge University Press, Cambridge, third ed., 2007. Theory, fast solvers, and applications in elasticity theory, Translated from the German by Larry L. Schumaker.

[8] J. H. Bramble, J. E. Pasciak, and A. H. Schatz, *The construction of preconditioners for elliptic problems by substructuring. I*, Math. Comp., 47 (1986), pp. 103–134.

[9] ——, *The construction of preconditioners for elliptic problems by substructuring. II*, Math. Comp., 49 (1987), pp. 1–16.

[10] ——, *The construction of preconditioners for elliptic problems by substructuring. III*, Math. Comp., 51 (1988), pp. 415–430.

[11] ——, *The construction of preconditioners for elliptic problems by substructuring. IV*, Math. Comp., 53 (1989), pp. 1–24.

[12] S. C. Brenner and L. R. Scott, *The mathematical theory of finite element methods*, vol. 15 of Texts in Applied Mathematics, Springer, New York, third ed., 2008.

[13] S. C. Brenner and L.-Y. Sung, *BDDC and FETI-DP without matrices or vectors*, Comput. Methods Appl. Mech. Engrg., 196 (2007), pp. 1429–1435.

[14] F. Brezzi and M. Fortin, *Mixed and hybrid finite element methods*, vol. 15 of Springer Series in Computational Mathematics, Springer-Verlag, New York, 1991.

[15] Z. Cai, R. Lazarov, T. A. Manteuffel, and S. F. McCormick, *First-order system least squares for second-order partial differential equations. I*, SIAM J. Numer. Anal., 31 (1994), pp. 1785–1799.

[16] S. H. Christiansen and R. Winther, *Smoothed projections in finite element exterior calculus*, Math. Comp., 77 (2008), pp. 813–829.

[17] R. Dautray and J.-L. Lions, *Mathematical analysis and numerical methods for science and technology. Vol. 3*, Springer-Verlag, Berlin, 1990. Spectral theory and applications, With the collaboration of Michel Artola and Michel Cessenat, Translated from the French by John C. Amson.

[18] C. R. Dohrmann, *A preconditioner for substructuring based on constrained energy minimization*, SIAM J. Sci. Comput., 25 (2003), pp. 246–258 (electronic).

[19] C. R. Dohrmann and O. B. Widlund, *An overlapping Schwarz algorithm for almost incompressible elasticity*, SIAM J. Numer. Anal., 47 (2009), pp. 2897–2923.

[20] ——, *Hybrid domain decomposition algorithms for compressible and almost incompressible elasticity*, Internat. J. Numer. Methods Engrg., 82 (2010), pp. 157–183.

[21] ——, *An iterative substructuring algorithm for two-dimensional problems in H(curl)*, Tech. Report TR-936, Courant Institue of Mathematical Sciences, Dec. 2010. Department of Computer Science.

[22] M. Dryja, *An additive Schwarz algorithm for two- and three-dimensional finite element elliptic problems*, in Domain decomposition methods (Los Angeles, CA, 1988), SIAM, Philadelphia, PA, 1989, pp. 168–172.

[23] M. Dryja, B. F. Smith, and O. B. Widlund, *Schwarz analysis of iterative substructuring algorithms for elliptic problems in three dimensions*, SIAM J. Numer. Anal., 31 (1994), pp. 1662–1694.

[24] M. Dryja and O. B. Widlund, *Some domain decomposition algorithms for elliptic problems*, in Iterative methods for large linear systems (Austin, TX, 1988), Academic Press, Boston, MA, 1990, pp. 273–291.

[25] ——, *Towards a unified theory of domain decomposition algorithms for elliptic problems*, in Third International Symposium on Domain Decomposition Methods for Partial Differential Equations (Houston, TX, 1989), SIAM, Philadelphia, PA, 1990, pp. 3–21.

[26] ——, *Schwarz methods of Neumann-Neumann type for three-dimensional elliptic finite element problems*, Comm. Pure Appl. Math., 48 (1995), pp. 121–155.

[27] C. Farhat, M. Lesoinne, P. LeTallec, K. Pierson, and D. Rixen, *FETI-DP: a dual-primal unified FETI method. I. A faster alternative to the two-level FETI method*, Internat. J. Numer. Methods Engrg., 50 (2001), pp. 1523–1544.

[28] C. Farhat and F. X. Roux, *A method of finite element tearing and interconnecting its parallel solution algorithm*, Internat. J. Numer. Methods Engrg., 32 (1991), pp. 1205–1227.

[29] V. Girault, *Incompressible finite element methods for Navier-Stokes equations with nonstandard boundary conditions in* $\mathbf{R}^3$, Math. Comp., 51 (1988), pp. 55–74.

[30] V. Girault and P.-A. Raviart, *Finite element methods for Navier-Stokes equations*, vol. 5 of Springer Series in Computational Mathematics, Springer-Verlag, Berlin, 1986. Theory and algorithms.

[31] R. Glowinski and M. F. Wheeler, *Domain decomposition and mixed finite element methods for elliptic problems*, in First International Symposium on Domain Decomposition Methods for Partial Differential Equations (Paris, 1987), SIAM, Philadelphia, PA, 1988, pp. 144–172.

[32] R. Hiptmair, *Multigrid method for Maxwell's equations*, SIAM J. Numer. Anal., 36 (1999), pp. 204–225 (electronic).

[33] R. Hiptmair, J. Li, and J. Zou, *Universal extension for Sobolev spaces of dfferential forms and applications*, Tech. Report TR2009-22, ETH Zurich, Jul. 2009. Seminar for Applied Mathematics.

[34] R. Hiptmair and A. Toselli, *Overlapping and multilevel Schwarz methods for vector valued elliptic problems in three dimensions*, in Parallel solution of partial differential equations (Minneapolis, MN, 1997), vol. 120 of IMA Vol. Math. Appl., Springer, New York, 2000, pp. 181–208.

[35] R. Hiptmair and J. Xu, *Nodal auxiliary space preconditioning in* $\mathbf{H}(\mathbf{curl})$ *and* $\mathbf{H}(\mathrm{div})$ *spaces*, SIAM J. Numer. Anal., 45 (2007), pp. 2483–2509 (electronic).

[36] A. Klawonn, O. Rheinbach, and O. B. Widlund, *An analysis of a FETI-DP algorithm on irregular subdomains in the plane*, SIAM J. Numer. Anal., 46 (2008), pp. 2484–2504.

[37] P. Le Tallec, *Domain decomposition methods in computational mechanics*, Comput. Mech. Adv., 1 (1994), pp. 121–220.

[38] J. H. Lee, *Domain decomposition methods for Reissner-Mindlin plates discretized with the Falk-Tu elements*, PhD thesis, Courant Institue of Mathematical Sciences, Jan. 2011.

[39] J. Li and O. Widlund, *BDDC algorithms for incompressible Stokes equations*, SIAM J. Numer. Anal., 44 (2006), pp. 2432–2455.

[40] J. Li and O. B. Widlund, *FETI-DP, BDDC, and block Cholesky methods*, Internat. J. Numer. Methods Engrg., 66 (2006), pp. 250–271.

[41] P. Lin, *A sequential regularization method for time-dependent incompressible Navier-Stokes equations*, SIAM J. Numer. Anal., 34 (1997), pp. 1051–1071.

[42] J. Mandel and M. Brezina, *Balancing domain decomposition for problems with large jumps in coefficients*, Math. Comp., 65 (1996), pp. 1387–1401.

[43] J. Mandel and C. R. Dohrmann, *Convergence of a balancing domain decomposition by constraints and energy minimization*, Numer. Linear Algebra Appl., 10 (2003), pp. 639–659.

[44] J. Mandel, C. R. Dohrmann, and R. Tezaur, *An algebraic theory for primal and dual substructuring methods by constraints*, Appl. Numer. Math., 54 (2005), pp. 167–193.

[45] J. Mandel and R. Tezaur, *Convergence of a substructuring method with Lagrange multipliers*, Numer. Math., 73 (1996), pp. 473–487.

[46] ——, *On the convergence of a dual-primal substructuring method*, Numer. Math., 88 (2001), pp. 543–558.

[47] T. P. Mathew, *Schwarz alternating and iterative refinement methods for mixed formulations of elliptic problems. I. Algorithms and numerical results*, Numer. Math., 65 (1993), pp. 445–468.

[48] ——, *Schwarz alternating and iterative refinement methods for mixed formulations of elliptic problems. II. Convergence theory*, Numer. Math., 65 (1993), pp. 469–492.

[49] A. M. Matsokin and S. V. Nepomnyashchikh, *The Schwarz alternation method in a subspace*, Izv. Vyssh. Uchebn. Zaved. Mat., (1985), pp. 61–66, 85.

[50] P. Monk, *Finite element methods for Maxwell's equations*, Numerical Mathematics and Scientific Computation, Oxford University Press, New York, 2003.

[51] S. V. Nepomnyaschikh, *Domain Decomposition and Schwarz Methods in a Subspace for the Approximate Solution of Elliptic Boundary Value Problems*, PhD thesis, Computing Center of the Siberian Branch of the USSR Academy of Sciences, Novosibirsk, USSR, 1986.

[52] D.-S. Oh, *An overlapping Schwarz algorithm for Raviart-thomas vector fields with discontinuous coefficients*, Tech. Report TR-933, Courant Institue of Mathematical Sciences, Sep. 2010. Department of Computer Science.

[53] D. P. O'Leary and O. Widlund, *Capacitance matrix methods for the Helmholtz equation on general three-dimensional regions*, Math. Comp., 33 (1979), pp. 849–879.

[54] L. F. Pavarino, O. B. Widlund, and S. Zampini, *BDDC preconditioners for spectral element discretizations of almost incompressible elasticity in three dimensions*, SIAM J. Sci. Comput., 32 (2010), pp. 3604–3626.

[55] C. Pechstein and R. Scheichl, *Analysis of FETI methods for multiscale PDEs*, Numer. Math., 111 (2008), pp. 293–333.

[56] A. Quarteroni and A. Valli, *Numerical approximation of partial differential equations*, vol. 23 of Springer Series in Computational Mathematics, Springer-Verlag, Berlin, 1994.

[57] Y. Saad, *Iterative methods for sparse linear systems*, Society for Industrial and Applied Mathematics, Philadelphia, PA, second ed., 2003.

[58] J. Schöberl, *Commuting quasi-interpolation operators for mixed finite elements*, Tech. Report ISC-01-10-MATH, Texas A & M University, 2001. Institute for Scientific Computation.

[59] ——, *A posteriori error estimates for Maxwell equations*, Math. Comp., 77 (2008), pp. 633–649.

[60] L. R. Scott and S. Zhang, *Finite element interpolation of nonsmooth functions satisfying boundary conditions*, Math. Comp., 54 (1990), pp. 483–493.

[61] B. F. Smith, P. E. Bjørstad, and W. D. Gropp, *Domain decomposition*, Cambridge University Press, Cambridge, 1996. Parallel multilevel methods for elliptic partial differential equations.

[62] A. Toselli, *Domain decomposition methods for vector value field problems*, PhD thesis, Courant Institue of Mathematical Sciences, May. 1999.

[63] ——, *Neumann-Neumann methods for vector field problems*, Electron. Trans. Numer. Anal., 11 (2000), pp. 1–24.

[64] ——, *Overlapping Schwarz methods for Maxwell's equations in three dimensions*, Numer. Math., 86 (2000), pp. 733–752.

[65] ——, *Dual-primal FETI algorithms for edge finite-element approximations in 3D*, IMA J. Numer. Anal., 26 (2006), pp. 96–130.

[66] A. Toselli and A. Klawonn, *A FETI domain decomposition method for edge element approximations in two dimensions with discontinuous coefficients*, SIAM J. Numer. Anal., 39 (2001), pp. 932–956 (electronic).

[67] A. Toselli and O. Widlund, *Domain decomposition methods—algorithms and theory*, vol. 34 of Springer Series in Computational Mathematics, Springer-Verlag, Berlin, 2005.

[68] X. Tu, *A BDDC algorithm for a mixed formulation of flow in porous media*, Electron. Trans. Numer. Anal., 20 (2005), pp. 164–179 (electronic).

[69] ——, *A BDDC algorithm for flow in porous media with a hybrid finite element discretization*, Electron. Trans. Numer. Anal., 26 (2007), pp. 146–160.

[70] B. I. Wohlmuth, *Discretization methods and iterative solvers based on domain decomposition*, vol. 17 of Lecture Notes in Computational Science and Engineering, Springer-Verlag, Berlin, 2001.

[71] B. I. Wohlmuth, A. Toselli, and O. B. Widlund, *An iterative substructuring method for Raviart-Thomas vector fields in three dimensions*, SIAM J. Numer. Anal., 37 (2000), pp. 1657–1676 (electronic).