# Overlapping Schwarz Preconditioners for Spectral Nédélec Elements for a Model Problem in H(curl)

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# Overlapping Schwarz Preconditioners for Spectral Nédélec Elements for a Model Problem in H(curl)

Bernhard Hientzsch *

**Abstract**

A two-level overlapping domain decomposition method is analyzed for a Nédélec spectral element approximation of a model problem appearing in the solution of Maxwell's equations. The overlap between subdomains can consist of entire spectral elements or rectangular subsets of spectral elements. For fixed relative overlap and overlap made from entire elements, the condition number of the method is bounded, independently of the mesh size, the number of subregions, the coefficients and the degree of the spectral elements. In the case of overlap including just parts of spectral elements, a bound linear in the degree of the elements is proven. It is assumed that the coarse and fine mesh are quasi-uniform and shape-regular and that the domain is convex. Arguments that would not require quasi-uniformity of the coarse mesh and convexity of the domain are mentioned. Our work generalizes results obtained for lower-order Nédélec elements in Toselli [*Numer. Math. (2000) 86:733-752*]. Numerical results for the two-level algorithm in two dimensions are also presented, supporting our analysis.

## 1 Introduction

In the solution of time-discretized Maxwell's equations, the electric or magnetic field $u$ satisfies an equation of the form

$$\alpha \mathbf{u} + \mathbf{curl}(\beta \, \mathbf{curl}\, \mathbf{u}) = \mathbf{f} \qquad \text{in } \Omega. \tag{1}$$

$\alpha$ and $\beta$ are functions of the material properties $\epsilon$, $\mu$, $\sigma$ and the time-step $\Delta t$, while the right hand side contains the solution from previous time-steps and forcing terms such as applied currents. For a detailed derivation of the above model problem in the context of the time-discretization of Maxwell's equations and in the solution of the

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time-harmonic Maxwell's equations, see HIENTZSCH [16, Chapter 3]. For a general discussion of Maxwell's equations, we refer to DAUTRAY AND LIONS [12]. In this article, we will assume that $\alpha$ and $\beta$ are positive real constants.

We will work with perfectly conducting boundaries,

$$\mathbf{u} \times \mathbf{n}|_{\partial\Omega} = 0, \tag{2}$$

but similar results can be proven for natural boundary conditions as well.

We assume that $\Omega \subset \mathbb{R}^2$ or $\Omega \subset \mathbb{R}^3$ is a bounded and convex polyhedron of diameter $H_\Omega$ of order 1.

We have to introduce appropriate Sobolev spaces for the analysis of this problem. The space $H(\mathbf{curl}, \Omega)$ of square integrable vectors with square integrable curls, is a Hilbert space with the scalar product and norm

$$(\mathbf{u}, \mathbf{v})_{\mathbf{curl},\Omega} := (\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v}) + (\mathbf{u}, \mathbf{v}) \qquad ||\mathbf{u}||_{\mathbf{curl},\Omega} := \sqrt{(\mathbf{u}, \mathbf{u})_{\mathbf{curl},\Omega}}$$

We denote by $(\cdot, \cdot)$ the scalar product in $L^2(\Omega)$ or in Cartesian products thereof. $||\cdot|| = ||\cdot||_0$ is the corresponding norm. For Lipschitz continuous boundaries, we can define a tangential trace in $H^{-1/2}$, and the functions for which this trace vanishes are collected in the space $H_0(\mathbf{curl})$.

We introduce the bilinear form $a(\cdot, \cdot)$, defined on $H(\mathbf{curl}) \times H(\mathbf{curl})$, by

$$a(\mathbf{u}, \mathbf{v}) := \alpha(\mathbf{u}, \mathbf{v}) + \beta(\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v}) \tag{3}$$

The variational formulation of the boundary value problem (1) and (2) is: find $\mathbf{u} \in H_0(\mathbf{curl})$ such that for all $\mathbf{v} \in H_0(\mathbf{curl})$

$$a(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{u}). \tag{4}$$

The spectral element discretization approximates this problem in a computational subspace $V(\mathrm{curl})$ of $H_0(\mathbf{curl})$, and solves the problem: find $\mathbf{u} \in V(\mathrm{curl})$ such that for all $\mathbf{v} \in V(\mathrm{curl})$

$$a(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{u}). \tag{5}$$

The study and analysis of preconditioners for Nédélec and Raviart-Thomas discretizations started only recently, even for the $h-$version of the elements. Two-level overlapping Schwarz preconditioners for $H(\mathsf{div})$ were developed by ARNOLD, FALK AND WINTHER [5]. They were further investigated by TOSELLI in the $H(\mathbf{curl})$ case in [28, 30] and by HIPTMAIR AND TOSELLI [22] for both $H(\mathsf{div})$ and $H(\mathbf{curl})$.

Multigrid and multilevel methods for $H(\mathsf{div})$ and $H(\mathbf{curl})$ were considered in ARNOLD, FALK AND WINTHER [7, 6], HIPTMAIR AND TOSELLI [22] and HIPTMAIR [21, 20, 19]. Iterative substructuring methods are treated in ALONSO AND VALLI [3]; TOSELLI [28]; TOSELLI, WIDLUND AND WOHLMUTH [33]; and WOHLMUTH, TOSELLI AND WIDLUND [34], a Neumann-Neumann method is considered in

TOSELLI [29] and FETI preconditioners are proposed in TOSELLI AND RAPETTI [32], and TOSELLI AND KLAWONN [31]. We are not aware of any work on domain decomposition preconditioners for spectral element discretizations for Maxwell's equations or the model problem.

Our original analysis is contained in HIENTZSCH [16, Chapter 11], to which we also refer for more information about spectral Nédélec elements and the implementation of fast solvers and domain decomposition preconditioners for such elements for Maxwell's equation. We presented some results in HIENTZSCH [15], but the analysis in this paper improves on the analysis in [16] and [15]. The proof follows the general outline in TOSELLI [30]. Several changes were necessitated by our use of spectral elements and the need for bounds explicit in the degree of polynomials $N$ and number of colors $N_C$.

We define two operators, an additive one-level operator given by

$$T_{as1} = \sum_{i=1}^{J} T_i$$

and an additive two-level operator

$$T_{as2} = T_0 + \sum_{i=1}^{J} T_i = T_0 + T_{as1}$$

Denoting by $h$ the size of the elements of the fine mesh, by $H$ the size of the subregions, by $\delta$ the size of the overlap, by $N_C$ the number of colors needed to color the overlapping regions so that no two regions of the same color overlap, and by $N$ the degree of the spectral Nédélec elements, we prove a condition number estimate of the form

$$\kappa(T_{as2})) \leq C(N_C+1)\left(1+N_C\left(1+\left(\frac{H}{\delta}\right)^2\right)\right)$$

for generous overlap (i.e., $\delta$ being a multiple of $h$), and of the form

$$\kappa(T_{as2}) \leq C(N_C+1)N\left(1+N_C\left(1+\left(\frac{H}{\delta}\right)^2\right)\right)$$

for minimal overlap. Possibly the power of $\frac{H}{\delta}$ and of $N$ can be decreased, but we observe in numerical experiments that the condition number does grow for small overlap and increasing degree.

With similar techniques and the same kind of estimates, results for $T_{as1}$, multiplicative, and hybrid methods could also be proven, see, e.g., SMITH, BJØRSTAD, AND GROPP [26, pages 155–158].

In the second section, we introduce the continuous spaces needed, followed by a section about spectral Nédélec and Raviart-Thomas elements. The fourth section serves as a

short presentation of the spectral element discretization of the model problem, and the fifth section introduces the overlapping Schwarz methods that we are considering. The three sections that follow present the analysis of these methods. First, some required estimates are discussed, then, some technical results are established, and finally the condition number estimate is proven. In the final section of the paper, we present some numerical results in two dimensions that support our analysis.

## 2 Continuous spaces

$L^p(\Omega)$ is the space of Lebesgue measurable functions $u$ with $||u||_{L^p(\Omega)} = ||u||_{0,p,\Omega} < \infty$ where

$$||u||^p_{0,p,\Omega} := \int_\Omega |u|^p$$

$$||u||_{0,\infty,\Omega} := \text{ess sup}_\Omega |u|$$

The Sobolev space $W^{k,p}(\Omega)$ for integer $k$ consists of all locally summable functions $u$ such that for each multi-index $\alpha$ with $|\alpha| \leq k$, $D^\alpha u \in L^p(\Omega)$. Its norm is defined by

$$||u||_{W^{k,p}(\Omega)} = ||u||_{k,p,\Omega} := \left( \sum_{|\alpha|\leq k} ||D^\alpha u||^p_{0,p,\Omega} \right)^{1/p}$$

$$||u||_{k,\infty,\Omega} := \max_{|\alpha|\leq k} ||D^\alpha u||_{0,\infty,\Omega}$$

The spaces $H^k(\Omega) := W^{k,2}(\Omega)$ are Hilbert spaces, their norms are denoted by $||\cdot||_{H^k(\Omega)} = ||\cdot||_{k,\Omega}$ and their inner product by $(\cdot,\cdot)_{k,\Omega}$. We introduce the spaces $H^s(\Omega)$ for $s$ real and positive by Hilbert space interpolation between spaces $H^k(\Omega)$ for integer $k$ (see, e.g. ADAMS [1] or BERGH AND LÖFSTRÖM [8]).

For integer $k$, one defines $W^{k,p}(\Omega)$ semi-norms as follows:

$$|u|^p_{k,p,\Omega} := \sum_{|\alpha|=k} ||D^\alpha u||^p_{0,p,\Omega}$$

$$|u|_{k,\infty,\Omega} := \max_{|\alpha|=k} ||D^\alpha u||_{0,\infty,\Omega}$$

The semi-norms on the spaces $H^k$ are denoted $|u|_k$.

We define the two curl operators

$$\mathbf{curl}\, q := (\partial_{x_2} q, -\partial_{x_1} q) \qquad\qquad \text{curl}\, \mathbf{u} := \partial_{x_1} u_2 - \partial_{x_2} u_1$$

in two dimensions, and

$$\mathbf{curl}\, \mathbf{u} := (\partial_{x_2} u_3 - \partial_{x_3} u_2, \partial_{x_3} u_1 - \partial_{x_1} u_3, \partial_{x_1} u_2 - \partial_{x_2} u_1)^T$$

in three dimensions.

$H(\mathsf{div},\Omega)$ is the graph space of $\mathsf{div}$ over $L^2$, i.e.,

$$H(\mathsf{div},\Omega) = \{\mathbf{u} \in (L^2(\Omega))^n | \ \mathsf{div}\,\mathbf{u} \in L^2(\Omega)\}.$$

It is a Hilbert space under the graph norm

$$(\mathbf{u},\mathbf{v})_{\mathsf{div},\Omega} = (\mathbf{u},\mathbf{v})_0 + (\mathsf{div}\,\mathbf{u},\mathsf{div}\,\mathbf{v})_0 \qquad ||\mathbf{u}||^2_{\mathsf{div},\Omega} = (\mathbf{u},\mathbf{u})_{\mathsf{div},\Omega}$$

The trace operator $\gamma_n$ that maps a vector field to its normal component on the boundary, can be extended to a continuous (and surjective) operator

$$\gamma_n : H(\mathsf{div},\Omega) \mapsto H^{-\frac{1}{2}}(\partial\Omega)$$

We also need the subspaces

$$H_0(\mathsf{div},\Omega) = \{\mathbf{u} \in H(\mathsf{div},\Omega) | \gamma_n(\mathbf{u}) = 0\}$$

$$H(\mathsf{div}_0,\Omega) = \{\mathbf{u} \in H(\mathsf{div},\Omega) | \ \mathsf{div}\,\mathbf{u} = 0\}$$

$$H_0(\mathsf{div}_0,\Omega) = \{\mathbf{u} \in H_0(\mathsf{div},\Omega) | \ \mathsf{div}\,\mathbf{u} = 0\}$$

In two dimensions, $H(\mathrm{curl},\Omega)$ is defined as the graph space of curl over $L^2(\Omega)$, and is a Hilbert space:

$$H(\mathrm{curl},\Omega) = \{\mathbf{u} \in (L^2(\Omega))^2 | \ \mathrm{curl}\,\mathbf{u} \in L^2(\Omega)\},$$

$$(\mathbf{u},\mathbf{v})_{\mathrm{curl},\Omega} = (\mathbf{u},\mathbf{v})_0 + (\mathrm{curl}\,\mathbf{u},\mathrm{curl}\,\mathbf{v})_0 \qquad ||\mathbf{u}||^2_{\mathrm{curl},\Omega} = (\mathbf{u},\mathbf{u})_{\mathrm{curl},\Omega}$$

A vector $\mathbf{u} = (u_1,u_2)$ belongs to $H(\mathrm{curl},\Omega)$ if and only if $\mathbf{v} = (-u_2,u_1)$ belongs to $H(\mathsf{div},\Omega)$. Denoting the unit tangent vector on $\partial\Omega$ with $\mathbf{t}$, we have $\mathbf{v}\cdot\mathbf{n} = -\mathbf{u}\cdot\mathbf{t}$ and $\mathrm{curl}\,\mathbf{u} = \mathsf{div}\,\mathbf{v}$. The trace operator $\gamma_t(\mathbf{u}) = \mathbf{u}\cdot\mathbf{t}|_{\partial\Omega}$ can be extended to a continuous (and surjective) operator

$$\gamma_t : H(\mathrm{curl},\Omega) \mapsto H^{-\frac{1}{2}}(\partial\Omega)$$

In three dimensions, $H(\mathbf{curl})$ is a Hilbert space with the graph norm and inner product:

$$H(\mathbf{curl},\Omega) = \{\mathbf{u} \in (L^2(\Omega))^3 | \ \mathbf{curl}\,\mathbf{u} \in (L^2(\Omega))^3\},$$

$$(\mathbf{u},\mathbf{v})_{\mathbf{curl},\Omega} = (\mathbf{u},\mathbf{v})_0 + (\mathbf{curl}\,\mathbf{u},\mathbf{curl}\,\mathbf{v})_0 \qquad ||\mathbf{u}||^2_{\mathbf{curl},\Omega} = (\mathbf{u},\mathbf{u})_{\mathbf{curl},\Omega}$$

The tangential components on the boundary $\partial\Omega$ can be defined as

$$\gamma_t(\mathbf{u}) = \mathbf{u} - (\mathbf{u}\cdot\mathbf{n})\mathbf{n} = (\mathbf{n}\times\mathbf{u})\times\mathbf{n}$$

By extending $\gamma_t$, which certainly is well-defined and continuous for smooth enough $\mathbf{u}$, we can find a continuous operator

$$\gamma_t : H(\mathbf{curl},\Omega) \mapsto (H^{-\frac{1}{2}}(\partial\Omega))^3$$

which is not surjective. Its range has been fully characterized (see, for instance, ALONSO AND VALLI [2]).

To allow us to state some results concisely for the two-dimensional and three-dimensional case using the same formulas, we denote $\underset{2}{\mathbf{curl}} := \mathrm{curl}$ and $\underset{3}{\mathbf{curl}} := \mathbf{curl}$. We refer to DAUTRAY AND LIONS [12] for proofs and further results for the rest of this section.

We define the subspaces

$$H_0(\underset{n}{\mathbf{curl}}, \Omega) = \{\mathbf{u} \in H(\underset{n}{\mathbf{curl}}, \Omega) | \gamma_t(\mathbf{u}) = 0\}$$

$$H(\underset{n}{\mathbf{curl}}_0, \Omega) = \{\mathbf{u} \in H(\underset{n}{\mathbf{curl}}, \Omega) | \underset{n}{\mathbf{curl}}\, \mathbf{u} = 0\}$$

$$H_0(\underset{n}{\mathbf{curl}}_0, \Omega) = \{\mathbf{u} \in H_0(\underset{n}{\mathbf{curl}}, \Omega) | \underset{n}{\mathbf{curl}}\, \mathbf{u} = 0\}$$

**Theorem 2.1 (Orthogonal decompositions of $(L^2(\Omega))^n$)**
*The space $(L^2(\Omega))^n$ allows the following orthogonal decompositions*

$$(L^2(\Omega))^n = H(\mathsf{div}_0, \Omega) \oplus \mathbf{grad}\, H_0^1(\Omega)$$

$$(L^2(\Omega))^n = H_0(\mathsf{div}_0, \Omega) \oplus \mathbf{grad}\, H^1(\Omega)$$

$$(L^2(\Omega))^n = H_0(\mathsf{div}_0, \Omega) \oplus \mathbf{grad}\, \mathcal{H}^1(\Omega) \oplus \mathbf{grad}\, H_0^1(\Omega)$$

*where $\mathcal{H}^1(\Omega)$ is the space of harmonic functions in $H^1(\Omega)$.*

**Theorem 2.2 (Orthogonal decompositions of $H(\underset{n}{\mathbf{curl}}, \Omega)$)**
*$H(\underset{n}{\mathbf{curl}}, \Omega)$ allows the following orthogonal decompositions*

$$H(\underset{n}{\mathbf{curl}}, \Omega) = \mathbf{grad}\, H_0^1(\Omega) \oplus H^\perp(\underset{n}{\mathbf{curl}}, \Omega)$$

$$H_0(\underset{n}{\mathbf{curl}}, \Omega) = \mathbf{grad}\, H^1(\Omega) \oplus H_0^\perp(\underset{n}{\mathbf{curl}}, \Omega)$$

*with*

$$H^\perp(\underset{n}{\mathbf{curl}}, \Omega) = H(\mathsf{div}, \Omega) \cap H_0(\underset{n}{\mathbf{curl}}, \Omega)$$

$$H_0^\perp(\underset{n}{\mathbf{curl}}, \Omega) = H_0(\mathsf{div}, \Omega) \cap H(\underset{n}{\mathbf{curl}}, \Omega)$$

*The two decompositions are orthogonal with respect to both $(\cdot,\cdot)_{0,\Omega}$ and $(\cdot,\cdot)_{\mathbf{curl},\Omega}$.*

For simply connected $\Omega$, we have

$$H(\underset{n}{\mathbf{curl}}_0, \Omega) = \mathbf{grad}\, H^1(\Omega)$$

For multiply connected $\Omega$, this statement holds with a $\subset$, and the complement has been characterized (see, e.g., DAUTRAY AND LIONS [12]).

**Theorem 2.3 (Friedrichs' inequality for $H^{\perp}(\mathbf{curl},\Omega)$)**
*If $\Omega$ is simply connected, then the following inequality holds:*

$$\forall \mathbf{u} \in H^{\perp}(\underset{n}{\mathbf{curl}},\Omega) : ||\mathbf{u}||_{0,\Omega} \leq C H_{\Omega} || \underset{n}{\mathbf{curl}}\, \mathbf{u}||_{0,\Omega}$$

*If, in addition, $\partial\Omega$ is connected, the same inequality holds for the space with vanishing tangential components:*

$$\forall \mathbf{u} \in H_0^{\perp}(\underset{n}{\mathbf{curl}},\Omega) : ||\mathbf{u}||_{0,\Omega} \leq C H_{\Omega} || \underset{n}{\mathbf{curl}}\, \mathbf{u}||_{0,\Omega}$$

For some extensions to multiply connected domains and to domains with boundaries consisting of several connected components, see DAUTRAY AND LIONS [12] and AMROUCHE, BERNARDI, DAUGE, AND GIRAULT [4].

We also define

$$H(\Omega) = H(\underset{n}{\mathbf{curl}},\Omega) \cap H(\mathsf{div},\Omega)$$

with the norm

$$||\mathbf{u}||^2_{H(\Omega)} := ||\mathbf{u}||^2_{0,\Omega} + || \underset{n}{\mathbf{curl}}\, \mathbf{u}||^2_{0,\Omega} + || \mathsf{div}\, \mathbf{u}||^2_{0,\Omega}$$

and the following spaces with different boundary behavior:

$$H_T(\Omega) = \{\mathbf{u} \in H(\Omega) | \gamma_t(\mathbf{u}) = 0\}$$

$$H_N(\Omega) = \{\mathbf{u} \in H(\Omega) | \gamma_n(\mathbf{u}) = 0\}$$

**Theorem 2.4 (Smooth boundaries or convex domains)**
*Let $\Omega \subset \mathbb{R}^3$. If $\partial\Omega$ is of class $C^{1,1}$ or if $\Omega$ is convex, then $H_T(\Omega)$ and $H_N(\Omega)$ are continuously imbedded in $(H_0^1(\Omega))^3$.*

# 3 Spectral Nédélec and Raviart-Thomas spaces

In this section, we will introduce $H(\mathrm{curl})$ and $H(\mathsf{div})$ conforming elements. We use high order polynomial local spaces and tensorial bases.

We denote by $\Lambda$ the interval $[-1,1]$, by $K$ any element, and by $\hat{K}$ the reference element $[-1,1]^2$ or $[-1,1]^3$, respectively.

We define $\mathbb{P}_N(\Lambda)$ as the space of polynomials of degree of at most $N$ on the interval $\Lambda$, $\mathbb{Q}_N(K)$ as the space of polynomials of total degree $N$ on $K$, $\mathbb{P}_N(K)$ as the space of polynomials of degree at most $N$ in each variable, and $\mathbb{P}_{L,M}(K)$ and $\mathbb{P}_{L,M,N}(K)$ as the space of polynomials on $K$ with degree $L$ in $x$, $M$ in $y$ and $N$ in $z$. Since we want to work with tensor product bases, we will work with $\mathbb{P}$-type spaces.

High order polynomial local element spaces for vector fields in two dimensions are of the form

$$L_{((p_1,q_1),(p_2,q_2))}(K) := \mathbb{P}_{p_1,q_1}(K) \times \mathbb{P}_{p_2,q_2}(K)$$

and in three dimensions,

$$L_{((p_1,q_1,r_1),(p_2,q_2,r_2),(p_3,q_3,r_3))}(K) \\ := \mathbb{P}_{p_1,q_1,r_1}(K) \times \mathbb{P}_{p_2,q_2,r_2}(K) \times \mathbb{P}_{p_3,q_3,r_3}(K)$$

To ensure that the resulting global space is conforming in $H(\mathbf{curl})$, tangential components at the element interfaces have to match; for $H(\mathsf{div})$ conforming spaces, normal components are matched.

## 3.1 The $H(\mathrm{curl})$ conforming case: spectral Nédélec elements

$H(\mathbf{curl})$ conforming elements were first introduced by Nédélec. Nédélec elements of the first kind (NÉDÉLEC [24]) of order $k$ on rectangles use the following local spaces:

$$ND_k^I(K) := L_{((k-1,k),(k,k-1))}(K) \supset \mathbf{grad}\,\mathbb{P}_k(K)$$

while those of the second kind use

$$ND_k^{II}(K) := L_{((k,k),(k,k))}(K)$$

In three dimensions, the analogous element spaces are

$$ND_k^I(K) := L_{((k-1,k,k),(k,k-1,k),(k,k,k-1))}(K) \supset \mathbf{grad}\,\mathbb{P}_k(K)$$

$$ND_k^{II}(K) := L_{((k,k,k),(k,k,k),(k,k,k))}(K)$$

One can also define anisotropic version of these elements, with different degrees in different directions:

$$ND_{m,n}^I(K) := L_{((m-1,n),(m,n-1))}(K) \supset \mathbf{grad}\,\mathbb{P}_{m,n}(K)$$

$$ND_{m,n}^{II}(K) := L_{((m,n),(m,n))}(K)$$

$$ND_{l,m,n}^I(K) := L_{((l-1,m,n),(l,m-1,n),(l,m,n-1))}(K) \supset \mathbf{grad}\,\mathbb{P}_{m,n,n}(K)$$

$$ND_{l,m,n}^{II}(K) := L_{(((l,m,n),(l,m,n),(l,m,n))}(K)$$

The global spaces corresponding to these local spaces will be denoted by the same symbol, but set in an outline style, i.e., $\mathbb{ND}_\cdot^\cdot(\Omega)$ for the global space (with proper matching) associated to the local space $ND_\cdot^\cdot(K)$. $\mathbb{ND}_\cdot^{\cdot\,0}(\Omega)$ is the subspace of the global space that in addition satisfies homogeneous tangential boundary conditions.

We can find potentials (in the Helmholtz decomposition of the Nédélec spaces) in local spectral element spaces, which makes certain operations, such as curl-free corrections, numerically more accessible. (See, for instance, HIPTMAIR [21].)

To completely specify the spectral element method, we need to specify the degrees of freedom used. Nédélec introduced interior moments, face moments, and edge moments; to ensure properties needed in his analysis. We will write down the degrees of

freedom for the elements of the first kind. For the general case the only differences are the degrees of the polynomials in the test ($L$) space.

In two dimensions, for $ND_k^I$, the degrees of freedom are ($\mathbf{t}_e$ is the direction vector of the edge $e$):

$$\int_e \mathbf{u}\cdot\mathbf{t}_e p \quad p\in\mathbb{P}_{k-1}(e) \quad \text{for all edges } e \text{ of } K. \tag{6}$$

$$\int_K \mathbf{u}\cdot\mathbf{p} \quad \mathbf{p}\in L_{((k-1,k-2),(k-2,k-1))}(K) \quad \text{for } k>1. \tag{7}$$

In three dimensions, we also have face moments. The set of degrees of freedom is then:

$$\int_e \mathbf{u}\cdot\mathbf{t}_e p \quad p\in\mathbb{P}_{k-1}(e) \quad \text{for all edges } e \text{ of } K. \tag{8}$$

$$\int_F (\mathbf{u}\times\mathbf{n})\cdot\mathbf{p} \quad \mathbf{p}\in L_{((k-2,k-1),(k-1,k-2))}(F) \quad \text{for all faces } F \text{ of } K. \tag{9}$$

$$\int_K \mathbf{u}\cdot\mathbf{p} \quad \mathbf{p}\in L_{((k-1,k-2,k-2),(k-2,k-1,k-2),(k-2,k-2,k-1))}(K) \quad \text{for } k>1. \tag{10}$$

Associated to these degrees of freedom is an interpolation operator, which we will denote by $\mathbf{\Pi}^{ND,kind}_{degrees}$, which is defined element by element using the element versions $\Pi^{ND,kind}_{degrees}$. This interpolation operator is local, respects tangential boundary conditions and satisfies the commuting diagram property (see Lemma 3.1), but it is not defined for all vector fields in $H(\mathrm{curl})$. To wit, the interior degrees of freedom are defined for all of $H(\mathrm{curl})$, but the edge moments (and the face moments in three dimensions) need more regularity. There are different spaces used in the literature on which the moments are defined; the spaces used most often are $(H^{1+\epsilon}(\Omega))^d$, $(W^{1,s}(\Omega))^d$,

$$X^p(\Omega) := \{\mathbf{u}\in(L^p(\Omega))^2, \operatorname{curl}\mathbf{u}\in L^p(\Omega), \mathbf{u}\cdot\mathbf{t}\in L^p(\partial\Omega)\}$$

for $\Omega\subset\mathbb{R}^2$, and for $\Omega\subset\mathbb{R}^3$

$$X^p(\Omega) := \{\mathbf{u}\in(L^p(\Omega))^3, \mathbf{curl}\,\mathbf{u}\in(L^p(\Omega))^3, \mathbf{u}\times\mathbf{n}\in(L^p(\partial\Omega))^3\}.$$

(See for instance in GIRAULT AND RAVIART [14], ARNOLD, FALK, AND WINTHER [7], and AMROUCHE, BERNARDI, DAUGE, AND GIRAULT [4]). If we opt for modified degrees of freedom on the edges (and possibly on the faces in three dimensions), we either need to invoke a nontrivial matching or a non-local definition.

No interpolation operator is known that satisfies the commuting diagram property and that is also defined on all of $H(\mathbf{curl})$, works on quadrilateral meshes, and in both the two-dimensional and the three-dimensional case. Very recently, there has been some progress on such an operator in two dimensions on triangular elements satisfying the commuting diagram property defined on a less regular space, $(H^\epsilon(\Omega))^2\cap H(\mathrm{curl})$, see DEMKOWICZ AND BABUŠKA [13], which is optimal in $N$ (except for an arbitrarily small $\delta$, on which the bound depends) with respect to the $H(\mathbf{curl})$ norm.

To avoid the added technical difficulties in this approach – since, to the best of our knowledge, all known convergence proofs use the commuting diagram property – we will use more regular spaces.

These interpolation operators can also be used to restrict functions that are locally of high degree to global low order spaces, as needed when defining coarse spaces in multilevel methods or domain decomposition algorithms. The different definitions of the degrees of freedom and the interpolants so constructed will lead to different operators with different properties.

We want to construct spectral element type discretizations. Therefore we will use tensorized nodal basis functions built from interpolants on a Gauss-Lobatto-Legendre (GLL) mesh. For directions in which we do not have to impose boundary conditions directly, we could also use Gauss-Legendre (GL) meshes, but in our implementations we have always used GLL meshes.

The spectral element type degrees of freedom for $L_{((m_1,n_1),(m_2,n_2))}$ in two dimensions are the nodal values at

$$(\mathrm{GLL}_{m_1+1} \otimes \mathrm{GLL}_{n_1+1}) \times (\mathrm{GLL}_{m_2+1} \otimes \mathrm{GLL}_{n_2+1}) \tag{11}$$

where the normal components on the boundary are defined as the appropriate one-sided limit from the inside of the element, and $\mathrm{GLL}_m$ denotes a GLL mesh with $m$ points.

Similarly, in three dimensions, the method uses as degrees of freedom the values on the GLL mesh in all components and directions:

$$\begin{aligned}(\mathrm{GLL}_{l_1+1} \otimes \mathrm{GLL}_{m_1+1} \otimes \mathrm{GLL}_{n_1+1}) \times (\mathrm{GLL}_{l_2+1} \otimes \mathrm{GLL}_{m_2+1} \otimes \mathrm{GLL}_{n_2+1}) \\ \times (\mathrm{GLL}_{l_3+1} \otimes \mathrm{GLL}_{m_3+1} \otimes \mathrm{GLL}_{n_3+1}).\end{aligned} \tag{12}$$

## 3.2 The $H(\mathsf{div})$ conforming case: spectral Raviart-Thomas elements

We will now discuss the $H(\mathsf{div})$ conforming spectral Raviart-Thomas elements. The local spaces for (generalized) Raviart-Thomas elements in two dimensions are

$$RT_{m,n}(K) := L_{((m,n-1),(m-1,n))}(K) \qquad RT_k(K) := RT_{k,k}(K)$$

and in three dimensions

$$\begin{aligned}RT_{l,m,n}(K) &:= L_{((l,m-1,n-1),(l-1,m,n-1),(l-1,m-1,n))}(K) \\ RT_k(K) &:= RT_{k,k,k}(K)\end{aligned}$$

As for the Nédélec elements, in spectral element methods we will usually work with degrees of freedom that correspond to point values of polynomial interpolants on Cartesian products of GL or GLL meshes. For the GLL-only method we use exactly the same mesh as before, see (11) and (12).

The standard nodal interpolation operator can be defined as soon as point values are defined. As in the $H(\mathrm{curl})$ case, we can extend its domain to include functions of lower regularity. We can easily make it respect boundary and interface values, but it does not have the commuting diagram property (see Lemma 3.1), i.e., there is no nodal interpolation operator on the Nédélec spaces known, so that the $\mathbf{curl}$ of it is equal to the nodal interpolation operator on the Raviart-Thomas spaces of the $\mathbf{curl}$ of the interpolated function.

To obtain an interpolation operator that makes the diagram commute, we define alternative degrees of freedom as we did for Nédélec elements.

We define edge moments

$$\int_e \mathbf{u}\cdot\mathbf{n}p \quad p\in\mathbb{P}_{k-1}(e) \quad \text{for all edges } e \text{ of } K.$$

and interior moments

$$\int_K \mathbf{u}\cdot\mathbf{p} \quad \mathbf{p}\in L_{((k-2,k-1),(k-1,k-2))}(K) \quad \text{for } k>1.$$

These two sets of moments uniquely determine a function $\mathbf{u}\in RT_k(K)$.

In three dimensions face moments are used instead of edge moments:

$$\int_F \mathbf{u}\cdot\mathbf{n}p \quad p\in\mathbb{P}_{k-1,k-1}(F) \quad \text{for all faces } F \text{ of } K.$$

and the interior moments are defined with the appropriate space:

$$\int_K \mathbf{u}\cdot\mathbf{p} \quad \mathbf{p}\in L_{((k-2,k-1,k-1),(k-1,k-2,k-1),(k-1,k-1,k-2))}(K) \text{ for } k>1.$$

The extension of these degrees of freedom to the case of different degrees in different direction, as in $RT_{m,n}$ and $RT_{l,m,n}$, is straightforward.

Associated to these degrees of freedom on an element is an interpolation operator which will be denoted by $\Pi_k^{RT}$, $\Pi_{m,n}^{RT}$, or $\Pi_{l,m,n}^{RT}$, and which is used element by element to define the global interpolation operator $\mathbf{\Pi}_k^{RT}$, $\mathbf{\Pi}_{m,n}^{RT}$ or $\mathbf{\Pi}_{l,m,n}^{RT}$. These interpolation operators are not defined for all of $H(\mathsf{div})$, since the edge moments (in two dimensions) or the face moments (in three dimensions) are not defined for general functions $\mathbf{u}\in H(\mathsf{div})$. They are certainly well-defined when the normal trace of $\mathbf{u}$ is sufficiently regular; $\mathbf{u}\in(H^r(\Omega))^d$ with $r>\frac{1}{2}$ is enough. We can rotate the interpolation operator of DEMKOWICZ AND BABUŠKA [13] on triangles to obtain an interpolation operator on $H(\mathsf{div})$ in two dimensions that is defined on $(H^\epsilon(\Omega))^2\cap H(\mathsf{div})$ and bounded, and arbitrarily close to optimal in $N$. There is no interpolation operator known that is defined in all of $H(\mathsf{div})$ and satisfies the commuting diagram property with some interpolation operator in $H(\mathrm{curl})$.

### 3.3 Nédélec interpolation on spectral Nédélec-Raviart-Thomas spaces

For our analysis we need to compute the form of the Nédélec type interpolation operators on Nédélec and Raviart-Thomas spectral elements. We will give details for the two-dimensional case on Nédélec elements on the reference element $\hat{K}$, for complete details we refer to HIENTZSCH [16, 18].

We will give the explicit form of the Nédélec type interpolants on $\mathbb{RT}$ and $\mathbb{ND}$ between local spaces of general degree. All the interpolants in the two-dimensional and three-dimensional case can be written as tensor products of two types of terms, one corresponding to a $L^2$-projection, while the other corresponds to a modified $H^1$-projection.

First, in the two-dimensional case, we can restrict our attention to the first component of the interpolant, the form of the second follows by symmetry considerations. By a standard rotation argument, we can derive the form of the interpolant for the Raviart-Thomas-Nédélec spaces in two dimensions.

We will construct the interpolation operator from the GLL spectral element degrees of freedom on $\mathbb{P}_{p_1,q_1}(\hat{K})$ to the Nédélec type degrees of freedom on $\mathbb{P}_{m_1,n_1}(\hat{K})$. We will also assume $p_i \geq m_i$, $q_i \geq n_i$. The case $p_i = m_i$ and $q_i = n_i$ gives us the mapping between the GLL degrees of freedom and the Nédélec degrees of freedom on $L_{((m_1,n_1),(m_2,n_2)}(\hat{K})$, and taking the inverse and applying it, we obtain the Nédélec interpolation operator as an operator on the spectral element degrees of freedom. (The case $p_i \leq m_i$, $q_i \leq n_i$ can be treated by lifting $\mathbb{P}_{p_1,q_1}(\hat{K}) \times \mathbb{P}_{p_2,q_2}(\hat{K})$ to $\mathbb{P}_{m_1,n_1}(\hat{K}) \times \mathbb{P}_{m_2,n_2}(\hat{K})$ by the standard polynomial interpolation $(I_{p_1}^{m_1} \otimes I_{q_1}^{n_1}) \times (I_{p_2}^{m_2} \otimes I_{q_2}^{n_2})$ and using the result for the $p_i = m_i$ and $q_i = n_i$ case. $I_n^m$ here stands for the one-dimensional linear operator that takes the values of the polynomial on $GLL_{n+1}$ and returns the values on $GLL_{m+1}$.)

We organize the degrees of freedom according to components: for $u_1$ there are

$$\int_K u_1 \cdot p_1^I \qquad p_1^I \in \mathbb{P}_{m_1,n_1-2}$$

$$\int_{\{y=-1\}} u_1 p_1^A \qquad p_1^A \in \mathbb{P}_{m_1}$$

$$\int_{\{y=1\}} u_1 p_1^B \qquad p_1^B \in \mathbb{P}_{m_1}$$

and for $u_2$

$$\int_K u_2 \cdot p_2^I \qquad p_2^I \in \mathbb{P}_{m_2-2,n_2}$$

$$\int_{\{x=-1\}} u_2 p_2^A \qquad p_2^A \in \mathbb{P}_{n_2}$$

$$\int_{\{x=1\}} u_2 p_2^B \qquad p_2^B \in \mathbb{P}_{n_2}$$

To derive tensor product forms mapping to Nédélec degrees of freedom, we have to arrange them in two two-dimensional arrays $p_1$ and $p_2$:

$$p_1(i,1) = p_1^A(i) \qquad p_1(i,j) = p_1^I(i,j-1) \qquad p_1(i,n_1) = p_1^B(i)$$
$$p_2(1,j) = p_2^A(j) \qquad p_2(i,j) = p_1^I(i-1,j) \qquad p_2(m_2,j) = p_1^B(j)$$

or with self-explanatory notation

$$p_1^A = (I_{m_1} \otimes e_1^{n_1})p_1 \qquad p_1^B = (I_{m_1} \otimes e_{n_1}^{n_1})p_1 \qquad p_1^I = (I_{m_1} \otimes R_{I,n_1})p_1$$
$$p_2^A = (e_1^{m_1} \otimes I_{n_1})p_2 \qquad p_2^B = (e_{m_1}^{m_1} \otimes I_{n_1})p_2 \qquad p_2^I = (R_{I,m_1} \otimes I_{n_1})p_2$$

The degrees of freedom for $u_1$ can now be computed by Gaussian quadrature of high enough order. In the following we use besides the interpolation matrix $I_n^m$ already introduced, the mass matrix $M_n^m$, which corresponds to the mass matrix on $GLL_{n+1}$ computed by GLL integration on $GLL_{m+1}$.

$$\begin{aligned}\int_K u_1 \cdot p_1^I &= p_1^T((I_{m_1} \otimes R_{I,n_1}^T)(I_{m_1}^{p_1,T} \otimes I_{n_1-2}^{q_1,T})(M_{p_1}^{p_1+1} \otimes M_{q_1}^{q_1+1})u_1 \\ &= p_1^T((I_{m_1}^{p_1,T} M_{p_1}^{p_1+1}) \otimes (R_{I,n_1}^T I_{n_1-2}^{q_1,T} M_{q_1}^{q_1+1}))u_1\end{aligned}$$

$$\begin{aligned}\int_{\{y=-1\}} u_1 p_1^A &= p_1^T(I_{m_1} \otimes e_1^{n_1,T})(I_{m_1}^{p_1,T} \otimes 1)(M_{p_1}^{p_1+1} \otimes 1)(I_{p_1} \otimes e_1^{p_1})u_1 \\ &= p_1^T((I_{m_1}^{p_1,T} M_{p_1}^{p_1+1}) \otimes (e_1^{n_1,T} e_1^{p_1}))u_1\end{aligned}$$

$$\begin{aligned}\int_{\{y=1\}} u_1 p_1^B &= p_1^T(I_{m_1} \otimes e_{n_1}^{n_1,T})(I_{m_1}^{p_1,T} \otimes 1)(M_{p_1}^{p_1+1} \otimes 1)(I_{p_1} \otimes e_{p_1}^{p_1})u_1 \\ &= p_1^T((I_{m_1}^{p_1,T} M_{p_1}^{p_1+1}) \otimes (e_{n_1}^{n_1,T} e_{p_1}^{p_1}))u_1\end{aligned}$$

Adding up these expressions, we find that the first component of the Nédélec interpolant from the spectral element degrees of freedom to the Nédélec degrees of freedom is:

$$((I_{m_1}^{p_1,T} M_{p_1}^{p_1+1}) \otimes (e_1^{n_1,T} e_1^{p_1} + R_{I,n_1}^T I_{n_1-2}^{q_1,T} M_{q_1}^{q_1+1} + e_{n_1}^{n_1,T} e_{p_1}^{p_1}))$$

We will introduce the following notation for the two types of terms, since they will appear in all our interpolants:

$$L_{p_1}^{m_1} := I_{m_1}^{p_1,T} M_{p_1}^{p_1+1} \qquad H_{q_1}^{n_1} := e_1^{n_1,T} e_1^{p_1} + R_{I,n_1}^T I_{n_1-2}^{q_1,T} M_{q_1}^{q_1+1} + e_{n_1}^{n_1,T} e_{p_1}^{p_1}$$

To obtain the version of the Nédélec interpolant that maps between spectral element degrees of freedom, we multiply this by the inverse of the same mapping for the case $m_1 = p_1$ and $n_1 = q_1$:

$$\mathcal{L}_{p_1}^{m_1} := (L_{m_1}^{m_1})^{-1} L_{p_1}^{m_1} \qquad \mathcal{H}_{q_1}^{n_1} := (H_{n_1}^{n_1})^{-1} H_{q_1}^{n_1}$$

and finally obtain that $\Pi^{ND}_{m_1,n_1;m_2,n_2}\mathbf{u}$ on $L_{((p_1,q_1),(p_2,q_2))}(\hat{K})$ has the form

$$\Pi^{ND}_{m_1,n_1;m_2,n_2}\mathbf{u} = \begin{pmatrix} (\mathcal{L}^{m_1}_{p_1} \otimes \mathcal{H}^{n_1}_{q_1})u_1 \\ (\mathcal{H}^{m_2}_{p_2} \otimes \mathcal{L}^{n_2}_{q_2})u_2 \end{pmatrix} \tag{13}$$

Proceeding analogously in the three-dimensional case (see HIENTZSCH [16, Chapter 7]), we obtain the Nédélec interpolation operator on $L_{((p_1,q_1,r_1),(p_2,q_2,r_2),(p_3,q_3,r_3))}(\hat{K})$ as:

$$\Pi^{ND}_{l_1,m_1,n_1;l_2,m_2,n_2;l_3,m_3,n_3}\mathbf{u} = \begin{pmatrix} (\mathcal{L}^{l_1}_{p_1} \otimes \mathcal{H}^{m_1}_{q_1} \otimes \mathcal{H}^{n_1}_{r_1})u_1 \\ (\mathcal{H}^{l_2}_{p_2} \otimes \mathcal{L}^{m_2}_{q_2} \otimes \mathcal{H}^{n_2}_{r_2})u_2 \\ (\mathcal{H}^{l_3}_{p_3} \otimes \mathcal{H}^{m_3}_{q_3} \otimes \mathcal{L}^{n_3}_{r_3})u_3 \end{pmatrix} \tag{14}$$

Similarly, we define the layout of the coefficients and discretize the Nédélec interpolation operator for the Raviart-Thomas elements, and find in two dimensions that $\Pi^{RT}_{m_1,n_1;m_2,n_2}\mathbf{u}$ on $L_{((p_1,q_1),(p_2,q_2))}(\hat{K})$ has the form:

$$\Pi^{RT}_{m_1,n_1;m_2,n_2}\mathbf{u} = \begin{pmatrix} (\mathcal{H}^{m_1}_{p_1} \otimes \mathcal{L}^{n_1}_{q_1})u_1 \\ (\mathcal{L}^{m_2}_{p_2} \otimes \mathcal{H}^{n_2}_{q_2})u_2 \end{pmatrix} \tag{15}$$

and that in three dimensions $\Pi^{RT}_{l_1,m_1,n_1;l_2,m_2,n_2;l_3,m_3,n_3}\mathbf{u}$ on $L_{((p_1,q_1,r_1),(p_2,q_2,r_2),(p_3,q_3,r_3))}(\hat{K})$ can be written as:

$$\Pi^{RT}_{l_1,m_1,n_1;l_2,m_2,n_2;l_3,m_3,n_3}\mathbf{u} = \begin{pmatrix} (\mathcal{H}^{l_1}_{p_1} \otimes \mathcal{L}^{m_1}_{q_1} \otimes \mathcal{L}^{n_1}_{r_1})u_1 \\ (\mathcal{L}^{l_2}_{p_2} \otimes \mathcal{H}^{m_2}_{q_2} \otimes \mathcal{L}^{n_2}_{r_2})u_2 \\ (\mathcal{L}^{l_3}_{p_3} \otimes \mathcal{L}^{m_3}_{q_3} \otimes \mathcal{H}^{n_3}_{r_3})u_3 \end{pmatrix} \tag{16}$$

It can be shown that on polynomials, $\mathcal{L}^m_n$ acts as the $L^2$ projection $\pi_m$ on $\mathbb{P}_n$, and $\mathcal{H}^m_n$ is the modified $H^1$ projection $\tilde{\pi}^1_m$ on $\mathbb{P}_n$. Therefore, also in the continuous setting, we have that the interpolation operator is a tensor product of $L^2$ and modified $H^1$ projections, if the vector field is regular enough.

## 3.4 The commuting diagram property

We turn now to the commuting diagram property. We assume that the domain $\Omega$ is a simply connected polygon or polyhedron, with a connected boundary. In the statement of the commuting diagram properties, and in the analysis of the spaces $\mathbb{ND}$ and $\mathbb{RT}$, we need the standard scalar piecewise polynomial spaces.

The $H^1$-conforming space with continuity across the interfaces is:

$$\mathbb{S}_N(\Omega) := \{q \in H^1(\Omega) | q|_K \in \mathbb{P}_N(K) \quad \forall K\}$$

We also define the space with enforced zero boundary values $\mathbb{S}^0_N(\Omega) \subset H^1_0(\Omega)$. We denote the standard nodal interpolation operator onto $\mathbb{S}_N(\Omega)$ by $\mathbf{\Pi}^S_N$.

The $L^2$-conforming space, in which no continuity is required across the interfaces, is defined analogously:

$$\mathbb{W}_N(\Omega) := \{q \in L^2(\Omega) | q|_K \in \mathbb{P}_N(K) \quad \forall K\}$$

The interpolation operator $\mathbf{\Pi}^W_N$ is the $L^2$-projection onto $\mathbb{W}_N(\Omega)$.

**Lemma 3.1 (Commuting diagram properties)**
*Assume that $q$, $\mathbf{u}$ and $\mathbf{v}$ are sufficiently regular. Then the following identities hold*

$$\mathbf{grad}\left(\mathbf{\Pi}_N^S q\right) = \mathbf{\Pi}_N^{ND,I}\left(\mathbf{grad}\, q\right),$$

$$\mathrm{curl}\left(\mathbf{\Pi}_N^{ND,I}\mathbf{u}\right) = \mathbf{\Pi}_N^{W}\left(\mathrm{curl}\,\mathbf{u}\right),$$

$$\mathbf{curl}\left(\mathbf{\Pi}_N^S q\right) = \mathbf{\Pi}_N^{RT}\left(\mathbf{curl}\, q\right),$$

$$\mathbf{curl}\left(\mathbf{\Pi}_N^{ND,I}\mathbf{u}\right) = \mathbf{\Pi}_N^{RT}\left(\mathbf{curl}\,\mathbf{u}\right),$$

$$\mathsf{div}\left(\mathbf{\Pi}_N^{RT}\mathbf{v}\right) = \mathbf{\Pi}_N^{W}\left(\mathsf{div}\,\mathbf{v}\right).$$

*Proof:* See HIPTMAIR [19, Theorem 2.30]. ∎

This lemma also holds for the anisotropic case with the obvious changes.

## 3.5 The discrete Helmholtz decomposition

**Lemma 3.2 (Kernel of $\mathbf{curl}$)**
*If $\Omega$ is simply connected, with a connected boundary, the kernels of the curl operator defined in $\mathbb{ND}_N^I(\Omega)$ and $\mathbb{ND}_N^{I,0}(\Omega)$ are $\mathbf{grad}\,\mathbb{S}_N(\Omega)$ and $\mathbf{grad}\,\mathbb{S}_N^0(\Omega)$, respectively.*

We can now state the following discrete analog of the Helmholtz decomposition for the Nédélec spaces into a $\mathbf{curl}$-free part and a $\mathsf{div}$-free part:

$$\mathbb{ND}_N^I(\Omega) = \mathbf{grad}\,\mathbb{S}_N(\Omega) \oplus \mathbb{ND}_N^{I,+}(\Omega)$$

$$\mathbb{ND}_N^{I,0}(\Omega) = \mathbf{grad}\,\mathbb{S}_N^0(\Omega) \oplus \mathbb{ND}_N^{I,0,+}(\Omega)$$

with the orthogonal complements

$$\mathbb{ND}_N^{I,+}(\Omega) := \{\mathbf{u} \in \mathbb{ND}_N^I(\Omega) | (\mathbf{u}, \mathbf{grad}\, p_N)_0 = 0 \quad \forall p_N \in \mathbb{S}_N(\Omega)\} \tag{17}$$

$$\mathbb{ND}_N^{I,0,+}(\Omega) := \{\mathbf{u} \in \mathbb{ND}_N^{I,0}(\Omega) | (\mathbf{u}, \mathbf{grad}\, p_N)_0 = 0 \quad \forall p_N \in \mathbb{S}_N^0(\Omega)\} \tag{18}$$

In general, the spaces $\mathbb{ND}_N^{I,+}(\Omega)$ and $\mathbb{ND}_N^{I,0,+}(\Omega)$ are not included in $H^{\perp}(\mathbf{curl};\Omega)$ and $H_0^{\perp}(\mathbf{curl};\Omega)$, the analogous spaces in the continuous Helmholtz decomposition.

A discrete version of Friedrichs' inequality holds:

**Theorem 3.3 (Discrete Friedrichs' inequality for the $hN$ case)**
*Assume that the bounded and convex domain $\Omega$ with $H_\Omega = O(1)$ has a Lipschitz boundary and is covered with a shape regular and quasi-uniform mesh of elements of size $h$. Let $\mathbf{\Phi}_N$ be discretely divergence free of degree $N$, i.e., assume $\mathbf{\Phi}_N \in \mathbb{ND}_N^{I,+}$. Then there exist constants $C$ and $C'$ such that*

$$||\mathbf{\Phi}_N||_0 \leq C\left(1 + h\left(1 + C_{1,2}(\epsilon)N^{-1+\epsilon}\right)\right)||\,\mathbf{curl}\,\mathbf{\Phi}_N||_0 \leq C'||\,\mathbf{curl}\,\mathbf{\Phi}_N||_0$$

*Proof:* see, e.g., HIENTZSCH [16, Theorem 7.18] or MONK [23, Section 4]

*Remark 3.1:* We assume quasi-uniformity here and in the proof of Estimate 6.1, to obtain an explicit estimate in $N$ and for simplicity of presentation. Without this assumption, we could proceed as in ARNOLD, FALK, AND WINTHER [7, inequality (2.4)] for the convex case or in AMROUCHE, BERNARDI, DAUGE, AND GIRAULT [4, Proposition 4.6, Lemma 4.7] for the general case. Then we will have to replace the finite dimensional space arguments in these references by explicit analyses for the polynomial spaces. We leave these extensions to future work.

# 4 Spectral element discretizations

We choose a local space with arbitrary degrees in each component and direction as $V = V_N(\Omega)$, in which the variational problem is discretized. In our analysis we only treat $V = \mathbb{ND}_N^{I,0}(\Omega, T_h)$, for ease of presentation.

We approximate for $\mathbf{u}, \mathbf{v} \in V(\mathrm{curl})$

$$(\alpha \mathbf{u}, \mathbf{v})_0 + (\beta \operatorname{curl} \mathbf{u}, \operatorname{curl} \mathbf{v})_0 = (\mathbf{f}_{(1)}, \mathbf{v})_0 + (\mathbf{f}_{(2)}, \operatorname{curl} \mathbf{v})_0 \tag{19}$$

We use Gaussian integration of arbitrary degree to discretize this formulation, and we obtain in the two-dimensional case on one element

$$(M_1^x \otimes A^y)\underline{u}_1 + (B^x \otimes C^y)\underline{u}_2 = (M_1^x \otimes M_1^y)\underline{f}_1 - (F_1^x \otimes F_1^y)\underline{f}_3 \tag{20}$$

$$(C^x \otimes B^y)\underline{u}_1 + (A^x \otimes M_2^y)\underline{u}_2 = (M_2^x \otimes M_2^y)\underline{f}_2 + (F_2^x \otimes F_2^y)\underline{f}_3 \tag{21}$$

with ($D_n$ denotes the matrix representation of the differentiation operator on $GLL_{n+1}$)

$$M_1^x = M_{m_1}^{M_1} \qquad M_1^y = M_{n_1}^{N_1} \qquad M_2^x = M_{m_2}^{M_2} \qquad M_2^y = M_{n_2}^{N_2}$$

$$A^x = \alpha M_{m_2}^{M_2} + \beta D_{m_2}^T M_{m_2}^{M_4} D_{m_2} = \alpha M_{m_2}^{M_2} + \beta K_{m_2}^{M_4}$$

$$A^y = \alpha M_{n_1}^{N_1} + \beta D_{n_1}^T M_{n_1}^{N_3} D_{n_1} = \alpha M_{n_1}^{N_1} + \beta K_{n_1}^{N_3}$$

$$B^x = -\beta I_{m_1}^{M_5,T} M_{M_5} I_{m_2}^{M_5} D_{m_2}$$

$$B^y = I_{n_2}^{N_5,T} M_{N_5} I_{n_1}^{N_5} D_{n_1}$$

$$C^x = -\beta D_{m_2}^T I_{m_2}^{M_5,T} M_{M_5} I_{m_1}^{M_5} = B^{x,T}$$

$$C^y = D_{n_1}^T I_{n_1}^{N_5,T} M_{N_5} I_{n_2}^{N_5} = B^{y,T}$$

$$F_1^x = I_{m_1}^{m_3,T} M_{m_3} \qquad F_1^y = D_{n_1}^T I_{n_1}^{n_3,T} M_{n_3}$$

$$F_2^x = D_{m_2}^T I_{m_2}^{m_3,T} M_{m_3} \qquad F_2^y = I_{n_2}^{n_3,T} M_{n_3}$$

$A^x$ and $A^y$ are scaled discretizations of one-dimensional Helmholtz operators, and they contain $K_n^N$, which is a spectral discretization of an one-dimensional Laplace operator. (Slightly different decompositions into tensor product factors could be used, since $A \otimes B = (cA) \otimes (c^{-1}B)$.)

These element matrices are subassembled into the global system, as is standard in finite element codes. If the global domain is a rectangle, and we have matching degrees in the elements, the entire, global, system is also of this block tensor product structure.

In three dimensions, we obtain a similar block-tensor system.

For further details in the two-dimensional case, we refer to HIENTZSCH [16, Chapter 8]; we will give details for the three-dimensional case in future work.

For general geometries, mapped elements are used. If affine mappings suffice, we obtain the same discretization on the elements, up to a scaling. For the more general case, we assume that there is an invertible map from the mapped element $K_i$ to the reference element $\hat{K} = [-1,1]^2$

$$(x,y) \mapsto (P_i(x,y), Q_i(x,y))$$

with the inverse mapping

$$(p,q) \mapsto (X_i(p,q), Y_i(p,q))$$

Integrals over $K_i$ are computed by mapping to $\hat{K}$

$$\int_{K_i} \int I(x,y)dxdy = \int_{\hat{K}} \int I(X_i(p,q), Y_i(p,q)) J_i(p,q) dpdq$$

with the Jacobian

$$J_i(p,q) = \frac{\partial X_i}{\partial p}(p,q)\frac{\partial Y_i}{\partial q}(p,q) - \frac{\partial X_i}{\partial q}(p,q)\frac{\partial Y_i}{\partial p}(p,q)$$

Derivatives with respect to $x$ and $y$ are rewritten as combinations of derivatives with respect to $p$ and $q$ using the chain rule.

The discretization on one mapped element can be written – we assume equal degrees for both components and diagonal mass matrices, the general case of arbitrary degrees in the elements and arbitrary degree of integration has a similar form –

$$(\alpha JM + \beta G_y^T JMG_y)\underline{u}_1 + \beta G_y^T JMG_x \underline{u}_2 = \tilde{\underline{f}}_1 \quad (22)$$

$$\beta G_x^T JMG_y \underline{u}_1 + (\alpha JM + \beta G_x^T JMG_x)\underline{u}_2 = \tilde{\underline{f}}_2 \quad (23)$$

with $M = M^p \otimes M^q$ and

$$G_x = P_x(D_p \otimes I) + Q_x(I \otimes D_q) \qquad G_q = P_y(D_p \otimes I) + Q_y(I \otimes D_q)$$

$M^p$ and $M^q$ are diagonal one-dimensional mass matrices, containing the GLL integration weights on $GLL_{m+1}$ and $GLL_{n+1}$. $D_p$ and $D_q$ are one-dimensional spectral differentiation matrices on the reference element. $J$, $P_x$, $P_y$, $Q_x$, and $Q_y$ are diagonal matrices, the diagonal entries are $J_i$, $\partial P_i/\partial x$, $\partial P_i/\partial y$, $\partial Q_i/\partial x$, and $\partial Q_i/\partial y$, respectively, evaluated at the GLL grid points in the reference element. Multiplication with these matrices can be implemented by multiplication with tensor product matrices and by elementwise multiplication of two matrices, therefore a fast matrix-vector multiplication can be implemented for this discretization. The discretization is also symmetric,

so that the conjugate gradient method can still be used. There are no longer fast direct solvers in the general case. Following work on the Laplace, Stokes and Navier-Stokes equation, we use preconditioners for the affine case also for the mapped case, constructing the affine preconditioner on an affine average of the element geometry.

Our theory continues to hold for sufficently benign mappings between the mapped elements and the reference element. For simplicity, we only treat the affine case. We intend to present theoretical and numerical results for the isoparametric case in future work.

# 5 Overlapping methods

The domain $\Omega$ is covered by a shape-regular and quasi-uniform mesh $T_H$ of quadrilateral elements of size $H$. These elements are further subdivided into spectral elements of size $h$ and degree $N$, constituting a shape-regular fine mesh $T_h$. There are several ways of obtaining the overlapping subregions. In one of them, the subdomains $\Omega_i$, $i = 1, \cdots, J$, correspond to the elements of the coarse mesh, and they are extended by some distance $\delta_i$ to yield the overlapping regions $\Omega_i'$.

See figure 1 for an example which shows one of the $\Omega_i'$, with some surrounding elements from $T_h$.

A second way is to combine elements (and parts of elements) in such a way that the constructed subregion still has a diameter of $O(H)$ and overlaps other subregions with a geometric overlap $\delta$. We have chosen such a setting for the implementation in HIENTZSCH [16, Chapter 10].

The overlap parameter $\delta$ is the minimal distance between $\partial\Omega_i'$ and $\Omega_i$, and therefore equal to $\min_i \delta_i$. In the element-wise overlap case it will be a multiple of $h$ for uniform meshes, or the local sum of $h_i$ for a fine mesh of varying mesh size $h_i$..

In the element-wise overlap case, the local spaces $V_i$ are the subspaces of functions in $V$ that have support in $\Omega_i'$. For general (smaller) overlap, $V_i$ is the subspace of functions in $V$ spanned by the spectral element basis functions in $\mathbb{ND}_N^I(\Omega, T_h)$ that are associated to GLL points that are inside $\Omega_i'$. (Or equivalently, the subspace of functions in $V$ with spectral element degrees of freedom outside of $\Omega_i'$ set to zero.) In the element-wise overlap case, the support of functions in $V_i$ is $\Omega_i'$; for general overlap, the support is $\Omega_i^*$, equal to the union of all the elements in $T_h$ that intersect $\Omega_i'$.

The coarse space $V_0$ is chosen as $\mathbb{ND}_{N_0}^{I,0}(\Omega, T_H) \subset \mathbb{ND}_N^{I,0}(\Omega, T_h)$. (Different choices for the fine-to-coarse mapping correspond either to a different space $V_0$, or to a different system, i.e., different $a_0(\cdot,\cdot)$, posed on $V_0$.) For any fixed $N_0$ we have certain estimates given in Lemma 7.2. For the sake of simplicity, we will not try to explicate the dependence of the condition number of the operator on $N_0$.

The global space $V$ admits a non-unique decomposition $V = \sum_{i=0}^{J} V_i$.

We will use exact solvers in the subspaces, i.e., the bilinear form for all problems will be $a(\cdot,\cdot)$. The proof could be extended to inexact solvers by standard arguments. (See,

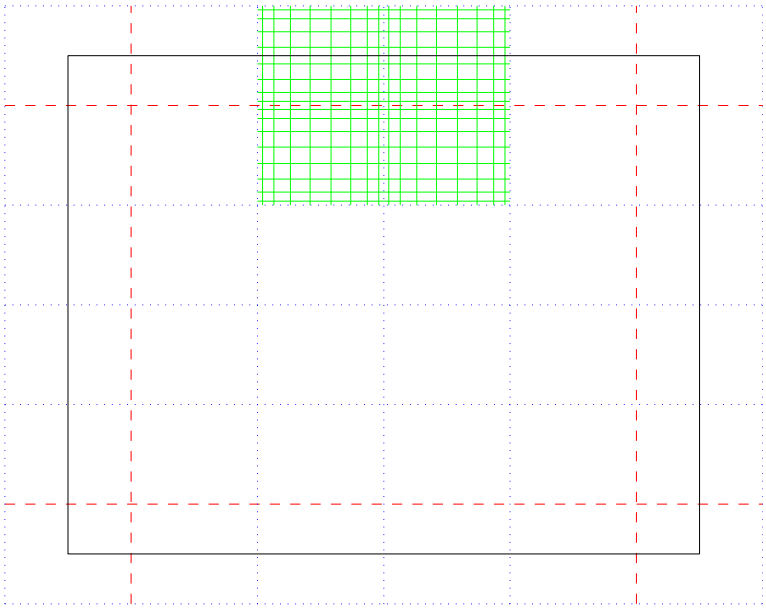


Figure 1: An overlapping subregion for the domain decomposition method. $h = H/4$, $\delta = h/2$, $N = 10$. Broken lines: subdomain mesh $T_H$. Dotted lines: element mesh $T_h$. Solid enclosure: $\partial\Omega_i'$. We also show the GLL mesh associated to a degree 10 spectral element in four of the elements of size $h$.

e.g., SMITH, BJØRSTAD, AND GROPP [26].)

We introduce the local projections $T_i : V \to V_i, u \mapsto T_i u$ defined by

$$?T_i u : \forall \mathbf{v} \in V_i : a(T_i \mathbf{u}, \mathbf{v}) = a(\mathbf{u}, \mathbf{v})$$

Using these projections, many domain decomposition methods can be defined (see, for instance, [26, pages 149–153]). We recall that we defined a one- and a two-level additive method in section 1. We refer to HIENTZSCH[16, Chapter 10], for an implementation of these methods in the two-dimensional case.

We will prove a condition number estimate for $T_{as2}$. With similar techniques and the same kind of estimates, results for $T_{as1}$, multiplicative and hybrid methods could also be proven, see, e.g., SMITH, BJØRSTAD, AND GROPP [26, pages 155–158].

To use the standard coloring arguments (see, e.g., [26, bottom of page 165 and proof of theorem 1 on page 167]), we need an assumption about the covering of $\Omega$ by the overlapping regions $\Omega_i^*$:

**Coloring assumption:** *The overlapping regions $\{\Omega_i^*\}$ can be colored using $N_C$ colors, in such a way that regions with the same color do not intersect.*

# 6 Required estimates

In our analysis, we depend on four estimates, three concerning properties of the Nédélec interpolation operator, and one concerning the partition of unity. Since we know how to analyze the Nédélec interpolation operator on polynomial spaces, we need to use a polynomial partition of unity, for which we will have to ascertain the standard estimates for partitions of unity. The first of the three other estimates, an interpolation property for divergence-free vector fields with spectral element curls, is needed in the analysis of the projection into the coarse space. The other two estimates state the $L^2$- and $\mathbf{curl}$-stability of the Nédélec interpolation on the local spaces. In the proof of two of the estimates, we use the commuting diagram property of the Nédélec interpolation operator and the Raviart-Thomas interpolation operator.

**Estimate 6.1 (Interpolation property for divergence-free vector fields)** *There is a constant $C$ independent of $N$, $h$, and $\mathbf{u}$, and a function $f_1(N,h)$ of the form $h^s N^t$, $s = 1+\epsilon$, $t = -1/2+\epsilon$ for the two-dimensional case, and $h(C+N^t)$ for the three-dimensional case, such that for all $\mathbf{u} \in H_0^{\perp}(\mathbf{curl})$ with $\mathbf{curl}\,\mathbf{u} \in \mathbb{W}_N(\Omega, T_h)$ there is a bound*

$$||\mathbf{u} - \mathbf{\Pi}_N^{ND,I}\mathbf{u}||_0 \leq C f_1(N,h)||\,\mathbf{curl}\,\mathbf{u}||_0 \tag{24}$$

*Proof:* In the proof, we need to use more regular spaces than $H^1$, since the Nédélec interpolation operator needs more regularity to be defined, and there is no interpolation operator known that both satisfies the commuting diagram property and is defined on less regular spaces. The Nédélec interpolation operator is well-defined on $H^{1+\epsilon}$ or on $W^{1,s}$ for the proper choice of $s$. Here, we will use the $H^{1+\epsilon}$ spaces. We need stability and interpolation error estimates for the Nédélec interpolation. There are several such estimates known, but the exponents in them can most likely be improved. However, we will not attempt to do so in this article.

We refer to HIENTZSCH [16, Lemma 7.17] for the $W^{1,s}$ approach and more details. The proof here and in Hientzsch [16] requires quasi-uniform meshes, at least as written. So do the proofs in ARNOLD, FALK, AND WINTHER [7], GIRAULT AND RAVIART [14] and MONK [23]. For possible alternatives, see Remark 3.1 at the end of section 3.

We will present different proofs for the three-dimensional and the two-dimensional case. In three dimensions, we have a stability estimate (see MONK [23, Theorem 3.3]):

$$||\mathbf{u} - \mathbf{\Pi}_N^{ND,I}\mathbf{u}||_0 \leq Ch\left(N^{-1}||\mathbf{u}||_{1+\epsilon} + ||\mathbf{u}||_1\right) \tag{25}$$

Under our assumptions, $\mathbf{u}$ is a solution of a $\mathbf{curl}$ potential problem:

$$\mathbf{curl}\,\mathbf{u} \in (\mathbb{Q}_{N,N,N}(K))^3 \subset (H^{\epsilon}(\Omega))^3 \qquad \mathsf{div}\,\mathbf{u} = 0 \qquad \mathbf{u}\times n|_{\partial\Omega} = 0 \tag{26}$$

Here, $\epsilon \in [0, 1/2)$. This potential problem is regular from $(H^{\epsilon})^3$ to $(H^{1+\epsilon})^3$ for $\epsilon \in [0, \epsilon_0)$ for convex domains, and therefore

$$||\mathbf{u}||_{1+\epsilon} \leq C_2(\epsilon)||\,\mathbf{curl}\,\mathbf{u}||_{\epsilon}$$

$\mathbf{curl}\,\mathbf{u}$ is a piecewise polynomial, and so an inverse estimate holds

$$||\,\mathbf{curl}\,\mathbf{u}||_\epsilon \leq CN^{2\epsilon}||\,\mathbf{curl}\,\mathbf{u}||_0$$

Together this implies

$$||\mathbf{u}||_{1+\epsilon} \leq C_1C_2(\epsilon)N^{2\epsilon}||\,\mathbf{curl}\,\mathbf{u}||_0$$

The second term is bounded for convex domains by the embedding theorem for $H_N(\Omega)$ (see section 2)

$$||\mathbf{u}||_1 \leq C(||\mathbf{u}||_0 + ||\,\mathbf{curl}\,\mathbf{u}||_0)$$

and using the continuous Friedrichs' inequality (recall that $H_\Omega = 1$)

$$||\mathbf{u}||_0 \leq C||\,\mathbf{curl}\,\mathbf{u}||_0$$

we obtain

$$||\mathbf{u}||_1 \leq C||\,\mathbf{curl}\,\mathbf{u}||_0.$$

(25) now implies

$$||\mathbf{u} - \mathbf{\Pi}_N^{ND,I}\mathbf{u}||_0 \leq C_1h(C_2(\epsilon)N^{-1+2\epsilon} + 1)||\,\mathbf{curl}\,\mathbf{u}||_0 \tag{27}$$

that is, $f_1(N,h)$ in the estimate is equals to $h(C_2(\epsilon)N^{-1+2\epsilon} + 1) \leq C_2(\epsilon)C_3h$ with $C_3$ independent of $N$ and $\epsilon$.

For two dimensions, the results of SURI [27] and a rotation by 90 degrees imply the interpolation estimate

$$||\mathbf{u} - \mathbf{\Pi}_N^{ND,I}\mathbf{u}||_0 \leq Ch^{1+\epsilon}N^{-(1/2+\epsilon)}||\mathbf{u}||_{1+\epsilon} \tag{28}$$

Proceeding as before, we obtain for $\epsilon \in (0,\epsilon_0)$ for some $\epsilon_0$ depending on the domain,

$$||\mathbf{u} - \mathbf{\Pi}_N^{ND,I}\mathbf{u}||_0 \leq C_2'(\epsilon)C_4h^{1+\epsilon}N^{-1/2+\epsilon}||\,\mathbf{curl}\,\mathbf{u}||_0 \tag{29}$$

implying $f_1(N,h) = C_2'(\epsilon)C_4h^{1+\epsilon}N^{-1/2+\epsilon}$ in two dimensions. ∎

The next three estimates are needed to analyze the local splitting. In the splitting, we first remove a component $\mathbf{u}_0$ in the coarse space, $\mathbf{u} = \mathbf{u}_0 + \mathbf{v}$, with $\mathbf{u}_0 = Q_0(P_N\mathbf{u})$, use a partition of unity $\chi_i$ to localize the splitting, and then use the Nédélec interpolant to obtain the contributions in the local spaces $\mathbf{v}_i := \mathbf{\Pi}_N^{ND,I}(\chi_i\mathbf{v})$. We will need to analyze the Nédélec and the Raviart-Thomas interpolation operator on $\chi_i\mathbf{v}$ and $\mathbf{curl}\,\chi_i\mathbf{v}$. If we use the standard piecewise linear partition of unity $\chi_i^{PL}$, $\chi_i^{PL}\mathbf{v}$ will be in a piecewise polynomial space. The analysis of the interpolation operators on such spaces presents new technical difficulties requiring new approaches, and we will not attempt it here. If we use instead a polynomial interpolant of $\chi_i^{PL}$, interpolating it at least on the GLL mesh, we have to study the interpolation operators on polynomial spaces. We recently obtained such results and will therefore use polynomial interpolants of partitions of unity. The piece-wise linear partition of unity $\chi^{PL}$ satisfies $\sum_i \chi^{PL} = 1$ and $0 \leq \chi^{PL}(x) \leq 1$, its polynomial interpolation only satisfies the first property.

Since we are working on rectangular or hexahedral elements, we construct the partition of unity $\chi_i$ as a tensor product $\chi^M_{k(i)} \otimes \chi^M_{l(i)}$ or $\chi^M_{k(i)} \otimes \chi^M_{l(i)} \otimes \chi^M_{m(i)}$, and discuss in the following only the one-dimensional polynomial partition of unity, $\chi^M_i$.

We have to prove that the interpolated partition of unity satisfies the standard bounds:

**Estimate 6.2 (Bounds on the partition of unity)** *There are constants $C_1$ and $C_2$ independent of $\delta$ and $M$ such that for $\chi^M_i \in \mathbb{P}_M([-1,1])$ the following estimate holds*

$$||\chi^M_i||_{0,\infty} \leq C \qquad ||\,\mathbf{grad}\,\chi^M_i||_{0,\infty} \leq \frac{C}{\delta} \tag{30}$$

We will prove this estimate in two special cases later, and first present some observations. In the general case, we can only prove a weaker result.

In the previous section, we considered two layouts for overlapping subdomains: a vertex centered one, and one obtained extending the coarse subdomains. In the one-dimensional case, these two cases are shown in figure 2. Partitions of unity with overlap that is at most one element can be build in the two cases from the functions shown in figure 3. These functions are given by

$$\chi^e_{2\delta} = \frac{1}{2}\left(\frac{x-1+\delta}{\delta}\right)_+ \tag{31}$$

$$\chi^v_{2\delta} = \min\left\{\left(\frac{x+\delta}{2\delta}\right)_+, 1\right\}. \tag{32}$$

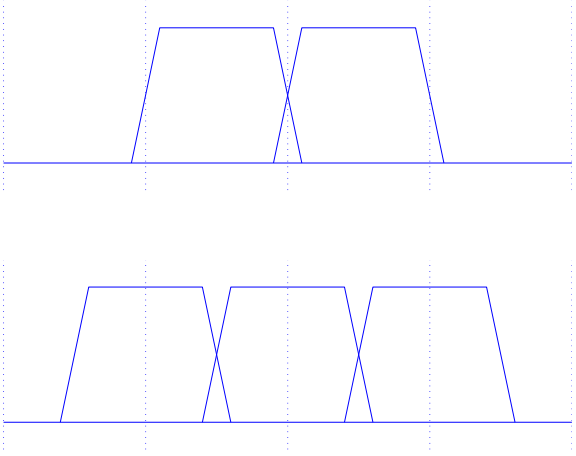


Figure 2: Partitions of unity: Vertex centered subdomains and extended subdomains

In the vertex centered layout, the partition of unity restricted to an element can be written $\chi^v_\delta(a+x)$ or $\chi^v_\delta(b-x)$; for the extended subdomain layout we have $\chi^e_\delta(a+x)$,

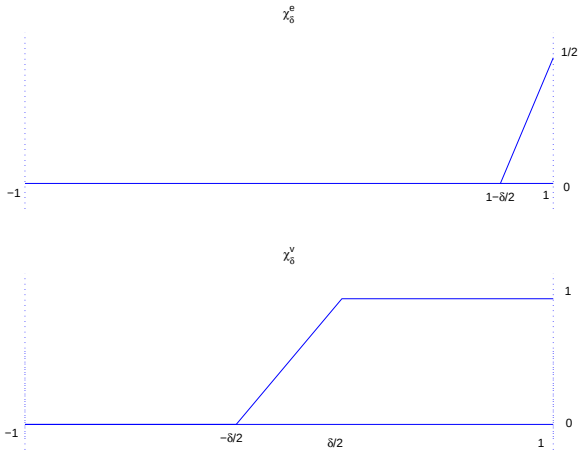



Figure 3: Partitions of unity: Building blocks for vertex centered subdomains and extended subdomains

$\chi^e_\delta(b-x)$, and $1-\chi^e_\delta(a+x)-\chi^e_\delta(b-x)$. Furthermore, $\chi^v_{2\delta}$ can be written as

$$\chi^v_{2\delta}(x) = \frac{x+\delta}{2\delta} + \frac{\delta}{1-\delta}\left(\chi^e_{1-2\delta}(-x) - \chi^e_{1-2\delta}(x)\right)$$

Therefore it is obviously enough to analyze the interpolation and approximation of $\chi^e_\delta$. Without loss of generality, we can assume that $\chi^e_\delta = I^N_h \chi^e_\delta$, i.e., $\chi^e_\delta$ agrees with its piecewise linear interpolant on the GLL points.

In figure 4, we show the GLL interpolants for $\delta = 0.2$ and $N = 10$. We see that there is some overshoot and undershoot in the interpolation, but that the interpolants are still comparable in magnitude with the piecewise partition of unity.

We ran extensive numerical tests both for fixed $\delta$ and varying $N$ (for two examples with $\delta = 0.5$ and $\delta = 0.01$, see figures 5 and 6), and for overlap involving only a few GLL points, translating into an $N$-dependent overlap $\delta \sim N^{-2}$ or $\delta \sim N^{-1}$ (see figure 7 for two examples). In all cases, we observed that the constants for the interpolated partition of unity are at most a factor of 2 worse than those for the piecewise linear case.

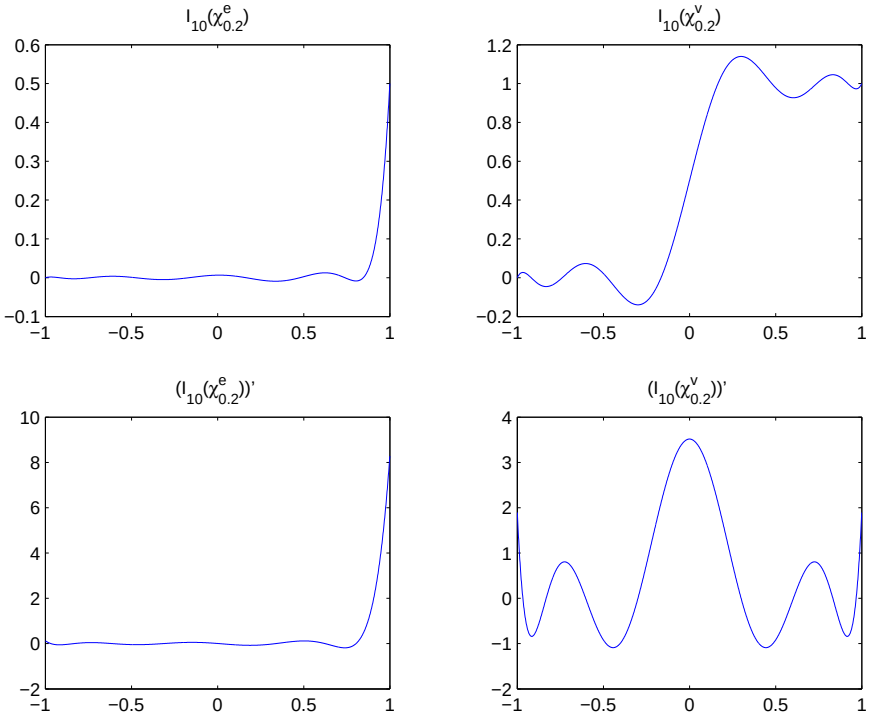



Figure 4: Partitions of unity: GLL interpolants of building blocks and their derivatives, $\delta = 0.2$, $N = 10$.

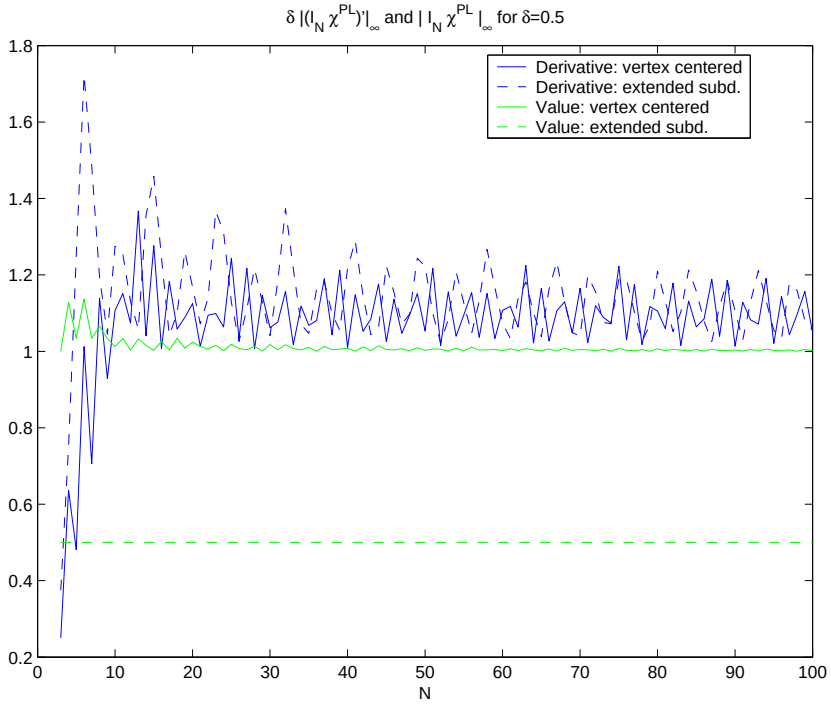
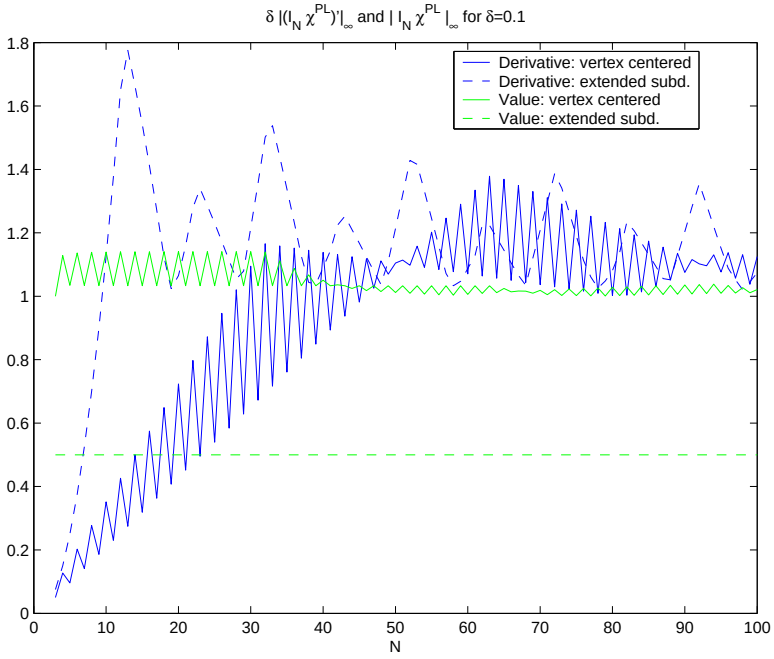



Figure 5: Partitions of unity: GLL interpolants and derivatives: $\delta = 0.5$, $N = 3, \cdots, 100$.



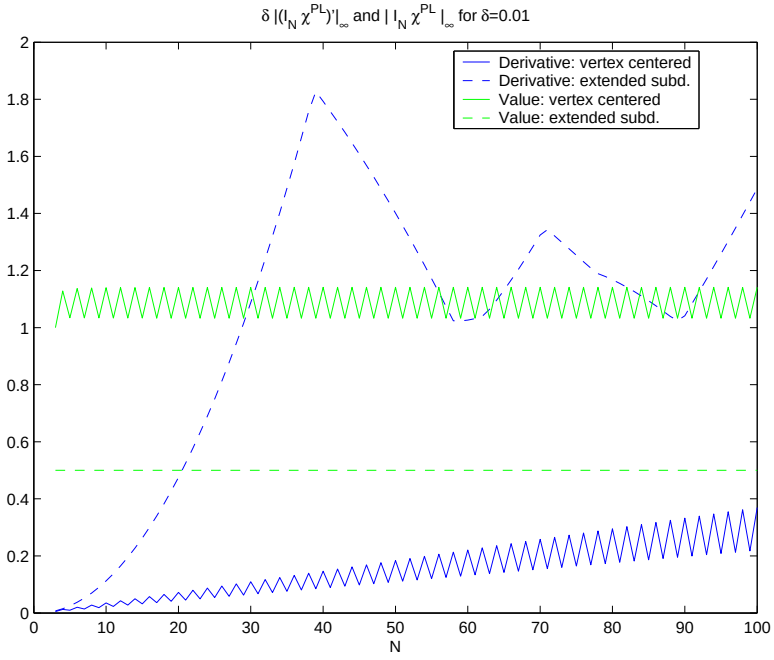
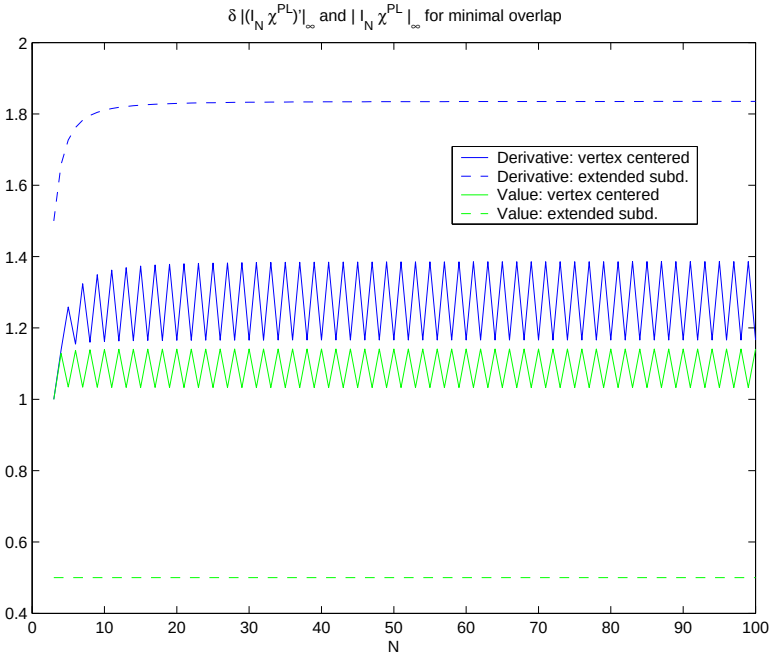




Figure 6: Partitions of unity: GLL interpolants and derivatives: Top: $\delta = 0.1$, Bottom:$\delta = 0.01$; $N = 3, \cdots, 100$.



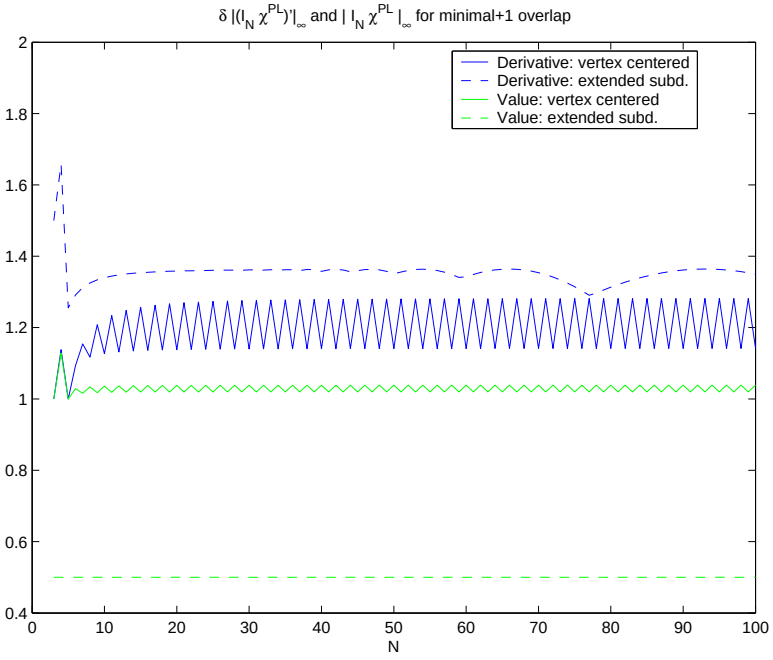




Figure 7: Partitions of unity: GLL interpolants and derivatives: Top: minimal overlap, Bottom: second smallest overlap; $N = 5, \cdots, 100$.

*Proof of Estimate 6.2:*

It is easy to check that[1]

$$|\chi_\delta^e|_1 \approx \delta^{-1/2} \qquad |\chi_\delta^e|_0 \approx \delta^{1/2}$$

For the estimate of the absolute value, we start with a special case of the Gagliardo-Nirenberg inequality (see, e.g., NIRENBERG [25])

$$|I_N\chi_\delta^e|_{0,\infty} \lesssim |I_N\chi_\delta^e|_1^{1/2}|I_N\chi_\delta^e|_0^{1/2}$$

Since the $H^1$-seminorm and the $L^2$-norm of a function $\Phi$ in $\mathbb{P}_N(\Lambda)$ and of its piecewise linear GLL interpolant $I_h^N\Phi$ are equivalent (see, e.g., CANUTO [9, Propositions 2.1 and 2.2]), we obtain (noting that $I_h^N I_N$ is the identity on $\chi_\delta^e$) the desired estimate

$$|I_N\chi_\delta^e|_{0,\infty} \lesssim |\chi_\delta^e|_1^{1/2}|\chi_\delta^e|_0^{1/2} \lesssim 1$$

For a bound on the gradient of the GLL interpolant of $\chi_\delta^e$, we can use an inverse inequality and the equivalence of the $|\cdot|_1$ norm between $\mathbb{P}_N(\Lambda)$ and the piecewise linear GLL space to obtain

$$|I_N\chi_\delta^e|_{1,\infty} \lesssim N|I_N\chi_\delta^e|_1 \lesssim N|\chi_\delta^e|_1 \lesssim N\delta^{-1/2}$$

For fixed $\delta$, this estimate is better by $\delta^{1/2}$ in $\delta$ and worse by $N$ in $N$. If $\delta \approx N^{-\alpha}$, the bound is $N^{1+\alpha/2}$ and we obtain the desired bound for $\alpha \geq 2$, i.e. minimal overlap, else the estimate is worse by $N^{1-\alpha/2}$.

For $\delta = h$, the piecewise linear partition of unity is actually linear in each element, and so polynomial interpolation will reproduce it, and therefore the original bounds also hold trivially for the polynomially interpolated partition. ∎

We have not been able to prove that the GLL interpolant of the piece-wise linear partition of unity is stable in the $|.|_{1,\infty}$ seminorm for general overlap, but have obtained the result only for some special cases. Instead of proving stability, one could also try to analyze the relationship between $|I_N u|_{1,\infty}$ and $|u|_{1,\infty}$ for piecewise linear $u$ on the GLL mesh, which would imply a $L^\infty$ bound for the gradient as well. All the approaches mentioned above require interpolation estimates, approximation results and analysis that do not seem to be known or at least not published yet.

As a feasible alternative, we can instead define an interpolant $I_N^\star\chi_\delta^e$ of degree proportional to $N$ which interpolates $\chi_\delta^e$ at the GLL grid points, but also guarantees an uniform approximation of its derivative (that is, the derivative of a smoothed version of $\chi_\delta^e$). Our analysis of the Nedélec type interpolant on polynomial spaces gives the same order in the estimate when the degree of the polynomial partition of unity is $2N$ or $3N$ instead of $N$. An extended Lagrangian interpolation, starting from the zeros of orthogonal polynomials and adding more zeros and endpoints to the interpolation mesh, and which is guaranteed to approximate both function and derivative uniformly, can be found in CRISCUOLO, MASTROIANNI, AND VERTESI [11]. We will not follow this line of argument in this paper and will instead refer for the details to future work.

[1] We use the notation $a \approx b$ and $a \lesssim b$ meaning that there exist $c$ and $C$ independent of the parameters of interest such that $cb \leq a \leq Cb$ and $a \leq Cb$, respectively.

**Estimate 6.3 ($L_2$-stability of the Nédélec interpolant on the local spaces)** *Let $\chi_i^M$ be the interpolated partition of unity used to define the local splitting. Then, there exist a constant $C$ independent of $N$, $h$, and $\mathbf{u}$, and a function $f_2(N)$ independent of $h$ and $\mathbf{u}$ such that for all $\mathbf{u} \in \mathbb{ND}_N^I$*

$$||\mathbf{\Pi}_N^{ND,I}(\chi_i^M \mathbf{u})||_0 \leq C f_2(N) ||\chi_i^M \mathbf{u}||_0 \tag{33}$$

**Estimate 6.4 ($\mathbf{curl}$-stability of the Nédélec interpolant on the local splaces)** *Let $\chi_i^M$ be the interpolated partition of unity used to define the local splitting. Then, there exist a constant $C$ independent of $N$, $h$ and $\mathbf{u}$, and a function $f_3(N)$ independent of $h$ and $\mathbf{u}$ such that for all $\mathbf{u} \in \mathbb{ND}_N^I$*

$$||\,\mathbf{curl}\left(\mathbf{\Pi}_N^{ND,I}(\chi_i^M \mathbf{u})\right)||_0 \leq C f_3(N) ||\,\mathbf{curl}(\chi_i^M \mathbf{u})||_0 \tag{34}$$

*Proofs:*

The properties of the local splitting (33) and (34) are usually proven in a way that makes as little use of the special form of $\chi_i$ as possible, and are based on estimates for $\mathbf{\Pi}_N^{ND,I}$ on certain polynomial spaces of higher degree in which $\chi_i^M \mathbf{u}$ lies. If that space is denoted $\mathbb{V}_{N+}$, then it would be enough to prove

$$\forall v_{N+} \in \mathbb{V}_{N+}: \qquad ||\mathbf{\Pi}_N^{ND,I} v_{N+}||_0 \leq C f_2(N) ||v_{N+}||_0$$

$$\forall v_{N+} \in \mathbb{V}_{N+}: \quad ||\,\mathbf{curl}\left(\mathbf{\Pi}_N^{ND,I} v_{N+}\right)||_0 \leq C f_3(N) ||\,\mathbf{curl}\, v_{N+}||_0$$

The second estimate is reduced to an estimate of a different interpolation operator on a different space using the commuting diagram property: Let $\mathbb{C}_{N+}$ be a space containing $\mathbf{curl}\,\mathbb{V}_{N+}$. Set $\mathbb{T}_N = \mathbb{W}_N(\Omega, T_h)$ for the two-dimensional case, and $\mathbb{T}_N = \mathbb{RT}_N(\Omega, T_h)$ for the three-dimensional case and let $\mathbf{\Pi}_N^T$ be the commuting interpolant in $\mathbb{T}_N$. Then the commuting diagram property implies that $\mathbf{curl}\left(\mathbf{\Pi}_N^{ND,I} v_{N+}\right) = \mathbf{\Pi}_N^T(\mathbf{curl}\, v_{N+})$. Therefore proving

$$\forall v_{N+} \in \mathbb{V}_{N+}: \qquad ||\mathbf{\Pi}_N^{ND,I} v_{N+}||_0 \leq C f_2(N) ||v_{N+}||_0$$

$$\forall w_{N+} \in \mathbb{C}_{N+}: \qquad ||\mathbf{\Pi}_N^T w_{N+}||_0 \leq C f_3(N) ||w_{N+}||_0$$

implies the estimates (33) and (34).

These bounds can be reduced to bounds on the components of the interpolation operators, and from there to bounds on the factors in the tensor products constituting the components. Therefore, if we can bound the $L^2$ and modified $H^1$ projection between polynomial spaces, we can analyze the local splitting. The $L^2$ projection has norm at most 1 in the $L^2$ norm, so we are just left with the modified $H^1$ projection. Numerical experiments in HIENTZSCH [16, Section 7.6] and a theoretical analysis in HIENTZSCH [17] allow us to bound the modified $H^1$ projection between polynomial spaces.

We choose $\chi_i$ as a polynomial inside each small element $K_h$ in $T_h$, interpolating the piecewise linear partition of unity at the GLL points. In the case of element-wise

overlap, we can just choose a linear function inside each element, and $\chi_i \mathbf{u}$ would be in $\mathbb{V}_{N+} = \mathbb{ND}^I_{N+1}$, and $\mathbb{C}_{N+} = \mathbb{T}_{N+1}$. For this case our numerical and theoretical results (see HIENTZSCH [16, 17]) show that (33) and (34) are satisfied with $f_1(N) = f_2(N) = 1$ and a small constant $C$.

To use $\chi_i^M$ in the proof, we need $L^2$-bounds for the interpolation operators on the spaces $\mathbb{V}_{N+} = \mathbb{ND}^I_{N+M}$ and $\mathbb{C}_{N+} = \mathbb{T}^I_{N+M}$. We refer to HIENTZSCH [16, Section 7.6] where we computed such bounds numerically. For $M$ constant, $f_2(N) = f_3(N) = 1$. For $M = N$ or $M = cN$, we obtained $f_2(N) = \sqrt{N}$ and $f_3(N) = \sqrt{N}$. In HIENTZSCH [17], we prove these bounds. For general overlap, we need $M = N$ or $M = cN$, and therefore $f_2(N) = f_3(N) = \sqrt{N}$. ∎

# 7 Technical tools

To introduce and analyze a stable projection into the coarse space, we need several operators. One of them is the orthogonal projection into the weakly divergence-free space

$$\Theta : H_0(\mathbf{curl}) \to H_0^{\perp}(\mathbf{curl})$$

defined by

$$\Theta \mathbf{u} := \mathbf{u} - \mathbf{grad}\, q$$

where $q \in H_0^1(\Omega)$ is the unique solution of

$$?q : \forall p \in H_0^1(\Omega) : (\mathbf{grad}\, q, \mathbf{grad}\, p) = (\mathbf{u}, \mathbf{grad}\, p)$$

It follows easily that $\Theta$ leaves the $\mathbf{curl}$ of its argument unchanged, and is also an orthogonal projection in $(L^2(\Omega))^3$.

We use $\Theta$ now to define the finite dimensional subspace

$$V^{\perp} = \Theta(\mathbb{ND}^{I,+}_N(\Omega, T_h)) \subset H_0^{\perp}(\mathbf{curl}).$$

Even though $V^{\perp}$ is not a spectral element space, the $\mathbf{curl}$ of functions in $V^{\perp}$ is a piecewise polynomial vector field. We recall that we showed in Estimate 6.1 that the Nédélec interpolant has better bounds on such functions.

Next, we define a projection $P_N$ onto $V^{\perp}$:

$$\begin{aligned} P_N : H_0(\mathbf{curl}) &\to V^{\perp} \\ ?P_N\mathbf{u} \in V^{\perp} : \forall \mathbf{v} \in V^{\perp} &: (\mathbf{curl}(P_N\mathbf{u} - \mathbf{u}), \mathbf{curl}\, \mathbf{v}) = 0 \end{aligned}$$

Since $||\, \mathbf{curl} \cdot ||_0$ is an equivalent norm to $|| \cdot ||_0$ on $H_0(\mathbf{curl})$, $P_N$ is well-defined.

*Remark* On $\mathbb{ND}^{I,+}_N(\Omega, T_h)$, $P_N\mathbf{u}$ coincides with $\Theta\mathbf{u}$, and it is clear that

$$\mathbf{curl}\, P_N\mathbf{u} = \mathbf{curl}\, \Theta\mathbf{u} = \mathbf{curl}\, \mathbf{u}$$

Next, we prove an error estimate for $P_N$ using the interpolation properties of the Nédélec interpolant. This lemma corresponds to [30, Lemma 3.3]. We follow the idea of the proof from that paper, but instead of invoking [30, Lemma 3.1], we use Estimate 6.1 .

**Lemma 7.1** *Let $\Omega$ be convex. Then, the operator $P_N$ satisfies the following error estimate for all $\mathbf{u} \in \mathbb{ND}_N^{I,+}(\Omega, T_h)$ with $C$ independent of $h$, $N$, and $\mathbf{u}$:*

$$||\mathbf{u} - P_N\mathbf{u}||_0 \leq C f_1(N,h)||\,\mathbf{curl}\,\mathbf{u}||_0 \tag{35}$$

*Proof:* Let $\mathbf{u} \in \mathbb{ND}_N^{I,+}(\Omega, T_h)$. Using the remark after the definition of $P_N\mathbf{u}$, $\mathbf{curl}(\mathbf{u} - P_N\mathbf{u}) = 0$, and therefore

$$\mathbf{u} - P_N\mathbf{u} = \mathbf{grad}\, q$$

with some $q \in H_0^1(\Omega)$. Now the appropriate version of the commuting diagram property guarantees that

$$\mathbf{u} - \mathbf{\Pi}_N^{ND,I} P_N\mathbf{u} = \mathbf{\Pi}_N^{ND,I}(\mathbf{u} - P_N\mathbf{u}) = \mathbf{grad}\, q_N \tag{36}$$

with some $q_N \in \mathbb{W}_N(\Omega)$.

We rewrite

$$||\mathbf{u} - P_N\mathbf{u}||_0^2 = (\mathbf{u} - P_N\mathbf{u}, \mathbf{u} - \mathbf{\Pi}_N^{ND,I} P_N\mathbf{u} + \mathbf{\Pi}_N^{ND,I} P_N\mathbf{u} - P_N\mathbf{u})$$

and use that $\mathbf{u}$ and $P_N\mathbf{u}$, by (36), are orthogonal to $\mathbf{u} - \mathbf{\Pi}_N^{ND,I} P_N\mathbf{u}$ :

$$\begin{aligned} ||\mathbf{u} - P_N\mathbf{u}||_0^2 &= (\mathbf{u} - P_N\mathbf{u}, \mathbf{\Pi}_N^{ND,I} P_N\mathbf{u} - P_N\mathbf{u}) \\ &\leq ||\mathbf{u} - P_N\mathbf{u}||_0 ||P_N\mathbf{u} - \mathbf{\Pi}_N^{ND,I} P_N\mathbf{u}||_0 \end{aligned}$$

to obtain

$$||\mathbf{u} - P_N\mathbf{u}||_0 \leq ||P_N\mathbf{u} - \mathbf{\Pi}_N^{ND,I} P_N\mathbf{u}||_0$$

We use Estimate 6.1 and the remark to estimate

$$\begin{aligned} ||P_N\mathbf{u} - \mathbf{\Pi}_N^{ND,I} P_N\mathbf{u}||_0 &\leq C f_1(N,h)||\,\mathbf{curl}\, P_N\mathbf{u}||_0 \\ &\leq C f_1(N,h)||\,\mathbf{curl}\,\mathbf{u}||_0 \end{aligned}$$

and thus obtain the estimate in the lemma. ∎

We also need the $L^2$-projection

$$Q_0 : (L^2(\Omega))^3 \to V_0$$

onto the coarse space $V_0$. We require some estimates for $Q_0$, which can be proven exactly as in TOSELLI [30]:

**Lemma 7.2** *Let $T_H$ be shape-regular and quasi-uniform. Then, the following estimates hold with constants independent of $\mathbf{u}$ and $H$:*

$$\forall \mathbf{u} \in (H^1(\Omega))^3 : \qquad ||\,\mathbf{curl}\, Q_0\mathbf{u}||_0 \leq C|\mathbf{u}|_1 \tag{37}$$

$$\forall \mathbf{u} \in (H^1(\Omega))^3 : \qquad ||\mathbf{u} - Q_0\mathbf{u}||_0 \leq C H|\mathbf{u}|_1 \tag{38}$$

The use of the $L^2$-projection as the projection into the coarse space requires the coarse triangulation to be quasi-uniform. We could instead use a locally defined quasi-interpolant to weaken the conditions on the coarse triangulation, analogous to the techniques developed in the scalar elliptic case. We will not do so in this article, and hope to present such results in future work.

# 8 Condition number estimate

We will use the abstract Schwarz theory; see SMITH, BJØRSTAD, AND GROPP [26] for an introduction in textbook form, discussing the $h$-versions of standard algorithms.

We will give an upper bound for the inverse of the smallest eigenvalue by the standard decomposition argument, and $C_0^{-2}$ will then be a lower bound for the smallest eigenvalue. Using a standard argument (see, e.g., SMITH, BJØRSTAD, AND GROPP [26, proof of theorem 1 on page 167]), the coloring assumption implies an upper bound for the eigenvalues of $T_{as1}$ of $N_C$ and of $T_{as2}$ of $N_C+1$. Therefore, the bound $C_0^2$ proven in the next theorem will imply a bound of $(N_C+1)C_0^2$ for the condition number of $T_{as2}$.

We will first prove the theorem using the general forms of the required estimates. Afterwards we will discuss the estimate for specific cases, and give shorter forms.

**Theorem 8.1 (Lower bound)** *For every $\mathbf{u} \in V$ there is a splitting $\mathbf{u} = \sum \mathbf{u}_i$ with $\sum a(\mathbf{u}_i,\mathbf{u}_i) \leq C_0^2 a(\mathbf{u},\mathbf{u})$, $C_0^2 = CC_1$ with a $C_1$ of the form*

$$\max\left\{ N_C\left(1+\frac{H}{\delta}\right), \max\left(1+N_C f_2^2(N) \quad , \right.\right.$$
$$\left.\left. 1+N_C f_3^2(N)\left(1 \quad + \quad \left(\frac{H+f_1(N,h)}{\delta}\right)^2\right)\right)\right\}$$

*Proof:* First, we use the discrete Helmholtz decomposition (see section 3) in $\mathbb{ND}_N^I$ to split $\mathbf{u}$ into a sum $\mathbf{grad}\, q + \mathbf{w}$, where $q \in \mathbb{S}_N$ and $\mathbf{w} \in \mathbb{ND}_N^{I,+}$. The two parts are orthogonal in $H(\mathbf{curl})$ and also with respect to the bilinear form $a(\cdot,\cdot)$, so that we can decompose and estimate them separately. For gradients, the second term in $a(\cdot,\cdot)$ vanishes, and $a(\mathbf{grad}\, q, \mathbf{grad}\, q) = (\mathbf{grad}\, q, \mathbf{grad}\, q)_0$ is the bilinear form for the Laplace operator in $q$. Therefore, we can use the domain decomposition theory for scalar elliptic operators and results for the spectral element case for the Laplace equation. CASARIN [10, Theorem 3.5.2] proves a bound on the condition number of the additive two-level overlapping Schwarz preconditioner that corresponds to our preconditioner on the $\mathbf{grad}\, q$ part. His result implies that there is a decomposition $\sum_i q_i$ of $q$ such that

$$\begin{aligned}\sum_i a(\mathbf{grad}\, q_i, \mathbf{grad}\, q_i) &= \eta_1 \sum_i |q_i|_1^2 \leq CN_C\left(1+\frac{H}{\delta}\right)\eta_1 |q|_1^2 \\ &\leq CN_C\left(1+\frac{H}{\delta}\right) a(\mathbf{grad}\, q, \mathbf{grad}\, q) \qquad (39)\end{aligned}$$

We now decompose $\mathbf{w}$. We start with the coarse space. To define $w_0$, we first use the projection $P_N$ into the semicontinuous divergence-free space $V^\perp$, followed by the $L^2$-projection $Q_0$ into the coarse space $\mathbb{ND}^I_{N_0}$:

$$\mathbf{w} = \mathbf{w}_0 + \mathbf{v} \qquad \mathbf{w}_0 := Q_0(P_N\mathbf{w}) \in V_0 \qquad \mathbf{v} = (I - Q_0P_N)\mathbf{w}$$

We will decompose the remainder $\mathbf{v}$ by multiplying it with the partition of unity $\chi_i$ and using the Nédélec interpolant to project it back into the local space:

$$\mathbf{w}_i := \mathbf{\Pi}^{ND,I}_N(\chi_i\mathbf{v}) \in V_i$$

First, we estimate $||\,\mathbf{curl}\,\mathbf{w}_0||^2_0$:

$$\begin{aligned} ||\,\mathbf{curl}\,\mathbf{w}_0||^2_0 &= ||\,\mathbf{curl}\,Q_0(P_N\mathbf{w})||^2_0 \lesssim |P_N\mathbf{w}|^2_1 \\ &\lesssim ||\,\mathbf{curl}\,P_N\mathbf{w}||^2_0 \lesssim ||\,\mathbf{curl}\,\mathbf{w}||^2_0 \end{aligned} \tag{40}$$

The first inequality uses property (37) from Lemma 7.2 on the $L^2$-projection, the second follows from the imbedding of $H_T(\Omega)$ in $H^1(\Omega)$ (see section 2), and the last one by noticing that $P_N$ leaves the $\mathbf{curl}$ of its argument unchanged.

We then bound $||\,\mathbf{curl}\,\mathbf{w}_i||^2_0$ (here, we need the assumptions on the partition of unity in Estimate 6.2):

$$\begin{aligned} ||\,\mathbf{curl}\,\mathbf{w}_i||^2_0 &= ||\,\mathbf{curl}\,\mathbf{\Pi}^{ND,I}_N(\chi_i\mathbf{v})||^2_0 \\ &\lesssim f_3^2(N)||\,\mathbf{curl}(\chi_i\mathbf{v})||^2_0 \\ &\lesssim f_3^2(N)\left(||\,\mathbf{grad}\,\chi_i \times \mathbf{v} + \chi_i\,\mathbf{curl}\,\mathbf{v}||^2_0\right) \\ &\lesssim f_3^2(N)\left(||\,\mathbf{grad}\,\chi_i||^2_{0,\infty}||\mathbf{v}||^2_{0,\Omega_i'} + ||\chi_i||^2_{0,\infty}||\,\mathbf{curl}\,\mathbf{v}||^2_{0,\Omega_i'}\right) \\ &\lesssim f_3^2(N)\left(\delta^{-2}||\mathbf{v}||^2_0 + ||\,\mathbf{curl}\,\mathbf{v}||^2_0\right) \end{aligned} \tag{41}$$

We realize that we have to bound $||\mathbf{v}||^2_0$ and $||\,\mathbf{curl}\,\mathbf{v}||^2_0$ to complete this estimate.

To bound the $L^2$-norm of $\mathbf{v} = \mathbf{w} - Q_0P_N\mathbf{w}$, we write $\mathbf{v} = \mathbf{w} - P_N\mathbf{w} + P_N\mathbf{w} - Q_0P_N\mathbf{w}$ and use the triangle inequality to obtain

$$||\mathbf{v}||^2_0 \lesssim ||\mathbf{w} - P_N\mathbf{w}||^2_0 + ||P_N\mathbf{w} - Q_0P_N\mathbf{w}||^2_0$$

We can estimate the first term by Lemma 7.1 from the last section, and the second term by property (38) from Lemma 7.2 on the $L^2$-projection and the arguments in (40):

$$\begin{aligned} ||\mathbf{v}||^2_0 &\lesssim f_1^2(N,h)||\,\mathbf{curl}\,\mathbf{w}||^2_0 + CH^2||\,\mathbf{curl}\,\mathbf{w}||^2_0 \\ &\lesssim (H + f_1(N,h))^2||\,\mathbf{curl}\,\mathbf{w}||^2_0 \end{aligned}$$

To estimate $||\,\mathbf{curl}\,\mathbf{v}||^2_0$, we rewrite $\mathbf{curl}\,\mathbf{v} = \mathbf{curl}(\mathbf{w} - \mathbf{w}_0) = \mathbf{curl}\,\mathbf{w} - \mathbf{curl}\,\mathbf{w}_0$, use (40) and the triangle inequality to obtain

$$||\,\mathbf{curl}\,\mathbf{v}||^2_0 \lesssim ||\,\mathbf{curl}\,\mathbf{w}||^2_0$$

Substituting these bounds into (41) yields

$$|| \mathbf{curl}\, \mathbf{w}_i||_0^2 \lesssim f_3^2(N) \left(1 + \left(\frac{H + f_1(N,h)}{\delta}\right)^2\right) || \mathbf{curl}\, \mathbf{w}||_0^2 \tag{42}$$

The bound on $||\mathbf{w}_0||_0^2$ follows from the definitions of $Q_0$ and $\theta$ and the remark after their definition:

$$||\mathbf{w}_0||_0^2 = ||Q_0 P_N \mathbf{w}||_0^2 \leq ||P_N \mathbf{w}||_0^2 = ||\Theta \mathbf{w}||_0^2 \leq ||\mathbf{w}||_0^2 \tag{43}$$

Finally, using $\mathbf{w}_i = \mathbf{\Pi}_N^{ND,I}(\chi_i(\mathbf{w} - \mathbf{w}_0))$, (33), the triangle inequality, and (43), we obtain

$$\begin{aligned} ||\mathbf{w}_i||_0^2 &= ||\mathbf{\Pi}_N^{ND,I}(\chi_i(\mathbf{w} - \mathbf{w}_0))||_0^2 \\ &\lesssim f_2^2(N)||\mathbf{w} - \mathbf{w}_0||_0^2 \\ &\lesssim f_2^2(N)||\mathbf{w}||_0^2 \end{aligned} \tag{44}$$

Adding up (40) and (42) for all $i$, and using the coloring assumption, one shows

$$\sum_{i=0}^{J} || \mathbf{curl}\, \mathbf{w}_i||_0^2 \lesssim C_1(N_C, N, \delta, h, H)|| \mathbf{curl}\, \mathbf{w}_0||_0^2 \tag{45}$$

with

$$C_1(N_C, N, \delta, h, H) := \left(1 + N_C f_3^2(N) \left(1 + \left(\frac{H + f_1(N,h)}{\delta}\right)^2\right)\right) \tag{46}$$

Similarly, adding (43) and (44) shows

$$\sum_{i=0}^{J} ||\mathbf{w}_i||_0^2 \lesssim (1 + N_C f_2^2(N))||\mathbf{w}||_0^2 \tag{47}$$

Now we put the bounds back together, to estimate

$$\begin{aligned} \sum_{i=0}^{J} a(\mathbf{w}_i, \mathbf{w}_i) &= \alpha \sum_{i=0}^{J} ||\mathbf{w}_i||_0^2 + \beta \sum_{i=0}^{J} || \mathbf{curl}\, \mathbf{w}_i||^2 \\ &\lesssim C_2(N_C, N, \delta, h, H)(\alpha||\mathbf{w}||_0^2 + \beta|| \mathbf{curl}\, \mathbf{w}||_0^2) \\ &\lesssim C_2(N_C, N, \delta, h, H) a(w, w) \end{aligned}$$

with

$$C_2(N_C, N, \delta, h, H) := \max\left((1 + N_C f_2^2(N)), C_1(N_C, N, \delta, h, h)\right) \tag{48}$$

Finally, we combine this bound with the bound from the decomposition of $\mathbf{grad}\, q$ in (39) to obtain the bound given in the theorem. ∎

Domain decomposition methods for spectral elements are often used with the spectral elements constituting the subdomains, and therefore $H = h$.

In both two and three dimensions, we then have

$$\left(1 + \left(\frac{H + f_1(N,h)}{\delta}\right)^2\right) \leq C\left(1 + \left(\frac{H}{\delta}\right)^2\right).$$

In two dimensions, $f_1(N,h)$ goes to zero with increasing $N$.

For the element-wise overlap case $f_2(N) = f_3(N) = 1$, and therefore we obtain after some easy computations:

**Corollary 8.2 (Element-wise overlap)** *In the case of element-wise overlap, the condition number of $T_{as2}$ is bounded by*

$$\kappa(T_{as2}) \leq C(N_C + 1)\left(1 + N_C\left(1 + \left(\frac{H}{\delta}\right)^2\right)\right)$$

This result corresponds to the result in TOSELLI [30] and differs only in that it is explicit in $N$ and $N_C$.

For the minimal overlap case, we obtain

**Corollary 8.3 (Minimal overlap)** *For general $\delta$, an upper bound of the condition number of $T_{as2}$ is given by*

$$\kappa(T_{as2}) \leq C(N_C + 1)N\left(1 + N_C\left(1 + \left(\frac{H}{\delta}\right)^2\right)\right)$$

# 9 Numerical results

We have performed numerical experiments on the square $[-1,1]^2$ split into $M \times M$ spectral elements of degree $(N \times N, N \times N)$ (corresponding to Nédélec elements of the second kind). We take as overlapping subdomains vertex centered domains $\Omega_i$ that overlap each other by $\delta$. For $\delta = h$, this gives us extended subdomains consisting out of $2 \times 2$ spectral elements around each inner vertex of the spectral element mesh, see figure 8, left panel. For $\delta < h$, we work on rectangular parts of spectral elements, see figure 8, right panel. We chose the spectral elements as subdomains, $h = H$ and use Nédélec elements of the second kind of degree $N_0$ as the coarse space. We also performed experiments with Nédélec elements of the first kind and with the anisotropic version and obtain similar results. The number of colors for $\delta \leq h$ is $N_C = 4$, so that we have an upper bound on the spectrum of $T_{as2}$ of 5.

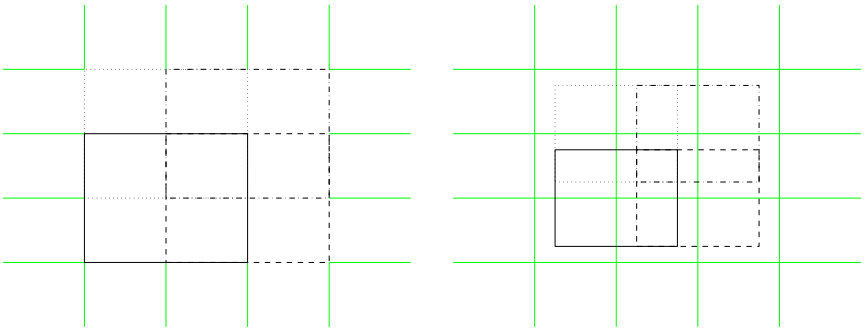


Figure 8: Four overlapping subregions in the vertex centered case. Left: Element-wise overlap, overlapping subdomains are 2x2 spectral elements. Right: Overlap $\delta/h = 1/2$, overlapping subdomains consist out of 4 rectangular patches from different spectral elements.

We first present results for the element-wise overlapping case, $\delta = h$. The first figure, figure 9, shows a representative result for the one-level method for $\alpha = \beta = 1$, for the element-wise overlap case. We see that we have uniform bounds with respect to $N$, and growth with respect to $M$, because of the lack of a coarse space to transmit information fast enough for fast convergence.

The second figure, figure 10, shows the result for the same problem, but with the two-level method. We see that not only the condition numbers are considerably smaller, they are bounded with respect to both $N$ and $M$, as our theoretical results predict.

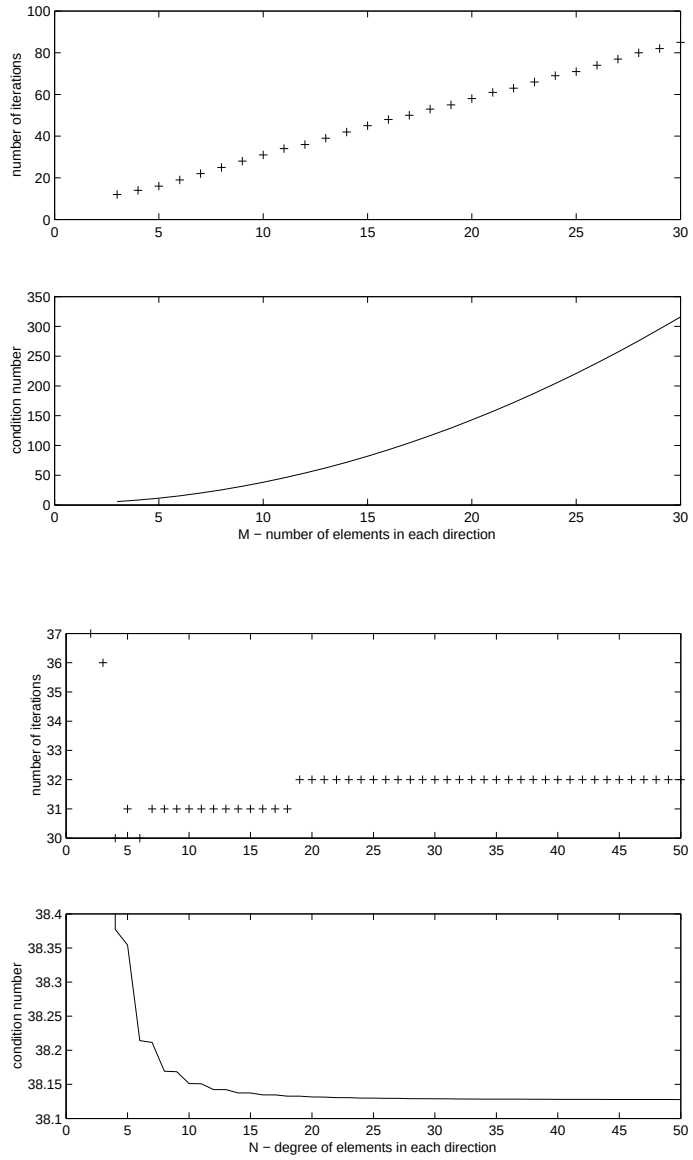



Figure 9: One-level method, $\alpha = 1$, $\beta = 1$. Top: Scaling with respect to $M$, the number of subdomains of degree $10 \times 10$. Bottom: Scaling with respect to $N$, the degree inside the $10 \times 10$ spectral elements.

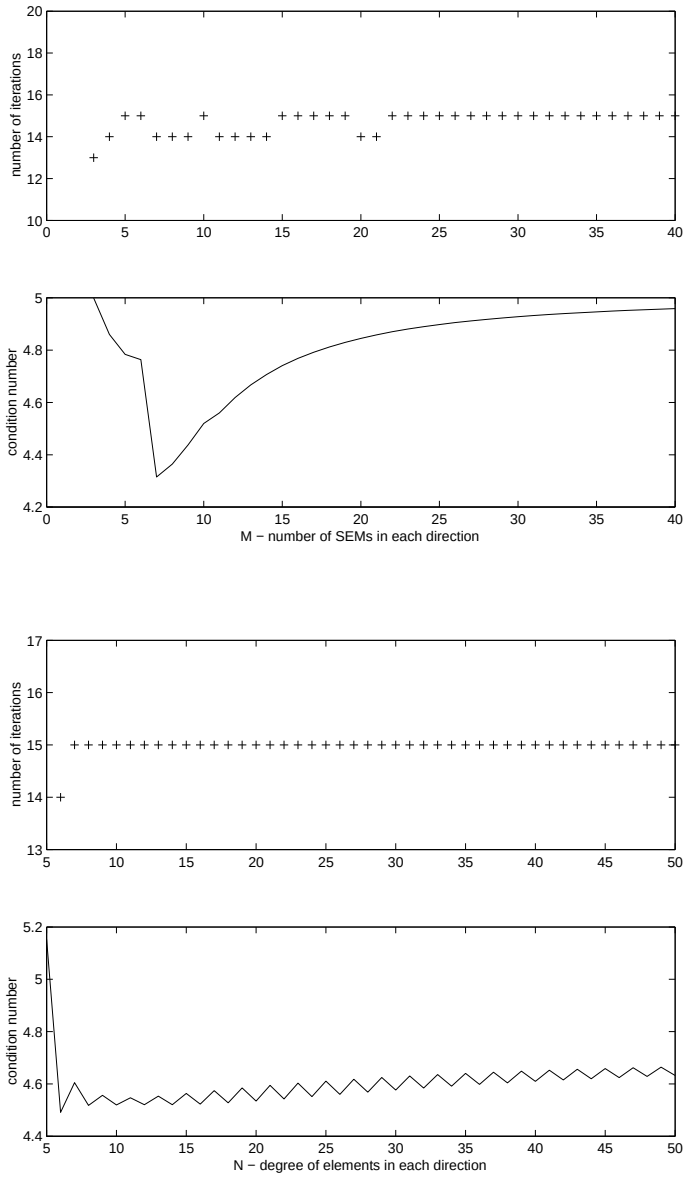



Figure 10: Two-level method, $\alpha = 1$, $\beta = 1$, $N_0 = 3$. Top: Scaling with respect to $M$, the number of subdomains of degree $10 \times 10$. Bottom: Scaling with respect to $N$, the degree inside the $10 \times 10$ spectral elements.

In the next experiment we test the robustness of the method against changes in $\alpha$ and $\beta$. We fix $\alpha = 1$ and vary $\beta$ over 16 orders of magnitude, and see that the condition numbers and the iteration numbers do not only stay bounded, but they actually depend only very slightly on $\beta$.

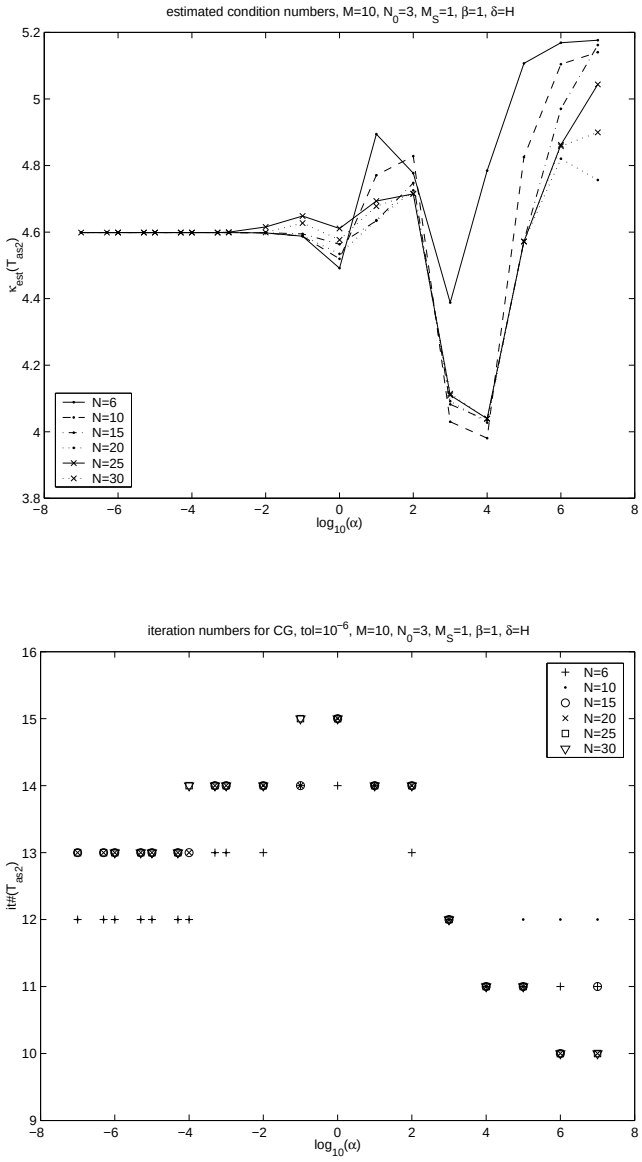



Figure 11: Two-level method, $10 \times 10$ spectral elements of degree $N \times N$, $n_0 = 3$, $\delta = H$, $\beta = 1$. Dependence on $\alpha$. Top: Condition number estimates. Bottom: Iteration numbers

Next, we present two examples in figure 12 for fixed $M$, $N$ and $N_0$ (and $\alpha = \beta = 1$), and investigate the dependence of the iteration and condition number on the overlap. We see that the condition number grows with decreasing overlap, but there is not enough data to estimate if it behaves quadratically or linearly in the inverse of the relative overlap.

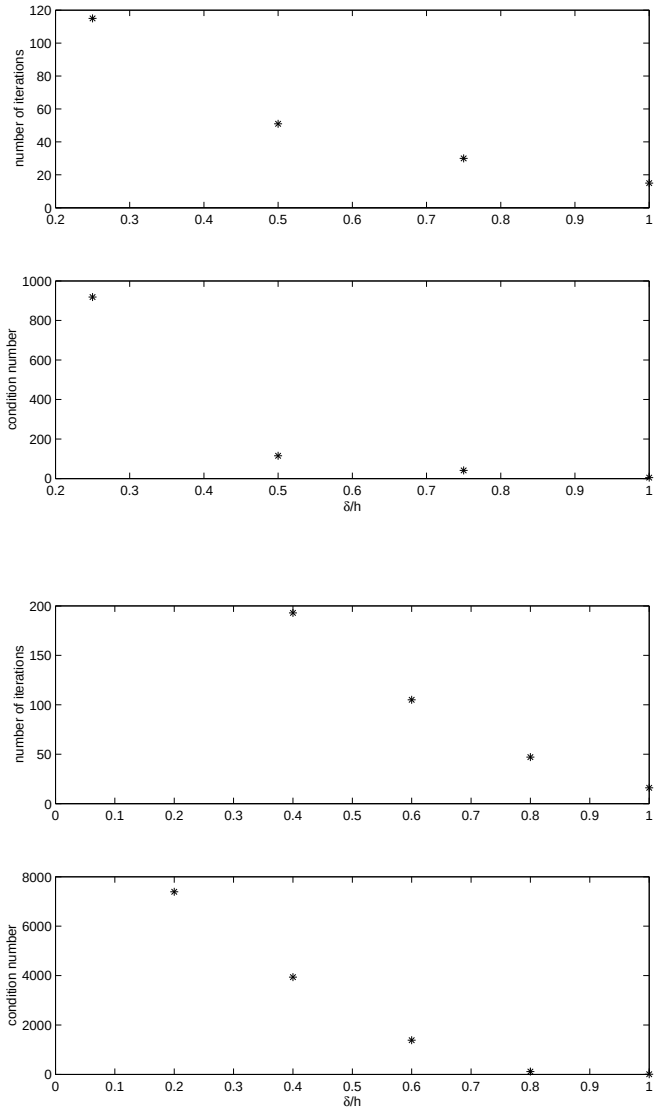



Figure 12: Two-level method, $10 \times 10$ spectral elements, $n_0 = 2$, varying overlap. Top: Spectral element degree $10 \times 10$. Bottom: Spectral element degree $20 \times 20$

Finally, we show some experiments for fixed overlap smaller than one element. We see that both condition number and iteration number grow with increasing degrees, seemingly in steps. (Since only overlaps corresponding to GLL points can be realized, some stepwise behavior should be expected.)

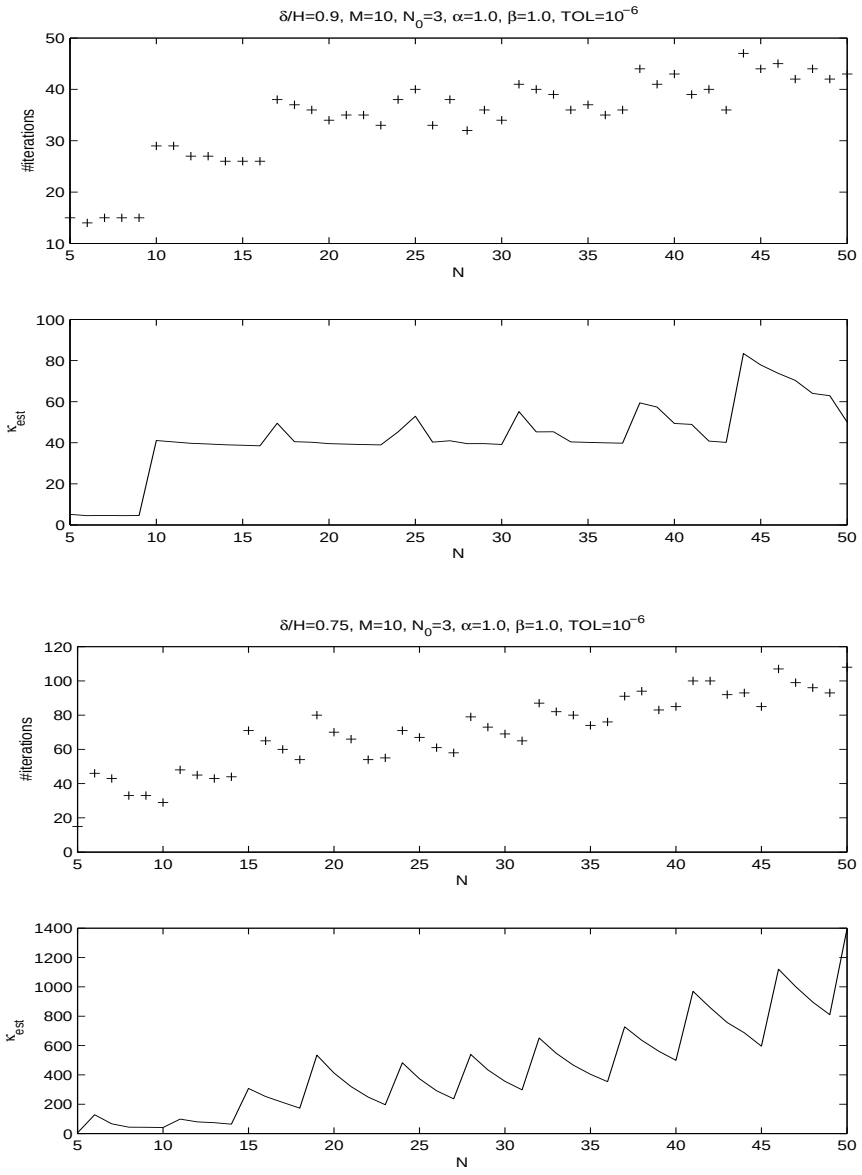



Figure 13: Two-level method, $10 \times 10$ spectral elements of degree $N \times N$, $n_0 = 3$. Top: $\delta = 0.9H$ Bottom: $\delta = 0.75H$.

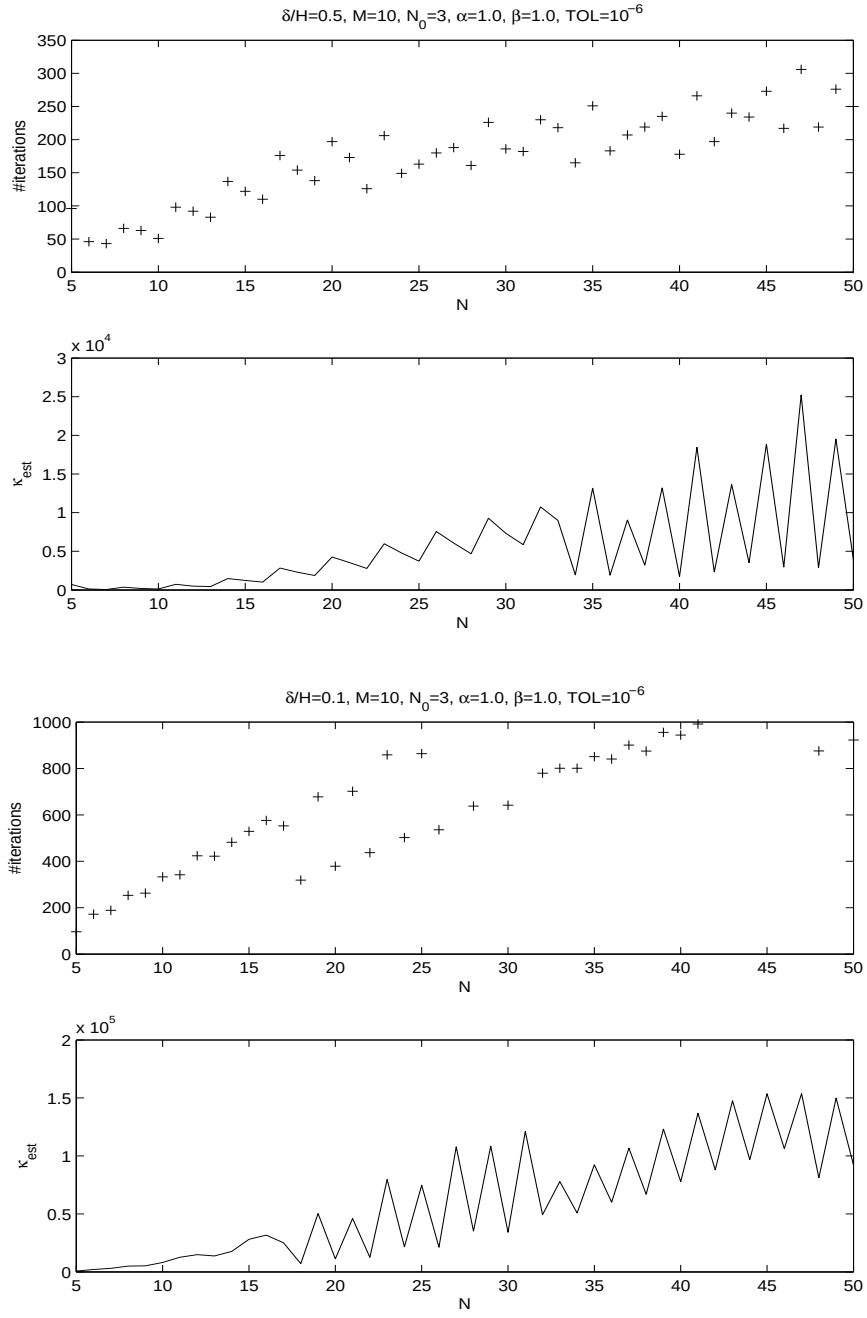



Figure 14: Two-level method, $10 \times 10$ spectral elements of degree $N \times N$, $n_0 = 3$. Top: $\delta = 0.5H$ Bottom: $\delta = 0.1H$.

The results are consistent with a linear growth in $N$ as predicted in our analysis. We will explore the behavior of our methods for less than element-wise overlap and increasing $N$ in more detail in future work.

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