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Dual-Primal FETI Methods for Stationary Stokes and Navier-Stokes Equations

by

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This thesis is dedicated to my grandmother, Xiu-Wen.

ABSTRACT

Finite element tearing and interconnecting (FETI) type domain decomposition methods are first extended to solving incompressible Stokes equations. One-level, two-level, and dual-primal FETI algorithms are proposed. Numerical experiments show that these FETI type algorithms are scalable, i.e., the number of iterations is independent of the number of subregions into which the given domain is subdivided. A convergence analysis is then given for dual-primal FETI algorithms both in two and three dimensions.

Extension to solving linearized nonsymmetric stationary Navier-Stokes equations is also discussed. The resulting linear system is no longer symmetric and a GMRES method is used to solve the preconditioned linear system. Eigenvalue estimates show that, for small Reynolds number, the nonsymmetric preconditioned linear system is a small perturbation of that in the symmetric case. Numerical experiments also show that, for small Reynolds number, the convergence of GMRES method is similar to the convergence of solving symmetric Stokes equations with the conjugate gradient method. The convergence of GMRES method depends on the Reynolds number; the larger the Reynolds number, the slower the convergence.

Dual-primal FETI algorithms are further extended to nonlinear stationary Navier-Stokes equations, which are solved by using a Picard iteration. In each iteration step, a linearized Navier-Stokes equation is solved by using a dual-primal FETI algorithm. Numerical experiments indicate that convergence of the Picard iteration depends on the Reynolds number, but is independent of both the number of subdomains and the subdomain problem size.

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Chapter 1

Introduction

1.1 An overview

Solving partial differential equations numerically is often reduced to solving large linear systems of equations. The size of a problem can often make the problem impossible to be solved on a single machine. Solving large linear systems on parallel computers is therefore imperative. The efficiency of such an approach depends on the degree of parallelism of the algorithm, i.e., the portion of the works that can be implemented in parallel. Direct solvers of solving a linear system are robust, but not efficient on parallel computers. Iterative methods are therefore often used in parallel computing for solving large linear systems. A very important type of iterative methods are the Krylov subspace methods. It is important that a Krylov method incur a constant (or nearly a constant) number of iterations, when the size of the problem (or correspondingly the number of processors) increases, in order to make the algorithm scalable on a parallel computer.

When using a Krylov subspace method to solve a linear system $Ax = b$, we just need to implement the multiplication of the matrix A times vectors. In each iteration step, we are looking for an approximation x_n of x, which minimizes a norm of the error, $x - x_n$, or the residual, $b - Ax_n$, in a certain Krylov subspace.

The convergence of a Krylov subspace method, for a symmetric positive definite problem, depends on the condition number of the matrix A. The condition number of A often depends on the mesh. In order to make the convergence of a Krylov subspace method independent of the mesh, we often solve a preconditioned linear system $M^{-1}Ax = M^{-1}b$, where M^{-1} is called the preconditioner. We expect the operator $M^{-1}A$ to be better conditioned and therefore faster convergence to be achieved.

Domain decomposition methods are a type of preconditioned Krylov subspace methods, where we use subdomain solvers (often coupled with a coarse level solver) to construct the preconditioner for solving the original global problem. In each iteration step, we solve small subdomain problems in parallel, and we also solve a coarse level problem. In most cases, a coarse level solver is crucial to make the parallel performance insensitive to the number of processors. A powerful framework of analysis has been well established in the past two decades for the study of symmetric positive definite elliptic PDEs; see the proceedings of annual international domain decomposition meetings, cf. [2], [14], [15], [38], [39], [40], [45], [46], [54], [58], [67]. The book [74] and the references therein also give a good introduction to this field.

There are two main different types of domain decomposition methods: overlapping Schwarz methods and iterative substructuring methods. In an overlapping Schwarz method, the domain is decomposed into overlapping subdomains. For a two-level algorithm of this type, the preconditioner is assembled from subdomain solvers and a coarse level solver. In each iteration step, only subdomain problems and a coarse level problem are solved. It has been proved that the condition number of a two-level overlapping Schwarz method, for symmetric positive definite problems and with generous overlap between subdomains, is bounded from above independently of the number of subdomains and the mesh size, cf. [18].

In an iterative substructuring method, the global problem is reduced to a problem for the subdomain interface variables, by using a Schur complement procedure. This interface problem is then solved by a preconditioned Krylov subspace method. The scalability of the nonoverlapping substructuring methods has also been established, cf. [19] and [21].

The Finite Element Tearing and Interconnecting (FETI) methods form a special family of domain decomposition methods. They are iterative substructuring methods of dual type. The first FETI method was proposed in [29] for solving positive definite elliptic partial differential equations. In this method, the spatial domain is decomposed into nonoverlapping subdomains, and the interior subdomain variables are eliminated to form a Schur problem for the interface variables. Lagrange multipliers are then introduced to enforce continuity across the interface, and a symmetric positive semi-definite linear system for the Lagrange multipliers is solved by using a preconditioned conjugate gradient (PCG) method.

There are extensive applications of domain decomposition methods to saddle point problems, especially for incompressible Stokes problems. Previous domain decomposition methods for incompressible Stokes equations have been based on primal iterative substructuring methods, cf. [1], [7], [11], [12], [13], [35], [59], [60], [61], [64], [66], [69], [76], on overlapping Schwarz methods, cf. [34], [36], [48], [70], and on block preconditioners, cf. [47], [49]. A discussion of overlapping Schwarz methods and iterative substructuring methods for solving incompressible Stokes problems is given in Chapter 2.

In this thesis, FETI type domain decomposition methods are extended to solving incompressible Stokes equations. In Chapter 3, one-level, two-level, and dual-primal FETI algorithms are proposed to solve incompressible Stokes problems in two dimensions. We only consider approximations with discontinuous pressure. Therefore, only continuity of the velocity component is enforced across the subdomain interface by introducing Lagrange multipliers. The pressure component is not required to be continuous either across the subdomain interface or inside each subdomain. By a Schur complement procedure, the indefinite Stokes problem is reduced to a symmetric positive definite problem for the dual variables, i.e., the Lagrange multipliers. A conjugate gradient method is used to solve this dual problem. Numerical experiments show that these FETI algorithms are scalable.

Convergence analyses for the dual-primal FETI algorithms are given in Chapter 4, in both two and three dimensions. We prove that the condition number of the preconditioned linear system is independent of the number of subdomains and bounded from above by the product of the square of the logarithm of the number of unknowns in each subdomain and a factor that depends on the inf-sup constants of the discrete Stokes problems.

Extension to solving linearized nonsymmetric stationary Navier-Stokes equation is discussed in Chapter 5. The resulting linear system is no longer symmetric and a GMRES method is used to solve the preconditioned linear system. Eigenvalue estimates show that, for small Reynolds number, the non-symmetric preconditioned linear system is a small perturbation of that in the symmetric case. Numerical experiments also show that, for small Reynolds number, the convergence of GMRES method is similar to that of the symmetric Stokes equation case. The convergence of GMRES method depends on the Reynolds number; the larger the Reynolds number, the slower the convergence.

A nonlinear Navier-Stokes equation solver, using Picard iteration, is also considered in Chapter 5. In each iteration step, a linearized Navier-Stokes equation is solved by using a dual-primal FETI algorithm. Numerical experiments show that the convergence of Picard iteration depends on the Reynolds number, but is independent of either the number of subdomains or the subdomain problem size.

The following two sections give a review of some introductory material, including Sobolev spaces and mixed finite element methods for solving saddle point problems.

1.2 Sobolev Spaces

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$, $d = 2$ or 3 . The space $L^2(\Omega)$ is the space of all real valued measurable functions u for which

$$\|u\|_{L^2(\Omega)} = \left(\int_\Omega |u|^2 dx \right)^{1/2} < \infty.$$

$L_0^2(\Omega)$ is the subspace of the functions $u \in L^2(\Omega)$, which satisfy $\int_\Omega u dx = 0$. The space $H^1(\Omega)$ is the space of all locally integrable functions u, for which

$$\int_\Omega |u|^2 dx < \infty, \text{ and } \int_\Omega \nabla u \cdot \nabla u dx < \infty.$$

The scaled H^1-norm of u is given by

$$\|u\|^2_{H^1(\Omega)} = \int_\Omega \nabla u \cdot \nabla u dx + \frac{1}{H_\Omega^2} \int_\Omega |u|^2 dx,$$

where H_Ω is the diameter of Ω; this scaling factor is obtained by dilation from a region of unit diameter. The H^1-seminorm of u is defined by

$$|u|^2_{H^1(\Omega)} = \int_\Omega \nabla u \cdot \nabla u dx.$$

$H^1_0(\Omega)$ is the subspace of $H^1(\Omega)$ of functions vanishing on the boundary $\partial\Omega$.

The trace space of $H^1(\Omega)$ is $H^{1/2}(\partial\Omega)$, where for any $\Gamma \subset \partial\Omega$, the corresponding semi-norm and norm are given by

$$|u|^2_{H^{1/2}(\Gamma)} = \int_\Gamma \int_\Gamma \frac{|u(x) - u(y)|^2}{|x - y|^d} dxdy, \tag{1.1}$$

and

$$\|u\|^2_{H^{1/2}(\Gamma)} = |u|^2_{H^{1/2}(\Gamma)} + \frac{1}{H_\Gamma}\|u\|^2_{L^2(\Gamma)}, \tag{1.2}$$

where H_Γ is the diameter of Γ. The maximal subspace of $H^{1/2}(\Gamma)$, for which the extension by zero to the complement of Γ is a bounded operator in the $H^{1/2}$-norm, is known as $H^{1/2}_{00}(\Gamma)$, cf. [82]. This space can be defined in terms of a norm given by

$$\|u\|^2_{H^{1/2}_{00}(\Gamma)} = |u|^2_{H^{1/2}(\Gamma)} + \int_\Gamma \frac{u^2(x)}{d(x, \partial\Gamma)} dx, \tag{1.3}$$

where $d(x, \partial\Gamma)$ is the distance from x to the boundary $\partial\Gamma$.

The following two lemmas can be found in [27, Section 5.8.1] and in [16, Theorem 6.1], respectively.

Lemma 1 (Poincaré's inequality) *Let*

$$\hat{u} = \frac{1}{volume(\Omega)} \int_\Omega u dx,$$

be the average of u over Ω. Then there exits a constant $C(\Omega)$, which depends only on the Lipschitz constant of $\partial\Omega$, such that

$$\|u - \hat{u}\|_{L^2(\Omega)} \leq C(\Omega) H_\Omega |u|_{H^1(\Omega)}, \quad \forall u \in H^1(\Omega).$$

Lemma 2 (Poincaré-Friedrichs' inequality) *Let $\partial\Omega_D$ be an open subset of $\partial\Omega$ with positive measure. There exists a constant $C(\Omega, \partial\Omega_D)$, which depends only on the Lipschitz constants of $\partial\Omega$ and on the measure of $\partial\Omega_D$ relative to $\partial\Omega$, such that*

$$\|u\|^2_{L^2(\Omega)} \leq C(\Omega, \partial\Omega_D) H_\Omega^2 \left(|u|^2_{H^1(\Omega)} + \frac{1}{H_\Omega} \left(\int_{\partial\Omega_D} u dx \right)^2 \right), \quad \forall u \in H^1(\Omega).$$

1.3 Mixed finite element methods for saddle point problems

Let W be a Hilbert space with the norm $\|\cdot\|_W$ and inner product $(\cdot,\cdot)_W$. We consider a continuous bilinear form $a(u,v)$ on $W \times W$, i.e.,

$$|a(u,v)| \leq \|a\| \|u\|_W \|v\|_W.$$

This bilinear form defines a continuous linear operator $A : W \to W'$ given by

$$\langle Au, v \rangle_{W' \times W} = a(u,v), \quad \forall v \in W, u \in W.$$

Let Q be another Hilbert space with the norm $\|\cdot\|_Q$ and inner product $(\cdot,\cdot)_Q$, and let $b(w,q)$ be a continuous bilinear form on $W \times Q$, i.e.,

$$|b(w,q)| \leq \|b\| \|w\|_W \|q\|_Q.$$

Again, we can introduce a linear operator $B : W \to Q'$, and its transpose $B' : Q \to W'$ by

$$\langle Bw, q \rangle_{Q' \times Q} = \langle w, B'q \rangle_{W \times W'} = b(w,q), \quad \forall w \in W, q \in Q.$$

We consider the solution of the following saddle point problem: for any $f \in W'$, $g \in Q'$, find $u \in W$ and $p \in Q$, such that,

$$\begin{cases} a(u,v) + b(v,p) & = & \langle f, v \rangle, \quad \forall \mathbf{v} \in W \ , \\ b(u,q) & = & \langle g, q \rangle, \quad \forall q \in Q \ . \end{cases} \tag{1.4}$$

We have the following theorem about the existence, uniqueness, and stability of problem (1.4), cf. [10, Proposition 1.3],

Lemma 3 *If there exit positive constants* α_0 *and* β_0 *such that*

$$a(v,v) \geq \alpha_0 \|v\|_W^2, \quad \forall v \in KerB, \tag{1.5}$$

and

$$\sup_{v \in W} \frac{b(v,q)}{\|v\|_W} \geq \beta_0 \|q\|_{Q/KerB'}, \quad \forall q \in Q, \tag{1.6}$$

then there exists a unique solution (u,p) *to problem (1.4) for any* $f \in W'$ *and* $g \in ImB$, *and*

$$\begin{aligned} \|u\|_W &\leq \tfrac{1}{\alpha_0}\|f\|_{W'} + \tfrac{1}{\beta_0}\left(1 + \tfrac{\|a\|}{\alpha_0}\right)\|g\|_{Q'}, \\ \|p\|_{Q/KerB'} &\leq \tfrac{1}{\beta_0}\left(1 + \tfrac{\|a\|}{\alpha_0}\right)\|f\|_{W'} + \tfrac{\|a\|}{\beta_0^2}\left(1 + \tfrac{\|a\|}{\alpha_0}\right)\|g\|_{Q'}. \end{aligned}$$

Let W^h and Q^h be finite element subspaces of W and Q, respectively, where h refers to the element size. The discrete variational problem is to find $u^h \in W^h$ and $p^h \in Q^h$ such that

$$\left\{ \begin{array}{rcl} a(u^h, v^h) + b(v^h, p^h) &=& \langle f, v^h \rangle, \quad \forall v^h \in W^h \ , \\ b(u^h, q^h) &=& \langle g, q^h \rangle, \quad \forall q^h \in Q^h \ . \end{array} \right. \tag{1.7}$$

We have the following approximation result, cf. [37, Theorem 1.1, Chapter II],

Theorem 1 *If there exit positive constants* α *and* β *such that*

$$a(v^h, v^h) \geq \alpha \|v^h\|_W^2, \quad \forall v \in KerB^h, \tag{1.8}$$

and

$$\sup_{v^h \in W^h} \frac{b(v^h, q^h)}{\|v^h\|_W} \geq \beta \|q^h\|_{Q/KerB^{h\prime}}, \quad \forall q^h \in Q^h, \tag{1.9}$$

then there exists a unique solution (u^h, p^h) *to problem (1.7), and*

$$\|u - u^h\|_W + \|p - p^h\|_Q \leq c \left(\inf_{v^h \in W^h} \|u - v^h\|_W + \inf_{q^h \in Q^h} \|p - q^h\|_Q \right), \qquad (1.10)$$

where c *depends on* β*, but is independent of* h*. The solution* (u^h, p^h) *is bounded as follows:*

$$\|u^h\|_W \leq \tfrac{1}{\alpha}\|f\|_{W'} + \tfrac{1}{\beta}\left(1 + \tfrac{\|a\|}{\alpha}\right)\|g\|_{Q'}, \qquad (1.11)$$

$$\|p^h\|_{Q/Ker B'} \leq \tfrac{1}{\beta}\left(1 + \tfrac{\|a\|}{\alpha}\right)\|f\|_{W'} + \tfrac{\|a\|}{\beta^2}\left(1 + \tfrac{\|a\|}{\alpha}\right)\|g\|_{Q'}. \qquad (1.12)$$

Condition (1.9) is called the inf-sup condition and is often referred to in the literature as the Ladyzhenskaya-Babuška-Brezzi (LBB) condition. It is a sufficient but not a necessary condition for the estimate (1.10).

Chapter 2

Domain decomposition methods for incompressible Stokes equations

2.1 Krylov subspace methods and preconditioners

Krylov subspace methods are a very important type of iterative methods to solve the linear system $Ax = b$. The idea is to project an m-dimensional problem into a lower-dimensional Krylov subspace. Given a matrix A and a vector b, we denote by $\mathcal{K}_n$ the Krylov subspace spanned by the vectors b, Ab, ..., $A^{n-1}b$. In the nth iteration of a Krylov subspace method, we look for an approximate x_n of x, which minimizes a norm of the error, $e_n = x - x_n$, or the residual, $b - Ax_n$, in the Krylov subspace $\mathcal{K}_n$.

A very important Krylov subspace method to solve $Ax = b$, when A is symmetric, positive definite, is the conjugate gradient method, cf. [43]. In the nth iteration step of the conjugate gradient method, the approximate solution x_n is the unique element in the space $x_0 + \mathcal{K}_n$, which minimize $\|e_n\|_A$. Here x_0 is the initial guess. We have the following theorem about the convergence rate of the conjugate gradient method, cf. [78, Theorem 38.5],

Theorem 2 *Let the CG method be applied to a symmetric, positive definite problem $Ax = b$. Let κ be the 2-norm condition number of A, i.e., the ratio of the largest eigenvalue to the smallest. Then the A-norm of the error satisfies*

$$\frac{\|e_n\|_A}{\|e_0\|_A} \leq 2\left(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}\right)^n.$$

Thus, the convergence rate of a conjugate gradient method depends on the condition number, i.e., the distribution of the eigenvalues of the matrix A.

When A is not symmetric, positive definite, the generalized minimal residuals (GMRES) method can be used to solve $Ax = b$. In the nth step, we approximate the solution x by a vector $x_n \in x_0 + \mathcal{K}_n$, for which the norm of the residual, $r_n = b - Ax_n$, is minimized, cf. [72]. The following theorem gives the convergence rate of the GMRES method, cf. [78, Theorem 35.2],

Theorem 3 *In step n of the GMRES iteration, the residual r_n satisfies*

$$\frac{\|r_n\|}{\|r_0\|} \leq \inf_{p_n \in P_n} \|p_n(A)\| \leq \kappa(V) \inf_{p_n \in P_n} \|p_n\|_{\Lambda(A)},$$

where $\Lambda(A)$ is the set of eigenvalues of A, V is a nonsingular matrix of eigenvectors (assuming A is diagonalizable), P_n is the set of polynomials p of degree $\leq n$ with $p(0) = 1$, and $\|p_n\|_{\Lambda(A)} = \sup_{z \in \Lambda(A)} |p_n(z)|$.

From this theorem, we can see that, if A is not too far from normal in the sense that $\kappa(V)$ is not too large, and if a properly normalized degree n polynomial can be found with rapidly decreasing values on the spectrum $\Lambda(A)$ with increasing n, then

GMRES converges quickly. When the spectrum of A is tightly clustered away from the imaginary axis, we can expect that $\inf_{p_n \in P_n} ||p_n||_{\Lambda(A)}$ decreases quickly with n.

We can see that the distribution of the eigenvalues of A often determines the rate of convergence of a Krylov subspace method. When the linear system $Ax = b$ comes from the discretization of a PDE, in most cases, the spectrum of A depends on the mesh: the finer the mesh, the more scattered the eigenvalues. In order to make the convergence of a Krylov subspace method insensitive to the mesh, we need to build a preconditioner of the matrix A, and instead of solving $Ax = b$, we solve the preconditioned linear system

$$M^{-1}Ax = M^{-1}b, \tag{2.1}$$

where M^{-1} is called the preconditioner. We expect that the spectrum of the preconditioned operator $M^{-1}A$ be more tightly clustered than that of A, so that faster convergence can be achieved by solving the preconditioned problem (2.1).

The choice of $M^{-1} = A^{-1}$ makes the condition number of $M^{-1}A$ equal to one, and the convergence is achieved in one step. However, we need to multiply M^{-1}, A^{-1} in this case, by a vector in each iteration step, and the computation of A^{-1} times a vector is just as expensive as to solve $Ax = b$. Therefore, not only is the condition number of the preconditioned operator $M^{-1}A$ required to improve, but it is also important to make the multiplication of M^{-1} times a vector inexpensive.

2.2 Domain decomposition procedures for Stokes equations

2.2.1 Discrete incompressible Stokes equations

We consider solving the following incompressible Stokes problem on a bounded, polyhedral domain Ω in two or three dimensions with a Dirichlet boundary condition,

$$\left\{ \begin{array}{rcll} -\Delta \mathbf{u} + \nabla p & = & \mathbf{f}, & \text{in } \Omega\ , \\ -\nabla \cdot \mathbf{u} & = & 0, & \text{in } \Omega\ , \\ \mathbf{u} & = & \mathbf{g}, & \text{on } \partial\Omega\ , \end{array} \right. \tag{2.2}$$

where the boundary data $\mathbf{g}$ satisfies the compatibility condition $\int_{\partial\Omega} \mathbf{g} \cdot \mathbf{n} = 0$. For the sake of simplicity, we choose $\mathbf{g} = \mathbf{0}$ in our discussions.

The solution $(\mathbf{u}, p)$ of equation (2.2) satisfies the variational problem (1.4) with the velocity space $(H_0^1(\Omega))^d$ and the pressure space $Q = L_0^2(\Omega)$, and

$$a(\mathbf{u}, \mathbf{v}) = \int_\Omega \nabla \mathbf{u} \cdot \nabla \mathbf{v} = \sum_{i,j=1}^{d} \int_\Omega \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j}, \qquad b(\mathbf{v}, q) = -\int_\Omega (\nabla \cdot \mathbf{v}) q.$$

Later, the domain Ω will be decomposed into subdomains, and we will use $a_i(\cdot,\cdot)$ and $b_i(\cdot,\cdot)$ to denote these bilinear forms restricted on each subdomain Ω^i.

Both bilinear forms, $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$, are continuous:

$$|a(\mathbf{u}, \mathbf{v})| \leq C \|\mathbf{u}\|_{H^1} \|\mathbf{v}\|_{H^1}, \quad \forall \mathbf{u}, \mathbf{v} \in (H_0^1(\Omega))^d,$$

$$|b(\mathbf{v}, q)| \leq C \|\mathbf{v}\|_{H^1} \|q\|_{L^2}, \quad \forall \mathbf{v} \in (H_0^1(\Omega))^d\ ,\ q \in L_0^2(\Omega),$$

and the bilinear form $a(\cdot,\cdot)$ is elliptic on $(H_0^1(\Omega))^d$, i.e., there exists an α_0 such that

$$|a(\mathbf{v}, \mathbf{v})| \geq \alpha_0 \|\mathbf{v}\|_{H^1}^2, \quad \forall \mathbf{v} \in (H_0^1(\Omega))^d.$$

We also know that there exists a positive constant β such that

$$\sup_{\mathbf{v}\in(H_0^1(\Omega))^d} \frac{b(\mathbf{v},q)}{\|\mathbf{v}\|_{H^1}} \geq \beta\|q\|_{L^2}, \quad \forall q \in L_0^2(\Omega),$$

and therefore, we know, from Theorem 1, that there exists a unique solution $(\mathbf{u}, p) \in (H_0^1(\Omega))^d \times L_0^2(\Omega)$ to problem (2.2), for any $\mathbf{f} \in \left((H_0^1(\Omega))^d\right)'$, cf. [10, Section II.1.1].

We solve this variational problem (1.4), corresponding to the incompressible Stokes equations (2.2), by using finite element methods. We triangulate the domain Ω into shape-regular elements of characteristic size h. From now on, $\mathbf{W}$ and Q are used to denote finite element subspaces of $(H_0^1(\Omega))^d$ and $L_0^2(\Omega)$, respectively. The discrete variational problem is: find $\mathbf{u} \in \mathbf{W}$ and $p \in Q$ such that

$$\left\{ \begin{array}{rcll} (\nabla\mathbf{u}, \nabla\mathbf{v}) - (p, \nabla\cdot\mathbf{v}) & = & (\mathbf{f}, \mathbf{v}), & \forall \mathbf{v} \in \mathbf{W}\ , \\ -(\nabla\cdot\mathbf{u}, q) & = & 0, & \forall q \in Q\ . \end{array} \right. \tag{2.3}$$

We assume an inf-sup stability of the chosen mixed finite element space $\mathbf{W} \times Q$, i.e., there exists a positive constant β, independent of h, such that

$$\sup_{\mathbf{w}\in\mathbf{W}} \frac{b(\mathbf{w},q)}{\|\mathbf{w}\|_{H^1}} \geq \beta\|q\|_{L^2}, \quad \forall q \in Q. \tag{2.4}$$

After discretizing the integrals in (2.3), we have the following matrix form:

$$K \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} = \begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ 0 \end{pmatrix}. \tag{2.5}$$

2.2.2 Finite element methods with discontinuous pressure

We assume that the domain Ω can be decomposed into quadrilateral (hexagonal in three dimensions) subdomains and that each subdomain is further refined into a fine quadrilateral or hexagonal finite element mesh. Among the many choices of the inf-sup stable mixed finite elements for incompressible Stokes equations, we consider several. In each case, we use a discontinuous pressure approximation; this is allowed because the pressure is the space of $L_0^2(\Omega)$.

The first choice is the popular $Q2 - P1$ mixed finite elements for the two-dimensional case, where the velocity is a quadratic function and the pressure is a linear function on each quadrilateral element; see Figure 2.1 for illustration. We know that this choice satisfies the inf-sup stability condition (2.4), cf. [10].

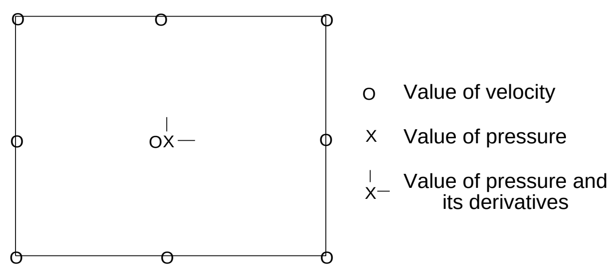

Figure 2.1: Q2-P1 mixed finite element in two dimensions

The second choice for the two-dimensional case is illustrated in Figure 2.2, where the velocity is linear on each triangle and the pressure is a common constant on these four triangles. The inf-sup stability of this choice can be easily proved by using the macroelement technique given in [75].

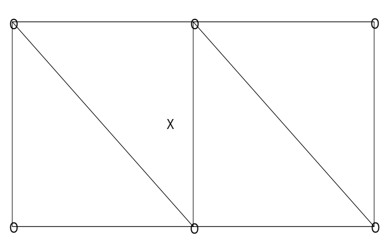

Figure 2.2: A mixed finite element in two dimensions

The finite elements used in our three-dimensional numerical experiments are illustrated in Figure 2.3, where the velocity is spanned by 1, x, y, z, zx, zy on each prism, and the pressure is a constant on these eight prisms. It is also easy to prove the inf-sup stability of this choice, by using the macroelement technique.

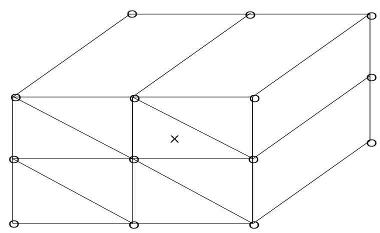

Figure 2.3: A mixed finite element in three dimensions

2.2.3 Decomposition of the solution space

The domain Ω is decomposed into N_s nonoverlapping polyhedral subdomains Ω^i, $i = 1, 2, ..., N_s$, of characteristic size H. $\Gamma = \overline{(\cup \partial\Omega^i) \backslash \partial\Omega}$ is the subdomain interface. We decompose the discrete velocity space $\mathbf{W}$ and the pressure space Q into

$$\begin{aligned} \mathbf{W} &= \mathbf{W}_I \oplus \mathbf{W}_\Gamma, \\ Q &= Q_I \oplus Q_0, \end{aligned} \tag{2.6}$$

where $\mathbf{W}_I$ and Q_I are the direct sums of subdomain interior velocity space $\mathbf{W}_I^i$, and of subdomain interior pressure space Q_I^i, respectively, i.e.,

$$\mathbf{W}_I = \oplus_{i=1}^{N_s} \mathbf{W}_I^i, \quad Q_I = \oplus_{i=1}^{N_s} Q_I^i.$$

Each subdomain interior velocity component $\mathbf{w}_I^i \in \mathbf{W}_I^i$ has its support inside the subdomain Ω^i and equals zero on the subdomain interface $\Gamma \cap \partial\Omega^i$. Each subdomain pressure component $q_I^i \in Q_I^i$ has zero average on subdomain Ω^i and equals zero outside. $q_0 \in Q_0$ is the subdomain constant pressure part, which has a constant value q_0^i in the subdomain Ω^i. These q_0^i satisfy

$$\sum_{i}^{N_s} q_0^i \, m(\Omega^i) = 0, \tag{2.7}$$

to make the global Stokes problem have a unique solution. Here $m(\Omega^i)$ is the measure of the subdomain Ω^i. We can see that the space $\mathbf{W}_I$ and the space Q_I are direct sums of independent subdomain spaces, which means that there are no direct communication between different subdomain components. Therefore, these parts can be easily processed on different processors in parallel for the subdomain solvers. The space Q_0 is not subdomain independent, because each its component has to satisfy a global condition in equation (2.7). Therefore, they cannot be processed the same as the subdomain interior velocity and pressure. Instead, these subdomain constant pressures appear in a coarse level problem. Solving this coarse level problem in each iteration step makes the subdomain boundary velocities satisfy the incompressibility condition on each subdomain, cf. [7], [48], and [64].

$\mathbf{W}_\Gamma$ is the subdomain interface velocity space, and it is decomposed differently in different algorithms. In the primal iterative substructuring methods, the neighboring subdomains share degrees of freedom at each interface node. Each function $\mathbf{w}_\Gamma \in \mathbf{W}_\Gamma$ is continuous across the subdomain interface Γ. In this case, $\mathbf{W}_\Gamma$ cannot be written as the direct sums of subdomain interface velocity space $\mathbf{W}_\Gamma^i$; however we use $\mathbf{W}_\Gamma^i$ to denote the component of $\mathbf{W}_\Gamma$ on $\partial\Omega^i$. In the one-level and two-level FETI algorithms, the neighboring subdomains are completely torn apart, which means, at each interface node, different degrees of freedom are assigned to the neighboring subdomains which share this node. In this way, jumps across the subdomain interface are allowed for the interface velocity $\mathbf{w}_\Gamma \in \mathbf{W}_\Gamma$, and the space $\mathbf{W}_\Gamma$ is the direct sum of independent subdomain interface velocity space $\mathbf{W}_\Gamma^i$. In the FETI-DP algorithm, the neighboring subdomains share degrees of freedom only at a few interface nodes, for example at subdomain vertices, and have different degrees of freedom at all other common interface nodes. In this case, $\mathbf{W}_\Gamma$ is decomposed into

$$\mathbf{W}_\Gamma = \mathbf{W}_\Pi \oplus \mathbf{W}_\Delta = \mathbf{W}_\Pi \oplus \left(\oplus_{i=1}^{N_s} \mathbf{W}_\Delta^i\right) \ , \tag{2.8}$$

where each function $\mathbf{w}_\Pi \in \mathbf{W}_\Pi$ is continuous across Γ, and each function $\mathbf{w}_\Delta \in \mathbf{W}_\Delta$ allows jumps. We also use $\mathbf{W}_\Gamma^i$ to denote the component of $\mathbf{W}_\Gamma$ on $\partial\Omega^i$. See Figure 2.4 for illustration of the interface velocity in different nonoverlapping domain decomposition methods in two dimensions. In all cases, we use $\mathbf{W}^i$ to denote the space of velocity on subdomain Ω^i, i.e., $\mathbf{W}^i = \mathbf{W}_I^i \cup \mathbf{W}_\Gamma^i$.

Using the decomposition of the solution space given in (2.6), the linear system (2.5) can be written as

$$\begin{pmatrix} A_{II} & B_{II}^T & A_{\Gamma I}^T & 0 \\ B_{II} & 0 & B_{I\Gamma} & 0 \\ A_{\Gamma I} & B_{I\Gamma}^T & A_{\Gamma\Gamma} & B_0^T \\ 0 & 0 & B_0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ p_I \\ \mathbf{u}_\Gamma \\ p_0 \end{pmatrix} = \begin{pmatrix} \mathbf{f}_I \\ 0 \\ \mathbf{f}_\Gamma \\ 0 \end{pmatrix} . \tag{2.9}$$

We define a Schur complement operator S_Γ as

$$S_\Gamma = A_{\Gamma\Gamma} - (A_{\Gamma I} \ \ B_{I\Gamma}^T) \begin{pmatrix} A_{II} & B_{II}^T \\ B_{II} & 0 \end{pmatrix}^{-1} \begin{pmatrix} A_{\Gamma I}^T \\ B_{I\Gamma} \end{pmatrix} . \tag{2.10}$$

The linear system (2.9) can then be reduced to a problem for the interface velocity $\mathbf{u}_\Gamma$ and subdomain constant pressure p_0:

$$\begin{pmatrix} S_\Gamma & B_0^T \\ B_0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_\Gamma \\ p_0 \end{pmatrix} = \begin{pmatrix} \mathbf{f}_\Gamma^* \\ 0 \end{pmatrix} , \tag{2.11}$$

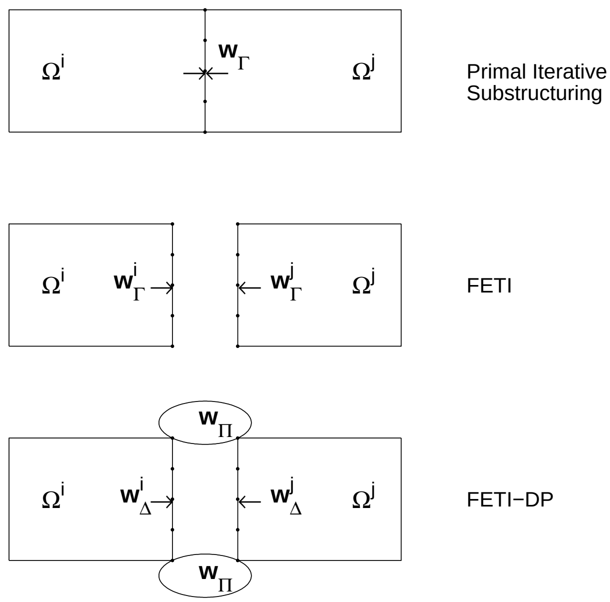

Figure 2.4: Interface velocity of different nonoverlapping methods

where

$$\mathbf{f}_\Gamma^* = \mathbf{f}_\Gamma - \begin{pmatrix} A_{\Gamma I} & B_{I\Gamma}^T \end{pmatrix} \begin{pmatrix} A_{II} & B_{II}^T \\ B_{II} & 0 \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{f}_I \\ 0 \end{pmatrix}. \tag{2.12}$$

2.2.4 Discrete Stokes extensions

The Schur complement operator S_Γ, defined in (2.10), can be subassembled from the subdomain Schur complement operators S_Γ^i, where S_Γ^i is defined on the subdomain Ω^i by

$$S_\Gamma^i = A_{\Gamma\Gamma}^i - (A_{\Gamma I}^i \;\; B_{I\Gamma}^{i^T}) \begin{pmatrix} A_{II}^i & B_{II}^{i^T} \\ B_{II}^i & 0 \end{pmatrix}^{-1} \begin{pmatrix} A_{\Gamma I}^{i^T} \\ B_{I\Gamma}^i \end{pmatrix}. \tag{2.13}$$

S_Γ^i can be viewed in terms of an operator applied on the interface velocity vector $\mathbf{u}_\Gamma^i$, i.e., given any interface vector $\mathbf{u}_\Gamma^i$, the vector $S_\Gamma^i \mathbf{u}_\Gamma^i$ satisfies

$$\begin{pmatrix} A_{II}^i & B_{II}^{iT} & A_{\Gamma I}^{iT} \\ B_{II}^i & 0 & B_{I\Gamma}^i \\ A_{\Gamma I}^i & B_{I\Gamma}^{iT} & A_{\Gamma\Gamma}^i \end{pmatrix} \begin{pmatrix} \mathbf{u}_I^i \\ p_I^i \\ \mathbf{u}_\Gamma^i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ S_\Gamma^i \mathbf{u}_\Gamma^i \end{pmatrix}. \tag{2.14}$$

We know, from the definition (2.13), that the action of S_Γ^i can be evaluated by solving a Dirichlet problem on the subdomain Ω^i. From equation (2.14), we see that to compute the action of $S_\Gamma^{i^{-1}}$, we need to solve a Neumann problem on the subdomain Ω^i. In the following lemma, we show that the resulting subdomain Schur complements S_Γ^i are positive semi-definite.

Lemma 4 *The subdomain Schur complements S_Γ^i defined in (2.14) are symmetric, positive semi-definite, and they are singular for any subdomain with a boundary that does not intersect $\partial\Omega$.*

Proof: It is easy to see that S_Γ^i is symmetric. Therefore we just need to show that ${\mathbf{u}_\Gamma^i}^T S_\Gamma^i \mathbf{u}_\Gamma^i \geq 0$, for any nonzero vector $\mathbf{u}_\Gamma^i$. For any given vector $\mathbf{u}_\Gamma^i$, we can always find a vector

$$\begin{pmatrix} \mathbf{u}_I^i \\ p_I^i \end{pmatrix} = - \begin{pmatrix} A_{II}^i & B_{II}^{iT} \\ B_{II}^i & 0 \end{pmatrix}^{-1} \begin{pmatrix} A_{\Gamma I}^{iT} \\ B_{I\Gamma}^i \end{pmatrix} \mathbf{u}_\Gamma^i, \tag{2.15}$$

by solving a subdomain incompressible Stokes problem with Dirichlet boundary data $\mathbf{u}_\Gamma^i$. Therefore,

$$\mathbf{u}_\Gamma^{iT} S_\Gamma^i \mathbf{u}_\Gamma^i = \begin{pmatrix} \mathbf{u}_I^i \\ p_I^i \\ \mathbf{u}_\Gamma^i \end{pmatrix}^T \begin{pmatrix} A_{II}^i & B_{II}^{iT} & A_{\Gamma I}^{iT} \\ B_{II}^i & 0 & B_{I\Gamma}^i \\ A_{\Gamma I}^i & B_{I\Gamma}^{iT} & A_{\Gamma\Gamma}^i \end{pmatrix} \begin{pmatrix} \mathbf{u}_I^i \\ p_I^i \\ \mathbf{u}_\Gamma^i \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{u}_I^i \\ \mathbf{u}_\Gamma^i \end{pmatrix}^T \begin{pmatrix} A_{II}^i & A_{\Gamma I}^{i^T} \\ A_{\Gamma I}^i & A_{\Gamma\Gamma}^i \end{pmatrix} \begin{pmatrix} \mathbf{u}_I^i \\ \mathbf{u}_\Gamma^i \end{pmatrix} + 2p_I^{i^T} \begin{pmatrix} B_{II}^i & B_{I\Gamma}^i \end{pmatrix} \begin{pmatrix} \mathbf{u}_I^i \\ \mathbf{u}_\Gamma^i \end{pmatrix}$$
$$= \begin{pmatrix} \mathbf{u}_I^i \\ \mathbf{u}_\Gamma^i \end{pmatrix}^T \begin{pmatrix} A_{II}^i & A_{\Gamma I}^{i^T} \\ A_{\Gamma I}^i & A_{\Gamma\Gamma}^i \end{pmatrix} \begin{pmatrix} \mathbf{u}_I^i \\ \mathbf{u}_\Gamma^i \end{pmatrix},$$

where the last equality results from $B_{II}^i \mathbf{u}_I^i + B_{I\Gamma}^i \mathbf{u}_\Gamma^i = 0$, because the vector $(\mathbf{u}_I^i, p_I^i, \mathbf{u}_\Gamma^i)$ satisfies equation (2.14). Since the matrix

$$\begin{pmatrix} A_{II}^i & A_{\Gamma I}^{i^T} \\ A_{\Gamma I}^i & A_{\Gamma\Gamma}^i \end{pmatrix}$$

is just the discretization of a direct sum of Laplace operators on the subdomain Ω^i, it is symmetric positive semi-definite, and we find that $\mathbf{u}_\Gamma^{i\,T} S_\Gamma^i \mathbf{u}_\Gamma^i \geq 0$, for any nonzero vector $\mathbf{u}_\Gamma^i$. The inequality is strict when the boundary of subdomain Ω^i intersects $\partial\Omega$.

$\square$

Each subdomain Schur complement operator S_Γ^i corresponds to the standard discrete Stokes extension operator $\mathcal{SH}^i$: $\mathbf{W}_\Gamma^i \rightarrow \mathbf{W}^i$, defined by: given a velocity $\mathbf{u}_\Gamma^i \in \mathbf{W}_\Gamma^i$, find $\mathcal{SH}^i \mathbf{u}_\Gamma^i \in \mathbf{W}^i$ and $p_I^i \in Q_I^i$ such that

$$\begin{cases} a_i(\mathcal{SH}^i \mathbf{u}_\Gamma^i, \mathbf{v}^i) + b_i(\mathbf{v}^i, p_I^i) &= 0, \quad \forall \mathbf{v}^i \in \mathbf{W}^i, \\ b_i(\mathcal{SH}^i \mathbf{u}_\Gamma^i, q_I^i) &= 0, \quad \forall q_I^i \in Q_I^i, \\ \mathcal{SH}^i \mathbf{u}_\Gamma^i &= \mathbf{u}_\Gamma^i. \end{cases} \tag{2.16}$$

The S_Γ^i-seminorm is defined on the finite element function space $\mathbf{W}_\Gamma^i$ by

$$|\mathbf{u}_\Gamma^i|_{S_\Gamma^i}^2 = \mathbf{u}_\Gamma^{i^T} S_\Gamma^i \mathbf{u}_\Gamma^i = a_i(\mathcal{SH}^i \mathbf{u}_\Gamma^i, \mathcal{SH}^i \mathbf{u}_\Gamma^i), \quad \forall \mathbf{u}_\Gamma^i \in \mathbf{W}_\Gamma^i.$$

A global discrete Stokes extension operator $\mathcal{SH}$: $\mathbf{W}_\Gamma \rightarrow \mathbf{W}$, is defined such that $\mathcal{SH}^i$ is its restriction to the subspace $\mathbf{W}_\Gamma^i$. The S_Γ-seminorm is defined on the finite element function space $\mathbf{W}_\Gamma$, by

$$|\mathbf{u}_\Gamma|^2_{S_\Gamma} = \mathbf{u}_\Gamma^T S_\Gamma \mathbf{u}_\Gamma = a(\mathcal{SH}\mathbf{u}_\Gamma, \mathcal{SH}\mathbf{u}_\Gamma), \quad \forall \mathbf{u}_\Gamma \in \mathbf{W}_\Gamma.$$

The interface problem (2.11) can be written in the following variational form: find $\mathbf{u}_\Gamma \in \mathbf{W}_\Gamma$ and $p_0 \in Q_0$, such that

$$\left\{ \begin{array}{lcll} a(\mathcal{SH}\mathbf{u}_\Gamma, \mathcal{SH}\mathbf{v}_\Gamma) + b(\mathcal{SH}\mathbf{v}_\Gamma, p_0) & = & < \mathbf{f}_\Gamma, \mathbf{v}_\Gamma >, & \forall \mathbf{v}_\Gamma \in \mathbf{W}_\Gamma, \\ b(\mathcal{SH}\mathbf{u}_\Gamma, q_0) & = & 0, & \forall q_0 \in Q_0. \end{array} \right. \tag{2.17}$$

The discrete Stokes extension operators $\mathcal{SH}^i$ and $\mathcal{SH}$ are the counterparts of the discrete harmonic extension operators $\mathcal{H}^i$ and $\mathcal{H}$, respectively. The subdomain discrete harmonic extension operator $\mathcal{H}^i : \mathbf{W}^i_\Gamma \rightarrow \mathbf{W}^i$, is defined by: given $\mathbf{u}^i_\Gamma \in \mathbf{W}^i_\Gamma$, find $\mathcal{H}^i \mathbf{u}^i_\Gamma \in \mathbf{W}^i$ such that

$$\left\{ \begin{array}{lcl} a_i(\mathcal{H}^i \mathbf{u}^i_\Gamma, \mathbf{v}^i) & = & 0, \quad \forall \mathbf{v}^i \in \mathbf{W}^i, \\ \mathcal{H}^i \mathbf{u}^i_\Gamma & = & \mathbf{u}^i_\Gamma. \end{array} \right. \tag{2.18}$$

The global discrete harmonic extension operator $\mathcal{H}$: $\mathbf{W}_\Gamma \rightarrow \mathbf{W}$, is defined such that $\mathcal{H}^i$ is its restriction on the subspace $\mathbf{W}^i_\Gamma$.

The following lemma can be found in [3] for the case of piecewise linear elements and two dimensions. The tools necessary to extend this result to more general finite elements are provided in [81].

Lemma 5 *There exist positive constants C_1 and C_2, independent of H and h, such that*

$$C_1 |\mathbf{w}_\Gamma|_{H^{1/2}(\Gamma)} \leq a(\mathcal{H}\mathbf{w}_\Gamma, \mathcal{H}\mathbf{w}_\Gamma) \leq C_2 |\mathbf{w}_\Gamma|_{H^{1/2}(\Gamma)}, \quad \forall \mathbf{w}_\Gamma \in \mathbf{W}_\Gamma,$$
$$C_1 |\mathbf{w}^i_\Gamma|_{H^{1/2}(\partial\Omega^i)} \leq a_i(\mathcal{H}^i \mathbf{w}^i_\Gamma, \mathcal{H}^i \mathbf{w}^i_\Gamma) \leq C_2 |\mathbf{w}^i_\Gamma|_{H^{1/2}(\partial\Omega^i)}, \quad \forall \mathbf{w}^i_\Gamma \in \mathbf{W}^i_\Gamma.$$

The following lemma gives a comparison of the energies of the discrete Stokes and discrete harmonic extensions, cf. [7, Theorem 4.1]:

Lemma 6 *There exist positive constants C_1 and C_2, independent of H and h, such that*

$$C_1\beta^2 a(\mathcal{SH}\mathbf{w}_\Gamma, \mathcal{SH}\mathbf{w}_\Gamma) \leq a(\mathcal{H}\mathbf{w}_\Gamma, \mathcal{H}\mathbf{w}_\Gamma) \leq C_2 a(\mathcal{SH}\mathbf{w}_\Gamma, \mathcal{SH}\mathbf{w}_\Gamma), \ \ \forall \mathbf{w}_\Gamma \in \mathbf{W}_\Gamma,$$

$$C_1\beta^2 a_i(\mathcal{SH}^i\mathbf{w}_\Gamma^i, \mathcal{SH}^i\mathbf{w}_\Gamma^i) \leq a_i(\mathcal{H}^i\mathbf{w}_\Gamma^i, \mathcal{H}^i\mathbf{w}_\Gamma^i) \leq C_2 a_i(\mathcal{SH}^i\mathbf{w}_\Gamma^i, \mathcal{SH}^i\mathbf{w}_\Gamma^i), \ \ \forall \mathbf{w}_\Gamma^i \in \mathbf{W}_\Gamma^i,$$

$$C_1\beta^2 a(\mathcal{SH}\mathbf{w}_\Pi, \mathcal{SH}\mathbf{w}_\Pi) \leq a(\mathcal{H}\mathbf{w}_\Pi, \mathcal{H}\mathbf{w}_\Pi) \leq C_2 a(\mathcal{SH}\mathbf{w}_\Pi, \mathcal{SH}\mathbf{w}_\Pi), \ \ \forall \mathbf{w}_\Pi \in \mathbf{W}_\Pi,$$

i.e.,

$$C_1\beta|\mathbf{w}_\Gamma|_{S_\Gamma} \leq |\mathbf{w}_\Gamma|_{H^{1/2}(\Gamma)} \leq C_2|\mathbf{w}_\Gamma|_{S_\Gamma}, \quad \forall \mathbf{w}_\Gamma \in \mathbf{W}_\Gamma,$$

$$C_1\beta|\mathbf{w}_\Gamma^i|_{S_\Gamma^i} \leq |\mathbf{w}_\Gamma^i|_{H^{1/2}(\partial\Omega^i)} \leq C_2|\mathbf{w}_\Gamma^i|_{S_\Gamma^i}, \quad \forall \mathbf{w}_\Gamma^i \in \mathbf{W}_\Gamma^i,$$

$$C_1\beta|\mathbf{w}_\Pi|_{S_\Gamma} \leq |\mathbf{w}_\Pi|_{H^{1/2}(\Gamma)} \leq C_2|\mathbf{w}_\Pi|_{S_\Gamma}, \quad \forall \mathbf{w}_\Pi \in \mathbf{W}_\Pi,$$

where β is the inf-sup stability defined in (2.4).

2.3 Overlapping Schwarz methods

In this and the following sections, we will discuss some domain decomposition methods that have previously been developed for the incompressible Stokes equations (2.2).

In [48], an overlapping Schwarz method was proposed for solving the linear system (2.5). Each nonoverlapping subdomain Ω^i is extended to a larger subdomain $\Omega^{i'}$. The preconditioner M^{-1} for K is based on solutions of local saddle point problems on the overlapping subdomains $\Omega^{i'}$ and on solution of a coarse saddle point problem on the coarse subdomain mesh, i.e.,

$$M^{-1} = \sum_{i=1}^{N_s} R_i^T K_i^{-1} R_i + R_0^T K_0^{-1} R_0, \tag{2.19}$$

where $R_i^T K_i^{-1} R_i$ represents a local problem solver and $R_0^T K_0^{-1} R_0$ represents a coarse problem solver. A GMRES method is used to solve the following preconditioned linear system,

$$\left(\sum_{i=1}^{N_s} R_i^T K_i^{-1} R_i K + R_0^T K_0^{-1} R_0 K \right) \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} = M^{-1} \begin{pmatrix} \mathbf{f} \\ 0 \end{pmatrix}.$$

Thus, in each iteration step, we need to solve subdomain saddle point problems and a coarse level problem. Numerical experiments show that the convergence of the GMRES method is independent of the mesh size h and the number of subdomains N_s, cf. [48]. Theoretical analysis of this method is still missing.

2.4 Primal iterative substructuring methods

Iterative substructuring is the other major type of domain decomposition methods. We recall that, when an iterative substructuring method is used to solve a partial differential equation, we first decompose the domain Ω into nonoverlapping subdomains and set up subdomain stiffness matrices. We then eliminate subdomain interior variables and form an interface problem. This interface problem is solved by a preconditioned Krylov subspace method to obtain the solution on the interface. A back solve is then applied to obtain the interior part of the solution.

A balancing Neumann-Neumann method was proposed in [64] to solve the interface linear system (2.11). The intuition is that solving the indefinite problem (2.11) is equivalent to solving the following symmetric positive definite problem: find

$\mathbf{u}_\Gamma \in \mathbf{W}_{\Gamma,B}$ such that

$$S_\Gamma \mathbf{u}_\Gamma = \mathbf{f}_\Gamma^*, \tag{2.20}$$

where S_Γ and $\mathbf{f}_\Gamma^*$ are defined in (2.10) and (2.12), respectively. $\mathbf{W}_{\Gamma,B}$ is the subspace of *benign* velocities defined by

$$\mathbf{W}_{\Gamma,B} = KerB_0 = \{\mathbf{w}_\Gamma \in \mathbf{W}_\Gamma \mid B_0 \mathbf{w}_\Gamma = 0\},$$

The preconditioner used in the balancing Neumann-Neumann method is of the form:

$$Q^H + \left(I - Q^H \begin{pmatrix} S_\Gamma & B_0^T \\ B_0 & 0 \end{pmatrix}\right) \sum_{i=1}^{N_s} Q^i \left(I - \begin{pmatrix} S_\Gamma & B_0^T \\ B_0 & 0 \end{pmatrix} Q^H\right),$$

where Q^H is a coarse level saddle point problem solver and Q^i requires solving subdomain saddle point problems.

By utilizing this coarse level operator Q^H, we can keep the interface velocity $\mathbf{u}_\Gamma$ in the benign subspace $\mathbf{W}_{\Gamma,B}$ throughout the computation, and therefore reduce the indefinite saddle point interface problem (2.11) to the positive definite problem (2.20). A conjugate gradient method is used to solve the preconditioned linear system, and it was proved in [64] that the condition number of the preconditioned operator is bounded from above by

$$C\left(1 + \frac{1}{\beta_0}\right) \frac{1}{\beta^2} \left(1 + \log \frac{H}{h}\right)^2,$$

where β is the inf-sup constant of the original discrete Stokes problem, β_0 is the inf-sup constant of the coarse level saddle point problem, and C is a positive constant independent of the mesh size H and h. Further applications of balancing Neumann-Neumann methods to linear elasticity problems can be found in [41] and [42].

2.5 Some other methods

In [7], a nonoverlapping domain decomposition approach was directly applied to solving the discrete system (2.5). The algorithm only requires the solutions of smaller discrete Stokes systems on the subdomains and another reduced system. This reduced system is a saddle problem which involves the subdomain interface velocity and the mean value of the pressure on each subdomain. This reduced system is solved by an iterative method. We note that Lemma 6 first appeared in [7].

In [76], a nonoverlapping domain decomposition method was applied to solving incompressible Stokes equations modeled by the hp version of finite element methods. In this approach, the original indefinite problem is reduced to a positive definite interface problem by the introduction of a discrete divergence-free harmonic extension map. A preconditioned conjugate gradient strategy is then used for solving this interface problem. Two preconditioners were introduced there. It was proved that the condition number is bounded from above by

$$C\frac{H}{h}\left(1+\frac{1}{\beta}\right)^2\left(1+\log\frac{Hk}{h}\right)(1+\log k),$$

when a hierarchical hp preconditioner was used, cf. [76, Theorem 4], and by

$$C(k)\left(1+\frac{1}{\beta}\right)\left(1+\log\frac{H}{h}\right)^2,$$

when a Neumann-Neumann preconditioner was used, cf. [76, Theorem 5]. Here β is the inf-sup constant of the original discrete Stokes problem and k is the local degree of the finite elements.

Block-diagonal and block-triangular preconditioners, are considered in [47] and [49] for solving saddle point problems. with diagonal blocks approximated by an overlapping Schwarz technique with positive definite local and coarse problems. It was proved that the condition number of the block-diagonal preconditioned operator is bounded from above by $C(1 + \frac{H}{\delta})$, where δ is the width of the overlap, cf. [49, Theorem 4.2]. When a block-triangular preconditioner was used, it was proved that the spectrum of the preconditioned system is contained in a real, positive interval, which is independent of the discretization, cf. [47].

In [11], an iterative substructuring method of dual type was used to solve the incompressible Stokes equations. There the pressure is continuous inside each subdomain, but jumps are allowed across subdomain interfaces. A preconditioned conjugate gradient method is used to solve a linear system of Lagrange multipliers, which constrain the continuity of velocity on the subdomain interface. Since no coarse level solver is implemented in each iteration step, the convergence of CG depends on the number of subdomains.

Some other domain decomposition methods for incompressible Stokes equations can be found in [1], [12], [13], [34], [35], [36], [59], [60], [61], [66], [69], [70].

Other approaches for the iterative solution of saddle point problems are Uzawa's algorithm, cf. [6], [22], [55]; multigrid methods, cf. [4], [5], [9], [79], [83]; preconditioned conjugate gradient methods for a positive definite equivalent problem, cf. [8]; block-diagonal preconditioners, cf. [50], [62], [71], [73], [80]; and block-triangular preconditioners, cf. [24], [26], [47], [63].

Chapter 3

FETI type algorithms for incompressible Stokes equations

3.1 Introduction

The Finite Element Tearing and Interconnecting (FETI) methods were first proposed in [29] for positive definite elliptic partial differential equations. In this method, the spatial domain is decomposed into nonoverlapping subdomains, and the interior subdomain variables are eliminated to form a Schur problem for the interface variables. Lagrange multipliers are then introduced to enforce continuity across the interface, and a symmetric positive semi-definite linear system for the Lagrange multipliers is solved by using a preconditioned conjugate gradient (PCG) method. This method has been shown to be numerically scalable for second order elliptic problems. For fourth-order problems, a two-level FETI method was developed in [28]. The main idea in this variant is that an extra set of Lagrange multipliers are used to enforce the continuity at the subdomain corners in every step of the PCG iterations. A similar idea was used in [31] and [65], to introduce the Dual-Primal FETI (FETI-DP) methods in which the continuity of the iterates are enforced directly at the corners, i.e., the degrees of freedom at the subdomain corners are shared by neighboring subdomains. The FETI-DP methods were further refined to solve three-dimensional problems by introducing Lagrange multipliers to enforce continuity constraints for the averages of the solution on interface edges or faces. This set of Lagrange multipliers, together with the corner variables, form the coarse problem of this FETI-DP method. This richer, primal problem is necessary to obtain satisfactory convergence rates in three dimensions. It was proved in [57] that the condition number of a FETI-DP algorithm grows at most as $C(1 + log(H/h))^2$ for two-dimensional second order and fourth order positive definite elliptic equations. New preconditioners were proposed in [51] and it was proved that the condition numbers can be bounded from above by $C(1 + log(H/h))^2$ for three-dimensional problems; these bounds are also independent of possible jumps of the coefficients of the elliptic problem.

In this chapter, we extend the FETI algorithms to solving incompressible Stokes problems. In our FETI algorithms, we only consider finite elements with discontinuous pressure component. This makes it possible to only enforce continuity of the velocity component across the subdomain interface using Lagrange multipliers. The pressure component is not required to be continuous either across the subdomain interface or inside each subdomain. By reducing the original problem to a dual problem on the Lagrange multiplier variables, the solution of the original indefinite saddle point problem is reduced to solving a positive definite problem for the Lagrange multiplier variables, and therefore a preconditioned conjugate gradient method can be applied. In each iteration step, we solve indefinite subdomain saddle point problems and coarse level problems directly. In one-level FETI and two-level FETI algorithms, the coarse level problems are positive definite, while in the FETI-DP algorithm, the coarse level problem is a saddle point problem. In the following sections, we develop these three different FETI algorithms for two dimensional incompressible Stokes equations. In the next chapter, we prove the condition number bounds for our FETI-DP algorithms, both in two and three dimensions.

3.2 One-level FETI algorithms

In this section, we develop a FETI algorithm for solving the incompressible Stokes equations (2.2). Let us recall that the problem has been reduced to solving the interface problem (2.11), where the solution $\mathbf{u}_\Gamma \in \mathbf{W}_\Gamma$ and $p_0 \in Q_0$. For each interface grid point, we assign different degrees of freedom to the neighboring subdomains. Therefore, the velocity field allows jumps across the subdomain interface. The continuity constraint on the interface velocity $\mathbf{u}_\Gamma$ is enforced by

$$B_\Gamma \mathbf{u}_\Gamma = 0 \tag{3.1}$$

where B_Γ is a matrix constructed from $\{0, 1, -1\}$ such that the values of $\mathbf{u}_\Gamma$ coincide across the subdomain interface Γ when $B_\Gamma \mathbf{u}_\Gamma = 0$.

We know from section 2.2.4 that the global Schur complement operator S_Γ is positive semi-definite. Therefore, the problem (2.11) and (3.1) can now be written as a saddle point problem subject to the continuity constraints: find the stationary point of

$$\mathcal{L}(\mathbf{w}_\Gamma, q_0) = \inf_{\mathbf{w}_\Gamma \in \mathbf{W}_\Gamma} \sup_{q_0 \in Q_0} \frac{1}{2}\mathbf{w}_\Gamma^T S_\Gamma \mathbf{w}_\Gamma + q_0^T B_0 \mathbf{w}_\Gamma - \mathbf{f}_\Gamma^{*^T} \mathbf{w}_\Gamma,$$

subject to

$$B_\Gamma \mathbf{w}_\Gamma = 0.$$

By introducing a vector of Lagrange multipliers μ to enforce all the constraints $B_\Gamma \mathbf{w}_\Gamma = 0$, we obtain the following linear system:

$$\begin{pmatrix} S_\Gamma & B_0^T & B_\Gamma^T \\ B_0 & 0 & 0 \\ B_\Gamma & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_\Gamma \\ p_0 \\ \mu \end{pmatrix} = \begin{pmatrix} \mathbf{f}_\Gamma^* \\ 0 \\ 0 \end{pmatrix}, \tag{3.2}$$

which can be written as

$$\begin{pmatrix} S_\Gamma & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_\Gamma \\ \lambda \end{pmatrix} = \begin{pmatrix} \mathbf{f}_\Gamma^* \\ 0 \end{pmatrix}, \tag{3.3}$$

if we use λ to incorporate both μ and p_0, and use B to denote the block matrix built from B_0 and B_Γ.

To guarantee the solvability of equation (3.3), we require that

$$\mathbf{f}_\Gamma^* - B^T \lambda \perp Ker(S_\Gamma), \tag{3.4}$$

where $Ker(S_\Gamma)$ is the kernel of S_Γ. The solution $\mathbf{u}_\Gamma$ of equation (3.3) then has the form

$$\mathbf{u}_\Gamma = S_\Gamma^\dagger (\mathbf{f}_\Gamma^* - B^T \lambda) - R\alpha, \tag{3.5}$$

where $S_\Gamma^\dagger$ is a pseudoinverse of S_Γ, and the range of R is the null space of the operator S_Γ. Notice that $B\mathbf{u}_\Gamma = 0$ and $R^T \left(\mathbf{f}_\Gamma^* - B^T \lambda\right) = 0$, and we have

$$\begin{pmatrix} F & G \\ G^T & 0 \end{pmatrix} \begin{pmatrix} \lambda \\ \alpha \end{pmatrix} = \begin{pmatrix} d \\ e \end{pmatrix}, \tag{3.6}$$

where $F = BS_\Gamma^\dagger B^T$, $G = BR$, $d = BS_\Gamma^\dagger \mathbf{f}_\Gamma^*$, and $e = R^T \mathbf{f}_\Gamma^*$.

Solving this FETI interface problem is equivalent to solving the following constrained optimization problem

$$\begin{aligned} \text{Minimize} \quad & \tfrac{1}{2}\lambda^T F \lambda - \lambda^T d, \\ \text{subject to} \quad & G^T \lambda = e. \end{aligned} \tag{3.7}$$

When we use a conjugate gradient method to solve this optimization problem, we need to make λ satisfy the constraint $G^T\lambda = e$, in each iteration. We choose the initial guess λ_0 as $\lambda_0 = G\left(G^T G\right)^{-1} e$, which satisfies this constraint. And then we apply a conjugate gradient method to solve

$$PF\lambda = Pd, \quad \lambda \in \lambda_0 + Range(P). \tag{3.8}$$

where the projection operator P is defined by

$$P = I - G\left(G^T G\right)^{-1} G^T.$$

We can see that the λ always satisfies the constraint $G^T\lambda = e$.

Two different type preconditioners have been proposed in the literature for the FETI methods for positive definite elliptic problems: the computationally economical lumped preconditioners, cf. [32], and the mathematically optimal Dirichlet preconditioners, cf. [30] and [33]. A condition number bound, $C(1 + log(H/h))^3$, has been proved in [56] for the FETI method with Dirichlet preconditioner, both in two and three dimensions. A scaled Dirichlet preconditioner is proposed in [53], and a condition number bound $C(1 + log(H/h))^2$ is proved. This bound is also independent of possible jumps of the coefficients of the elliptic problem.

In our numerical experiments in section 3.5, no preconditioner is used to solve the FETI system (3.8) of the incompressible Stokes equations. We have tested Dirichlet preconditioners in this FETI algorithm for solving Stokes problems, and found no improvement in the convergence.

3.3 Two-level FETI algorithms

In the two-level FETI methods, we are solving

$$\begin{array}{rl} \text{Minimize} & \frac{1}{2}\lambda^T F\lambda - \lambda^T d, \\ \text{subject to} & G^T\lambda = e \;\; \textit{and} \;\; C^T P(F\lambda - d) = 0, \end{array} \tag{3.9}$$

where C is a chosen matrix. We can see that

$$\begin{aligned} C^T P(F\lambda - d) &= C^T\left(I - G\left(G^T G\right)^{-1} G^T\right)(F\lambda - d) \\ &= C^T\left(F\lambda - d - G\left(G^T G\right)^{-1} G^T (F\lambda - d)\right) \\ &= C^T\left(F\lambda - d - G\alpha\right) \\ &= -C^T B\mathbf{u}_\Gamma. \end{aligned}$$

Notice that $B\mathbf{u}_\Gamma$ is the jump of the velocity components across the subdomain interface. By requiring $C^T P(F\lambda - d) = 0$, we make this jump orthogonal to the columns of the matrix C. If we choose a matrix C such that each column corresponds to a subdomain corner mode, then the velocity will always be continuous at the subdomain corners when we solve problem (3.9) by using a conjugate gradient method.

3.4 Dual-primal FETI algorithms

In this section, we develop a dual-primal FETI algorithm for solving the incompressible Stokes equations in two dimensions. Our algorithms for three-dimensional problems will be discussed in the next chapter.

We start by considering equation (2.9), where we need to find $(\mathbf{u}_I, p_I, \mathbf{u}_\Gamma, p_0) \in (\mathbf{W}_I, Q_I, \mathbf{W}_\Gamma, Q_0)$ such that the equation (2.9) is satisfied.

We decompose $\mathbf{W}_\Gamma$ into a subdomain corner velocity subspace $\mathbf{W}_C$ and the remaining interface velocity subspace $\mathbf{W}_\Delta$, i.e.,

$$\mathbf{W}_\Gamma = \mathbf{W}_C \oplus \mathbf{W}_\Delta \ . \tag{3.10}$$

The continuity of each function in $\mathbf{W}_C$ is enforced directly, i.e., the degrees of freedom at a cornerpoint are common to all subdomains sharing this corner. The continuity constraint for any function $\mathbf{w}_\Delta \in \mathbf{W}_\Delta$ is of the form

$$B_\Delta \mathbf{w}_\Delta = 0, \tag{3.11}$$

where the matrix B_Δ is constructed from {0,1,-1} such that the values of $\mathbf{w}_\Delta$ coincide across the subdomain interface Γ when $B_\Delta \mathbf{w}_\Delta = 0$. We also introduce redundant continuity constraints of the form

$$Q_\Delta^T B_\Delta \mathbf{w}_\Delta = 0, \tag{3.12}$$

for all $\mathbf{w}_\Delta \in \mathbf{W}_\Delta$. This continuity condition is enforced in each iteration step of our algorithm, while all the continuity constraints of (3.11) are not satisfied until convergence. The matrix Q_Δ, in equation (3.12), is constructed such that, for any function $\mathbf{w}_\Delta \in \mathbf{W}_\Delta$, $Q_\Delta^T B_\Delta \mathbf{w}_\Delta = 0$ implies that,

$$\int_{\Gamma^{ij}} (\mathbf{w}_\Delta^i - \mathbf{w}_\Delta^j) = 0, \quad \forall \Gamma^{ij}. \tag{3.13}$$

By introducing Lagrange multipliers λ and μ to enforce the continuity constraints

(3.11) and (3.12) for the functions in $\mathbf{W}_\Delta$, equation (2.9) can be written as

$$
\begin{pmatrix}
A_{II} & B_{II}^T & A_{\Delta I}^T & A_{CI}^T & 0 & 0 & 0 \\
B_{II} & 0 & B_{I\Delta} & B_{IC} & 0 & 0 & 0 \\
A_{\Delta I} & B_{I\Delta}^T & A_{\Delta\Delta} & A_{C\Delta}^T & B_{\Delta 0}^T & B_\Delta^T Q_\Delta & B_\Delta^T \\
A_{CI} & B_{IC}^T & A_{C\Delta} & A_{CC} & B_{C0}^T & 0 & 0 \\
0 & 0 & B_{\Delta 0} & B_{C0} & 0 & 0 & \\
0 & 0 & Q_\Delta^T B_\Delta & 0 & 0 & 0 & 0 \\
0 & 0 & B_\Delta & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\mathbf{u}_I \\ p_I \\ \mathbf{u}_\Delta \\ \mathbf{u}_C \\ p_0 \\ \mu \\ \lambda
\end{pmatrix}
=
\begin{pmatrix}
\mathbf{f}_I \\ 0 \\ \mathbf{f}_\Delta \\ \mathbf{f}_C \\ 0 \\ 0 \\ 0
\end{pmatrix} . \tag{3.14}
$$

We notice that two sets of Lagrange multipliers λ and μ are introduced to enforce the continuity constraints of the velocity across the interface Γ given by equations (3.11) and (3.12). In fact, μ is redundant because $B_\Delta \mathbf{u}_\Delta = 0$ implies $Q_\Delta^T B_\Delta \mathbf{u}_\Delta = 0$. But in our algorithm, λ and μ are treated differently. We iterate on the dual variable λ, and the continuity condition $B_\Delta \mathbf{u}_\Delta = 0$ is not satisfied until convergence. However, by solving an augmented coarse problem exactly in each iteration step, $Q_\Delta^T B_\Delta \mathbf{u}_\Delta = 0$ is satisfied throughout. This extra set of Lagrange multipliers μ is in fact grouped together with the primal variables and it augments the corner velocities to form the coarse level velocity component. This coarse level velocity component forms the coarse level saddle point problem together with the subdomain constant pressures. We will prove the inf-sup stability of this coarse level problem in section 4.3. Without this extra set of Lagrange multiplier μ to augment the subdomain corner velocities, the coarse level saddle point problem is very similar to the $Q1 - P0$ mixed finite element, which is not inf-sup stable.

By using the notations

$$
\tilde{\mathbf{u}}_r = \begin{pmatrix} \mathbf{u}_I \\ p_I \\ \mathbf{u}_\Delta \end{pmatrix}, \; \tilde{\mathbf{u}}_c = \begin{pmatrix} \mathbf{u}_C \\ p_0 \\ \mu \end{pmatrix}, \tag{3.15}
$$

equation (3.14) can be written as,

$$\begin{pmatrix} K_{rr} & K_{rc} & B_r^T \\ K_{rc}^T & K_{cc} & 0 \\ B_r & 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{\mathbf{u}}_r \\ \tilde{\mathbf{u}}_c \\ \lambda \end{pmatrix} = \begin{pmatrix} \tilde{\mathbf{f}}_r \\ \tilde{\mathbf{f}}_c \\ 0 \end{pmatrix} , \tag{3.16}$$

where $K_{rr}, K_{rc}, K_{cc}, B_r, \tilde{\mathbf{f}}_r$, and $\tilde{\mathbf{f}}_c$, are the corresponding block matrices and block vectors.

Our algorithm results from two consecutive elimination procedures applied to equation (3.16). We first eliminate the subdomain independent variables $\tilde{\mathbf{u}}_r$ and obtain

$$\begin{pmatrix} \tilde{K}_{cc} & \tilde{K}_{cl} \\ \tilde{K}_{cl}^T & \tilde{K}_{ll} \end{pmatrix} \begin{pmatrix} \tilde{\mathbf{u}}_c \\ \lambda \end{pmatrix} = \begin{pmatrix} \tilde{\mathbf{f}}_c^* \\ d_l \end{pmatrix} , \tag{3.17}$$

where

$$\tilde{K}_{cc} = K_{cc} - K_{rc}^T K_{rr}^{-1} K_{rc} , \quad \tilde{K}_{ll} = -B_r K_{rr}^{-1} B_r^T , \quad \tilde{K}_{cl} = -K_{rc}^T K_{rr}^{-1} B_r^T ,$$

and

$$\tilde{\mathbf{f}}_c^* = \tilde{\mathbf{f}}_c - K_{rc}^T K_{rr}^{-1} \tilde{\mathbf{f}}_r , \quad d_l = -B_r K_{rr}^{-1} \tilde{\mathbf{f}}_r .$$

We then eliminate $\tilde{\mathbf{u}}_c$ from equation (3.17), and obtain a linear system for the Lagrange multipliers λ,

$$(\tilde{K}_{ll} - \tilde{K}_{cl}^T \tilde{K}_{cc}^{-1} \tilde{K}_{cl})\lambda = d_l - \tilde{K}_{cl}^T \tilde{K}_{cc}^{-1} \tilde{\mathbf{f}}_c^* . \tag{3.18}$$

This is the dual problem in our nonpreconditioned FETI-DP algorithm for Stokes equations. We will show in section 4.1 that this dual problem is symmetric, positive definite, and therefore a conjugate gradient method can be used to solve (3.18). After obtaining λ, we solve equations (3.17) and (3.16) to obtain $\tilde{\mathbf{u}}_c$ and $\tilde{\mathbf{u}}_r$, respectively.

The preconditioner used in our FETI-DP algorithm, is the standard Dirichlet preconditioner, which is given by $B_\Delta S_\Gamma B_\Delta^T$, where S_Γ is defined in (2.10). The preconditioner involves solving subdomain incompressible Stokes problems with Dirichlet boundary conditions. We will show in section 4.4 that this preconditioned linear system can also be solved by a conjugate gradient method.

3.5 Numerical experiments

One-level FETI, two-level FETI, nonpreconditioned FETI-DP, and preconditioned FETI-DP algorithms were tested by solving the two-dimensional incompressible Stokes equation (2.2) with $\Omega = [0,1] \times [0,1]$, $\mathbf{f} = \mathbf{0}$, the boundary conditions $\mathbf{g} = (1,0)$ on the upper side $y = 1$, and $\mathbf{g} = \mathbf{0}$ on the three other sides.

We choose the initial guess λ to satisfy the constraints in (3.7) and (3.9), for the one-level and two-level FETI algorithms, respectively. We use a zero initial guess $\lambda = 0$, for the FETI-DP algorithms. In each case, the conjugate gradient method is stopped when the residual satisfies $\|r_k\|_2 / \|r_0\|_2 \leq 10^{-6}$, where r_k is the residual of the Lagrange multiplier equation in the kth iteration.

In Figure 3.1, we give the CG iterations counts, to achieve convergence in each algorithm, for different mesh sizes. We see, from the left figure, that the convergence of each algorithm is insensitive to the number of subdomains, and that the preconditioned and nonpreconditioned FETI-DP algorithms converge faster than the one-level and two-level FETI algorithms. The right figure shows that the convergence of CG iteration, for each algorithm, depends on the subdomain problem size. The growth of the CG iteration count for the preconditioned FETI-DP algorithm is the slowest. It is interesting to see that for smaller problem, the nonpreconditioned FETI-DP algorithm behaves better, but for bigger problem the preconditioned FETI-DP becomes advantageous. The reason is that the condition number of the preconditioned algorithm is bounded from above by the square of the logarithm of H/h, while the condition number of the nonpreconditioned algorithm is most likely bounded by a linear function of H/h.

Figure 3.2 plots the approximate velocity and pressure of this lid driven cavity problem, solved by using the preconditioned FETI-DP algorithm in the case of 10×10 subdomains and $H/h = 12$.

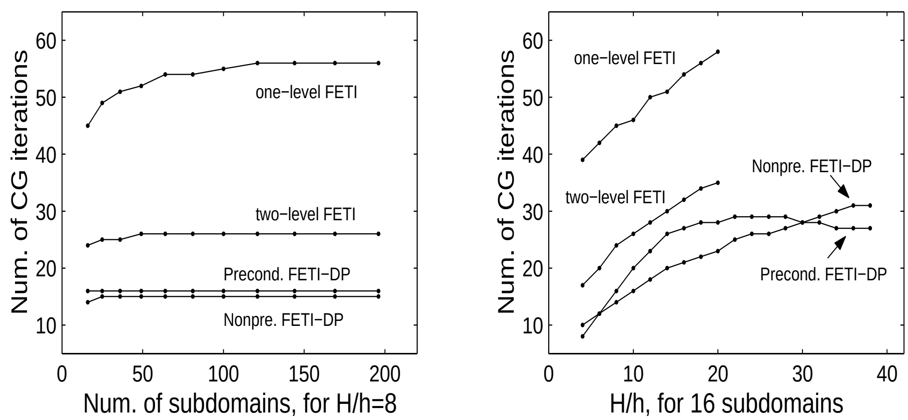

Figure 3.1: Scalability of FETI type algorithms

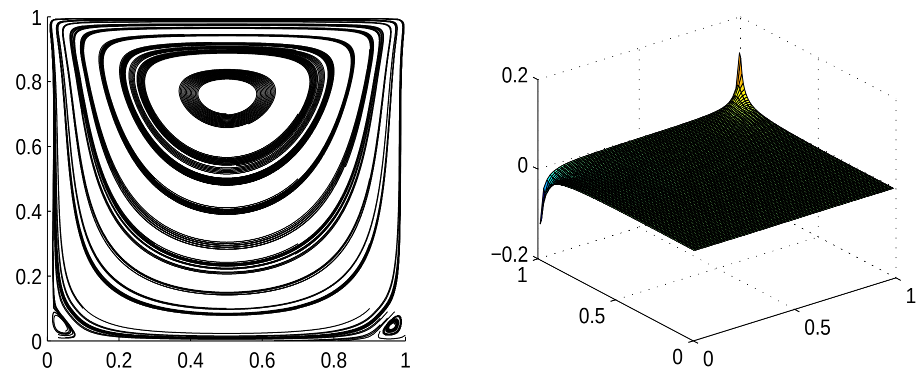

Figure 3.2: Approximate velocity and pressure of FETI-DP algorithm for the case of 10×10 subdomains and $H/h = 12$

Chapter 4

Convergence analysis for FETI-DP algorithms for Stokes equations

In this chapter, we will give a convergence analysis of our dual-primal FETI algorithms for solving incompressible Stokes equations in both two and three dimensions.

4.1 Dual-primal FETI algorithms in general form

In section 3.4, we have developed a dual-primal FETI algorithm for solving two-dimensional incompressible Stokes equations. In this section, we give a general description of our dual-primal FETI algorithms, including the three-dimensional case.

We start by solving the global linear system (2.9), where we need to find a vector $(\mathbf{u}_I, p_I, \mathbf{u}_\Gamma, p_0) \in (\mathbf{W}_I, Q_I, \mathbf{W}_\Gamma, Q_0)$ such that equation (2.9) is satisfied. For convenience, we make no distinction between the symbols of a vector and its corresponding finite element function, and the same rule also applies to a vector space and its corresponding finite element function space. For example, $\mathbf{w}_\Gamma$ can denote either a vector in the vector space $\mathbf{W}_\Gamma$, or a finite element function in the finite element function space $\mathbf{W}_\Gamma$.

Let us recall that, the domain Ω is decomposed into N_s non-overlapping polyhedral subdomains Ω^i, $i = 1, 2, ..., N_s$, of characteristic size H. $\Gamma = \overline{(\cup \partial\Omega^i) \backslash \partial\Omega}$ is the subdomain interface, $\Gamma^{ij} = \overline{\partial\Omega^i \cap \partial\Omega^j}$ is the interface of two neighboring subdomains Ω^i and Ω^j. Each subdomain Ω^i is triangulated into shape-regular elements of characteristic size h, with the finite element nodes on the boundaries of the neighboring subdomains matching across the interface Γ. Γ_h and Γ_h^{ij} are used to denote the grid points on Γ and Γ^{ij}, respectively.

In three dimensions, each subdomain interface Γ^{ij} is decomposed into a subdomain face $\mathcal{F}^{ij}$, regarded as an open set, which is shared by two subdomains, edges $\mathcal{E}^{ik}$, which are shared by more than two subdomains, and vertices $\mathcal{V}^{il}$, which are endpoints of edges. We use $\mathcal{F}_h^{ij}$ and $\mathcal{E}_h^{ik}$ to denote the grid points on $\mathcal{F}^{ij}$ and $\mathcal{E}^{ik}$, respectively. $\theta_{\mathcal{F}^{ij}}$, $\theta_{\mathcal{E}^{ik}}$, and $\theta_{\mathcal{V}^{il}}$ are the standard finite element cutoff functions. The first two are the discrete harmonic functions which equal 1 on $\mathcal{F}_h^{ij}$ and $\mathcal{E}_h^{ik}$, respectively, and vanish on the rest of Γ_h; $\theta_{\mathcal{V}^{il}}$ denotes the piecewise discrete harmonic extension of the standard nodal basis function associated with the vertex $\mathcal{V}^{il}$. In two dimensions, we only have edges and vertices, with each edge shared by two neighboring subdomains and the vertices being the end points of edges.

When we were developing a dual-primal FETI algorithm in section 3.4 for the two-dimensional case, we introduced an extra set of Lagrange multipliers μ to enforce the redundant continuity constraint (3.12). In each iteration step of our algorithm, a coarse level problem is solved, which includes μ as one of its variables, and therefore the dual velocity part $\mathbf{w}_\Delta$ always satisfies the continuity condition (3.13) throughout the computation. The Lagrange multipliers μ therefore play a role as a coarse level velocity component, which is spanned by the cutoff functions of the subdomain interface edges. By viewing μ as a coarse level primal velocity variable, we can decompose the interface velocity space $\mathbf{W}_\Gamma$ as

$$\mathbf{W}_\Gamma = \mathbf{W}_\Pi \oplus \widetilde{\mathbf{W}}_\Delta \ , \tag{4.1}$$

where, in the two-dimensional case, the primal subspace $\mathbf{W}_\Pi$ is spanned by the subdomain vertex nodal finite element basis functions $\theta_{\mathcal{V}^{il}}$ and the cutoff functions $\theta_{\mathcal{E}^{ik}}$ associated with all the edges of the interface Γ. $\widetilde{\mathbf{W}}_\Delta$ is the dual part of the velocity space, and it is the direct sum of local subspaces $\widetilde{\mathbf{W}}_\Delta^i$, which are defined by

$$\widetilde{\mathbf{W}}_\Delta^i = \{\mathbf{w}_\Delta^i \in \mathbf{W}_\Gamma^i : \mathbf{w}_\Delta^i(\mathcal{V}^{il}) = 0, \overline{\mathbf{w}}^i_{\Delta,\mathcal{E}^{ik}} = 0,\ \forall \mathcal{V}^{il}, \mathcal{E}^{ik} \subset \partial\Omega^i\},$$

with $\overline{\mathbf{w}}^i_{\Delta,\mathcal{E}^{ik}}$ defined by

$$\overline{\mathbf{w}}^i_{\Delta,\mathcal{E}^{ik}} = \frac{\int_{\mathcal{E}^{ik}} \mathbf{w}_\Delta^i d\mathbf{x}}{\int_{\mathcal{E}^{ik}} d\mathbf{x}}.$$

In three dimensions, we can similarly introduce Lagrange multipliers to enforce continuity constraints for each subdomain edge and face such that the dual velocities have the same edge and face integrals across the interface. In this way, the primal velocity subspace $\mathbf{W}_\Pi$ is spanned by the subdomain vertex nodal finite element basis functions $\theta_{\mathcal{V}^{il}}$ and the cutoff functions $\theta_{\mathcal{E}^{ik}}$ and $\theta_{\mathcal{F}^{ij}}$ associated with all the edges and faces of the subdomain interface Γ. $\widetilde{\mathbf{W}}_\Delta$ is the dual part of the velocity space, and it is the direct sum of local subspaces $\widetilde{\mathbf{W}}_\Delta^i$, which are defined by

$$\widetilde{\mathbf{W}}_\Delta^i = \{\mathbf{w}_\Delta^i \in \mathbf{W}_\Gamma^i : \mathbf{w}_\Delta^i(\mathcal{V}^{il}) = 0, \overline{\mathbf{w}}^i_{\Delta,\mathcal{E}^{ik}} = 0, \overline{\mathbf{w}}^i_{\Delta,\mathcal{F}^{ij}} = 0,\ \forall \mathcal{V}^{il}, \mathcal{E}^{ik}, \mathcal{F}^{ij} \subset \partial\Omega^i\},$$

with $\overline{\mathbf{w}}^i_{\Delta,\mathcal{E}^{ik}}$ and $\overline{\mathbf{w}}^i_{\Delta,\mathcal{F}^{ij}}$ defined by

$$\overline{\mathbf{w}}^i_{\Delta,\mathcal{E}^{ik}} = \frac{\int_{\mathcal{E}^{ik}} \mathbf{w}^i_\Delta d\mathbf{x}}{\int_{\mathcal{E}^{ik}} d\mathbf{x}}, \qquad \text{and} \qquad \overline{\mathbf{w}}^i_{\Delta,\mathcal{F}^{ij}} = \frac{\int_{\mathcal{F}^{ij}} \mathbf{w}^i_\Delta d\mathbf{x}}{\int_{\mathcal{F}^{ij}} d\mathbf{x}}.$$

Using these notations, the discrete velocity and pressure spaces are decomposed as follows:

$$\mathbf{W} = \mathbf{W}_I \oplus \mathbf{W}_\Pi \oplus \widetilde{\mathbf{W}}_\Delta,$$

$$Q = Q_I \oplus Q_0,$$

and we need to solve the following problem: find a vector $(\mathbf{u}_I, p_I, \mathbf{u}_\Pi, \mathbf{u}_\Delta, p_0) \in (\mathbf{W}_I, Q_I, \mathbf{W}_\Pi, \widetilde{\mathbf{W}}_\Delta, Q_0)$ such that

$$\begin{pmatrix} A_{II} & B_{II}^T & A_{\Pi I}^T & A_{\Delta I}^T & 0 \\ B_{II} & 0 & B_{I\Pi} & B_{I\Delta} & 0 \\ A_{\Pi I} & B_{I\Pi}^T & A_{\Pi\Pi} & A_{\Delta\Pi}^T & B_{0\Pi}^T \\ A_{\Delta I} & B_{I\Delta}^T & A_{\Delta\Pi} & A_{\Delta\Delta} & B_{0\Delta}^T \\ 0 & 0 & B_{0\Pi} & B_{0\Delta} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ p_I \\ \mathbf{u}_\Pi \\ \mathbf{u}_\Delta \\ p_0 \end{pmatrix} = \begin{pmatrix} \mathbf{f}_I \\ 0 \\ \mathbf{f}_\Pi \\ \mathbf{f}_\Delta \\ 0 \end{pmatrix}, \tag{4.2}$$

where the dual velocity part $\mathbf{u}_\Delta$ is required to be continuous across the subdomain interface Γ.

A Lagrange multiplier space Λ is introduced to enforce the continuity of the velocities across Γ. We obtain the following discrete problem: find a vector $(\mathbf{u}_I, p_I, \mathbf{u}_\Pi, p_0, \mathbf{u}_\Delta, \lambda) \in (\mathbf{W}_I, Q_I, \mathbf{W}_\Pi, Q_0, \widetilde{\mathbf{W}}_\Delta, \Lambda)$ such that

$$\begin{pmatrix} A_{II} & B_{II}^T & A_{\Pi I}^T & 0 & A_{\Delta I}^T & 0 \\ B_{II} & 0 & B_{I\Pi} & 0 & B_{I\Delta} & 0 \\ A_{\Pi I} & B_{I\Pi}^T & A_{\Pi\Pi} & B_{0\Pi}^T & A_{\Delta\Pi}^T & 0 \\ 0 & 0 & B_{0\Pi} & 0 & B_{0\Delta} & 0 \\ A_{\Delta I} & B_{I\Delta}^T & A_{\Delta\Pi} & B_{0\Delta}^I & A_{\Delta\Delta} & B_\Delta^T \\ 0 & 0 & 0 & 0 & B_\Delta & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ p_I \\ \mathbf{u}_\Pi \\ p_0 \\ \mathbf{u}_\Delta \\ \lambda \end{pmatrix} = \begin{pmatrix} \mathbf{f}_I \\ 0 \\ \mathbf{f}_\Pi \\ 0 \\ \mathbf{f}_\Delta \\ 0 \end{pmatrix}, \tag{4.3}$$

where the matrix B_Δ is constructed from $\{0, 1, -1\}$ such that the values of $\mathbf{u}_\Delta$ coincide across subdomain interface Γ when $B_\Delta \mathbf{u}_\Delta = 0$. Here, we will exclusively work with fully redundant sets of Lagrange multipliers, i.e., all possible constraints are used for each node on Γ. The matrix B_Δ^T then has a null space and to assure uniqueness it is appropriate to restrict the choice of Lagrange multipliers to $Range(B_\Delta)$, i.e., $\Lambda = B_\Delta \widetilde{\mathbf{W}}_\Delta$. Also note that we are not requiring the pressure to be continuous across the subdomain interfaces in our algorithm, since we only consider finite elements with discontinuous pressure component.

By defining a Schur complement operator $\tilde{S}$ by

$$\begin{pmatrix} A_{II} & B_{II}^T & A_{\Pi I}^T & 0 & A_{\Delta I}^T \\ B_{II} & 0 & B_{I\Pi} & 0 & B_{I\Delta} \\ A_{\Pi I} & B_{I\Pi}^T & A_{\Pi\Pi} & B_{0\Pi}^T & A_{\Delta\Pi}^T \\ 0 & 0 & B_{0\Pi} & 0 & B_{0\Delta} \\ A_{\Delta I} & B_{I\Delta}^T & A_{\Delta\Pi} & B_{0\Delta}^T & A_{\Delta\Delta} \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ p_I \\ \mathbf{u}_\Pi \\ p_0 \\ \mathbf{u}_\Delta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \tilde{S}\mathbf{u}_\Delta \end{pmatrix}, \tag{4.4}$$

solving linear system (4.3) is reduced to solving the following linear system

$$\begin{pmatrix} \tilde{S} & B_\Delta^T \\ B_\Delta & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_\Delta \\ \lambda \end{pmatrix} = \begin{pmatrix} \mathbf{f}_\Delta^* \\ 0 \end{pmatrix} .$$

By using an additional Schur complement procedure, the problem is finally reduced to solving the following linear system with the Lagrange multipliers λ as variables:

$$B_\Delta \tilde{S}^{-1} B_\Delta^T \lambda = B_\Delta \tilde{S}^{-1} \mathbf{f}_\Delta^*. \tag{4.5}$$

This is the dual-primal FETI algorithm without preconditioning, equivalent to the linear system (3.18), for solving incompressible Stokes problems. The preconditioned algorithm and its convergence analysis will be discussed in section 4.4 for two dimensions and in section 4.5 for three dimensions. Here we show that $\tilde{S}$ is symmetric, positive definite on the space $\widetilde{\mathbf{W}}_\Delta$, and that we therefore can define an $\tilde{S}$-norm.

Lemma 7 *$\tilde{S}$ is symmetric, positive definite on the space $\widetilde{\mathbf{W}}_\Delta$.*

Proof: It is easy to see, from its definition (4.4), that $\tilde{S}$ is symmetric. We next just need to show that $(\tilde{S}\mathbf{u}_\Delta, \mathbf{u}_\Delta) > 0$, for any nonzero function $\mathbf{u}_\Delta \in \widetilde{\mathbf{W}}_\Delta$. For any given function $\mathbf{u}_\Delta \in \widetilde{\mathbf{W}}_\Delta$, since the coarse-level saddle point problem is inf-sup stable, we can always find a vector $(\mathbf{u}_I, p_I, \mathbf{u}_\Pi, p_0)$ such that equation (4.4) is satisfied. Therefore,

$$
\begin{aligned}
&(\tilde{S}\mathbf{u}_\Delta, \mathbf{u}_\Delta) = \mathbf{u}_\Delta^T \tilde{S}\mathbf{u}_\Delta \\
= &\begin{pmatrix} \mathbf{u}_I \\ p_I \\ \mathbf{u}_\Pi \\ p_0 \\ \mathbf{u}_\Delta \end{pmatrix}^T
\begin{pmatrix} A_{II} & B_{II}^T & A_{\Pi I}^T & 0 & A_{\Delta I}^T \\ B_{II} & 0 & B_{I\Pi} & 0 & B_{I\Delta} \\ A_{\Pi I} & B_{I\Pi}^T & A_{\Pi\Pi} & B_{0\Pi}^T & A_{\Delta\Pi}^T \\ 0 & 0 & B_{0\Pi} & 0 & B_{0\Delta} \\ A_{\Delta I} & B_{I\Delta}^T & A_{\Delta\Pi} & B_{0\Delta}^T & A_{\Delta\Delta} \end{pmatrix}
\begin{pmatrix} \mathbf{u}_I \\ p_I \\ \mathbf{u}_\Pi \\ p_0 \\ \mathbf{u}_\Delta \end{pmatrix} \\
= &\begin{pmatrix} \mathbf{u}_I \\ \mathbf{u}_\Pi \\ \mathbf{u}_\Delta \end{pmatrix}^T
\begin{pmatrix} A_{II} & A_{\Pi I}^T & A_{\Delta I}^T \\ A_{\Pi I} & A_{\Pi\Pi} & A_{\Delta\Pi}^T \\ A_{\Delta I} & A_{\Delta\Pi} & A_{\Delta\Delta} \end{pmatrix}
\begin{pmatrix} \mathbf{u}_I \\ \mathbf{u}_\Pi \\ \mathbf{u}_\Delta \end{pmatrix} \\
&+2 \begin{pmatrix} p_I \\ p_0 \end{pmatrix}^T \begin{pmatrix} B_{II} & B_{I\Pi} & B_{I\Delta} \\ 0 & B_{0\Pi} & B_{0\Delta} \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ \mathbf{u}_\Pi \\ \mathbf{u}_\Delta \end{pmatrix} + \begin{pmatrix} p_I \\ p_0 \end{pmatrix}^T \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} p_I \\ p_0 \end{pmatrix} \\
= &\begin{pmatrix} \mathbf{u}_I \\ \mathbf{u}_\Pi \\ \mathbf{u}_\Delta \end{pmatrix}^T
\begin{pmatrix} A_{II} & A_{\Pi I}^T & A_{\Delta I}^T \\ A_{\Pi I} & A_{\Pi\Pi} & A_{\Delta\Pi}^T \\ A_{\Delta I} & A_{\Delta\Pi} & A_{\Delta\Delta} \end{pmatrix}
\begin{pmatrix} \mathbf{u}_I \\ \mathbf{u}_\Pi \\ \mathbf{u}_\Delta \end{pmatrix},
\end{aligned}
$$

where the last equality results from $B_{II}\mathbf{u}_I + B_{I\Pi}\mathbf{u}_\Pi + B_{I\Delta}\mathbf{u}_\Delta = 0$ and $B_{0\Pi}\mathbf{u}_\Pi + B_{0\Delta}\mathbf{u}_\Delta = 0$, because the vector $(\mathbf{u}_I, p_I, \mathbf{u}_\Pi, p_0, \mathbf{u}_\Delta)$ satisfies equation (4.4). Since the matrix

$$
\begin{pmatrix} A_{II} & A_{\Pi I}^T & A_{\Delta I}^T \\ A_{\Pi I} & A_{\Pi\Pi} & A_{\Delta\Pi}^T \\ A_{\Delta I} & A_{\Delta\Pi} & A_{\Delta\Delta} \end{pmatrix}
$$

is just the symmetric, positive definite discretization of a direct sum of Laplace operators with Dirichlet boundary conditions, we find that $(\tilde{S}\mathbf{u}_\Delta, \mathbf{u}_\Delta) > 0$, for any nonzero function $\mathbf{u}_\Delta \in \widetilde{\mathbf{W}}_\Delta$.

$\square$

Since $\tilde{S}$ is symmetric, positive definite on the space $\widetilde{\mathbf{W}}_\Delta$, we can define an $\tilde{S}$-norm on $\widetilde{\mathbf{W}}_\Delta$, i.e.,

$$|\mathbf{w}_\Delta|_{\tilde{S}} = \sqrt{(\tilde{S}\mathbf{u}_\Delta, \mathbf{u}_\Delta)}, \quad \forall \mathbf{w}_\Delta \in \widetilde{\mathbf{W}}_\Delta.$$

We know, from the proof of Lemma 7, that for any given $\mathbf{w}_\Delta \in \widetilde{\mathbf{W}}_\Delta$, $|\mathbf{w}_\Delta|_{\tilde{S}} = |\mathbf{w}_\Gamma|_{S_\Gamma}$, where $\mathbf{w}_\Gamma = \mathbf{w}_\Pi + \mathbf{w}_\Delta$, with $\mathbf{w}_\Pi$ determined by equation (4.4). We have

$$|\mathbf{w}_\Delta|_{\tilde{S}} = \inf_{\mathbf{w}_\Pi \in \mathbf{W}_\Pi} \left\{ |\mathbf{w}_\Delta + \mathbf{w}_\Pi|_{S_\Gamma} \;\middle|\; \int_{\Omega^i} \nabla \cdot (\mathbf{w}_\Delta + \mathbf{w}_\Pi)^i = 0, \; \forall \Omega^i \right\}. \tag{4.6}$$

4.2 Technical tools

In this section, we give some lemmas which are necessary in our convergence analysis.

Lemma 8 can be found in [51, Lemma 7].

Lemma 8 *Define an interpolation operator* $I_{\Gamma\Pi} : \mathbf{W}_\Gamma \to \mathbf{W}_\Pi$ *by:*

$$I_{\Gamma\Pi}\mathbf{w}_\Gamma(\mathbf{x}) = \sum_{\mathcal{V}^{il}} \mathbf{w}_\Gamma(\mathcal{V}^{il})\theta_{\mathcal{V}^{il}}(x) + \sum_{\mathcal{E}^{ik}} \overline{\mathbf{w}}_{\mathcal{E}^{ik}}\theta_{\mathcal{E}^{ik}}(x) + \sum_{\mathcal{F}^{ij}} \overline{\mathbf{w}}_{\mathcal{F}^{ij}}\theta_{\mathcal{F}^{ij}}(x), \quad \text{in 3D},$$

and

$$I_{\Gamma\Pi}\mathbf{w}_\Gamma(\mathbf{x}) = \sum_{\mathcal{V}^{il}} \mathbf{w}_\Gamma(\mathcal{V}^{il})\theta_{\mathcal{V}^{il}}(x) + \sum_{\mathcal{E}^{ik}} \overline{\mathbf{w}}_{\mathcal{E}^{ik}}\theta_{\mathcal{E}^{ik}}(x), \quad \text{in 2D}.$$

We then have

$$|I_{\Gamma\Pi}\mathbf{w}_\Gamma|^2_{H^{1/2}(\Gamma)} \le C(1 + \log(H/h))|\mathbf{w}_\Gamma|^2_{H^{1/2}(\Gamma)}, \quad \forall \mathbf{w}_\Gamma \in \mathbf{W}_\Gamma,$$

and

$$\|\mathbf{w}_\Gamma - I_{\Gamma\Pi}\mathbf{w}_\Gamma\|^2_{L^2(\Gamma)} \le CH|\mathbf{w}_\Gamma|^2_{H^{1/2}(\Gamma)}, \quad \forall \mathbf{w}_\Gamma \in \mathbf{W}_\Gamma,$$

where C is a constant independent of H and h.

Lemma 9 can be found in [17, Lemma 4.5].

Lemma 9 *Let $\theta_{\mathcal{F}^{ij}}(x)$ be the cut-off function of the open face $\mathcal{F}^{ij}$, and let I^h denote the interpolation operator onto the finite element space $\mathbf{W}^i$. Then,*

$$\|I^h(\theta_{\mathcal{F}^{ij}}\mathbf{w}_\Gamma^i)\|^2_{H^{1/2}_{00}(\mathcal{F}^{ij})} \leq C(1+\log(H/h))^2(|\mathbf{w}_\Gamma^i|^2_{H^{1/2}(\mathcal{F}^{ij})}+\frac{1}{H}\|\mathbf{w}_\Gamma^i\|^2_{L^2(\mathcal{F}^{ij})}), \forall \mathbf{w}_\Gamma^i \in \mathbf{W}_\Gamma^i.$$

Lemma 10 can be found in [20, Lemma 3.3].

Lemma 10 *Let $\mathcal{E}^{ik}$ be any edge of Ω^i which forms part of the boundary of a face $\mathcal{F}^{ij} \subset \partial\Omega^i$. Then,*

$$\|\mathbf{w}_\Gamma^i\|^2_{L^2(\mathcal{E}^{ik})} \leq C(1+\log(H/h))(|\mathbf{w}_\Gamma^i|^2_{H^{1/2}(\mathcal{F}^{ij})} + \frac{1}{H}\|\mathbf{w}_\Gamma^i\|^2_{L^2(\mathcal{F}^{ij})}) \ , \ \forall \mathbf{w}_\Gamma^i \in \mathbf{W}_\Gamma^i \ .$$

For the next lemma, see [17, Lemma 4.7].

Lemma 11 *Let $\theta_{\mathcal{E}^{ik}}$ be the cutoff function associated with the edge $\mathcal{E}^{ik}$. Then,*

$$|I^h(\theta_{\mathcal{E}^{ik}}\mathbf{w}_\Gamma^i)|^2_{H^{1/2}(\partial\Omega^i)} \leq C\|\mathbf{w}_\Gamma^i\|^2_{L^2(\mathcal{E}^{ik})} \ , \ \forall \mathbf{w}_\Gamma^i \in \mathbf{W}_\Gamma^i \ .$$

The following lemma is for the two-dimensional case and can be found in [82, Lemma 3.3],

Lemma 12 *Let $\mathbf{w}_\Gamma \in \mathbf{W}_\Gamma$. $I^H\mathbf{w}_\Gamma$ is the linear interpolant from the values at subdomain vertices. Then*

$$\sum_{\mathcal{E}^{ik}\in\partial\Omega^i} |\mathbf{w}_\Gamma - I^H\mathbf{w}_\Gamma|^2_{1/2,2,\mathcal{E}^{ik}} \leq C(1+\log(H/h))^2|\mathbf{w}_\Gamma|^2_{1/2,2,\partial\Omega^i} \ .$$

4.3 Inf-sup stability of the coarse saddle point problem

In this section, we give an estimate of the inf-sup stability of the following coarse level saddle point problem,

$$\left\{\begin{array}{lcll} a(\mathcal{SH}\mathbf{u}_\Pi, \mathcal{SH}\mathbf{v}_\Pi) + b(\mathcal{SH}\mathbf{v}_\Pi, p_0) & = & < \mathbf{f}_\Pi, \mathbf{v}_\Pi >, & \forall \mathbf{v}_\Pi \in \mathbf{W}_\Pi, \\ b(\mathcal{SH}\mathbf{u}_\Pi, q_0) & = & 0, & \forall q_0 \in \Pi_0. \end{array}\right. \tag{4.7}$$

The method we are using here is the macroelement technique, introduced in [75]. This technique was also used in [64] to prove the inf-sup stability for a coarse level saddle point problem, introduced in the Balancing Neumann-Neumann method.

Theorem 4 *For the coarse level saddle point problem (4.7), we have*

$$\sup_{\mathbf{w}_\Pi \in \mathbf{W}_\Pi} \frac{b(\mathcal{SH}\mathbf{w}_\Pi, q_0)^2}{a(\mathcal{SH}\mathbf{w}_\Pi, \mathcal{SH}\mathbf{w}_\Pi)} \geq C\beta^2(1+\log(H/h))^{-1}||q_0||_{L^2}^2 = \beta_0^2||q_0||_{L^2}^2, \quad \forall q_0 \in \Pi_0,$$

where β is the inf-sup stability constant of the original Stokes problem, defined in (2.4). $\beta_0^2 = C\beta^2(1+\log(H/h))^{-1}$ is thus the inf-sup stability constant of the coarse level saddle point problem.

Proof: We know, from Lemma 6, that

$$C_1\beta^2 a(\mathcal{SH}\mathbf{u}_\Pi, \mathcal{SH}\mathbf{u}_\Pi) \leq a(\mathcal{H}\mathbf{u}_\Pi, \mathcal{H}\mathbf{u}_\Pi), \tag{4.8}$$

and we know that

$$b(\mathcal{SH}\mathbf{w}_\Pi, q_0) = b(\mathcal{H}\mathbf{w}_\Pi, q_0). \tag{4.9}$$

Therefore, we just need to estimate

$$\inf_{q_0 \in Q_0} \sup_{\mathbf{w}_\Pi \in \mathbf{W}_\Pi} \frac{b(\mathcal{H}\mathbf{w}_\Pi, q_0)^2}{a(\mathcal{H}\mathbf{w}_\Pi, \mathcal{H}\mathbf{w}_\Pi)},$$

which will provide the inf-sup stability estimate of the incompressible Stokes equation on the mixed finite element space $\mathbf{W}_\Pi \times Q_0$.

A macroelement technique is used to establish the inf-sup stability of the space $\mathbf{W}_\Pi \times Q_0$. A marcoelement M in this case consists of two neighboring subdomains, which share an edge in two dimensions and share a face in three dimensions; see Figure 4.1 for the two-dimensional case.

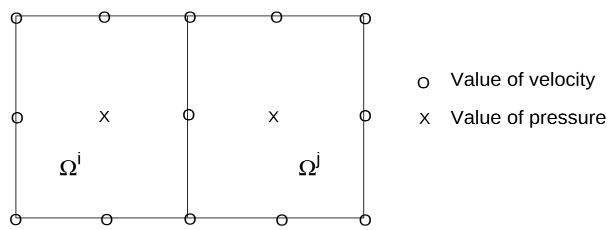

Figure 4.1: A marcoelement M being the union of two neighboring subdomains

It is easy to see that $p_0^i = p_0^j$, if $\int_M (\nabla \cdot \mathbf{v}) p_0 = 0$ for any finite element function $\mathbf{v}$ which vanishes on the boundary of M. The only thing we need to show is a result similar to [75, Lemma 2], i.e., we need to establish that

$$\sup_{\mathbf{w}_\Pi \in \mathbf{W}_\Pi} \frac{b(\mathcal{H}\mathbf{w}_\Pi, q_0)}{\sqrt{a(\mathcal{H}\mathbf{w}_\Pi, \mathcal{H}\mathbf{w}_\Pi)}} \geq C_1 \|q_0\|_{L^2} - C_2 \|q_0\|_h, \quad \forall q_0 \in Q_0,$$

where

$$\|q_0\|_h^2 = H \sum_{i=1}^{N_s} \int_{\partial \Omega^i} [q_0]^2 ds,$$

with $[q_0]$ the jump of q_0 across $\partial \Omega^i$.

We follow the procedure in the proof of [75, Lemma 2]. From the inf-sup stability of the corresponding continuous problem, we know that for any $q_0 \in Q_0$, there exists a function $\mathbf{w}_\Gamma \in H^{1/2}(\Gamma)$ such that

$$b(\mathcal{H}\mathbf{w}_\Gamma, q_0) \geq C_3 \|q_0\|_{L^2}^2, \tag{4.10}$$

and

$$\sqrt{a(\mathcal{H}\mathbf{w}_\Gamma, \mathcal{H}\mathbf{w}_\Gamma)} \leq \|q_0\|_{L^2}. \tag{4.11}$$

We choose $\mathbf{w}_\Pi$ as the interpolant of $\mathbf{w}_\Gamma$, i.e., $\mathbf{w}_\Pi = I_{\Gamma\Pi}\mathbf{w}_\Gamma$. We then know, from Lemma 8, that

$$\|\mathbf{w}_\Pi - \mathbf{w}_\Gamma\|_{L^2(\Gamma)} \leq CH^{1/2}|\mathbf{w}_\Gamma|_{H^{1/2}(\Gamma)}, \tag{4.12}$$

and

$$|\mathbf{w}_\Pi|_{H^{1/2}(\Gamma)} \leq C(1+\log(H/h))^{1/2}|\mathbf{w}_\Gamma|_{H^{1/2}(\Gamma)}. \tag{4.13}$$

Then, we have, from (4.10), (4.11), and (4.12),

$$\begin{aligned}
b(\mathcal{H}\mathbf{w}_\Pi, q_0) &= b(\mathcal{H}(\mathbf{w}_\Pi - \mathbf{w}_\Gamma), q_0) + b(\mathcal{H}\mathbf{w}_\Gamma, q_0) \\
&\geq b(\mathcal{H}(\mathbf{w}_\Pi - \mathbf{w}_\Gamma), q_0) + C_3\|q_0\|_{L^2}^2 \\
&= \sum_{i=1}^{N_s}(\mathcal{H}^i(\mathbf{w}_\Pi - \mathbf{w}_\Gamma), \nabla q_0)_{\Omega^i} + \sum_{i=1}^{N_s}\int_{\partial\Omega^i}[q_0](\mathbf{w}_\Pi - \mathbf{w}_\Gamma)\cdot\mathbf{n}ds \\
&\quad +C_3\|q_0\|_{L^2}^2 \\
&\geq -H^{-1/2}\|\mathbf{w}_\Pi - \mathbf{w}_\Gamma\|_{L^2(\Gamma)}\|q_0\|_h + C_3\|q_0\|_{L^2}^2 \\
&\geq -C_4|\mathbf{w}_\Gamma|_{H^{1/2}(\Gamma)}\|q_0\|_h + C_3\|q_0\|_{L^2}^2 \\
&\geq -C_4\|q_0\|_{L^2}\|q_0\|_h + C_3\|q_0\|_{L^2}^2 \\
&= \left(C_3\|q_0\|_{L^2} - C_4\|q_0\|_h\right)\|q_0\|_{L^2},
\end{aligned}$$

and, from (4.13) and (4.11), we have

$$\begin{aligned}
\sqrt{a(\mathcal{H}\mathbf{w}_\Pi, \mathcal{H}\mathbf{w}_\Pi)} &\leq C_5\left(1+\log(H/h)\right)^{1/2}\sqrt{a(\mathcal{H}\mathbf{w}_\Gamma, \mathcal{H}\mathbf{w}_\Gamma)} \\
&\leq C_5\left(1+\log(H/h)\right)^{1/2}\|q_0\|_{L^2}.
\end{aligned}$$

Therefore,

$$\sup_{\mathbf{w}_\Pi \in \mathbf{W}_\Pi} \frac{b(\mathcal{H}\mathbf{w}_\Pi, q_0)}{\sqrt{a(\mathcal{H}\mathbf{w}_\Pi, \mathcal{H}\mathbf{w}_\Pi)}} \geq \left(1+\log(H/h)\right)^{-1/2}\left(C_1\|q_0\|_{L^2} - C_2\|q_0\|_h\right),$$

where C_1 and C_2 are constants independent of the mesh size H and h. This is just the conclusion of [75, Lemma 2], except for the $(1+\log(H/h))^{-1/2}$ factor. We find, using the result of [75], that

$$\inf_{q_0 \in Q_0} \sup_{\mathbf{w}_\Pi \in \mathbf{W}_\Pi} \frac{b(\mathcal{H}\mathbf{w}_\Pi, q_0)^2}{a(\mathcal{H}\mathbf{w}_\Pi, \mathcal{H}\mathbf{w}_\Pi)} \geq C\,(1+\log(H/h))^{-1}\,||q_0||_{L^2}^2,$$

and therefore, from (4.8) and (4.9), we have

$$\inf_{q_0 \in Q_0} \sup_{\mathbf{w}_\Pi \in \mathbf{W}_\Pi} \frac{b(\mathcal{S}\mathcal{H}\mathbf{w}_\Pi, q_0)^2}{a(\mathcal{S}\mathcal{H}\mathbf{w}_\Pi, \mathcal{S}\mathcal{H}\mathbf{w}_\Pi)} \geq C\beta^2\,(1+\log(H/h))^{-1}\,||q_0||_{L^2}^2.$$

□

4.4 Condition number bounds in two dimensions

Our preconditioner in two dimensions is $B_\Delta S_\Gamma B_\Delta^T$, and the preconditioned linear system is

$$B_\Delta S_\Gamma B_\Delta^T B_\Delta \tilde{S}^{-1} B_\Delta^T \lambda = B_\Delta S_\Gamma B_\Delta^T B_\Delta \tilde{S}^{-1} \mathbf{f}_\Delta^* \,. \tag{4.14}$$

When we use a Krylov subspace iterative method to solve equation (4.14), both S_Γ and $\tilde{S}^{-1}$ are applied to vectors in $range(B_\Delta^T)$. In order to use a conjugate gradient method to solve the linear system (4.14), we have to show that both S_Γ and $\tilde{S}^{-1}$ are symmetric, positive definite on $range(B_\Delta^T)$. From Lemma 4, we know that S_Γ is symmetric, positive semi-definite on the space $\mathbf{W}_\Gamma$ and it is singular because of the interior floating subdomains. Now on the range of B_Δ^T, each interior floating subdomain becomes fixed, because the velocity equals zero at each subdomain vertex. Therefore S_Γ is nonsingular on the space $range(B_\Delta^T)$ and therefore becomes positive definite. We also know, from Lemma 7, that $\tilde{S}^{-1}$ is symmetric, positive definite. Therefore we can use a conjugate gradient method to solve (4.14). In the remainder of this section, we give a condition number bound for the preconditioned operator $B_\Delta S_\Gamma B_\Delta^T B_\Delta \tilde{S}^{-1} B_\Delta^T$.

Lemma 13 *For any* $\mathbf{w}_\Delta \in \widetilde{\mathbf{W}}_\Delta$*,* $B_\Delta^T B_\Delta \mathbf{w}_\Delta \in \widetilde{\mathbf{W}}_\Delta$*.*

Proof: Given $\mathbf{w}_\Delta \in \widetilde{\mathbf{W}}_\Delta$, we need to show that, on each subdomain interface edge $\mathcal{E}^{ij}$,

$$\int_{\mathcal{E}^{ij}} (B_\Delta^T B_\Delta \mathbf{w}_\Delta)^i = 0, \quad \text{and} \quad \int_{\mathcal{E}^{ij}} (B_\Delta^T B_\Delta \mathbf{w}_\Delta)^j = 0.$$

This is easily verified by noticing that,

$$(B_\Delta^T B_\Delta \mathbf{w}_\Delta)^i \mid_{\mathcal{E}^{ij}} = \left(\mathbf{w}_\Delta^i - \mathbf{w}_\Delta^j\right) \mid_{\mathcal{E}^{ij}}, \quad \text{and} \quad (B_\Delta^T B_\Delta \mathbf{w}_\Delta)^j \mid_{\mathcal{E}^{ij}} = \left(\mathbf{w}_\Delta^j - \mathbf{w}_\Delta^i\right) \mid_{\mathcal{E}^{ij}},$$

and that

$$\int_{\mathcal{E}^{ij}} \mathbf{w}_\Delta^i = \int_{\mathcal{E}^{ij}} \mathbf{w}_\Delta^j = 0,$$

because $\mathbf{w}_\Delta \in \widetilde{\mathbf{W}}_\Delta$.

$\square$

Lemma 14 *For any* $\mathbf{w}_\Delta \in \widetilde{\mathbf{W}}_\Delta$*,* $|\mathbf{w}_\Delta|_{\tilde{S}} \leq 2|\mathbf{w}_\Delta|_{S_\Gamma}$*.*

Proof: Given $\mathbf{w}_\Delta \in \widetilde{\mathbf{W}}_\Delta$, in order to compute its $\tilde{S}$-norm, we need to determine the element $\mathbf{w}_\Gamma = \mathbf{w}_\Pi + \mathbf{w}_\Delta \in \mathbf{W}_\Gamma$, such that (4.4) is satisfied, and we have $|\mathbf{w}_\Delta|_{\tilde{S}} = |\mathbf{w}_\Gamma|_{S_\Gamma}$. Therefore,

$$|\mathbf{w}_\Delta|_{\tilde{S}} = |\mathbf{w}_\Gamma|_{S_\Gamma} \leq |\mathbf{w}_\Delta|_{S_\Gamma} + |\mathbf{w}_\Pi|_{S_\Gamma}.$$

In the following, we bound $|\mathbf{w}_\Pi|_{S_\Gamma}$ by $|\mathbf{w}_\Delta|_{S_\Gamma}$.

We know that, given a $\mathbf{w}_\Delta$, to find a $\mathbf{w}_\Pi$ is equivalent to solving the following coarse level saddle point problem: find $\mathbf{w}_\Pi \in \mathbf{W}_\Pi$ and $p_0 \in Q_0$ such that

$$\begin{cases} a(\mathcal{SH}(\mathbf{w}_\Pi + \mathbf{w}_\Delta), \mathcal{SH}\mathbf{v}_\Pi) + b(\mathcal{SH}\mathbf{v}_\Pi, p_0) & = & 0, \quad \forall \mathbf{v}_\Pi \in \mathbf{W}_\Pi, \\ b(\mathcal{SH}(\mathbf{w}_\Pi + \mathbf{w}_\Delta), q_0) & = & 0, \quad \forall q_0 \in \Pi_0. \end{cases} \tag{4.15}$$

We know, from the inf-sup stability of this coarse level problem and Theorem 1, that

$$|\mathbf{w}_\Pi|_{S_\Gamma} \leq \sup_{\mathbf{v}_\Pi \in \mathbf{W}_\Pi} \frac{a(\mathcal{SH}\mathbf{w}_\Delta, \mathcal{SH}\mathbf{v}_\Pi)}{|\mathbf{v}_\Pi|_{S_\Gamma}} + \frac{2}{\beta_0} \sup_{q_0 \in Q_0} \frac{b(\mathcal{SH}\mathbf{w}_\Delta, q_0)}{\|q_0\|_{L^2}}.$$

Since $\mathbf{w}_\Delta \in \widetilde{\mathbf{W}}_\Delta$, we have $b(\mathcal{SH}\mathbf{w}_\Delta, q_0) = 0$. Therefore, we have, from the continuity of the bilinear form $a(\cdot, \cdot)$,

$$|\mathbf{w}_\Pi|_{S_\Gamma} \leq |\mathbf{w}_\Delta|_{S_\Gamma}.$$

Therefore

$$|\mathbf{w}_\Delta|_{\tilde{S}} \leq |\mathbf{w}_\Delta|_{S_\Gamma} + |\mathbf{w}_\Pi|_{S_\Gamma} \leq 2|\mathbf{w}_\Delta|_{S_\Gamma}.$$

□

Lemma 15 *For any $\mathbf{w}_\Delta \in \widetilde{\mathbf{W}}_\Delta$, we have,*

$$|B_\Delta^T B_\Delta \mathbf{w}_\Delta|_{S_\Gamma}^2 \leq C \frac{1}{\beta^2} (1 + \log(H/h))^2 |\mathbf{w}_\Delta|_{\tilde{S}}^2,$$

where $C > 0$ is independent of h and H.

Proof: We consider an arbitrary $\mathbf{w}_\Delta \in \widetilde{\mathbf{W}}_\Delta$. In order to compute its $\tilde{S}-$norm, we need to determine the element $\mathbf{w}_\Gamma = \mathbf{w}_\Pi + \mathbf{w}_\Delta \in \mathbf{W}_\Gamma$ with $\mathbf{w}_\Pi \in \mathbf{W}_\Pi$, which satisfies equation (4.4). Then, we know that $|\mathbf{w}_\Delta|_{\tilde{S}} = |\mathbf{w}_\Gamma|_{S_\Gamma}$. We next note that we can subtract any continuous function from $\mathbf{w}_\Delta$ without changing the values of

$B_\Delta^T B_\Delta \mathbf{w}_\Delta$; thus, $B_\Delta^T B_\Delta \mathbf{w}_\Delta = B_\Delta^T B_\Delta(\mathbf{w}_\Delta + \mathbf{w}_\Pi - I^H \mathbf{w}_\Pi)$, where $I^H \mathbf{w}_\Pi$ is the linear interpolant on the subdomain boundary, from the values of $\mathbf{w}_\Pi$ at the subdomain vertices.

We introduce the notation

$$(\mathbf{v}_\Gamma^i)_{i=1,\dots,N} := B_\Delta^T B_\Delta(\mathbf{w}_\Delta + \mathbf{w}_\Pi - I^H \mathbf{w}_\Pi) = B_\Delta^T B_\Delta(\mathbf{w}_\Gamma - I^H \mathbf{w}_\Gamma).$$

We then have to estimate

$$|B_\Delta^T B_\Delta \mathbf{w}_\Delta|_{S_\Gamma}^2 = |B_\Delta^T B_\Delta(\mathbf{w}_\Gamma - I^H \mathbf{w}_\Gamma)|_{S_\Gamma}^2 = \sum_{i=1}^{N_s} |\mathbf{v}_\Gamma^i|_{S_\Gamma^i}^2.$$

We can therefore focus on the estimate of a single subdomain contribution. Noticing that $\mathbf{v}_\Gamma^i$ vanishes at the subdomain corners, we can split $\mathbf{v}_\Gamma^i$ as

$$\mathbf{v}_\Gamma^i = \sum_{\mathcal{E}^{ik} \subset \partial\Omega^i} \mathbf{v}_{\Gamma,\mathcal{E}^{ik}}^i.$$

On each edge $\mathcal{E}^{ik}$, we know that

$$\mathbf{v}_{\Gamma,\mathcal{E}^{ik}}^i = \left(B_\Delta^T B_\Delta(\mathbf{w}_\Gamma - I^H \mathbf{w}_\Gamma)\right)_{\mathcal{E}^{ik}}^i = \left(\mathbf{w}_\Gamma - I^H \mathbf{w}_\Gamma\right)_{\mathcal{E}^{ik}}^i - \left(\mathbf{w}_\Gamma - I^H \mathbf{w}_\Gamma\right)_{\mathcal{E}^{ik}}^k.$$

We have to estimate its S_Γ^i-norm. We have, from Lemma 6,

$$\begin{aligned}
|\mathbf{v}_{\Gamma,\mathcal{E}^{ik}}^i|_{S_\Gamma^i}^2 &= |\left(\mathbf{w}_\Gamma - I^H \mathbf{w}_\Gamma\right)_{\mathcal{E}^{ik}}^i - \left(\mathbf{w}_\Gamma - I^H \mathbf{w}_\Gamma\right)_{\mathcal{E}^{ik}}^k|_{S_\Gamma^i}^2 \\
&\le \frac{1}{\beta^2} |\left(\mathbf{w}_\Gamma - I^H \mathbf{w}_\Gamma\right)^i - \left(\mathbf{w}_\Gamma - I^H \mathbf{w}_\Gamma\right)^k|_{H^{1/2}(\mathcal{E}^{ik})}^2 \\
&\le \frac{2}{\beta^2} |\left(\mathbf{w}_\Gamma - I^H \mathbf{w}_\Gamma\right)^i|_{H^{1/2}(\mathcal{E}^{ik})}^2 \\
&\quad + \frac{2}{\beta^2} |\left(\mathbf{w}_\Gamma - I^H \mathbf{w}_\Gamma\right)^k|_{H^{1/2}(\mathcal{E}^{ik})}^2,
\end{aligned}$$

where the two terms on the right can be bounded by $C\frac{1}{\beta^2}(1+\log(H/h))^2|\mathbf{w}_\Gamma^i|_{H^{1/2}(\partial\Omega^i)}^2$ and $C\frac{1}{\beta^2}(1+\log(H/h))^2|\mathbf{w}_\Gamma^k|_{H^{1/2}(\partial\Omega^k)}^2$, respectively, by using Lemma 12. Then, by using Lemma 6 again, we have

$$|\mathbf{v}_\Gamma^i|_{S_\Gamma^i}^2 \le C\frac{1}{\beta^2}(1+\log(H/h))^2 \left(|\mathbf{w}_\Gamma^i|_{S_\Gamma^i}^2 + \sum_{k\in\mathcal{N}^i} |\mathbf{w}_\Gamma^k|_{S_\Gamma^k}^2 \right) ,$$

where $\mathcal{N}^i$ is the set of indices of all the neighboring subdomains of Ω^i.

□

Theorem 5 *The condition number of the preconditioned linear system (4.14) is bounded from above by* $C\frac{1}{\beta^2}(1+\log(H/h))^2$*, where* C *is independent of* h *and* H*.*

Proof: We will show that

$$\lambda^T M\lambda \le \lambda^T F\lambda \le C\frac{1}{\beta^2}(1+\log(H/h))^2\lambda^T M\lambda, \quad \forall\lambda\in\Lambda \ , \tag{4.16}$$

where $M^{-1} = B_\Delta S_\Gamma B_\Delta^T, F = B_\Delta \tilde{S}^{-1} B_\Delta^T$, and C is independent of h and H.

Lower bound: We have, cf. [51] and [57],

$$\lambda^T F\lambda = \max_{0\neq\mathbf{w}_\Delta\in\widetilde{\mathbf{W}}_\Delta} \frac{|(\lambda, B_\Delta\mathbf{w}_\Delta)|^2}{|\mathbf{w}_\Delta|_{\tilde{S}}^2} .$$

From Lemma 13, we know that $B_\Delta^T B_\Delta \widetilde{\mathbf{W}}_\Delta \subset \widetilde{\mathbf{W}}_\Delta$. We also know from Lemma 14 that $|\mathbf{w}_\Delta|_{\tilde{S}} \le 2|\mathbf{w}_\Delta|_{S_\Gamma}$ for all $\mathbf{w}_\Delta \in B_\Delta^T B_\Delta \widetilde{\mathbf{W}}_\Delta$. Therefore, we have

$$\lambda^T F\lambda \ge \max_{0\neq\mathbf{w}_\Delta\in\widetilde{\mathbf{W}}_\Delta} \frac{|(\lambda, B_\Delta B_\Delta^T B_\Delta\mathbf{w}_\Delta)|^2}{|B_\Delta^T B_\Delta\mathbf{w}_\Delta|_{\tilde{S}}^2} \ge \max_{0\neq\mathbf{w}_\Delta\in\widetilde{\mathbf{W}}_\Delta} \frac{|(\lambda, B_\Delta\mathbf{w}_\Delta)|^2}{|B_\Delta^T B_\Delta\mathbf{w}_\Delta|_{S_\Gamma}^2},$$

where we have used that fact that $B_\Delta B_\Delta^T = 2I$.

Since for any $\nu\in\Lambda$ there is a $\mathbf{w}_\Delta \in \widetilde{\mathbf{W}}_\Delta$ such that $\nu = B_\Delta\mathbf{w}_\Delta$, we have

$$\lambda^T F\lambda \ge \frac{|(\lambda,\nu)|^2}{|B_\Delta^T\nu|_{S_\Gamma}^2} = \frac{|(\lambda,\nu)|^2}{(M^{-1}\nu,\nu)} .$$

We find that

$$\lambda^T M\lambda \le \lambda^T F\lambda,$$

by choosing $\nu = M\lambda$.

Upper bound: Using Lemma 15, we have

$$\begin{aligned}
\lambda^T F \lambda &= \max_{0 \neq \mathbf{w}_\Delta \in \widetilde{\mathbf{W}}_\Delta} \frac{(\lambda, B_\Delta \mathbf{w}_\Delta)^2}{|\mathbf{w}_\Delta|^2_{\tilde{S}}} \\
&\leq C \frac{1}{\beta^2}(1 + \log(H/h))^2 \max_{0 \neq \mathbf{w}_\Delta \in \widetilde{\mathbf{W}}_\Delta} \frac{(\lambda, B_\Delta \mathbf{w}_\Delta)^2}{|B_\Delta^T B_\Delta \mathbf{w}_\Delta|^2_{S_\Gamma}} \\
&= C \frac{1}{\beta^2}(1 + \log(H/h))^2 \max_{0 \neq \mathbf{w}_\Delta \in \widetilde{\mathbf{W}}_\Delta} \frac{(\lambda, B_\Delta \mathbf{w}_\Delta)^2}{(M^{-1} B_\Delta \mathbf{w}_\Delta, B_\Delta \mathbf{w}_\Delta)} \\
&= C \frac{1}{\beta^2}(1 + \log(H/h))^2 \max_{\nu \in \Lambda} \frac{(\lambda, \nu)^2}{(M^{-1}\nu, \nu)} \\
&= C \frac{1}{\beta^2}(1 + \log(H/h))^2 (M\lambda, \lambda) \ .
\end{aligned}$$

□

4.5 Condition number bounds in three dimensions

The preconditioner used for three-dimensional problem is $DB_\Delta S_\Gamma B_\Delta^T D$, where D is a diagonal scaling matrix. Each of its entries corresponds to a Lagrange multiplier and is given by $\mu^\dagger(\mathbf{x})$ at each interface point $\mathbf{x}$. $\mu^\dagger(\mathbf{x})$ is the pseudoinverse of the counting functions $\mu(\mathbf{x})$: at each interface node $\mathbf{x} \in \Gamma_h$, $\mu(\mathbf{x})$ equals the number of subdomains shared by that node, and $\mu^\dagger(\mathbf{x}) = 1/\mu(\mathbf{x})$.

The preconditioned linear system is

$$DB_\Delta S_\Gamma B_\Delta^T D B_\Delta \tilde{S}^{-1} B_\Delta^T \lambda = DB_\Delta S_\Gamma B_\Delta^T D B_\Delta \tilde{S}^{-1} \mathbf{f}^*_\Delta \ , \tag{4.17}$$

which defines our FETI-DP algorithm for solving three-dimensional incompressible Stokes equations.

We introduce two operators P_Δ and E_Δ: for any $\mathbf{w}_\Delta \in \widetilde{\mathbf{W}}_\Delta$,

$$P_\Delta \mathbf{w}_\Delta = B_\Delta^T D B_\Delta \mathbf{w}_\Delta,$$

and

$$E_\Delta \mathbf{w}_\Delta = \mathbf{w}_\Delta - P_\Delta \mathbf{w}_\Delta.$$

We know, from [51], that at any interface point $\mathbf{x} \in \Gamma_h \cap \partial\Omega_h^i$,

$$(P_\Delta \mathbf{w}_\Delta)^i(\mathbf{x}) = \sum_{j \in \mathcal{N}_\mathbf{x}^i} \mu^\dagger(\mathbf{x}) \left(\mathbf{w}_\Delta^i(\mathbf{x}) - \mathbf{w}_\Delta^j(\mathbf{x}) \right),$$

and

$$(E_\Delta \mathbf{w}_\Delta)^i(\mathbf{x}) = \mu^\dagger(\mathbf{x}) \left(\mathbf{w}_\Delta^i(\mathbf{x}) + \sum_{j \in \mathcal{N}_\mathbf{x}^i} \mathbf{w}_\Delta^j(\mathbf{x}) \right).$$

where $\mathcal{N}_\mathbf{x}^i$ is the set of indices of all the subdomains which share the interface point $\mathbf{x}$ with subdomain Ω^i. We note that $E_\Delta \mathbf{w}_\Delta(\mathbf{x})$ is the interface average of $\mathbf{w}_\Delta(\mathbf{x})$ and is continuous. $P_\Delta \mathbf{w}_\Delta(\mathbf{x})$ represents the jump of $\mathbf{w}_\Delta(\mathbf{x})$ across Γ and is not continuous in general.

Since $\Lambda = B_\Delta \widetilde{\mathbf{W}}_\Delta$, we know that, for any Lagrange multiplier $\lambda \in \Lambda$, there exists a function $\mathbf{w}_\Delta \in \widetilde{\mathbf{W}}_\Delta$, such that $\lambda = B_\Delta \mathbf{w}_\Delta$. The following lemma shows that we can choose a special $\mathbf{w}_\Delta \in \widetilde{\mathbf{W}}_\Delta$, such that its average $E_\Delta \mathbf{w}_\Delta \in \mathbf{W}_\Pi$ and $\lambda = B_\Delta \mathbf{w}_\Delta$.

Lemma 16 *For any Lagrange multiplier $\lambda \in \Lambda$, there exists a function $\mathbf{w}_\Delta \in \widetilde{\mathbf{W}}_\Delta$, such that $\lambda = B_\Delta \mathbf{w}_\Delta$ and the average of $\mathbf{w}_\Delta$, $E_\Delta \mathbf{w}_\Delta$, is in the primal space $\mathbf{W}_\Pi$.*

Proof: Let us denote the number of grid points on the face $\mathcal{F}_h^{ij}$, by $\#\mathcal{F}_h^{ij}$, and the number of grid points on the edge $\mathcal{E}_h^{ik}$, by $\#\mathcal{E}_h^{ik}$. We note that each face is shared by two neighboring subdomains, and each edge is shared by more than two neighboring subdomains. We use $m_{\mathcal{E}^{ik}}$ to denote the number of subdomains which share $\mathcal{E}^{ik}$.

In order to find a $\mathbf{w}_\Delta \in \widetilde{\mathbf{W}}_\Delta$, such that $\lambda = B_\Delta \mathbf{w}_\Delta$ and $E_\Delta \mathbf{w}_\Delta \in \mathbf{W}_\Pi$, we need to solve a linear system to determine the interface velocity values $\mathbf{w}_\Delta(\mathbf{x})$, $\forall \mathbf{x} \in \Gamma_h$, such that $\mathbf{w}_\Delta$ has zero integrals on each subdomain face and edge, $\lambda = B_\Delta \mathbf{w}_\Delta$, and $E_\Delta \mathbf{w}_\Delta$ is spanned by the face and edge cutoff functions $\theta_{\mathcal{F}^{ij}}$ and $\theta_{\mathcal{E}^{ik}}$. The faces contribute $6\sum_{\mathcal{F}^{ij}} \#\mathcal{F}_h^{ij}$ unknowns, and the edges $3\sum_{\mathcal{E}^{ik}} m_{\mathcal{E}^{ik}} \#\mathcal{E}_h^{ik}$, since the velocity $\mathbf{w}_\Delta$ is a vector valued function with three components.

Let us now check the number of linearly independent constraints in this linear system. The number of constraints, for the zero face and edge integrals, is $\sum_{\mathcal{F}^{ij}} 6 + 3\sum_{\mathcal{E}^{ik}} m_{\mathcal{E}^{ik}}$. In order that $E_\Delta \mathbf{w}_\Delta$ is spanned by $\theta_{\mathcal{F}^{ij}}$ and $\theta_{\mathcal{E}^{ik}}$, we need to have the same average value $E_\Delta \mathbf{w}_\Delta$ at different grid points, for each face and each edge. This will add $3\sum_{\mathcal{F}^{ij}}(\#\mathcal{F}_h^{ij} - 1) + 3\sum_{\mathcal{E}^{ik}}(\#\mathcal{E}_h^{ik} - 1)$ linearly independent constraints, where we have eliminated one grid point on each face and each edge to make these constraints independent, because we have already restricted $\mathbf{w}_\Delta$ to have zero integrals on each subdomain face and edge. The condition $\lambda = B_\Delta \mathbf{w}_\Delta$ will introduce another $3\sum_{\mathcal{F}^{ij}}(\#\mathcal{F}_h^{ij} - 1) + 3\sum_{\mathcal{E}^{ik}}(m_{\mathcal{E}^{ik}} - 1)(\#\mathcal{E}_h^{ik} - 1)$ linearly independent constraints, where we have used the fact that, at each edge point, there are $m_{\mathcal{E}^{ik}} - 1$ independent Lagrange multiplier constraints. We can see that we have in total

$$6\sum_{\mathcal{F}^{ij}} \#\mathcal{F}_h^{ij} + 3\sum_{\mathcal{E}^{ik}} m_{\mathcal{E}^{ik}} \#\mathcal{E}_h^{ik}$$

linearly independent constraints in the linear system to determine the same number of unknowns. Therefore, we can always find the $\mathbf{w}_\Delta \in \widetilde{\mathbf{W}}_\Delta$, such that $\lambda = B_\Delta \mathbf{w}_\Delta$

and $E_\Delta \mathbf{w}_\Delta \in \mathbf{W}_\Pi$.

As an extra check that these constraints are indeed linearly independent, we note that, if $\lambda = 0$, then $\mathbf{w}_\Delta$ is continuous and $E_\Delta \mathbf{w}_\Delta = \mathbf{w}_\Delta$. Since $E_\Delta \mathbf{w}_\Delta \in \mathbf{W}_\Pi$ and $\mathbf{w}_\Delta \in \widetilde{\mathbf{W}}_\Delta$, we know that $\mathbf{w}_\Delta$ is a constant on each face and edge, and the face and edge integrals equal zero. Therefore $\mathbf{w}_\Delta = 0$.

$\square$

Lemma 17 *For any $\mathbf{w}_\Delta \in \widetilde{\mathbf{W}}_\Delta$, if $E_\Delta \mathbf{w}_\Delta \in \mathbf{W}_\Pi$, then there exists a positive constant C, independent of h and H, such that $|\mathbf{w}_\Delta|_{\tilde{S}} \leq C(1 + \frac{1}{\beta_0})|P_\Delta \mathbf{w}_\Delta|_{S_\Gamma}$.*

Proof: Given $\mathbf{w}_\Delta \in \widetilde{\mathbf{W}}_\Delta$, we know from (4.6) that

$$
\begin{aligned}
|\mathbf{w}_\Delta|_{\tilde{S}} = \inf_{\mathbf{w}_\Pi \in \mathbf{W}_\Pi} & \left\{ |\mathbf{w}_\Delta + \mathbf{w}_\Pi|_{S_\Gamma} \;\middle|\; \int_{\Omega^i} \nabla \cdot (\mathbf{w}_\Delta + \mathbf{w}_\Pi)^i = 0, \;\; \forall \;\; \Omega^i \right\} \\
= \inf_{\mathbf{w}_\Pi \in \mathbf{W}_\Pi} & \left\{ |P_\Delta \mathbf{w}_\Delta + E_\Delta \mathbf{w}_\Delta + \mathbf{w}_\Pi|_{S_\Gamma} \;\middle|\; \int_{\Omega^i} \nabla \cdot (P_\Delta \mathbf{w}_\Delta + E_\Delta \mathbf{w}_\Delta + \mathbf{w}_\Pi)^i = 0 \right\} \\
= \inf_{\mathbf{w}_\Pi \in \mathbf{W}_\Pi} & \left\{ |P_\Delta \mathbf{w}_\Delta + \mathbf{w}_\Pi|_{S_\Gamma} \;\middle|\; \int_{\Omega^i} \nabla \cdot (P_\Delta \mathbf{w}_\Delta + \mathbf{w}_\Pi)^i = 0 \right\},
\end{aligned}
$$

where we have used the fact that $E_\Delta \mathbf{w}_\Delta \in \mathbf{W}_\Pi$.

Therefore,

$$
|\mathbf{w}_\Delta|_{\tilde{S}} \leq |P_\Delta \mathbf{w}_\Delta|_{S_\Gamma} + |\mathbf{w}_\Pi|_{S_\Gamma},
$$

where $\mathbf{w}_\Pi$ is determined by solving the following coarse level saddle point problem: given $P_\Delta \mathbf{w}_\Delta$, find $\mathbf{w}_\Pi \in \mathbf{W}_\Pi$ and $p_0 \in Q_0$ such that

$$
\left\{
\begin{array}{lcll}
a(\mathcal{SH}(\mathbf{w}_\Pi + P_\Delta \mathbf{w}_\Delta), \mathcal{SH}\mathbf{v}_\Pi) + b(\mathcal{SH}\mathbf{v}_\Pi, p_0) & = & 0, & \forall \mathbf{v}_\Pi \in \mathbf{W}_\Pi, \\
b(\mathcal{SH}(\mathbf{w}_\Pi + P_\Delta \mathbf{w}_\Delta), q_0) & = & 0, & \forall q_0 \in \Pi_0.
\end{array}
\right.
\tag{4.18}
$$

We know, from the inf-sup stability of this coarse level problem and Theorem 1, that

$$
|\mathbf{w}_\Pi|_{S_\Gamma} \leq \sup_{\mathbf{v}_\Pi \in \mathbf{W}_\Pi} \frac{a(\mathcal{SH}(P_\Delta \mathbf{w}_\Delta), \mathcal{SH}\mathbf{v}_\Pi)}{|\mathbf{v}_\Pi|_{S_\Gamma}} + \frac{2}{\beta_0} \sup_{q_0 \in Q_0} \frac{b(\mathcal{SH}(P_\Delta \mathbf{w}_\Delta), q_0)}{\|q_0\|_{L^2}}.
$$

Hence, from the continuity of the bilinear forms,

$$|\mathbf{w}_\Pi|_{S_\Gamma} \leq C\left(1+\frac{1}{\beta_0}\right)|P_\Delta \mathbf{w}_\Delta|_{S_\Gamma},$$

where C is a positive constant independent of h and H. Therefore, we have

$$|\mathbf{w}_\Delta|_{\tilde{S}} \leq C(1+\frac{1}{\beta_0})|P_\Delta \mathbf{w}_\Delta|_{S_\Gamma}.$$

□

The following lemma is the three-dimensional version of Lemma 15. Our proof is very similar to that of [51, Lemma 9].

Lemma 18 *For all* $\mathbf{w}_\Delta \in \widetilde{\mathbf{W}}_\Delta$, *we have,*

$$|P_\Delta \mathbf{w}_\Delta|^2_{S_\Gamma} \leq C\frac{1}{\beta^2}(1+\log(H/h))^2|\mathbf{w}_\Delta|^2_{\tilde{S}},$$

where $C>0$ *is independent of* h *and* H*.*

Proof: We consider an arbitrary $\mathbf{w}_\Delta \in \widetilde{\mathbf{W}}_\Delta$. In order to compute its $\tilde{S}-$norm, we need to determine the element $\mathbf{w}_\Gamma = \mathbf{w}_\Pi + \mathbf{w}_\Delta \in \mathbf{W}_\Gamma$, which satisfies the definition of $\tilde{S}$ in equation (4.4). We know that $|\mathbf{w}_\Delta|_{\tilde{S}} = |\mathbf{w}_\Gamma|_{S_\Gamma}$. We next note that we can subtract any continuous function from $\mathbf{w}_\Delta$ without changing the values of $B_\Delta^T D B_\Delta \mathbf{w}_\Delta$; thus, $B_\Delta^T D B_\Delta \mathbf{w}_\Gamma = B_\Delta^T D B_\Delta \mathbf{w}_\Delta$.

We introduce the notation $(\mathbf{v}_\Gamma^i)_{i=1,\ldots,N} := B_\Delta^T D B_\Delta \mathbf{w}_\Gamma$. Then, we have to estimate

$$|B_\Delta^T D B_\Delta \mathbf{w}_\Delta|^2_{S_\Gamma} = |B_\Delta^T D B_\Delta \mathbf{w}_\Gamma|^2_{S_\Gamma} = \sum_{i=1}^{N_s} |\mathbf{v}_\Gamma^i|^2_{S_\Gamma^i}.$$

We will focus on the estimate of the contribution from a single subdomain Ω^i. By noticing that $\mathbf{v}_\Gamma^i$ vanishes at the subdomain vertices, we can split the function $\mathbf{v}_\Gamma^i$

by using the cutoff functions $\theta_{\mathcal{F}^{ij}}$ and $\theta_{\mathcal{E}^{ik}}$,

$$\mathbf{v}_\Gamma^i = \sum_{\mathcal{F}^{ij}\subset\partial\Omega^i} I^h(\theta_{\mathcal{F}^{ij}}\mathbf{v}_\Gamma^i) + \sum_{\mathcal{E}^{ik}\subset\partial\Omega^i} I^h(\theta_{\mathcal{E}^{ik}}\mathbf{v}_\Gamma^i). \tag{4.19}$$

We need to estimate the S_Γ^i-norm of the function $\mathbf{v}_\Gamma^i$. From Lemma 6, we know that

$$|\mathbf{v}_\Gamma^i|_{S_\Gamma^i}^2 \leq \frac{1}{\beta^2}|\mathbf{v}_\Gamma^i|_{H^{1/2}(\partial\Omega^i)}^2,$$

therefore, we just need to estimate $|\mathbf{v}_\Gamma^i|_{H^{1/2}(\partial\Omega^i)}^2$. We have, from (4.19),

$$|\mathbf{v}_\Gamma^i|_{H^{1/2}(\partial\Omega^i)}^2 \leq C\sum_{\mathcal{F}^{ij}\subset\partial\Omega^i} \|I^h(\theta_{\mathcal{F}^{ij}}\mathbf{v}_\Gamma^i)\|_{H_{00}^{1/2}(\mathcal{F}^{ij})}^2 + C\sum_{\mathcal{E}^{ik}\subset\partial\Omega^i} |I^h(\theta_{\mathcal{E}^{ik}}\mathbf{v}_\Gamma^i)|_{H^{1/2}(\partial\Omega^i)}^2. \tag{4.20}$$

We first estimate $||I^h(\theta_{\mathcal{F}^{ij}}\mathbf{v}_\Gamma^i)||_{H_{00}^{1/2}(\mathcal{F}^{ij})}$. We note that $\mathbf{v}_\Gamma^i = \mu_{\mathcal{F}^{ij}}^\dagger(\mathbf{w}_\Gamma^i - \mathbf{w}_\Gamma^j)$ on $\mathcal{F}_h^{ij}$ and there $\mu_{\mathcal{F}^{ij}}^\dagger$ is constant, we have, by using Lemma 9,

$$\begin{aligned}
& ||I^h(\theta_{\mathcal{F}^{ij}}\mathbf{v}_\Gamma^i)||_{H_{00}^{1/2}(\mathcal{F}^{ij})} = \|I^h(\theta_{\mathcal{F}^{ij}}\mu_{\mathcal{F}^{ij}}^\dagger(\mathbf{w}_\Gamma^i - \mathbf{w}_\Gamma^j))\|_{H_{00}^{1/2}(\mathcal{F}^{ij})}^2 \\
= \ & \|I^h(\theta_{\mathcal{F}^{ij}}\mu_{\mathcal{F}^{ij}}^\dagger((\mathbf{w}_\Gamma^i - \overline{\mathbf{w}}_{\mathcal{F}^{ij}}^i) - (\mathbf{w}_\Gamma^j - \overline{\mathbf{w}}_{\mathcal{F}^{ij}}^j)))\|_{H_{00}^{1/2}(\mathcal{F}^{ij})}^2 \\
\leq \ & C\,(1+\log(H/h))^2\left(|\mathbf{w}_\Gamma^i - \mathbf{w}_\Gamma^j|_{H^{1/2}(\mathcal{F}^{ij})}^2 + \right. \\
& \left. + \frac{1}{H_i}\|(\mathbf{w}_\Gamma^i - \overline{\mathbf{w}}_{\mathcal{F}^{ij}}^i) - (\mathbf{w}_\Gamma^j - \overline{\mathbf{w}}_{\mathcal{F}^{ij}}^j)\|_{L_2(\mathcal{F}^{ij})}^2\right).
\end{aligned}$$

Using Poincaré's inequality, we have

$$||I^h(\theta_{\mathcal{F}^{ij}}\mathbf{v}_\Gamma^i)||_{H_{00}^{1/2}(\mathcal{F}^{ij})} \leq C(1+\log(H/h))^2\left(|\mathbf{w}_\Gamma^i|_{H^{1/2}(\mathcal{F}^{ij})}^2 + |\mathbf{w}_\Gamma^j|_{H^{1/2}(\mathcal{F}^{ij})}^2\right).$$

Let us now evaluate $|I^h(\theta_{\mathcal{E}^{ik}}\mathbf{v}_\Gamma^i)|_{H^{1/2}(\partial\Omega^i)}^2$, where $\mathcal{E}^{ik}$ is an edge on the subdomain boundary $\partial\Omega^i$. Let the edge $\mathcal{E}^{ik}$, e.g., be shared by the subdomains Ω^i, Ω^j, Ω^k, and Ω^l, where Ω^i shares a face with each of Ω^j and Ω^l, but only an edge with Ω^k.

From Lemma 11, we know that we just need to evaluate $|I^h(\theta_{\mathcal{E}^{ik}}\mathbf{v}_\Gamma^i)|^2_{L^2(\mathcal{E}^{ik})}$. Notice that on edge $\mathcal{E}^{ik}$,

$$\mathbf{v}_\Gamma^i = \mu^\dagger_{\mathcal{E}^{ik}}(\mathbf{w}_\Gamma^i - \mathbf{w}_\Gamma^j) + \mu^\dagger_{\mathcal{E}^{ik}}(\mathbf{w}_\Gamma^i - \mathbf{w}_\Gamma^k) + \mu^\dagger_{\mathcal{E}^{ik}}(\mathbf{w}_\Gamma^i - \mathbf{w}_\Gamma^l),$$

and that $\mu^\dagger_{\mathcal{E}^{ik}}$ is constant, therefore we just need to estimate

$$\begin{aligned} \|I^h\left(\theta_{\mathcal{E}^{ik}}(\mathbf{w}_\Gamma^i - \mathbf{w}_\Gamma^j)\right)\|^2_{L^2(\mathcal{E}^{ik})} &+ \|I^h\left(\theta_{\mathcal{E}^{ik}}(\mathbf{w}_\Gamma^i - \mathbf{w}_\Gamma^k)\right)\|^2_{L^2(\mathcal{E}^{ik})} \\ &+ \|I^h\left(\theta_{\mathcal{E}^{ik}}(\mathbf{w}_\Gamma^i - \mathbf{w}_\Gamma^l)\right)\|^2_{L^2(\mathcal{E}^{ik})}. \end{aligned}$$

The estimates for the first and the third terms can be reduced to face estimate, because Ω^i and Ω^j, as well as Ω^i and Ω^l, have a face in common. We just give the estimate for the first term here; the same procedure works for estimating the third term. We have, by using Lemma 10 and Poincaré's inequality

$$\begin{aligned} & ||I^h\left(\theta_{\mathcal{E}^{ik}}(\mathbf{w}_\Gamma^i - \mathbf{w}_\Gamma^j)\right)||^2_{L^2(\mathcal{E}^{ik})} \\ = \; & ||I^h\left(\theta_{\mathcal{E}^{ik}}(\mathbf{w}_\Gamma^i - \overline{\mathbf{w}}^i_{\mathcal{F}^{ij}})\right) - I^h\left(\theta_{\mathcal{E}^{ik}}(\mathbf{w}_\Gamma^j - \overline{\mathbf{w}}^j_{\mathcal{F}^{ij}})\right)||^2_{L^2(\mathcal{E}^{ik})} \\ \leq \; & ||I^h\left(\theta_{\mathcal{E}^{ik}}\left(\mathbf{w}_\Gamma^i - \overline{\mathbf{w}}^i_{\mathcal{F}^{ij}}\right)\right)||^2_{L^2(\mathcal{E}^{ik})} + ||I^h\left(\theta_{\mathcal{E}^{ik}}\left(\mathbf{w}_\Gamma^j - \overline{\mathbf{w}}^j_{\mathcal{F}^{ij}}\right)\right)||^2_{L^2(\mathcal{E}^{ik})} \\ \leq \; & C(1+\log(H/h))\left(\left(|\mathbf{w}_\Gamma^i|^2_{H^{1/2}(\mathcal{F}^{ij})} + \tfrac{1}{H}||\mathbf{w}_\Gamma^i - \overline{\mathbf{w}}^i_{\mathcal{F}^{ij}}||^2_{L^2(\mathcal{F}^{ij})}\right)\right. \\ & \qquad\qquad \left. + \left(|\mathbf{w}_\Gamma^j|^2_{H^{1/2}(\mathcal{F}^{kj})} + \tfrac{1}{H}||\mathbf{w}_\Gamma^j - \overline{\mathbf{w}}^j_{\mathcal{F}^{ij}}||^2_{L^2(\mathcal{F}^{ij})}\right)\right) \\ \leq \; & C(1+\log(H/h))\left(|\mathbf{w}_\Gamma^i|^2_{H^{1/2}(\mathcal{F}^{ij})} + |\mathbf{w}_\Gamma^j|^2_{H^{1/2}(\mathcal{F}^{ij})}\right). \end{aligned}$$

Let us now estimate the second term. Here, we use $|\overline{\mathbf{w}}^i_{\mathcal{E}^{ik}}|^2 \leq 1/H\|\mathbf{w}_\Gamma^i\|^2_{L_2(\mathcal{E}^{ik})}$ and $\|\theta_{\mathcal{E}^{ik}}\|^2_{L^2(\mathcal{E}^{ik})} \leq CH$. Using Lemma 10, and that $\mathbf{w}_\Gamma$ has common edge averages, i.e., $\overline{\mathbf{w}}^i_{\mathcal{E}^{ik}} = \overline{\mathbf{w}}^k_{\mathcal{E}^{ik}}$, we have

$$\begin{aligned} & ||I^h\left(\theta_{\mathcal{E}^{ik}}(\mathbf{w}_\Gamma^i - \mathbf{w}_\Gamma^k)\right)||^2_{L^2(\mathcal{E}^{ik})} \\ = \; & ||I^h\left(\theta_{\mathcal{E}^{ik}}(\mathbf{w}_\Gamma^i - \overline{\mathbf{w}}^i_{\mathcal{E}^{ik}})\right) - I^h\left(\theta_{\mathcal{E}^{ik}}(\mathbf{w}_\Gamma^k - \overline{\mathbf{w}}^k_{\mathcal{E}^{ik}})\right)||^2_{L^2(\mathcal{E}^{ik})} \\ \leq \; & ||I^h\left(\theta_{\mathcal{E}^{ik}}\left(\mathbf{w}_\Gamma^i - \overline{\mathbf{w}}^i_{\mathcal{E}^{ik}}\right)\right)||^2_{L^2(\mathcal{E}^{ik})} + ||I^h\left(\theta_{\mathcal{E}^{ik}}\left(\mathbf{w}_\Gamma^k - \overline{\mathbf{w}}^k_{\mathcal{E}^{ik}}\right)\right)||^2_{L^2(\mathcal{E}^{ik})} \end{aligned}$$

$$
\begin{aligned}
&\le C\left(\|\mathbf{w}_\Gamma^i\|^2_{L^2(\mathcal{E}^{ik})} + \|\mathbf{w}_\Gamma^k\|^2_{L^2(\mathcal{E}^{ik})}\right)\\
&\le C(1+\log(H/h))\left(\left(|\mathbf{w}_\Gamma^i|^2_{H^{1/2}(\mathcal{F}^{ij})} + \frac{1}{H}\|\mathbf{w}_\Gamma^i\|^2_{L^2(\mathcal{F}^{ij})}\right)\right.\\
&\qquad\qquad\left. + \left(|\mathbf{w}_\Gamma^k|^2_{H^{1/2}(\mathcal{F}^{kj})} + \frac{1}{H}\|\mathbf{w}_\Gamma^k\|^2_{L^2(\mathcal{F}^{kj})}\right)\right)\\
&\le C(1+\log(H/h))\left(|\mathbf{w}_\Gamma^i|^2_{H^{1/2}(\mathcal{F}^{ij})} + |\mathbf{w}_\Gamma^k|^2_{H^{1/2}(\mathcal{F}^{kj})}\right),
\end{aligned}
$$

with $\mathcal{F}^{ij}$ a face of Ω_i and $\mathcal{F}^{kj}$ a face of Ω_k, which have the edge $\mathcal{E}^{ik}$ in common. The last inequality follows from the shift invariance of the expressions on the third line, i.e., we can add constants to $\mathbf{w}_\Gamma^i$ and $\mathbf{w}_\Gamma^k$ without changing the value of the expressions and then use Poincaré's inequality. We have now bounded each term on the right side of (4.20), and we have

$$
|\mathbf{v}_\Gamma^i|^2_{H^{1/2}(\partial\Omega^i)} \le C(1+\log(H/h))^2\left(|\mathbf{w}_\Gamma^i|^2_{H^{1/2}(\partial\Omega^i)} + \sum_{j\in\mathcal{N}^i}|\mathbf{w}_\Gamma^j|^2_{H^{1/2}(\partial\Omega^j)}\right),
$$

and therefore

$$
|\mathbf{v}_\Gamma^i|^2_{S_\Gamma^i} \le C\frac{1}{\beta^2}(1+\log(H/h))^2\left(|\mathbf{w}_\Gamma^i|^2_{H^{1/2}(\partial\Omega^i)} + \sum_{j\in\mathcal{N}^i}|\mathbf{w}_\Gamma^j|^2_{H^{1/2}(\partial\Omega^j)}\right),
$$

where $\mathcal{N}^i$ is the set of indices of all the subdomains which surround the subdomain Ω^i. Then, by using Lemma 7, we have

$$
|\mathbf{v}_\Gamma^i|^2_{S_\Gamma^i} \le C\frac{1}{\beta^2}(1+\log(H/h))^2\left(|\mathbf{w}_\Gamma^i|^2_{S_\Gamma^i} + \sum_{j\in\mathcal{N}^i}|\mathbf{w}_\Gamma^j|^2_{S_\Gamma^j}\right).
$$

□

Theorem 6 *The condition number of the preconditioned linear system (4.17), in three dimensions, is bounded from above by* $C\left(1+\frac{1}{\beta_0}\right)^2\frac{1}{\beta^2}(1+\log(H/h))^2$, *where* C *is independent of* h *and* H.

Proof: We will show that

$$C_1\left(1+\frac{1}{\beta_0}\right)^{-2}\lambda^T M\lambda \le \lambda^T F\lambda \le C_2\frac{1}{\beta^2}(1+\log(H/h))^2\lambda^T M\lambda, \quad \forall\lambda\in\Lambda\ , \tag{4.21}$$

where $M^{-1} = DB_\Delta S_\Gamma B_\Delta^T D, F = B_\Delta \tilde{S}^{-1}B_\Delta^T$, and C_1, C_2 are positive constants independent of h and H.

The upper bound can be shown in the same way as in the proof of Theorem 5. Here we just give the proof of the lower bound.

We have, cf. [51] and [57],

$$\lambda^T F\lambda = \max_{0\neq\mathbf{w}_\Delta\in\widetilde{\mathbf{W}}_\Delta}\frac{|(\lambda, B_\Delta\mathbf{w}_\Delta)|^2}{|\mathbf{w}_\Delta|_{\tilde{S}}^2}\ .$$

Let $\mu\in\Lambda$ be arbitrary. It then follows from Lemma 16 that there exists a $\mathbf{w}_\Delta\in\widetilde{\mathbf{W}}_\Delta$, such that $\mu = B_\Delta\mathbf{w}_\Delta$ and $E_\Delta\mathbf{w}_\Delta\in\mathbf{W}_\Pi$. We also know from Lemma 17 that, for this $\mathbf{w}_\Delta$, $|\mathbf{w}_\Delta|_{\tilde{S}}\le C(1+\frac{1}{\beta_0})|P_\Delta\mathbf{w}_\Delta|_{S_\Gamma}$. Therefore, we have, by using the definitions of P_Δ and M^{-1}, that

$$\lambda^T F\lambda \ge \frac{|(\lambda, B_\Delta\mathbf{w}_\Delta)|^2}{|\mathbf{w}_\Delta|_{\tilde{S}}^2} \ge C(1+\frac{1}{\beta_0})^{-2}\frac{|(\lambda, B_\Delta\mathbf{w}_\Delta)|^2}{|P_\Delta\mathbf{w}_\Delta|_{S_\Gamma}^2} = C(1+\frac{1}{\beta_0})^{-2}\frac{|(\lambda,\mu)|^2}{(M^{-1}\mu,\mu)}.$$

It follows that

$$C_1\left(1+\frac{1}{\beta_0}\right)^{-2}\lambda^T M\lambda \le \lambda^T F\lambda,$$

by choosing $\mu = M\lambda$, and C_1 is independent of h and H.

□

4.6 A second three-dimensional FETI-DP algorithm

A decomposition of velocity space was given for the three-dimensional case in section 4.1. There the coarse level primal velocity space is spanned by the subdomain vertex nodal finite element basis functions $\theta_{\mathcal{V}^{il}}$ and the cutoff functions $\theta_{\mathcal{E}^{ik}}$ and $\theta_{\mathcal{F}^{ij}}$ associated with all the edges and faces of the interface Γ. A preconditioner is given in section 4.5, based on this decomposition, and the condition number bound is given in Theorem 6.

In this section, we propose a FETI-DP algorithm based on a different decomposition of the velocity space in three dimensions and prove the same condition number bound as in Theorem 6. The advantage of this second algorithm is that the size of coarse level problem is smaller and therefore the coarse level solver is less expensive.

We follow the procedure of section 4.1. The primal velocity space $\mathbf{W}_\Pi$ is now spanned by the subdomain vertex nodal finite element basis functions $\theta_{\mathcal{V}^{il}}$, and the pseudoinverse functions $\mu^\dagger_{\mathcal{F}^{ij}}$, corresponding to each subdomain face $\mathcal{F}^{ij}$. The counting function $\mu_{\mathcal{F}^{ij}}(\mathbf{x})$ equals 0 at the interface grid points outside the sets of $\mathcal{F}_h^{ij} \cup \left(\cup_{\mathcal{E}^{ik}\subset\partial\mathcal{F}^{ij}}\mathcal{E}_h^{ik}\right)$, while its value at any node on $\mathcal{F}_h^{ij}$ and its edges $\mathcal{E}_h^{ik}$ equals the number of subdomains shared by that node. $\mu^\dagger_{\mathcal{F}^{ij}}(\mathbf{x}) = 1/\mu_{\mathcal{F}^{ij}}(\mathbf{x})$ for all interface nodes where $\mu_{\mathcal{F}^{ij}}(\mathbf{x}) \neq 0$, and it vanishes at all other points. Also note that both $\mu_{\mathcal{F}^{ij}}$ and $\mu^\dagger_{\mathcal{F}^{ij}}$ vanish at the subdomain corners. $\widetilde{\mathbf{W}}_\Delta$ is the dual part of the velocity space, which is the direct sum of the local subspaces $\widetilde{\mathbf{W}}^i_\Delta$. In this three-dimensional case,

$$\widetilde{\mathbf{W}}^i_\Delta := \{\mathbf{w}^i_\Delta \in \mathbf{W}^i_\Gamma : \mathbf{w}^i_\Delta(\mathcal{V}^{il}) = 0, \overline{\mathbf{w}}^i_{\Delta,\mathcal{F}^{ij}} = 0,\ \forall \mathcal{V}^{il}, \mathcal{F}^{ij} \subset \partial\Omega^i\}. \tag{4.22}$$

We can see, from the definition of the dual velocity space $\widetilde{\mathbf{W}}_\Delta$, that we only require the face integrals across each subdomain interface Γ^{ij} to be zero, while there is no constraint on the edge integrals. Therefore the edge cutoff functions disappear from our coarse level primal velocity space $\mathbf{W}_\Pi$, and the size of the coarse level saddle point problem is reduced.

Before we give a condition number bound for this new FETI-DP algorithm, we need to discuss the inf-sup stability of the coarse level saddle point problem first. In section 4.3, we have given an inf-sup stability estimate for a pair of mixed finite element spaces, $\mathbf{W}_\Pi \times Q_0$, both in two and in three dimensions. There, in three-dimensional case, the coarse level velocity space $\mathbf{W}_\Pi$ is spanned by $\theta_{\mathcal{V}^{il}}$, $\theta_{\mathcal{E}^{ik}}$, and $\theta_{\mathcal{F}^{ij}}$, corresponding to each subdomain vertex, edge, and face. Now in this new FETI-DP algorithm, the space $\mathbf{W}_\Pi$ is reduced and it is spanned only by $\theta_{\mathcal{V}^{il}}$ and $\mu^\dagger_{\mathcal{F}^{ij}}$, while the coarse level pressure space Q_0 remains the same. The inf-sup stability of this new mixed finite elements is also proved by using a macroelement methods, like in the proof of Theorem 4. The only thing different is that, instead of using Lemma 8 in the proof, we use the following lemma, which can be found in a somewhat different form in [21].

Lemma 19 *Define an interpolation operator* $\mathcal{I}_{\Gamma\Pi} : \mathbf{W}_\Gamma \rightarrow \mathbf{W}_\Pi$ *by:*

$$\mathcal{I}_{\Gamma\Pi}\mathbf{w}_\Gamma(\mathbf{x}) = \sum_{\mathcal{V}^{il}} \mathbf{w}_\Gamma(\mathcal{V}^{il})\theta_{\mathcal{V}^{il}}(x) + \sum_{\mathcal{F}^{ik}} \overline{\mathbf{w}}_{\mathcal{F}^{ik}}\mu^\dagger_{\mathcal{F}^{ij}}(x), \quad \forall \mathbf{w}_\Gamma \in \mathbf{W}_\Gamma.$$

We then have

$$|\mathcal{I}_{\Gamma\Pi}\mathbf{w}_\Gamma|^2_{H^{1/2}(\Gamma)} \leq C(1+\log(H/h))|\mathbf{w}_\Gamma|^2_{H^{1/2}(\Gamma)},$$

and

$$\|\mathbf{w}_\Gamma - \mathcal{I}_{\Gamma\Pi}\mathbf{w}_\Gamma\|^2_{L^2(\Gamma)} \leq CH|\mathbf{w}_\Gamma|^2_{H^{1/2}(\Gamma)},$$

where C is a positive constant independent of H and h.

By using this lemma, we can choose $\mathbf{w}_\Pi = \mathcal{I}_{\Gamma\Pi}\mathbf{w}_\Gamma$ in the proof of Theorem 4, and have the same estimates as in (4.12) and (4.13). Everything else is the same, and the inf-sup stability estimate in Theorem 4 is therefore proved for this new pair of coarse level spaces $\mathbf{W}_\Pi \times Q_0$.

In order to obtain a condition number bound as in Theorem 6, we need to prove the counterparts of Lemmas 16 and 18 for the space $\widetilde{\mathbf{W}}_\Delta$ defined in (4.22). The proof of the counterpart of Lemma 16 is similar to the proof of Lemma 16 in section 4.5. The only difference is that no edge cutoff function is included in the primal velocity space $\mathbf{W}_\Pi$, and therefore there is no zero edge integral constraint in the dual velocity space $\widetilde{\mathbf{W}}_\Delta$. We can check that the number of unknowns and the number of linearly independent constraints are both $6\sum_{\mathcal{F}^{ij}} \#\mathcal{F}_h^{ij} + 3\sum_{\mathcal{E}^{ik}} m_{\mathcal{E}^{ik}}\#\mathcal{E}_h^{ik}$.

For the proof of the counterpart of Lemma 18, we follow the procedure in section 4.5 and need to bound the two terms on the right side of the inequality (4.20). The estimate of the first term, $||I^h(\theta_{\mathcal{F}^{ij}}\mathbf{v}_\Gamma^i)||_{H_{00}^{1/2}(\mathcal{F}^{ij})}$, is the same as in section 4.5. The estimate of the second term, $|I^h(\theta_{\mathcal{E}^{ik}}\mathbf{v}_\Gamma^i)|^2_{H^{1/2}(\partial\Omega^i)}$, needs to be changed. In the estimate in section 4.5, we used the fact that each function in the space $\widetilde{\mathbf{W}}_\Delta$ has common edge integrals. This constraint is not satisfied now, and here we use a technique from [52] to reduce the edge estimate to a face estimate.

As in section 4.5, we need to estimate

$$\begin{aligned} ||I^h\left(\theta_{\mathcal{E}^{ik}}(\mathbf{w}_\Gamma^i - \mathbf{w}_\Gamma^j)\right)||^2_{L^2(\mathcal{E}^{ik})} \;&+\; ||I^h\left(\theta_{\mathcal{E}^{ik}}(\mathbf{w}_\Gamma^i - \mathbf{w}_\Gamma^k)\right)||^2_{L^2(\mathcal{E}^{ik})} \\ &+\; ||I^h\left(\theta_{\mathcal{E}^{ik}}(\mathbf{w}_\Gamma^i - \mathbf{w}_\Gamma^l)\right)||^2_{L^2(\mathcal{E}^{ik})}. \end{aligned}$$

The estimates for the first and the third terms can be reduced to face estimates, in the same way as in section 4.5. The only difference is for the second term $||I^h\left(\theta_{\mathcal{E}^{ik}}(\mathbf{w}_\Gamma^i - \mathbf{w}_\Gamma^k)\right)||^2_{L_2(\mathcal{E}^{ik})}$, where the edge $\mathcal{E}^{ik}$ is shared by the subdomains Ω^i,

Ω^j, Ω^k, and Ω^l, and Ω^i shares a face with each of Ω^j and Ω^l, but only an edge with Ω^k. We find

$$\begin{aligned}
& ||I^h\left(\theta_{\mathcal{E}^{ik}}(\mathbf{w}_\Gamma^i - \mathbf{w}_\Gamma^k)\right)||^2_{L^2(\mathcal{E}^{ik})} \\
\leq\ & ||I^h\left(\theta_{\mathcal{E}^{ik}}(\mathbf{w}_\Gamma^i - \overline{\mathbf{w}}^i_{\mathcal{F}^{ij}})\right) - I^h\left(\theta_{\mathcal{E}^{ik}}(\mathbf{w}_\Gamma^j - \overline{\mathbf{w}}^j_{\mathcal{F}^{ij}})\right) \\
& \quad + I^h\left(\theta_{\mathcal{E}^{ik}}(\mathbf{w}_\Gamma^j - \overline{\mathbf{w}}^j_{\mathcal{F}^{jk}})\right) - I^h\left(\theta_{\mathcal{E}^{ik}}(\mathbf{w}_\Gamma^k - \overline{\mathbf{w}}^k_{\mathcal{F}^{jk}})\right)||^2_{L^2(\mathcal{E}^{ik})} \\
\leq\ & C\Big(||I^h\left(\theta_{\mathcal{E}^{ik}}(\mathbf{w}_\Gamma^i - \overline{\mathbf{w}}^i_{\mathcal{F}^{ij}})\right)||^2_{L^2(\mathcal{E}^{ik})} + I^h\left(\theta_{\mathcal{E}^{ik}}(\mathbf{w}_\Gamma^j - \overline{\mathbf{w}}^j_{\mathcal{F}^{ij}})\right)||^2_{L^2(\mathcal{E}^{ik})} + \\
& \quad ||I^h\left(\theta_{\mathcal{E}^{ik}}(\mathbf{w}_\Gamma^j - \overline{\mathbf{w}}^j_{\mathcal{F}^{jk}})\right)||^2_{L^2(\mathcal{E}^{ik})} + I^h\left(\theta_{\mathcal{E}^{ik}}(\mathbf{w}_\Gamma^k - \overline{\mathbf{w}}^k_{\mathcal{F}^{jk}})\right)||^2_{L^2(\mathcal{E}^{ik})}\Big).
\end{aligned}$$

It is sufficient to estimate the first term, the remaining three terms can be treated similarly. Using Lemma 10 and the Poincaré inequality, we have

$$\begin{aligned}
& \|I^h\left(\theta_{\mathcal{E}^{ik}}\left(\mathbf{w}_\Gamma^i - \overline{\mathbf{w}}^i_{\mathcal{F}^{ij}}\right)\right)\|^2_{L^2(\mathcal{E}^{ik})} \\
\leq\ & C(1+\log(H/h))\left(|\mathbf{w}_\Gamma^i|^2_{H^{1/2}(\mathcal{F}^{ij})} + \tfrac{1}{H}\|\mathbf{w}_\Gamma^i - \overline{\mathbf{w}}^i_{\mathcal{F}^{ij}}\|^2_{L^2(\mathcal{F}^{ij})}\right) \\
\leq\ & C(1+\log(H/h))|\mathbf{w}_\Gamma^i|^2_{H^{1/2}(\mathcal{F}^{ij})}.
\end{aligned}$$

Using these estimate, we have

$$\begin{aligned}
||I^h\left(\theta_{\mathcal{E}^{ik}}(\mathbf{w}_\Gamma^i - \mathbf{w}_\Gamma^k)\right)||^2_{L^2(\mathcal{E}^{ik})} \ \leq\ & C(1+\log(H/h))\Big(|\mathbf{w}_\Gamma^i|^2_{H^{1/2}(\mathcal{F}^{ij})} + |\mathbf{w}_\Gamma^j|^2_{H^{1/2}(\mathcal{F}^{ij})} \\
& + |\mathbf{w}_\Gamma^k|^2_{H^{1/2}(\mathcal{F}^{ij})}\Big).
\end{aligned}$$

We have now obtained the same bounds on the right two term of the inequality (4.20) as in section 4.5. Therefore the counterpart of Lemma 18 is proved for the space $\widetilde{\mathbf{W}}_\Delta$, defined in (4.22), and we therefore obtain the same condition number bound as in Theorem 6 for this three-dimensional FETI-DP algorithm without edge constraints.

4.7 Three-dimensional numerical experiments

We use this second FETI-DP algorithm, developed in section 4.6, to solve a three-dimensional lid driven cavity problem, described by the incompressible Stokes equations (2.2), with $\Omega = [0,1] \times [0,1] \times [0,1]$, $\mathbf{f} = \mathbf{0}$, $\mathbf{g} = (1,0,0)$ on the upper face $z = 1$, and $\mathbf{g} = \mathbf{0}$ elsewhere on the boundary. A conjugate gradient method is used to solve the preconditioned linear system (4.17), as well as the non-preconditioned linear system (4.5). The initial guess is $\lambda = 0$ and the stopping criterion is $||r_k||_2/||r_0||_2 \leq 10^{-6}$.

Figure 4.2 gives the number of CG iterations for different number of subdomains with a fixed subdomain problem size $H/h = 4$, and for different subdomain problem size H/h with 27 subdomains. We see, from the left figure, that the convergence of this FETI-DP method is independent of the number of subdomains, when the preconditioner is used. The right figure shows that the CG iteration count increases, in both the preconditioned and the non-preconditioned cases, with an increase of the subdomain problem size, but the growth is much slower with the Dirichlet preconditioner than without.

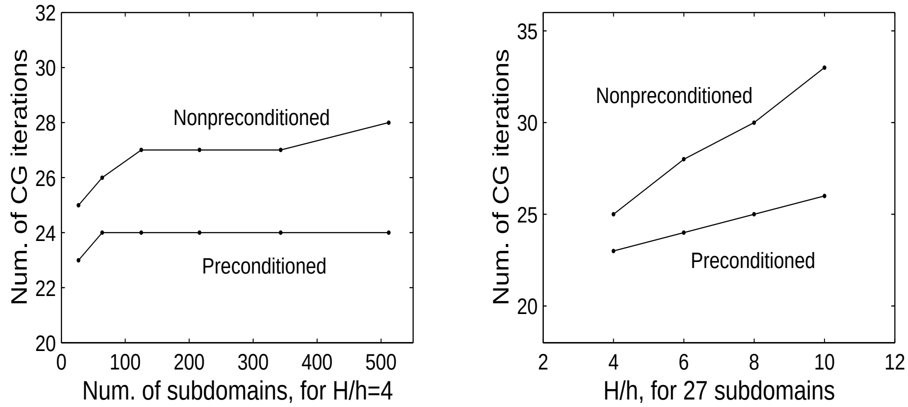

Figure 4.2: CG iteration counts for the 3D Stokes solver vs. number of subdomains for $H/h = 4$ (left) and vs. H/h for $4 \times 4 \times 4$ subdomains (right)

Chapter 5

FETI-DP algorithms for stationary Navier-Stokes equations

5.1 Introduction

In this chapter, dual-primal FETI algorithms are extended to solving stationary incompressible Navier-Stokes equations in two dimensions. For a linearized Navier-Stokes equation, the same procedure as in section 3.4 can be used to derive the FETI-DP algorithm, except that a stabilization term is added for problems with high Reynolds numbers. The preconditioned linear system is not symmetric, positive definite, and therefore a GMRES method is used to solve the preconditioned linear system for the Lagrange multipliers. We can show that, for small Reynolds number, the eigenvalues of the preconditioned operator are located in the right half plan and are bounded away from the imaginary axis. Numerical experiments show that the degree of clustering of these eigenvalues depends on the Reynolds number and the subdomain problem size, but is insensitive to the number of subdomains. For nonlinear Navier-Stokes equations, a Picard iteration is used. In each

Picard iteration, a linearized problem is solved by using the FETI-DP algorithm. Numerical experiments show that the convergence of Picard iteration depends on the Reynolds number, but is insensitive to the number of subdomains and the subdomain problem size.

5.2 A FETI-DP algorithm for linearized Navier-Stokes equations

In this section, we develop a FETI-DP algorithm for solving the following linearized incompressible Navier-Stokes equations on a bounded, polyhedral domain Ω in two dimensions,

$$\left\{\begin{array}{rcll} -\nu\Delta\mathbf{u} + (\mathbf{a}\cdot\nabla)\mathbf{u} + \nabla p &=& \mathbf{f}, & \text{in } \Omega\ , \\ -\nabla\cdot\mathbf{u} &=& 0, & \text{in } \Omega\ , \\ \mathbf{u}|_{\partial\Omega} &=& \mathbf{g}, & \text{on } \partial\Omega\ , \end{array}\right. \tag{5.1}$$

where ν is the viscosity, $\nabla\cdot\mathbf{a} = 0$, and the boundary data $\mathbf{g}$ satisfies the compatibility condition $\int_{\partial\Omega}\mathbf{g}\cdot\mathbf{n} = 0$.

The solution of equation (5.1) satisfies the following variational problem: find $\mathbf{u} \in \{\mathbf{u} \in (H^1(\Omega))^2 \mid \mathbf{u} = \mathbf{g} \text{ on } \partial\Omega\}$ and $p \in L_0^2(\Omega)$ such that,

$$\left\{\begin{array}{rcll} (\nabla\mathbf{u},\nabla\mathbf{v})_\Omega + \frac{1}{\nu}\left((\mathbf{a}\cdot\nabla)\mathbf{u},\mathbf{v}\right)_\Omega - (p,\nabla\cdot\mathbf{v})_\Omega &=& (\mathbf{f},\mathbf{v})_\Omega\,, & \forall\mathbf{v} \in (H_0^1(\Omega))^2, \\ -\left(\nabla\cdot\mathbf{u},q\right)_\Omega &=& 0, & \forall q \in L_0^2(\Omega)\ , \end{array}\right. \tag{5.2}$$

where $(\cdot,\cdot)_\Omega$ denotes the inner product in $L^2(\Omega)$, and the pressure variable p and the right side vector $\mathbf{f}$ have now been scaled by a factor of $1/\nu$. The reason that we put ν to the denominator is that we want to show that the nonsymmetric linearized Navier-Stokes problem (5.2) is a small perturbation of the symmetric incompressible Stokes problem (2.3), when ν is large.

Notice that since $\nabla \cdot \mathbf{a} = 0$ and $\mathbf{v} \in (H_0^1(\Omega))^2$, we can write the nonsymmetric bilinear term $((\mathbf{a}\cdot\nabla)\mathbf{u},\mathbf{v})_\Omega$ as a skew-symmetric term:

$$\begin{aligned}
((\mathbf{a}\cdot\nabla)\mathbf{u},\mathbf{v})_\Omega &= \int_\Omega (\mathbf{a}\cdot\nabla)\mathbf{u}\cdot\mathbf{v} \\
&= \frac{1}{2}\left(\int_\Omega (\mathbf{a}\cdot\nabla)\mathbf{u}\cdot\mathbf{v} - \int_\Omega (\mathbf{a}\cdot\nabla)\mathbf{v}\cdot\mathbf{u}\right) - \frac{1}{2}\int_\Omega (\nabla\cdot\mathbf{a})\mathbf{u}\cdot\mathbf{v} + \frac{1}{2}\int_{\partial\Omega}(\mathbf{a}\cdot\mathbf{n})\mathbf{u}\cdot\mathbf{v} \\
&= \frac{1}{2}\left(\int_\Omega (\mathbf{a}\cdot\nabla)\mathbf{u}\cdot\mathbf{v} - \int_\Omega (\mathbf{a}\cdot\nabla)\mathbf{v}\cdot\mathbf{u}\right) \\
&= \frac{1}{2}\left(((\mathbf{a}\cdot\nabla)\mathbf{u},\mathbf{v})_\Omega - ((\mathbf{a}\cdot\nabla)\mathbf{v},\mathbf{u})_\Omega\right).
\end{aligned}$$

Then the variational equation (5.2) can be written as

$$\left\{\begin{aligned}
(\nabla\mathbf{u},\nabla\mathbf{v})_\Omega + \tfrac{1}{2\nu}\left(((\mathbf{a}\cdot\nabla)\mathbf{u},\mathbf{v})_\Omega - ((\mathbf{a}\cdot\nabla)\mathbf{v},\mathbf{u})_\Omega\right) - (p,\nabla\cdot\mathbf{v})_\Omega &= (\mathbf{f},\mathbf{v})_\Omega\,, \\
-(\nabla\cdot\mathbf{u},q)_\Omega &= 0.
\end{aligned}\right. \tag{5.3}$$

We solve the variational problem (5.3) by using mixed finite element methods. The domain Ω is triangulated into shape-regular elements of characteristic size h. Again, $\mathbf{W}$ and Q are finite element subspaces of $(H^1(\Omega))^2$ and $L_0^2(\Omega)$, respectively. We have the following discrete variational problem: find the velocity $\mathbf{u} \in \mathbf{W}$, which equals $\mathbf{g}$ on $\partial\Omega$, and the pressure $p \in Q$ such that,

$$\left\{\begin{aligned}
(\nabla\mathbf{u},\nabla\mathbf{v}) + \tfrac{1}{2\nu}\left(((\mathbf{a}\cdot\nabla)\mathbf{u},\mathbf{v}) - ((\mathbf{a}\cdot\nabla)\mathbf{v},\mathbf{u})\right) - (p,\nabla\cdot\mathbf{v}) &= (\mathbf{f},\mathbf{v})\,, \quad \forall \mathbf{v}\in\mathbf{W}, \\
-(\nabla\cdot\mathbf{u},q) &= 0, \qquad\quad \forall q \in Q.
\end{aligned}\right. \tag{5.4}$$

A stabilization procedure is necessary to achieve better convergence for solving problems with large Reynolds number. We introduce an operator L to represent the differential operator for the velocity component:

$$L\mathbf{u} = -\Delta\mathbf{u} + \frac{1}{\nu}(\mathbf{a}\cdot\nabla)\mathbf{u} = L_s\mathbf{u} + L_{ss}\mathbf{u},$$

where $L_s\mathbf{u}$ and $L_{ss}\mathbf{u}$ are the symmetric and skew-symmetric parts, respectively.

A stabilized version of the discrete variational problem (5.4) is:

$$\left\{\begin{array}{rl} & (\nabla\mathbf{u},\nabla\mathbf{v})+\frac{1}{2\nu}\left(((\mathbf{a}\cdot\nabla)\mathbf{u},\mathbf{v})-((\mathbf{a}\cdot\nabla)\mathbf{v},\mathbf{u})\right) \\ & \quad +\sum_{K\in\tau_h}\delta\left(L\mathbf{u},\frac{h_K}{|\mathbf{a}|}(L_{ss}+\rho L_s)\mathbf{v}\right)_K-(p,\nabla\cdot\mathbf{v}) \\ = & (\mathbf{f},\mathbf{v})+\sum_{K\in\tau_h}\delta\left(\mathbf{f},\frac{h_K}{|\mathbf{a}|}(L_{ss}+\rho L_s)\mathbf{v}\right)_K, \quad \forall\mathbf{v}\in\mathbf{W}, \\ & \\ & -(\nabla\cdot\mathbf{u},q)=0, \quad \forall q\in Q, \end{array}\right. \tag{5.5}$$

where $\delta > 0$ and $\rho \in \mathbb{R}$ need to be chosen, cf. [68]. In our numerical experiments, we choose $\rho = 1$, which gives the Garlerkin/Least-Squares method originally developed in [44] for advection-diffusion problems. δ is chosen such that, on each element K, with diameter h_K,

$$\delta=\left\{\begin{array}{ll} \frac{\tau h_K}{2|\mathbf{a}|}, & \text{if } Pe_K\geq 1, \\ & \\ \frac{\tau h_K^2}{4\nu}, & \text{if } Pe_K<1, \end{array}\right.$$

where $Pe_K = \frac{h_K|\mathbf{a}|}{2\nu}$, $|\mathbf{a}| = |a_x|+|a_y|$. We choose $\tau = 1$ in our numerical experiments, cf. [77].

The same procedures as those used in Chapters 3 and 4 can now be used to solve the discrete variational problem (5.4) for small Reynolds number, and the stabilized discrete variational problem (5.5) for large Reynolds number. The preconditioned linear system, a nonsymmetric version of (4.14), is of the form:

$$B_\Delta S_\Gamma^N B_\Delta^T B_\Delta \tilde{S}^{N^{-1}} B_\Delta^T \lambda = B_\Delta S_\Gamma^N B_\Delta^T B_\Delta \tilde{S}^{N^{-1}} \mathbf{f}_\Delta^* \ , \tag{5.6}$$

where S_Γ^N and $\tilde{S}^N$ are defined by

$$\begin{pmatrix} A_{II}+\frac{1}{\nu}N_{II} & B_{II}^T & A_{\Delta I}^T+\frac{1}{\nu}N_{I\Delta} \\ B_{II} & 0 & B_{I\Delta} \\ A_{\Delta I}+\frac{1}{\nu}N_{\Delta I} & B_{I\Delta}^T & A_{\Delta\Delta}+\frac{1}{\nu}N_{\Delta\Delta} \end{pmatrix} \begin{pmatrix} \mathbf{u}_I^N \\ p_I^N \\ \mathbf{u}_\Delta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ S_\Gamma^N \mathbf{u}_\Delta \end{pmatrix}, \tag{5.7}$$

and

$$
\begin{pmatrix}
A_{II} + \frac{1}{\nu}N_{II} & B_{II}^T & A_{\Pi I}^T + \frac{1}{\nu}N_{I\Pi} & 0 & A_{\Delta I}^T + \frac{1}{\nu}N_{\Delta I} \\
B_{II} & 0 & B_{I\Pi} & 0 & B_{I\Delta} \\
A_{\Pi I} + \frac{1}{\nu}N_{\Pi I} & B_{I\Pi}^T & A_{\Pi\Pi} + \frac{1}{\nu}N_{\Pi\Pi} & B_{0\Pi}^T & A_{\Delta\Pi}^T + \frac{1}{\nu}N_{\Pi\Delta} \\
0 & 0 & B_{0\Pi} & 0 & B_{0\Delta} \\
A_{\Delta I} + \frac{1}{\nu}N_{\Delta I} & B_{I\Delta}^T & A_{\Delta\Pi} + \frac{1}{\nu}N_{\Delta\Pi} & B_{0\Delta}^T & A_{\Delta\Delta} + \frac{1}{\nu}N_{\Delta\Delta}
\end{pmatrix}
\begin{pmatrix}
\mathbf{u}_I^N \\ p_I^N \\ \mathbf{u}_\Pi^N \\ p_0^N \\ \mathbf{u}_\Delta
\end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \tilde{S}^N \mathbf{u}_\Delta \end{pmatrix}. \tag{5.8}
$$

Here the N matrices correspond to the discretization of the skew-symmetric term $\frac{1}{2}\left(\int_\Omega (\mathbf{a}\cdot\nabla)\mathbf{u}\cdot\mathbf{v} - \int_\Omega (\mathbf{a}\cdot\nabla)\mathbf{v}\cdot\mathbf{u}\right)$. The stabilization terms are not included here.

The conjugate gradient method cannot be used to solve the preconditioned linear system (5.6), because this problem is no longer symmetric, positive definite; instead we use GMRES.

5.3 Numerical experiments

In this section, we give some numerical experiments for solving the linearized incompressible Navier-Stokes Stokes equation (5.1) in a two-dimensional domain $\Omega = [0,1]\times[0,1]$, where $\mathbf{f} = \mathbf{0}$, the boundary condition $\mathbf{g} = (1,0)$ on the upper side $y = 1$, and $\mathbf{g} = \mathbf{0}$ on the three other sides. The convection coefficient is chosen as

$$
\mathbf{a} = \begin{pmatrix} 2(2y-1)(1-(2x-1)^2) \\ \\ -2(2x-1)(1-(2y-1)^2) \end{pmatrix};
$$

cf. [25].

We use GMRES method to solve the preconditioned linear system (5.6), with an initial guess $\lambda = 0$. The stopping criterion is $||r_k||_2/||r_0||_2 \leq 10^{-6}$, where r_k is the residual of the Lagrange multipliers at the kth iteration. A stabilization procedure is used as in section 5.2.

Figure 5.1 shows the convergence of GMRES method for solving the preconditioned linear system (5.6), on two different meshes with different Reynolds numbers. The left one is for 8×8 subdomains and $H/h = 10$; the right one is for 10×10 subdomains and $H/h = 16$. We can see that, on both meshes, the convergence of GMRES method depends on the Reynolds number: the larger the Reynolds number, the slower the convergence.

Figure 5.2 concerns the scalability of GMRES. The left one shows the change of GMRES iteration count with the number of subdomains, for different cases. We see that for small Reynolds number, the convergence is independent of the number of subdomains. For large Reynolds number the convergence becomes even better when we increase the number of subdomain. The right one shows that the convergence of the GMRES method depends on the subdomain problem size. The number of GMRES iterations appears to grow like a logarithmic function of H/h.

5.4 Eigenvalue estimates of the preconditioned linear system

We know, from Theorem 3, that the convergence of the GMRES iteration for solving a linear system, $Ax = b$, depends on $\kappa(V)$ and $\inf_{p_n \in P_n} ||p_n||_{\Lambda(A)}$. Here V is a nonsingular matrix of eigenvectors (assuming A is diagonalizable), $\Lambda(A)$ is the set of eigenvalues of A, P_n is the set of polynomials p of degree $\leq n$ with $p(0) = 1$,

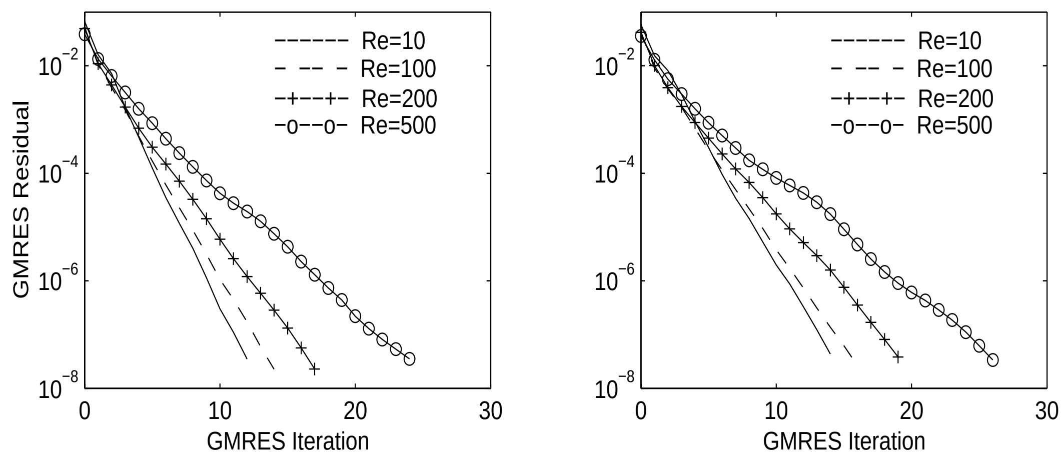

Figure 5.1: Dependence of the convergence of GMRES on the Reynolds number

and $\|p_n\|_{\Lambda(A)} = \sup_{z \in \Lambda(A)} |p_n(z)|$.

In the linearized Navier-Stokes case, we can see that $\kappa(V)$ depends on the Reynolds number. When the Reynolds number is large, the preconditioned operator $B_\Delta S_\Gamma^N B_\Delta^T B_\Delta \tilde{S}^{N^{-1}} B_\Delta^T$ is very far from normal, and therefore $\kappa(V)$ is large, and we have slow convergence. When the Reynolds number is not large, the preconditioned operator is not too far from normal, and therefore $\kappa(V)$ is small, and we will have fast convergence.

The value of $\inf_{p_n \in P_n} \|p_n\|_{\Lambda(A)}$ depends on the distribution of the eigenvalues of the preconditioned operator. When the spectrum is tightly clustered away from the imaginary axis, we expect that $\inf_{p_n \in P_n} \|p_n\|_{\Lambda(A)}$ will decrease quickly with n increasing.

In this section, we will show that the spectrum of the preconditioned operator $B_\Delta S_\Gamma^N B_\Delta^T B_\Delta \tilde{S}^{N^{-1}} B_\Delta^T$, without stabilization, is a perturbation of the spectrum of $B_\Delta S_\Gamma B_\Delta^T B_\Delta \tilde{S}^{-1} B_\Delta^T$ in the Stokes case, when the Reynolds number is not large;

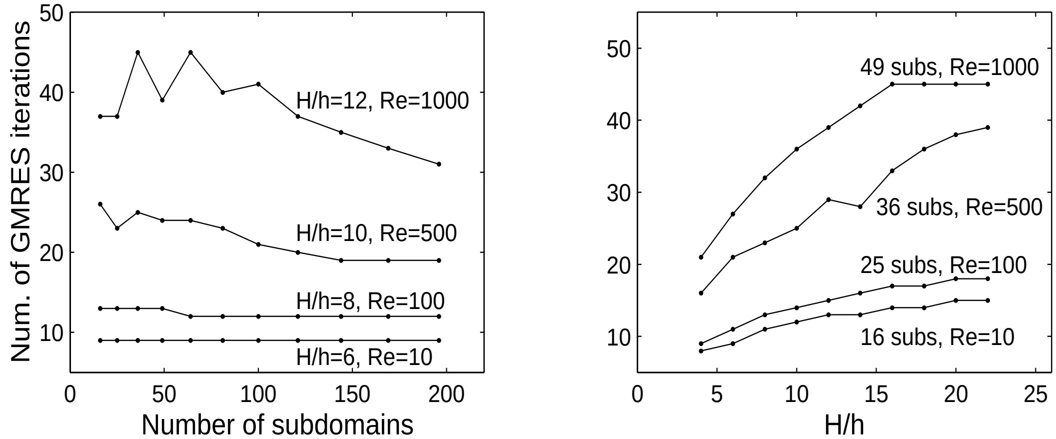

Figure 5.2: Scalability of our FETI-DP algorithm

this work has been inspired by [26].

It should be pointed out that, this section only explains the convergence behavior of the GMRES iterations for solving equation (5.6), but does not give a complete estimate of the rate of convergence.

In section 4.4, we have proved that for two-dimensional Stokes problems, the condition number of the preconditioned operator $B_\Delta S_\Gamma B_\Delta^T B_\Delta \tilde{S}^{-1} B_\Delta^T$ is bounded from above by $C\frac{1}{\beta^2}(1+log\frac{H}{h})^2$. We now have the following lemma:

Lemma 20 *For any* $\mathbf{w}_\Delta \in B_\Delta^T B_\Delta \widetilde{\mathbf{W}}_\Delta$*, we have*

$$\gamma \leq \frac{\mathbf{w}_\Delta^T B_\Delta^T B_\Delta S_\Gamma B_\Delta^T B_\Delta \mathbf{w}_\Delta}{\mathbf{w}_\Delta^T \tilde{S} \mathbf{w}_\Delta} \leq \Gamma,$$

where $\gamma = 1$ *and* $\Gamma = C\frac{1}{\beta^2}(1+log\frac{H}{h})^2$.

Proof: We know, from the inequality (4.21), that

$$\lambda^T\lambda \leq \lambda^T B_\Delta S_\Gamma B_\Delta^T B_\Delta \tilde{S}^{-1} B_\Delta^T \lambda \leq C\frac{1}{\beta^2}(1+log\frac{H}{h})^2\lambda^T\lambda, \quad \forall \lambda \in \Lambda\ .$$

Noticing that $B_\Delta B_\Delta^T = 2I$, we have

$$\begin{aligned}\lambda^T B_\Delta B_\Delta^T \lambda &\leq \lambda^T B_\Delta B_\Delta^T B_\Delta S_\Gamma B_\Delta^T B_\Delta \tilde{S}^{-1} B_\Delta^T \lambda \\ &\leq C\frac{1}{\beta^2}(1+log\frac{H}{h})^2 \lambda^T B_\Delta B_\Delta^T \lambda.\end{aligned}$$

For any $\mathbf{w}_\Delta \in B_\Delta^T B_\Delta \widetilde{\mathbf{W}}_\Delta$, there is always a $\lambda \in \Lambda$, such that $\mathbf{w}_\Delta = B_\Delta^T \lambda$. Therefore

$$\mathbf{w}_\Delta^T \mathbf{w}_\Delta \leq \mathbf{w}_\Delta^T B_\Delta^T B_\Delta S_\Gamma B_\Delta^T B_\Delta \tilde{S}^{-1} \mathbf{w}_\Delta \leq C\frac{1}{\beta^2}(1+log\frac{H}{h})^2 \mathbf{w}_\Delta^T \mathbf{w}_\Delta,$$

and we therefore know that the eigenvalues of the operator $B_\Delta^T B_\Delta S_\Gamma B_\Delta^T B_\Delta \tilde{S}^{-1}$ are located in the interval $[\gamma, \Gamma]$.

□

In the remainder of this section, we prove the following theorem, which gives a bound on the Rayleigh quotient corresponding to the nonsymmetric operator $B_\Delta^T B_\Delta S_\Gamma^N B_\Delta^T B_\Delta \tilde{S}^{N^{-1}}$.

Theorem 7 *For any $\mathbf{w}_\Delta \in B_\Delta^T B_\Delta \widetilde{\mathbf{W}}_\Delta$, we have*

$$\left(1+O(\nu^{-1})\right)\gamma \leq \frac{\mathbf{w}_\Delta^T B_\Delta^T B_\Delta S_\Gamma^N B_\Delta^T B_\Delta \mathbf{w}_\Delta}{\mathbf{w}_\Delta^T \tilde{S}^N \mathbf{w}_\Delta} \leq \left(1+O(\nu^{-1})\right)\Gamma,$$

where γ and Γ are given in Lemma 20.

Let $\mathbf{u}_\Delta = B_\Delta^T B_\Delta \mathbf{w}_\Delta$, and we have

$$\frac{\mathbf{w}_\Delta^T B_\Delta^T B_\Delta S_\Gamma^N B_\Delta^T B_\Delta \mathbf{w}_\Delta}{\mathbf{w}_\Delta^T \tilde{S}^N \mathbf{w}_\Delta} = \frac{\mathbf{u}_\Delta^T S_\Gamma^N \mathbf{u}_\Delta}{\mathbf{w}_\Delta^T \tilde{S}^N \mathbf{w}_\Delta} = \frac{\frac{\mathbf{u}_\Delta^T S_\Gamma^N \mathbf{u}_\Delta}{\mathbf{u}_\Delta^T S_\Gamma \mathbf{u}_\Delta} \cdot \frac{\mathbf{u}_\Delta^T S_\Gamma \mathbf{u}_\Delta}{\mathbf{w}_\Delta^T \tilde{S} \mathbf{w}_\Delta}}{\frac{\mathbf{w}_\Delta^T \tilde{S}^N \mathbf{w}_\Delta}{\mathbf{w}_\Delta^T \tilde{S} \mathbf{w}_\Delta}}.$$

We know, from Lemma 20, that

$$\gamma \leq \frac{\mathbf{u}_\Delta^T S_\Gamma \mathbf{u}_\Delta}{\mathbf{w}_\Delta^T \tilde{S} \mathbf{w}_\Delta} \leq \Gamma,$$

therefore we just need to show that both $\frac{\mathbf{u}_\Delta^T S_\Gamma^N \mathbf{u}_\Delta}{\mathbf{u}_\Delta^T S_\Gamma \mathbf{u}_\Delta}$ and $\frac{\mathbf{w}_\Delta^T \tilde{S}^N \mathbf{w}_\Delta}{\mathbf{w}_\Delta^T \tilde{S} \mathbf{w}_\Delta}$ are of the order $1 + O(\nu^{-1})$. Here we just give the estimate of $\frac{\mathbf{u}_\Delta^T S_\Gamma^N \mathbf{u}_\Delta}{\mathbf{u}_\Delta^T S_\Gamma \mathbf{u}_\Delta}$. The same argument works for $\frac{\mathbf{w}_\Delta^T \tilde{S}^N \mathbf{w}_\Delta}{\mathbf{w}_\Delta^T \tilde{S} \mathbf{w}_\Delta}$.

Recall, from the definition of S_Γ^N in equation (5.7), that

$$\mathbf{u}_\Delta^T S_\Gamma^N \mathbf{u}_\Delta = (\mathbf{u}_I^{N^T} \mathbf{u}_\Delta^T) \begin{pmatrix} A_{II} + \frac{1}{\nu} N_{II} & A_{\Delta I}^T + \frac{1}{\nu} N_{I\Delta} \\ A_{\Delta I} + \frac{1}{\nu} N_{\Delta I} & A_{\Delta\Delta} + \frac{1}{\nu} N_{\Delta\Delta} \end{pmatrix} \begin{pmatrix} \mathbf{u}_I^N \\ \mathbf{u}_\Delta \end{pmatrix}$$

$$\begin{aligned} &= (\mathbf{u}_I^{N^T} \mathbf{u}_\Delta^T) \begin{pmatrix} A_{II} & A_{\Delta I}^T \\ A_{\Delta I} & A_{\Delta\Delta} \end{pmatrix} \begin{pmatrix} \mathbf{u}_I^N \\ \mathbf{u}_\Delta \end{pmatrix} + \tfrac{1}{\nu} (\mathbf{u}_I^{N^T} \mathbf{u}_\Delta^T) \begin{pmatrix} N_{II} & N_{I\Delta} \\ N_{\Delta I} & N_{\Delta\Delta} \end{pmatrix} \begin{pmatrix} \mathbf{u}_I^N \\ \mathbf{u}_\Delta \end{pmatrix} \\ &= (\mathbf{u}_I^{N^T} \mathbf{u}_\Delta^T) \begin{pmatrix} A_{II} & A_{\Delta I}^T \\ A_{\Delta I} & A_{\Delta\Delta} \end{pmatrix} \begin{pmatrix} \mathbf{u}_I^N \\ \mathbf{u}_\Delta \end{pmatrix} \\ &\quad + \tfrac{1}{\nu} (\mathbf{u}_I^{N^T} \mathbf{u}_\Delta^T) \begin{pmatrix} A_{II} & A_{\Delta I}^T \\ A_{\Delta I} & A_{\Delta\Delta} \end{pmatrix}^{1/2} \tilde{N}_{I,\Delta} \begin{pmatrix} A_{II} & A_{\Delta I}^T \\ A_{\Delta I} & A_{\Delta\Delta} \end{pmatrix}^{1/2} \begin{pmatrix} \mathbf{u}_I^N \\ \mathbf{u}_\Delta \end{pmatrix}, \end{aligned}$$

where

$$\tilde{N}_{I,\Delta} = \begin{pmatrix} A_{II} & A_{\Delta I}^T \\ A_{\Delta I} & A_{\Delta\Delta} \end{pmatrix}^{-1/2} \begin{pmatrix} N_{II} & N_{I\Delta} \\ N_{\Delta I} & N_{\Delta\Delta} \end{pmatrix} \begin{pmatrix} A_{II} & A_{\Delta I}^T \\ A_{\Delta I} & A_{\Delta\Delta} \end{pmatrix}^{-1/2}.$$

We know from [23] that the spectral radius $\rho(\tilde{N}_{I,\Delta}) = O(1)$, therefore

$$\mathbf{u}_\Delta^T S_\Gamma^N \mathbf{u}_\Delta = \left(1 + O(\nu^{-1})\right) (\mathbf{u}_I^{N^T} \mathbf{u}_\Delta^T) \begin{pmatrix} A_{II} & A_{I\Delta} \\ A_{\Delta I} & A_{\Delta\Delta} \end{pmatrix} \begin{pmatrix} \mathbf{u}_I^N \\ \mathbf{u}_\Delta \end{pmatrix}.$$

At the same time, we know that

$$\mathbf{u}_\Delta^T S_\Gamma \mathbf{u}_\Delta = (\mathbf{u}_I^T \ \ \mathbf{u}_\Delta^T) \begin{pmatrix} A_{II} & A_{\Delta I}^T \\ A_{\Delta I} & A_{\Delta\Delta} \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ \mathbf{u}_\Delta \end{pmatrix}.$$

Therefore, in order to show that $\mathbf{u}_\Delta^T S_\Gamma^N \mathbf{u}_\Delta = (1 + O(\nu^{-1}))\, \mathbf{u}_\Delta^T S_\Gamma \mathbf{u}_\Delta$, we just need to show

$$\begin{aligned} &(\mathbf{u}_I^{N^T} \ \ \mathbf{u}_\Delta^T) \begin{pmatrix} A_{II} & A_{\Delta I}^T \\ A_{\Delta I} & A_{\Delta\Delta} \end{pmatrix} \begin{pmatrix} \mathbf{u}_I^N \\ \mathbf{u}_\Delta \end{pmatrix} \\ = \ &(1 + O(\nu^{-1}))\, (\mathbf{u}_I^T \ \ \mathbf{u}_\Delta^T) \begin{pmatrix} A_{II} & A_{\Delta I}^T \\ A_{\Delta I} & A_{\Delta\Delta} \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ \mathbf{u}_\Delta \end{pmatrix}, \end{aligned} \tag{5.9}$$

where $\mathbf{u}_I^N$ satisfies the following linear system, cf. (5.7),

$$\begin{pmatrix} A_{II} + \frac{1}{\nu} N_{II} & B_{II}^T \\ B_{II} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_I^N \\ p_I^N \end{pmatrix} = - \begin{pmatrix} A_{\Delta I}^T + \frac{1}{\nu} N_{I\Delta} \\ B_{I\Delta} \end{pmatrix} \mathbf{u}_\Delta, \tag{5.10}$$

and $\mathbf{u}_I$ satisfies the following linear system, cf. (2.14),

$$\begin{pmatrix} A_{II} & B_{II}^T \\ B_{II} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ p_I \end{pmatrix} = - \begin{pmatrix} A_{\Delta I}^T \\ B_{I\Delta} \end{pmatrix} \mathbf{u}_\Delta. \tag{5.11}$$

We know, from equations (5.10) and (5.11), that

$$\begin{pmatrix} A_{II} + \frac{1}{\nu} N_{II} & B_{II}^T \\ B_{II} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_I^N \\ p_I^N \end{pmatrix} = \begin{pmatrix} A_{II} & B_{II}^T \\ B_{II} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ p_I \end{pmatrix} - \begin{pmatrix} \frac{1}{\nu} N_{I\Delta} \mathbf{u}_\Delta \\ 0 \end{pmatrix},$$

which can be written as

$$\begin{pmatrix} A_{II} + \frac{1}{\nu} N_{II} & B_{II}^T \\ B_{II} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_I^N \\ p_I^N - p_I \end{pmatrix} = \begin{pmatrix} A_{II} & B_{II}^T \\ B_{II} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{1}{\nu} N_{I\Delta} \mathbf{u}_\Delta \\ 0 \end{pmatrix}. \tag{5.12}$$

We know, from (5.12), that

$$\begin{aligned} & p_I^N - p_I^\nu = \\ & - \left(B_{II}(A_{II} + \tfrac{1}{\nu} N_{II})^{-1} B_{II}^T\right)^{-1} \left(B_{II}\mathbf{u}_I - B_{II}(A_{II} + \tfrac{1}{\nu} N_{II})^{-1}(A_{II}\mathbf{u}_I - \tfrac{1}{\nu} N_{I\Delta}\mathbf{u}_\Delta)\right). \end{aligned}$$

We use the next lemma, cf. [26], to compute $\left(A_{II} + \frac{1}{\nu} N_{II}\right)^{-1}$.

Lemma 21

$$\begin{aligned} \left(A_{II} + \tfrac{1}{\nu} N_{II}\right)^{-1} &= A_{II}^{-1/2} \left(I - \tfrac{1}{\nu} \hat{N}_{II} + \tfrac{1}{\nu^2} E\right) A_{II}^{-1/2} \\ &= \left(A_{II}^{-1} - \tfrac{1}{\nu} A_{II}^{-1/2} \hat{N}_{II} A_{II}^{-1/2} + \tfrac{1}{\nu^2} A_{II}^{-1/2} E A_{II}^{-1/2}\right), \end{aligned}$$

where $\hat{N}_{II} = A_{II}^{-1/2} N_{II} A_{II}^{-1/2}$, $E = \hat{N}_{II}^2 \left(I + \frac{1}{\nu} \hat{N}_{II}\right)^{-1}$, *and the spectral radii* $\rho(\hat{N}_{II})$ *and* $\rho(E)$ *are of the order of* $O(1)$.

Using Lemma 21, we have

$$p_I^N - p_I = -\left(B_{II}\left(A_{II}^{-1} - \tfrac{1}{\nu}A_{II}^{-1/2}\hat{N}_{II}A_{II}^{-1/2} + \tfrac{1}{\nu^2}A_{II}^{-1/2}EA_{II}^{-1/2}\right)B_{II}^T\right)^{-1}$$
$$\left(B_{II}\mathbf{u}_I - B_{II}\left(A_{II}^{-1} - \tfrac{1}{\nu}A_{II}^{-1/2}\hat{N}_{II}A_{II}^{-1/2} + \tfrac{1}{\nu^2}A_{II}^{-1/2}EA_{II}^{-1/2}\right)\left(A_{II}\mathbf{u}_I - \tfrac{1}{\nu}N_{I\Delta}\mathbf{u}_\Delta\right)\right).$$

After dropping the $O(\frac{1}{\nu^2})$ term, we have

$$\begin{aligned} p_I^N - p_I &= -\left(B_{II}\left(A_{II}^{-1} - \tfrac{1}{\nu}A_{II}^{-1/2}\hat{N}_{II}A_{II}^{-1/2}\right)B_{II}^T\right)^{-1} \\ &\qquad \left(B_{II}\mathbf{u}_I - B_{II}\left(A_{II}^{-1} - \tfrac{1}{\nu}A_{II}^{-1/2}\hat{N}_{II}A_{II}^{-1/2}\right)\left(A_{II}\mathbf{u}_I - \tfrac{1}{\nu}N_{I\Delta}\mathbf{u}_\Delta\right)\right). \end{aligned}$$

Then use Lemma 21 again and drop the terms of order $O(\frac{1}{\nu^2})$. We have

$$p_I^N - p_I = -\tfrac{1}{\nu}\left(S_\infty^{-1} + \tfrac{1}{\nu}S_\infty^{-1/2}\hat{M}_{II}S_\infty^{-1/2}\right)B_{II}\left(A_{II}^{-1}N_{I\Delta}\mathbf{u}_\Delta + A_{II}^{-1/2}\hat{N}_{II}A_{II}^{1/2}\mathbf{u}_I\right),$$

where $S_\infty = B_{II}A_{II}^{-1}B_{II}^T$, and $\hat{M}_{II}$ is a matrix with spectral radius of $O(1)$.

From equation (5.12), we know that

$$\begin{aligned} \mathbf{u}_I^N &= \left(A_{II} + \tfrac{1}{\nu}N_{II}\right)^{-1}\left(A_{II}\mathbf{u}_I - \tfrac{1}{\nu}N_{I\Delta}\mathbf{u}_\Delta - B_{II}^T(p_I^N - p_I)\right) \\ &= \left(A_{II}^{-1} - \tfrac{1}{\nu}A_{II}^{-1/2}\hat{N}_{II}A_{II}^{-1/2}\right)\Big(A_{II}\mathbf{u}_I - \tfrac{1}{\nu}N_{I\Delta}\mathbf{u}_\Delta \\ &\qquad + \tfrac{1}{\nu}B_{II}^T\left(S_\infty^{-1} + \tfrac{1}{\nu}S_\infty^{-1/2}\hat{M}_{II}S_\infty^{-1/2}\right)B_{II}\left(A_{II}^{-1}N_{I\Delta}\mathbf{u}_\Delta + A_{II}^{-1/2}\hat{N}_{II}A_{II}^{1/2}\mathbf{u}_I\right)\Big) \\ &= \left(A_{II}^{-1} - \tfrac{1}{\nu}A_{II}^{-1/2}\hat{N}_{II}A_{II}^{-1/2}\right)\Big(A_{II}\mathbf{u}_I - \tfrac{1}{\nu}N_{I\Delta}\mathbf{u}_\Delta \\ &\qquad + \tfrac{1}{\nu}B_{II}^TS_\infty^{-1}B_{II}\left(A_{II}^{-1}N_{I\Delta}\mathbf{u}_\Delta + A_{II}^{-1/2}\hat{N}_{II}A_{II}^{1/2}\mathbf{u}_I\right)\Big) \\ &= \mathbf{u}_I - \tfrac{1}{\nu}\left(A_{II}^{-1} - A_{II}^{-1}B_{II}^TS_\infty^{-1}B_{II}A_{II}^{-1}\right)\left(N_{I\Delta}\mathbf{u}_\Delta + N_{II}\mathbf{u}_I\right). \end{aligned}$$

We have the following lemma:

Lemma 22 *The function* $\mathbf{u}_I^N$ *and* $\mathbf{u}_I$ *in equation (5.9) satisfy*

$$\mathbf{u}_I^N = \mathbf{u}_I - \tilde{\mathbf{u}}_I, \tag{5.13}$$

where

$$\tilde{\mathbf{u}}_I = -\frac{1}{\nu}A_{II}^{-1/2}\left(I - A_{II}^{-1/2}B_{II}^TS_\infty^{-1}B_{II}A_{II}^{-1/2}\right)A_{II}^{-1/2}\left(N_{I\Delta}\mathbf{u}_\Delta + N_{II}\mathbf{u}_I\right). \tag{5.14}$$

By using Lemma 22, Equation (5.9) can be easily proved, and so is the Theorem 7.

In the following, we describe some numerical experiments concerning the eigenvalues of the operator $B_\Delta S_\Delta^N B_\Delta^T B_\Delta \tilde{S}^{N^{-1}} B_\Delta^T$.

In Figure 5.3, we plot the eigenvalues of the preconditioned operator for different number of subdomains, with $H/h = 8$ and $\nu = 1/80$. We can see that the regions of eigenvalues are almost the same in the different cases. In Figure 5.4, we plot the eigenvalues of this preconditioned operator for different H/h, with 4×4 subdomains and $\nu = 1/80$. We see that the eigenvalue region expands when H/h increases. In Figure 5.5, we plot the eigenvalues of this preconditioned operator for different Reynolds number, with 4×4 subdomains and $H/h = 8$. We see that with an increase of the Reynolds number, the eigenvalues become more scattered, which is consistent with the estimate in Theorem 7. This explains the fact that the convergence of the GMRES method to solve (5.6) slows when the Reynolds number is increased.

5.5 Picard iterations for Navier-Stokes equations

In this section, we give some numerical results for the following incompressible Navier-Stokes equations solved by using Picard iteration.

We solve the nonlinear lid-driven-cavity problem:

$$\left\{\begin{array}{rcll} -\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p &=& \mathbf{f}, & \text{in } \Omega \ , \\ -\nabla\cdot\mathbf{u} &=& 0, & \text{in } \Omega \ , \\ \mathbf{u} &=& \mathbf{g}, & \text{on } \partial\Omega \ , \end{array}\right. \tag{5.15}$$

where $\Omega = [0,1]\times[0,1]$, $\mathbf{f} = \mathbf{0}$, the boundary condition $\mathbf{g} = (1,0)$ on the upper side

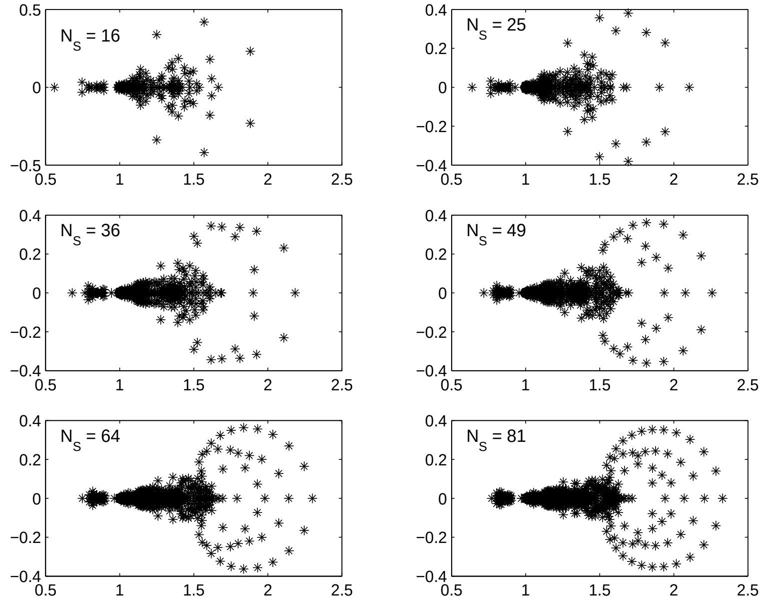

Figure 5.3: Plots of eigenvalues for different number of subdomains

$y = 1$, and $\mathbf{g} = \mathbf{0}$ on the three other sides. In each Picard iteration, a linearized Navier-Stokes problem:

$$\left\{ \begin{array}{rcl} -\nu\Delta\mathbf{u}^{n+1} + (\mathbf{u}^n \cdot \nabla)\mathbf{u}^{n+1} + \nabla p^{n+1} & = & \mathbf{f}, \\ -\nabla \cdot \mathbf{u}^{n+1} & = & 0, \\ \mathbf{u}^{n+1}|_{\partial\Omega} & = & \mathbf{g}. \end{array} \right. \tag{5.16}$$

is solved by using the dual-primal FETI algorithm developed in section 5.2.

In our experiments, we start from a zero initial guess $\mathbf{u}^0 = 0$, and the Picard iteration is stopped when the nonlinear residual is reduced by 10^{-6}. The residual of the Lagrange multipliers is reduced by 10^{-4}, in the GMRES solver, in each Picard iteration step when solving the linear equation (5.16).

Figure 5.6 shows that the convergence of the Picard iteration depends on the Reynolds number: the larger the Reynolds number, the slower the convergence. We

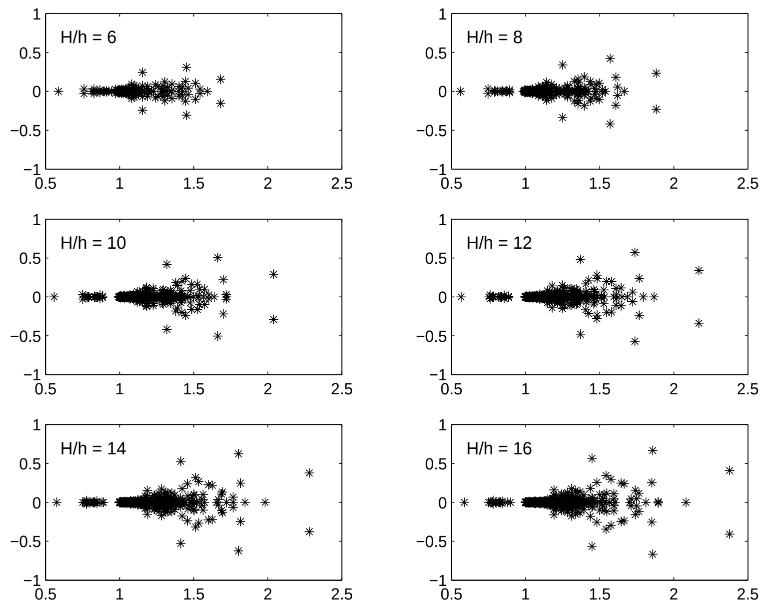

Figure 5.4: Plots of eigenvalues for different subdomain problem sizes

used two different mesh sizes: one is for 8×8 subdomains and $H/h = 10$; the other is for 10×10 subdomains and $H/h = 16$. Figure 5.7 indicates that the convergence of the Picard iteration is independent of the mesh size. From the left figure, we can see that the convergence is independent of the number of subdomains; from the right one, we can see that the convergence in also independent of the subdomain problem size, where for large Reynolds number, the mesh has to be fine enough to make the Picard iteration counts comparable.

We also check the dependence of GMRES convergence on the Reynolds number and on the mesh size, in each Picard iteration step. From the left graph in Figure 5.8, we can see that the convergence of GMRES iterations depends on the Reynolds number; from the right graph, we can see that the convergence of GMRES iterations is independent of the number of subdomains. Figure 5.9 shows that

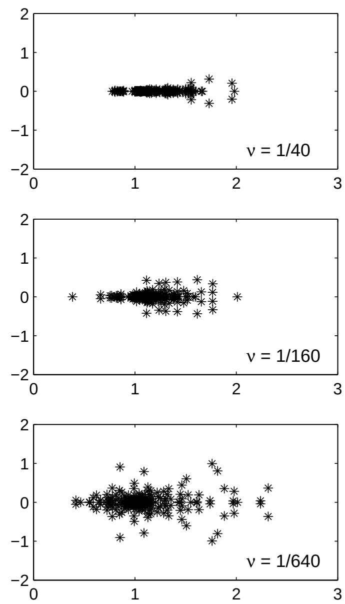

Figure 5.5: Plots of eigenvalues for different Reynolds numbers

the GMRES convergence depends on the subdomain problem size. In Figure 5.8 and Figure 5.9, the Y-axis is the average GMRES iteration count to reduce the residual by 10^{-4} in each Picard iteration.

Figure 5.10 gives plots of the numerical solutions of the nonlinear lid-driven-cavity problem (5.15) for viscosity $\nu = \frac{1}{5000}$, for these two mesh sizes. Figure 5.11 gives plots of the numerical solutions of the nonlinear lid-driven-cavity problem (5.15) when the viscosity $\nu = \frac{1}{10000}$, for 10×10 subdomains and $H/h = 16$.

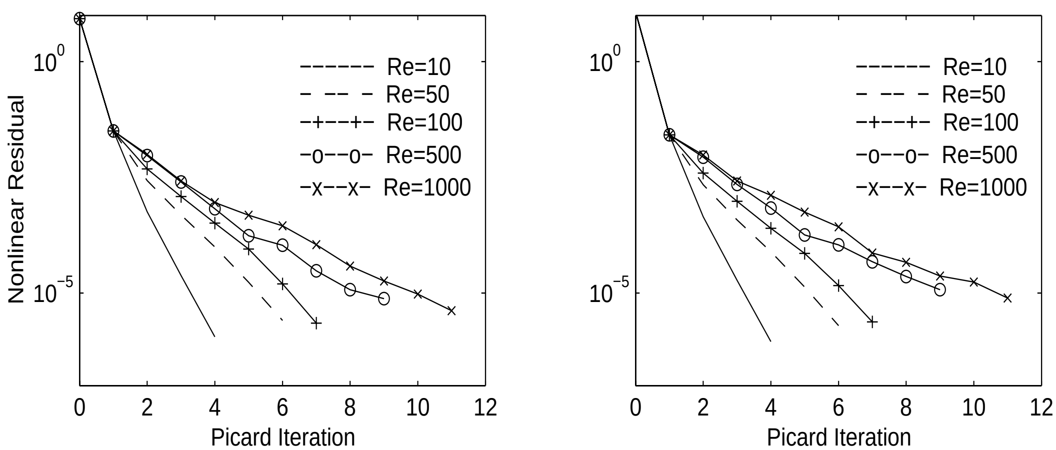

Figure 5.6: Dependence of the convergence of the Picard iteration on the Reynolds number

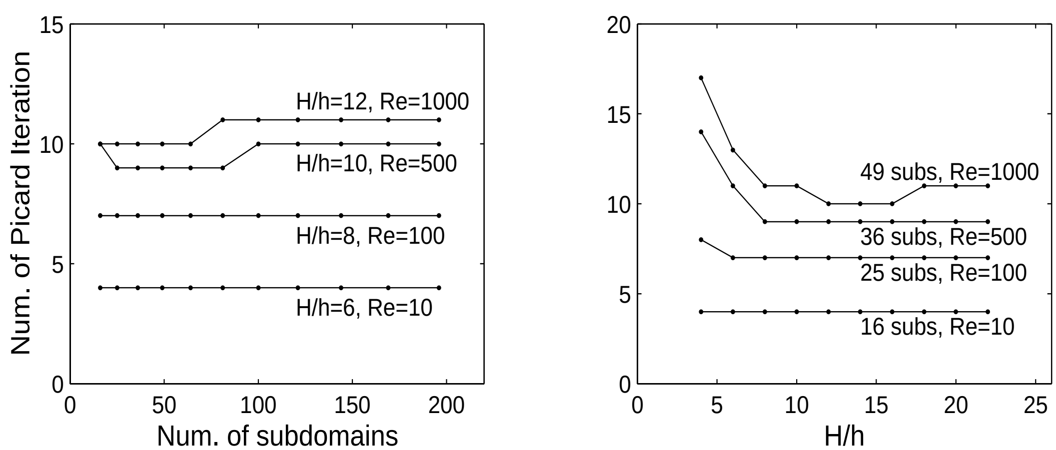

Figure 5.7: Scalability of the Picard iteration

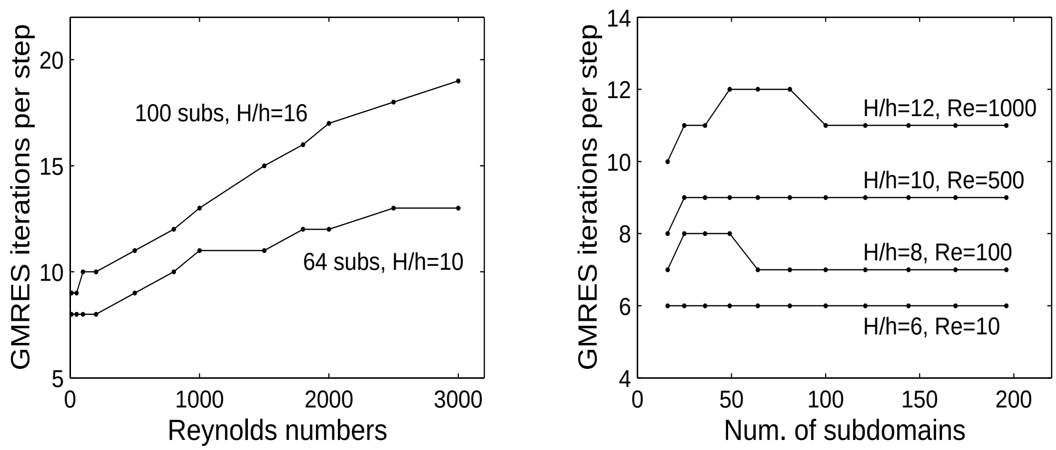

Figure 5.8: Dependence of the convergence of GMRES solver on the Reynolds number

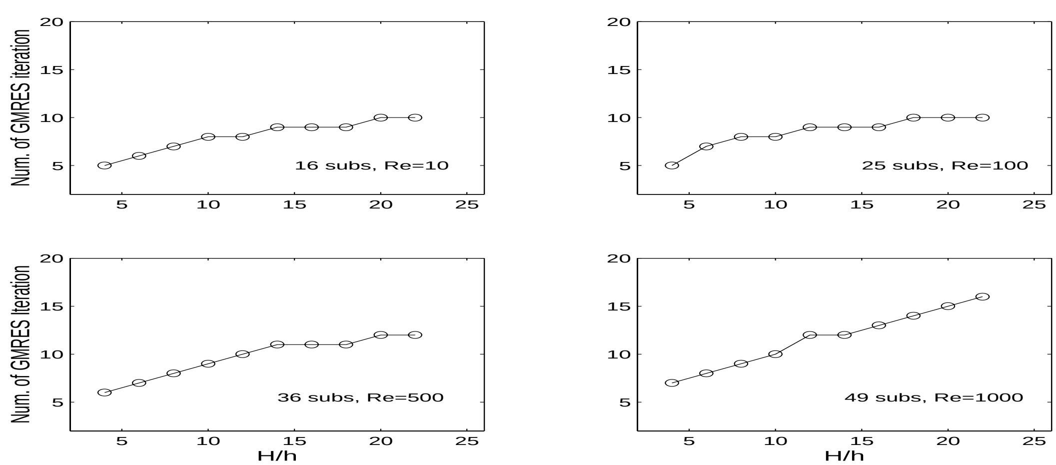

Figure 5.9: Scalability of GMRES solver

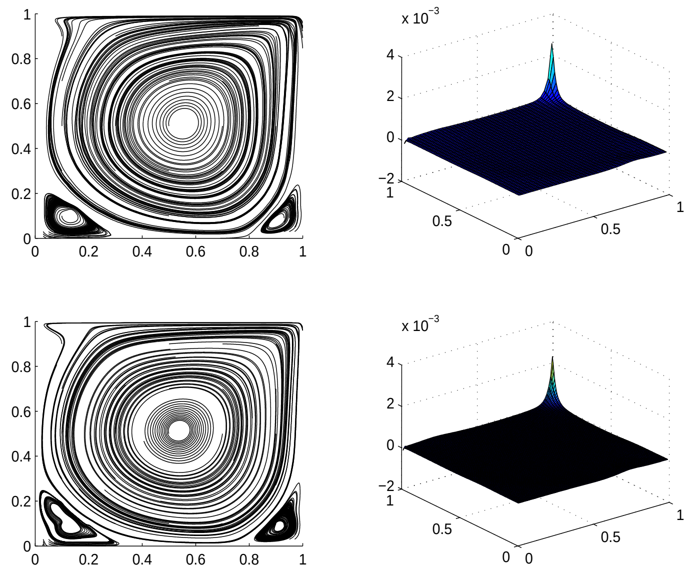

Figure 5.10: Approximate velocity and pressure for $\nu = \frac{1}{5000}$ and different meshes. Upper graphs: 8×8 subdomains and $H/h = 10$; lower graphs: 10×10 subdomains and $H/h = 16$

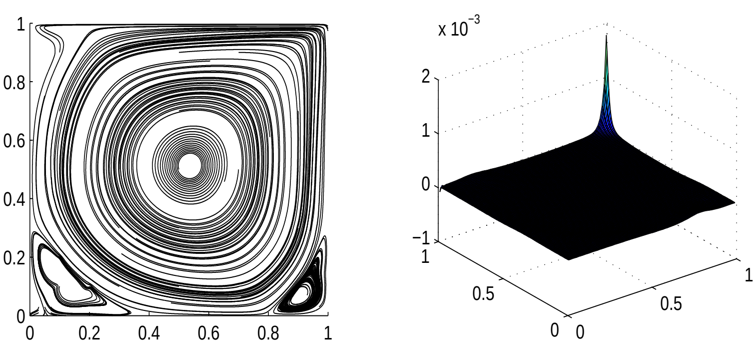

Figure 5.11: Approximate velocity and pressure for $\nu = \frac{1}{10000}$ with 10×10 subdomains and $H/h = 16$

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