A DUAL-PRIMAL FETI METHOD FOR INCOMPRESSIBLE STOKES EQUATIONS

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Abstract. In this paper, a dual-primal FETI method is developed for incompressible Stokes equations approximated by mixed finite elements with discontinuous pressures. The domain of the problem is decomposed into nonoverlapping subdomains, and the continuity of the velocity across the subdomain interface is enforced by introducing Lagrange multipliers. By a Schur complement procedure, solving the indefinite Stokes problem is reduced to solving a symmetric positive definite problem for the dual variables, i.e., the Lagrange multipliers. This dual problem is solved by a Krylov space method with a Dirichlet preconditioner. At each step of the iteration, both subdomain problems and a coarse problem on the coarse subdomain mesh are solved by a direct method. It is proved that the condition number of this preconditioned dual problem is independent of the number of subdomains and bounded from above by the product of the inverse of the inf-sup constant of the discrete problem and the square of the logarithm of the number of unknowns in the individual subdomain problems. Illustrative numerical results are presented by solving a lid driven cavity problem.

Key words. domain decomposition, Stokes, FETI, dual-primal methods

AMS subject classifications. 65N30, 65N55, 76D07

1. Introduction.

The finite elment tearing and interconnecting(FETI) methods were first proposed by Farhat and Roux [4] for elliptic partial differential equations. In this method, the spatial domain is decomposed into nonoverlapping subdomains, and the interior subdomain variables are eliminated to form a Schur problem for the interface variables. Lagrange multipliers are then introduced to enforce continuity across the interface, and a symmetric positive semi-definite linear system for the Lagrange multipliers is solved by using the preconditioned conjugate gradient (PCG) method. This method has been shown to be numerically scalable for second order elliptic problems if a Dirichlet preconditioner is used. Thus, Mandel and Tezaur [9] have proved that the condition number grows at most as $C(1+log(H/h))^3$ both in two and three dimensions, where H is the subdomain diameter and h is the element size. Klawonn and Widlund [7] proposed new preconditioners of this type and proved that the condition numbers are bounded from above by $C(1 + log(H/h))^2$; these bounds are also independent of possible jumps of the coefficients of the elliptic problem.

For fourth-order problems, a two-level FETI method was developed by Farhat and Mandel [5]. The main idea in this variant is that an extra set of Lagrange multipliers should be used to enforce the continuity at the subdomain corners in every step of the PCG algorithm. A similar idea was used by Farhat et al [6] to introduce the Dual-Primal FETI (FETI-DP) methods in which the continuity of the primal solution is enforced directly at the corners, i.e., the values of the degrees of freedom at the vertices of the subdomains remain the same. In [6], the FETI-DP methods were further refined to solve three-dimensional problems by introducing Lagrange multipliers to enforce a continuity constraint for the average of the solution on interface edges. This set of Lagrange multipliers, together with the corner variables, form the coarse problem of this FETI-DP method. This coarse, primal problem is necessary to obtain

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a satisfactory convergence rate for this method. A convergence analysis of dual-primal FETI methods was given by Mandel and Tezaur [10] for two-dimensional problems and by Klawonn et al. [8] for three dimensions.

In this paper, we develop a dual-primal FETI method for the Stokes problem in two dimensions and give a convergence analysis. In contrast to elliptic problems, the Stokes equation is an indefinite problem which involves pressure variables to impose the incompressibility condition for the velocity. In our algorithm, the pressure space is decomposed into two orthogonal parts; we exclusively deal with finite element approximations which use discontinuous pressures. The first part consists of subdomain interior pressures with zero average on each subdomain, and the second is spanned by the subdomain constant pressures with one average pressure for each subdomain. The velocity space is decomposed into three parts, the velocities interior to the subdomains, the velocities at subdomain corners and the velocities on the remaining part of the interface. By using this decomposition of the solution space, solving the original Stokes problem is replaced by solving subdomain Stokes problems, with a continuity constraint of the velocity field across the subdomain interface. The continuity of the velocities at the subdomain corners is enforced directly in our algorithm, the continuity of the velocities across the remaining interface is enforced by introducing a set of Lagrange multipliers, and the continuity of a weighted average of the velocities across each interface edge is enforced by using an additional set of Lagrange multipliers. By reducing the problem to a Schur complement, we obtain a symmetric positive definite problem for the dual variables, i.e., the first set of Lagrange multipliers. This dual problem is solved by an iterative method, either GMRES or the conjugate gradient method, with a Dirichlet preconditioner. We note that the additional set of Lagrange multipliers are important here because on the one hand it augments the corner velocities and the subdomain constant pressures to form a inf-sup stable coarse problem which is solved directly at each step of the iteration, and on the other hand it ensures that the subdomain Dirichlet Stokes problems, solved in the preconditioning procedure, are always compatible.

The remainder of this paper is organized as follows. In section 2, the Stokes problem is described in brief, and the domain decomposition method based on a decomposition of the solution space is proposed. The preconditioned augmented FETI-DP algorithm is derived in section 3. In section 4, an equivalent form of the algorithm is given in preparation of the convergence analysis, and an upper bound of the condition number of the algorithm is proved. In section 5, numerical experiments are presented for a lid driven cavity problem on a square.

2. Stokes problem and domain decomposition method. We are solving the following Stokes problem on a two-dimensional, bounded, polyhedral domain Ω,

$$\left\{\begin{array}{lll} -\Delta \mathbf{u} + \nabla p & = \mathbf{f}, & \text{in } \Omega \\ -\nabla \cdot \mathbf{u} & = 0, & \text{in } \Omega \\ \mathbf{u} & = \mathbf{g}, & \text{on } \partial\Omega \ , \end{array}\right. \tag{1}$$

where the boundary velocity $\mathbf{g}$ satisfies the compatibility condition $\int_{\partial\Omega} \mathbf{g} \cdot \mathbf{n} = 0$.

The equivalent variational problem is to find the velocity $\mathbf{u} \in \mathbf{W}$ and the pressure $p \in \Pi$ such that,

$$\left\{\begin{array}{lll} (\nabla \mathbf{u}, \nabla \mathbf{v})_\Omega - (p, \nabla \cdot \mathbf{v})_\Omega & = (\mathbf{f}, \mathbf{v})_\Omega, & \forall \mathbf{v} \in (H_0^1(\Omega))^2 \\ -(\nabla \cdot \mathbf{u}, q)_\Omega & = 0, & \forall q \in \Pi \ , \end{array}\right. \tag{2}$$

where $\mathbf{W} = \{\mathbf{u} \in (H^1(\Omega))^2 \mid \mathbf{u} = \mathbf{g} \text{ on } \partial\Omega\}$, $\Pi = \{p \in L^2(\Omega) \mid \int_\Omega p = 0\}$, and where $(.,.)_\Omega$ denotes the inner product in $L^2(\Omega)$.

The domain Ω is decomposed into N non-overlapping polyhedral subdomains Ω^i of characteristic size H. On each subdomain, function spaces $\mathbf{W}^i$ and Π^i are defined as, $\mathbf{W}^i = \{\mathbf{u}^i \in (H^1(\Omega^i))^2 \mid \mathbf{u}^i = \mathbf{g} \text{ on } \partial\Omega^i \cap \partial\Omega\}$, $\Pi^i = \{p^i \in L^2(\Omega^i) \mid \int_{\Omega^i} p^i = 0\}$. The subdomain interface is defined as $\Gamma = (\cup\partial\Omega^i)\backslash\partial\Omega$, and the interface edge $\Gamma^{i,j} = \partial\Omega^i \cap \partial\Omega^j$ is given for any two neighboring subdomains Ω^i and Ω^j. If we require that the subdomain velocities be continuous across Γ, then the variational problem (2) can be formulated as the following subdomain variational problems on a subspace of $\mathbf{W} \times \Pi$: find $\mathbf{u}^i \in \mathbf{W}^i$, $p^i \in \Pi^i$, and $p_0 \in \Pi_0$, such that

$$\left\{ \begin{array}{lll} (\nabla\mathbf{u}^i, \nabla\mathbf{v}^i)_{\Omega^i} - (p^i + p_0^i, \nabla\cdot\mathbf{v}^i)_{\Omega^i} & = (\mathbf{f}^i, \mathbf{v}^i)_{\Omega^i}, & \forall \mathbf{v}^i \in (H^1(\Omega^i))^2 \\ -(\nabla\cdot\mathbf{u}^i, q^i)_{\Omega^i} & = 0, & \forall q^i \in \Pi^i \ , \\ -(\nabla\cdot\mathbf{u}^i, q_0^i)_{\Omega^i} & = 0, & \forall q_0 \in \Pi_0 \ , \end{array} \right. \tag{3}$$

where the subdomain velocities, $\mathbf{u}^i$, are required to be continuous across the subdomain interface Γ, and $\Pi_0 = \{p_0;\ p_0(\Omega^i) = p_0^i,\ \sum_i (p_0^i m(\Omega^i)) = 0\}$ is a space for the subdomain constant pressures with $m(\Omega^i)$ the measure of the subdomain Ω^i.

Each subdomain Ω^i is triangulated into shape-regular elements of characteristic size h, with the finite element nodes on the boundaries of the neighboring subdomains matching across the interface Γ. A stable mixed finite element method is chosen for each subdomain saddle point problem. In our experiments, we are using the inf-sup stable $P_1(h) - P_0(2h)$ finite elements; see Brezzi and Fortin [3]. The velocities are continuous piecewise linear functions on a triangular mesh of size h, and the pressures are piecewise constant (discontinuous) functions on a coarser mesh of size $2h$. If we denote the subdomain interior velocities, of the subdomain Ω^i, by $\mathbf{u}_I^i$ and the subdomain interface velocities by $\mathbf{u}_\Gamma^i$, then the discrete linear system for solving problem (3) can be written as:

$$\begin{pmatrix} A_{II} & B_{II} & A_{I\Gamma} & B_{I0} \\ B_{II}^T & 0 & B_{\Gamma I}^T & 0 \\ A_{I\Gamma}^T & B_{\Gamma I} & A_{\Gamma\Gamma} & B_{\Gamma 0} \\ B_{I0}^T & 0 & B_{\Gamma 0}^T & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ p_I \\ \mathbf{u}_\Gamma \\ p_0 \end{pmatrix} = \begin{pmatrix} \mathbf{f}_I \\ 0 \\ \mathbf{f}_\Gamma \\ 0 \end{pmatrix} , \tag{4}$$

where $\mathbf{u}_I$, p_I, and $\mathbf{u}_\Gamma$ are direct sums of $\mathbf{u}_I^i$, p_I^i, $\mathbf{u}_\Gamma^i$, respectively, for $i = 1, 2, ..., N$. It follows from the divergence theorem that $B_{I0} = 0$.

In this paper we make no distinction, in our notations, between a finite element function and the corresponding vector, for example, $\mathbf{u}^i$ is used to denote either a finite element function or the corresponding vector, and the same applies to the notations $\mathbf{W}^i, \Pi^i, \Pi_0$, etc.

It still remains the problem of how to handle the continuity of the subdomain interface velocity $\mathbf{u}_\Gamma$ across Γ. Denote $\mathbf{W}_\Gamma$ as the function space of $\mathbf{u}_\Gamma$, and $\mathbf{W}_\Gamma$ is decomposed into a subdomain corner velocity part $\mathbf{W}_c$ and the remaining boundary velocity part $\mathbf{W}_\Delta$, i.e.,

$$\mathbf{W}_\Gamma = \mathbf{W}_c \oplus \mathbf{W}_\Delta \ .$$

The continuity of the element in $\mathbf{W}_c$ is enforced directly, i.e., the degrees of freedom at a cornerpoint are common to all subdomains sharing this corner. $\mathbf{W}_\Delta$ is decomposed into a direct sum of subdomain boundary velocity spaces $\mathbf{W}_\Delta^i$, i.e.,

$$\mathbf{W}_\Delta = \oplus_i \mathbf{W}_\Delta^i \ ,$$

and the continuity constraint is of the form

$$B_\Delta \mathbf{w}_\Delta = 0, \text{for any } \mathbf{w}_\Delta \in \mathbf{W}_\Delta \ , \tag{5}$$

where the matrix B_Δ is constructed from {0,1,-1} such that the values of $\mathbf{w}_\Delta$ coincide across the subdomain interface Γ when $B_\Delta \mathbf{w}_\Delta = 0$. We also introduce a redundant continuity constraint of the form

$$Q_\Delta^T B_\Delta \mathbf{w}_\Delta = 0, \text{for any } \mathbf{w}_\Delta \in \mathbf{W}_\Delta \ , \tag{6}$$

which will be enforced at each iteration step of our algorithm, while the equation (5) is not satisfied until convergence. The matrix Q_Δ, in equation (6), is constructed such that, for any function $\mathbf{w}_\Delta \in \mathbf{W}_\Delta$, $Q_\Delta^T B_\Delta \mathbf{w}_\Delta = 0$ implies that,

$$\int_{\Gamma^{i,j}} (\mathbf{w}_\Delta^i - \mathbf{w}_\Delta^j) = 0$$

for any edge $\Gamma^{i,j}$ between two neighboring subdomains Ω^i and Ω^j. We note that the matrix notations B_Δ and Q_Δ can also be used to denote the corresponding operators.

By introducing Lagrange multipliers λ and μ to enforce the continuity constraint equations (5) and (6) for the functions in $\mathbf{W}_\Delta$, equation (4) can be written as

$$\begin{pmatrix} A_{II} & B_{II} & A_{I\Delta} & A_{Ic} & 0 & 0 & 0 \\ B_{II}^T & 0 & B_{\Delta I}^T & B_{cI}^T & 0 & 0 & 0 \\ A_{I\Delta}^T & B_{\Delta I} & A_{\Delta\Delta} & A_{\Delta c} & B_{\Delta 0} & B_\Delta^T Q_\Delta & B_\Delta^T \\ A_{Ic}^T & B_{cI} & A_{\Delta c}^T & A_{cc} & B_{c0} & 0 & 0 \\ 0 & 0 & B_{\Delta 0}^T & B_{c0}^T & 0 & 0 & \\ 0 & 0 & Q_\Delta^T B_\Delta & 0 & 0 & 0 & 0 \\ 0 & 0 & B_\Delta & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ p_I \\ \mathbf{u}_\Delta \\ \mathbf{u}_c \\ p_0 \\ \mu \\ \lambda \end{pmatrix} = \begin{pmatrix} \mathbf{f}_I \\ 0 \\ \mathbf{f}_\Delta \\ \mathbf{f}_c \\ 0 \\ 0 \\ 0 \end{pmatrix} . \tag{7}$$

In the following section, we propose an augmented FETI-DP method for solving equation (7).

3. The augmented FETI-DP algorithm. In Section 2, two sets of Lagrange multiplies λ, μ were introduced to enforce the continuity of the velocity across the interface Γ, and equation (7) was formed. In fact, μ is redundant because $B_\Delta \mathbf{u}_\Delta = 0$ implies $Q_\Delta^T B_\Delta \mathbf{u}_\Delta = 0$. But in our algorithm, λ and μ are treated differently. We iterate on the dual problem variable λ, and the continuity condition $B_\Delta \mathbf{u}_\Delta = 0$ is not satisfied until convergence. The Lagrange multiplier μ, on the other hand, is treated together with the primal variables and it augments the corner velocities to form the coarse problem variables, together with the subdomain constant pressures p_0. By solving the augmented coarse problem exactly in each step of the iterations, $Q_\Delta^T B_\Delta \mathbf{u}_\Delta = 0$ will be satisfied throughout.

By using the notations

$$\tilde{\mathbf{u}}_r = \begin{pmatrix} \mathbf{u}_I \\ p_I \\ \mathbf{u}_\Delta \end{pmatrix} , \tilde{\mathbf{u}}_c = \begin{pmatrix} \mathbf{u}_c \\ p_0 \\ \mu \end{pmatrix} , \tag{8}$$

equation (7) can be written as,

$$\begin{pmatrix} K_{rr} & K_{rc} & B_r^T \\ K_{rc}^T & K_{cc} & 0 \\ B_r & 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{\mathbf{u}}_r \\ \tilde{\mathbf{u}}_c \\ \lambda \end{pmatrix} = \begin{pmatrix} \tilde{\mathbf{f}}_r \\ \tilde{\mathbf{f}}_c \\ 0 \end{pmatrix} , \tag{9}$$

where $K_{rr}, K_{rc}, K_{cc}, B_r, \tilde{\mathbf{f}}_r$, and $\tilde{\mathbf{f}}_c$, are the corresponding block matrices and block vectors.

Our algorithm results from two consecutive elimination procedures applied to equation (9). We first eliminate the subdomain variables $\tilde{\mathbf{u}}_r$ and obtain

$$\begin{pmatrix} \tilde{K}_{cc} & \tilde{K}_{cl} \\ \tilde{K}_{cl}^T & \tilde{K}_{ll} \end{pmatrix} \begin{pmatrix} \tilde{\mathbf{u}}_c \\ \lambda \end{pmatrix} = \begin{pmatrix} \tilde{\mathbf{f}}_c^* \\ d_l \end{pmatrix} , \tag{10}$$

where

$$\tilde{K}_{cc} = K_{cc} - K_{rc}^T K_{rr}^{-1} K_{rc} , \quad \tilde{K}_{ll} = -B_r K_{rr}^{-1} B_r^T , \quad \tilde{K}_{cl} = -K_{rc}^T K_{rr}^{-1} B_r^T ,$$

and

$$\tilde{\mathbf{f}}_c^* = \tilde{\mathbf{f}}_c - K_{rc}^T K_{rr}^{-1} \tilde{\mathbf{f}}_r , \quad d_l = -B_r K_{rr}^{-1} \tilde{\mathbf{f}}_r .$$

We then eliminate $\tilde{\mathbf{u}}_c$ from equation (10), and obtain a linear system for the Lagrange multipliers λ,

$$(\tilde{K}_{ll} - \tilde{K}_{cl}^T \tilde{K}_{cc}^{-1} \tilde{K}_{cl})\lambda = d_l - \tilde{K}_{cl}^T \tilde{K}_{cc}^{-1} \tilde{\mathbf{f}}_c^* . \tag{11}$$

Our preconditioned augmented FETI-DP algorithm solves equation (11) with a preconditioned CG or GMRES method to obtain λ, and we then obtain $\tilde{\mathbf{u}}_c$ and $\tilde{\mathbf{u}}_r$ from equations (10) and (9). The preconditioner involves solving subdomain incompressible Stokes problems with Dirichlet boundary conditions and will be discussed in the next section.

We note that K_{rr}^{-1}, $\tilde{K}_{ll}$, $\tilde{K}_{cl}$ and $\tilde{K}_{cl}^T$ require subdomain Dirichlet solvers with the corner velocities given. If a stable mixed finite element method is used for each subdomain, then we know that these problems are stable. Applying $\tilde{K}_{cc}^{-1}$ to a vector requires solving a coarse problem with the corner velocities, the subdomain constant pressures, and the Lagrange multipliers μ as variables. Solving this augmented coarse problem is similar to solving a Stokes problem on the coarse subdomain mesh by using the stable $Q2 - Q0$ mixed finite elements. Numerical evidence shows that this augmented coarse problem satisfies a discrete inf-sup condition.

4. Convergence analysis. Preparing for our convergence analysis, we derive equation (11) in another way. We reorder the unknowns in equation (7) to obtain

$$\begin{pmatrix} A_{II} & B_{II} & A_{Ic} & 0 & 0 & A_{I\Delta} & 0 \\ B_{II}^T & 0 & B_{cI}^T & 0 & 0 & B_{\Delta I}^T & 0 \\ A_{Ic}^T & B_{cI} & A_{cc} & B_{c0} & 0 & A_{\Delta c}^T & 0 \\ 0 & 0 & B_{c0}^T & 0 & 0 & B_{\Delta 0}^T & 0 \\ 0 & 0 & 0 & 0 & 0 & Q_\Delta^T B_\Delta & 0 \\ A_{I\Delta}^T & B_{\Delta I} & A_{\Delta c} & B_{\Delta 0} & B_\Delta^T Q_\Delta & A_{\Delta\Delta} & B_\Delta^T \\ 0 & 0 & 0 & 0 & 0 & B_\Delta & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ p_I \\ \mathbf{u}_c \\ p_0 \\ \mu \\ \mathbf{u}_\Delta \\ \lambda \end{pmatrix} = \begin{pmatrix} \mathbf{f}_I \\ 0 \\ \mathbf{f}_c \\ 0 \\ 0 \\ \mathbf{f}_\Delta \\ 0 \end{pmatrix} . \tag{12}$$

Define a subspace $\widetilde{\mathbf{W}}_\Delta$ of $\mathbf{W}_\Delta$ as,

$$\widetilde{\mathbf{W}}_\Delta = \{\mathbf{w}_\Delta \in \mathbf{W}_\Delta \mid Q_\Delta^T B_\Delta \mathbf{w}_\Delta = 0\} .$$

Solving (12) is then equivalent to solving the following problem: find $\mathbf{u}_\Delta \in \widetilde{\mathbf{W}}_\Delta, \lambda \in \Lambda = Range(B_\Delta \widetilde{\mathbf{W}}_\Delta)$, such that

$$\begin{pmatrix} \tilde{S} & B_\Delta^T \\ B_\Delta & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_\Delta \\ \lambda \end{pmatrix} = \begin{pmatrix} \mathbf{f}_\Delta^* \\ 0 \end{pmatrix} , \tag{13}$$

where the Schur complement $\tilde{S}$ is defined by

$$\begin{pmatrix} A_{II} & B_{II} & A_{Ic} & 0 & A_{I\Delta} \\ B_{II}^T & 0 & B_{cI}^T & 0 & B_{\Delta I}^T \\ A_{Ic}^T & B_{cI} & A_{cc} & B_{c0} & A_{\Delta c}^T \\ 0 & 0 & B_{c0}^T & 0 & B_{\Delta 0}^T \\ A_{I\Delta}^T & B_{\Delta I} & A_{\Delta c} & B_{\Delta 0} & A_{\Delta\Delta} \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ p_I \\ \mathbf{u}_c \\ p_0 \\ \mathbf{u}_\Delta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \tilde{S}\mathbf{u}_\Delta \end{pmatrix} . \tag{14}$$

We can show that the Schur complement $\tilde{S}$ defined in equation (14) can also be defined variationally on $\widetilde{\mathbf{W}}_\Delta$: for any $\mathbf{u}_\Delta \in \widetilde{\mathbf{W}}_\Delta$

$$\mathbf{u}_\Delta^T \tilde{S}\mathbf{u}_\Delta = \min_{\mathbf{v}_I} \min_{\mathbf{v}_c} \max_{p_I} \{\mathbf{v}^T K \mathbf{v} \mid \mathbf{v}_\Delta = \mathbf{u}_\Delta \text{ and } B_{c0}^T \mathbf{v}_c + B_{\Delta 0}^T \mathbf{v}_\Delta = 0\} \ , \tag{15}$$

where

$$K = \begin{pmatrix} A_{II} & B_{II} & A_{Ic} & A_{I\Delta} \\ B_{II}^T & 0 & B_{cI}^T & B_{\Delta I}^T \\ A_{Ic}^T & B_{cI} & A_{cc} & A_{\Delta c}^T \\ A_{I\Delta}^T & B_{\Delta I} & A_{\Delta c} & A_{\Delta\Delta} \end{pmatrix} , \mathbf{v} = \begin{pmatrix} \mathbf{v}_I \\ p_I \\ \mathbf{v}_c \\ \mathbf{v}_\Delta \end{pmatrix} .$$

LEMMA 1. *$\tilde{S}$ is symmetric, positive definite on $\widetilde{\mathbf{W}}_\Delta$.*

Proof: It is easy to see, from the definition (14) , that $\tilde{S}$ is symmetric. We next just need to show that $(\tilde{S}\mathbf{u}_\Delta, \mathbf{u}_\Delta) > 0$, for any nonzero function $\mathbf{u}_\Delta \in \widetilde{\mathbf{W}}_\Delta$. For any given function $\mathbf{u}_\Delta \in \widetilde{\mathbf{W}}_\Delta$, there is a vector $(\mathbf{u}_I, p_I, \mathbf{u}_c, p_0,)$ such that equation (14) is satisfied. Therefore,

$$\begin{aligned}
&(\tilde{S}\mathbf{u}_\Delta, \mathbf{u}_\Delta) = \mathbf{u}_\Delta^T \tilde{S}\mathbf{u}_\Delta \\
= &\begin{pmatrix} \mathbf{u}_I \\ p_I \\ \mathbf{u}_c \\ p_0 \\ \mathbf{u}_\Delta \end{pmatrix}^T \begin{pmatrix} A_{II} & B_{II} & A_{Ic} & 0 & A_{I\Delta} \\ B_{II}^T & 0 & B_{cI}^T & 0 & B_{\Delta I}^T \\ A_{Ic}^T & B_{cI} & A_{cc} & B_{c0} & A_{\Delta c}^T \\ 0 & 0 & B_{c0}^T & 0 & B_{\Delta 0}^T \\ A_{I\Delta}^T & B_{\Delta I} & A_{\Delta c} & B_{\Delta 0} & A_{\Delta\Delta} \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ p_I \\ \mathbf{u}_c \\ p_0 \\ \mathbf{u}_\Delta \end{pmatrix} \\
= &\begin{pmatrix} \mathbf{u}_I \\ \mathbf{u}_c \\ \mathbf{u}_\Delta \end{pmatrix}^T \begin{pmatrix} A_{II} & A_{Ic} & A_{I\Delta} \\ A_{Ic}^T & A_{cc} & A_{\Delta c}^T \\ A_{I\Delta}^T & A_{\Delta c} & A_{\Delta\Delta} \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ \mathbf{u}_c \\ \mathbf{u}_\Delta \end{pmatrix} \\
&+2 \begin{pmatrix} p_I \\ p_0 \end{pmatrix}^T \begin{pmatrix} B_{II}^T & B_{cI}^T & B_{\Delta I}^T \\ 0 & B_{c0}^T & B_{\Delta 0}^T \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ \mathbf{u}_c \\ \mathbf{u}_\Delta \end{pmatrix} + \begin{pmatrix} p_I \\ p_0 \end{pmatrix}^T \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} p_I \\ p_0 \end{pmatrix} \\
= &\begin{pmatrix} \mathbf{u}_I \\ \mathbf{u}_c \\ \mathbf{u}_\Delta \end{pmatrix}^T \begin{pmatrix} A_{II} & A_{Ic} & A_{I\Delta} \\ A_{Ic}^T & A_{cc} & A_{\Delta c}^T \\ A_{I\Delta}^T & A_{\Delta c} & A_{\Delta\Delta} \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ \mathbf{u}_c \\ \mathbf{u}_\Delta \end{pmatrix} ,
\end{aligned}$$

where the last equality results from $B_{II}^T \mathbf{u}_I + B_{cI}^T \mathbf{u}_c + B_{\Delta I}^T \mathbf{u}_\Delta = 0$ and $B_{c0}^T \mathbf{u}_c + B_{\Delta 0}^T \mathbf{u}_\Delta = 0$, because the vector $(\mathbf{u}_I, p_I, \mathbf{u}_c, p_0, \mathbf{u}_\Delta)$ satisfies equation (14) . Since the matrix

$$\begin{pmatrix} A_{II} & A_{Ic} & A_{I\Delta} \\ A_{Ic}^T & A_{cc} & A_{\Delta c}^T \\ A_{I\Delta}^T & A_{\Delta c} & A_{\Delta\Delta} \end{pmatrix}$$

is just a symmetric positive definite discretion of a direct sum of two Laplace operators, we find that $(\tilde{S}\mathbf{u}_\Delta, \mathbf{u}_\Delta) > 0$, for any nonzero function $\mathbf{u}_\Delta \in \widetilde{\mathbf{W}}_\Delta$.

$\square$

We therefore know, from Lemma 1, that the equivalent problem (13) can be reformulated as a minimization problem over $\widetilde{\mathbf{W}}_\Delta$, with the constraints given by the continuity requirement of the velocity across the subdomain interface Γ: find $\mathbf{u}_\Delta \in \widetilde{\mathbf{W}}_\Delta$, such that

$$\frac{1}{2}(\tilde{S}\mathbf{u}_\Delta, \mathbf{u}_\Delta) - (\mathbf{f}_\Delta^*, \mathbf{u}_\Delta) \to \text{ min }, \text{ with } B_\Delta \mathbf{u}_\Delta = 0 \; . \tag{16}$$

Equation (13) can be further reduced to a linear system for the Lagrange multipliers λ, which is of the form,

$$F\lambda = B_\Delta \tilde{S}^{-1} f_\Delta^* \; , \tag{17}$$

where $F = B_\Delta \tilde{S}^{-1} B_\Delta^T$. F is symmetric positive definite because we are using nonredundant Lagrange multipliers and the matrix B_Δ^T has full column rank. It is also easy to see that equation (17) is the same as equation (11).

We solve the dual system (17) using the preconditioned conjugate gradient method or GMRES with the preconditioner

$$M^{-1} = B_\Delta S_\Delta B_\Delta^T \; ,$$

where S_Δ is defined as

$$\mathbf{u}_\Delta^T S_\Delta \mathbf{u}_\Delta = \min_{\mathbf{v}_I} \max_{p_I} \{ \mathbf{v}^T K \mathbf{v} \mid \mathbf{v}_\Delta = \mathbf{u}_\Delta \text{ and } \mathbf{v}_c = 0 \} \; , \tag{18}$$

or in matrix form,

$$\begin{pmatrix} A_{II} & B_{II} & A_{I\Delta} \\ B_{II}^T & 0 & B_{\Delta I}^T \\ A_{I\Delta}^T & B_{\Delta I} & A_{\Delta\Delta} \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ p_I \\ \mathbf{u}_\Delta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ S_\Delta \mathbf{u}_\Delta \end{pmatrix} \; . \tag{19}$$

Then the preconditioned system is,

$$B_\Delta S_\Delta B_\Delta^T B_\Delta \tilde{S}^{-1} B_\Delta^T \lambda = B_\Delta S_\Delta B_\Delta^T B_\Delta \tilde{S}^{-1} f_\Delta^* \; , \tag{20}$$

In order to use the conjugate gradient method for this preconditioned system (20), we have to show that the preconditioner M^{-1} is symmetric positive definite. In fact we just need to show that S_Δ is symmetric positive definite on the space $B_\Delta^T B_\Delta \widetilde{\mathbf{W}}_\Delta$, because S_Δ is always applied to a vector in this space. We need the following lemma,

Lemma 2. *For any function $\mathbf{w}_\Delta \in \widetilde{\mathbf{W}}_\Delta$, $B_\Delta^T B_\Delta \mathbf{w}_\Delta \in \widetilde{\mathbf{W}}_\Delta$, and $B_\Delta^T B_\Delta \mathbf{w}_\Delta$ satisfies: $\int_{\Omega^i} \nabla \cdot \left(B_\Delta^T B_\Delta \mathbf{w}_\Delta\right)^i = 0$, for any subdomain Ω^i.*

Proof: Given a function $\mathbf{w}_\Delta \in \widetilde{\mathbf{W}}_\Delta$, we know that $Q_\Delta^T B_\Delta \mathbf{w}_\Delta = 0$, i.e., $\int_{\Gamma^{i,j}} (\mathbf{w}_\Delta^i - \mathbf{w}_\Delta^j) = 0$, for any edge $\Gamma^{i,j}$ common to the neighboring subdomains Ω^i and Ω^j. By using

$$(B_\Delta^T B_\Delta \mathbf{w}_\Delta)^i|_{\Gamma^{i,j}} = \pm(\mathbf{w}_\Delta^i|_{\Gamma^{i,j}} - \mathbf{w}_\Delta^j|_{\Gamma^{i,j}}) \; ,$$

we have

$$\int_{\Gamma^{i,j}} (B_\Delta^T B_\Delta \mathbf{w}_\Delta)^i = 0 \; . \tag{21}$$

In the same way, we have $\int_{\Gamma^{i,j}} (B_\Delta^T B_\Delta \mathbf{w}_\Delta)^j = 0$, and therefore we obtain that

$$\int_{\Gamma^{i,j}} ((B_\Delta^T B_\Delta \mathbf{w}_\Delta)^i - (B_\Delta^T B_\Delta \mathbf{w}_\Delta)^j) = 0 \ .$$

Therefore $Q_\Delta^T B_\Delta (B_\Delta^T B_\Delta \mathbf{w}_\Delta) = 0$, and $B_\Delta^T B_\Delta \mathbf{w}_\Delta \in \widetilde{\mathbf{W}}_\Delta$.

To prove $\int_{\Omega^i} \nabla \cdot \left(B_\Delta^T B_\Delta \mathbf{w}_\Delta\right)^i = 0$, we just need to use the divergence theorem and equality (21), and it follows,

$$\int_{\Omega^i} \nabla \cdot \left(B_\Delta^T B_\Delta \mathbf{w}_\Delta\right)^i = \int_{\partial\Omega^i} \left(B_\Delta^T B_\Delta \mathbf{w}_\Delta\right)^i \cdot \mathbf{n} = \sum_j \int_{\Gamma^{i,j}} \left(B_\Delta^T B_\Delta \mathbf{w}_\Delta\right)^i \cdot \mathbf{n}^{i,j} = 0 \ .$$

$\square$

LEMMA 3. *S_Δ is symmetric positive definite on the space $B_\Delta^T B_\Delta \widetilde{\mathbf{W}}_\Delta$.*

Proof: We first need to show that S_Δ is well defined on the space $B_\Delta^T B_\Delta \widetilde{\mathbf{W}}_\Delta$. From its definition in equation (19), we see that to apply S_Δ to a vector of the form $B_\Delta^T B_\Delta \mathbf{w}_\Delta$, where $\mathbf{w}_\Delta \in \widetilde{\mathbf{W}}_\Delta$, is reduced to solving subdomain incompressible Stokes problems with $B_\Delta^T B_\Delta \mathbf{w}_\Delta$ as the given subdomain boundary velocities. For these subdomain Dirichlet problems to be well posed, $B_\Delta^T B_\Delta \mathbf{w}_\Delta$ has to satisfy the compatible condition, $\int_{\Omega^i} \nabla \cdot \left(B_\Delta^T B_\Delta \mathbf{w}_\Delta\right)^i = 0$, in each subdomain Ω^i, which has just been proven in Lemma 2. Therefore, S_Δ is well defined.

Then, by arguments similar to those in the proof of Lemma 1, we find that S_Δ is symmetric positive definite on space $B_\Delta^T B_\Delta \widetilde{\mathbf{W}}_\Delta$.

$\square$

LEMMA 4. *$|B_\Delta^T B_\Delta \mathbf{w}_\Delta|_{\tilde{S}} \leq |B_\Delta^T B_\Delta \mathbf{w}_\Delta|_{S_\Delta}$, for any $\mathbf{w}_\Delta \in \widetilde{\mathbf{W}}_\Delta$.*

Proof: If we can prove that $B_{\Delta 0}^T (B_\Delta^T B_\Delta \mathbf{w}_\Delta) = 0$, for any $\mathbf{w}_\Delta \in \widetilde{\mathbf{W}}_\Delta$, then the constraints in equation (18), $\mathbf{v}_\Delta = B_\Delta^T B_\Delta \mathbf{w}_\Delta$ and $\mathbf{v}_c = 0$, implies the constraints $\mathbf{v}_\Delta = B_\Delta^T B_\Delta \mathbf{w}_\Delta$ and $B_{c0}^T \mathbf{v}_c + B_{\Delta 0}^T \mathbf{v}_\Delta = 0$ in equation (15). It then follows that $\mathbf{w}_\Delta^T B_\Delta^T B_\Delta \tilde{S} B_\Delta^T B_\Delta \mathbf{w}_\Delta \leq \mathbf{w}_\Delta^T B_\Delta^T B_\Delta S_\Delta B_\Delta^T B_\Delta \mathbf{w}_\Delta$.

In order to show that $B_{\Delta 0}^T (B_\Delta^T B_\Delta \mathbf{w}_\Delta) = 0$, we just need to note that the restriction of $B_{\Delta 0}^T (B_\Delta^T B_\Delta \mathbf{w}_\Delta)$ to any subdomain Ω^i is just $\int_{\Omega^i} \nabla \cdot \left(B_\Delta^T B_\Delta \mathbf{w}_\Delta\right)^i$, which is zero according to Lemma 2.

$\square$

In the remainder of this section, we give an upper bound for the condition number of the operator $M^{-1}F$. We start by introducing some notations as in Mandel and Tezaur [10]. We denote by $E^{i,j}$ the operator that extends a vector of values of the degrees of freedom on $\Gamma^{i,j}$, excluding the corners, by zero to a vector on $\partial\Omega^i$. Let $\mathcal{E}^i$ be the set of all indices of the neighbors Ω^j of the domain Ω^i with a common edge $\Gamma^{i,j}$. Denote by $V_h(\Omega^i)$ the linear finite element space on the subdomain Ω^i and by S^i the Schur complement on $\partial\Omega^i$ obtained by eliminating the interior degrees of freedom in the subdomain Ω^i, i.e.,

$${\mathbf{u}_{\Delta,c}^i}^T S^i \mathbf{u}_{\Delta,c}^i = \min_{\mathbf{v}_I^i} \max_{p_I^i} \{ {\mathbf{v}^i}^T K^i \mathbf{v}^i \mid \mathbf{v}_{\Delta,c}^i = \mathbf{u}_{\Delta,c}^i \} \ ,$$

where K^i is the block corresponding to subdomain Ω^i in the matrix K introduced in equation (15), and $\mathbf{u}_{\Delta,c}^i$ denotes the vector $\mathbf{u}_\Delta^i + \mathbf{u}_c^i$.

The following well-known estimate can be found in Widlund [12], and Bramble et al. [1]. Here we are using the version in Mandel and Tezaur [10].

LEMMA 5. *Let $w \in V_h(\Omega^i)$ such that $w = 0$ at the corners of Ω^i. Let $w_L \in V_h(\Omega^i)$ be linear on all edges $\Gamma^{i,j} \subset \Omega^i$, and for each $j \in \mathcal{E}^i$ let $w^{i,j}$ be defined by $w^{i,j} = w$ on $\Gamma^{i,j}$ and by $w^{i,j} = 0$ on $\partial\Omega^i \backslash \Gamma^{i,j}$. Then*

$$\sum_{j \in \mathcal{E}^i} |w^{i,j}|^2_{1/2,2,\partial\Omega^i} \leq C(1 + log\frac{H}{h})^2 |w + w_L|^2_{1/2,2,\partial\Omega^i} \ .$$

The following lemma can be found in Bramble and Pasciak [2],

LEMMA 6.

$$C_1 \beta |\mathbf{u}_{\Delta,c}|_{S^i} \leq |\mathbf{u}_{\Delta,c}|_{1/2,2,\partial\Omega^i} \leq C_2 |\mathbf{u}_{\Delta,c}|_{S^i} \ ,$$

where β is the inf-sup constant of the chosen mixed finite element space.

LEMMA 7. *For every $\mathbf{w}_{\Delta,c}$, and for all i, and $j \in \mathcal{E}^i$,*

$$|E^{i,j}(\mathbf{w}^s_\Delta - I^H \mathbf{w}^i_c)|_{S^i} \leq C\frac{1}{\beta}(1 + log\frac{H}{h})^2 |\mathbf{w}^s_{\Delta,c}|_{S^s} \ , \ s = i, j \ ,$$

where $I^H \mathbf{w}^i_c$ is the linear interpolant of $\mathbf{w}^i_c$ on the subdomain boundary.

Proof: Write $\mathbf{w}_{\Delta,c} = (\mathbf{w}_{\Delta,c} - I^H \mathbf{w}_c) + I^H \mathbf{w}_c$. It follows from Lemma 5 that

$$|E^{i,j}(\mathbf{w}^i_\Delta - I^H \mathbf{w}^i_c)|^2_{1/2,2,\partial\Omega^i} \leq C(1 + log\frac{H}{h})^2 |\mathbf{w}^i_{\Delta,c}|^2_{1/2,2,\partial\Omega^i} \ .$$

By using the uniform equivalence of the seminorms,

$$|\mathbf{v}|_{1/2,2,\partial\Omega^i} \approx |\mathbf{v}|_{1/2,2,\partial\Omega^j}, \text{ if } \mathbf{v} = 0 \text{ on } \partial\Omega^i \cup \partial\Omega^j \backslash \Gamma^{i,j} \ ,$$

we have

$$|E^{i,j}(\mathbf{w}^s_\Delta - I^H \mathbf{w}^i_c)|^2_{1/2,2,\partial\Omega^i} \leq C(1 + log\frac{H}{h})^2 |\mathbf{w}^s_{\Delta,c}|^2_{1/2,2,\partial\Omega^s}, \ s = i, j \ ,$$

and we then obtain from Lemma 6

$$\beta |E^{i,j}(\mathbf{w}^s_\Delta - I^H \mathbf{w}^i_c)|^2_{S^i} \leq C(1 + log\frac{H}{h})^2 |\mathbf{w}^s_{\Delta,c}|^2_{S^s} \ , \ s = i, j \ .$$

□

We next prove the following key estimate.

LEMMA 8. *For all $\mathbf{w}_\Delta \in \widetilde{\mathbf{W}}_\Delta$,*

$$|B^T_\Delta B_\Delta \mathbf{w}_\Delta|^2_{S_\Delta} \leq C\frac{1}{\beta}(1 + log(H/h))^2 |\mathbf{w}_\Delta|^2_{\tilde{S}} \ ,$$

where $C > 0$ is independent of H and h.

Proof: Given $\mathbf{w}_\Delta \in \widetilde{\mathbf{W}}_\Delta$, we know from the definition of $\tilde{S}$ in equation (14) that we can find $\mathbf{w}_c$ such that

$$|\mathbf{w}_\Delta|^2_{\tilde{S}} = \sum_{i=1}^{N} |\mathbf{w}^i_{\Delta,c}|^2_{S^i} \ .$$

It is also true that $B_\Delta^T B_\Delta \mathbf{w}_\Delta = B_\Delta^T B_\Delta(\mathbf{w}_\Delta - \widehat{\mathbf{w}}_\Delta)$, for any function $\widehat{\mathbf{w}}_\Delta$ which is continuous across the subdomain interfaces. If we choose $\widehat{\mathbf{w}}_\Delta$ as $I^H\mathbf{w}_c$, the linear interpolant of $\mathbf{w}_c$ on the coarse subdomain grid, we have

$$\begin{array}{ll} |B_\Delta^T B_\Delta \mathbf{w}_\Delta|_{S_\Delta}^2 & = |B_\Delta^T B_\Delta(\mathbf{w}_\Delta - \widehat{\mathbf{w}}_\Delta)|_{S_\Delta}^2 \\ & = |B_\Delta^T B_\Delta(\mathbf{w}_\Delta - I^H\mathbf{w}_c)|_{S_\Delta}^2 \\ & = \sum_{i=1}^N |\mathbf{v}_{\Delta,c}^i|_{S^i}^2 \ , \end{array}$$

with

$$\mathbf{v}_{\Delta,c}^i = \mathbf{v}_\Delta^i + \mathbf{v}_c^i \ ,$$

where

$$\mathbf{v}_\Delta^i = B_\Delta^T B_\Delta(\mathbf{w}_\Delta - I^H\mathbf{w}_c), \text{ and } \mathbf{v}_c^i = 0 \ .$$

Using the definition of $E^{i,j}$, $\mathbf{v}_{\Delta,c}^i$ can be written as,

$$\mathbf{v}_{\Delta,c}^i = \sum_{j\in\mathcal{E}^j} E^{i,j}\mathbf{v}_\Delta^i \ .$$

and from $(B_\Delta^T B_\Delta \mathbf{w}_\Delta)^i|_{\Gamma^{i,j}} = \pm(\mathbf{w}_\Delta^i|_{\Gamma^{i,j}} - \mathbf{w}_\Delta^j|_{\Gamma^{i,j}})$, we have

$$\begin{array}{lll} |\mathbf{v}_{\Delta,c}^i|_{S^i} & \leq & \sum_{j\in\mathcal{E}^i} |E^{i,j}\mathbf{v}_\Delta^i|_{S^i} \\ & = & \sum_{j\in\mathcal{E}^j} |E^{i,j}B_\Delta^T B_\Delta(\mathbf{w}_\Delta - I^H\mathbf{w}_c)|_{S^i} \\ & \leq & \sum_{j\in\mathcal{E}^i}(|E^{i,j}(\mathbf{w}_\Delta^i - I^H\mathbf{w}_c)|_{S^i} + |E^{i,j}(\mathbf{w}_\Delta^j - I^H\mathbf{w}_c)|_{S^i}) \ . \end{array}$$

By using Lemma 7, we have

$$|\mathbf{v}_{\Delta,c}^i|_{S^i}^2 \leq C\frac{1}{\beta}(1+log\frac{H}{h})^2 \sum_{j\in\mathcal{E}^i}(|\mathbf{w}_{\Delta,c}^i|_{S^i}^2 + |\mathbf{w}_{\Delta,c}^j|_{S^j}^2) \ ,$$

and therefore

$$\begin{array}{ll} |B_\Delta^T B_\Delta \mathbf{w}_\Delta|_{S_\Delta}^2 & = \sum_{i=1}^N |\mathbf{v}_{\Delta,c}^i|_{S^i}^2 \\ & \leq C\frac{1}{\beta}(1+log(H/h))^2 \sum_{i=1}^N \sum_{t\in\mathcal{E}^i}(|\mathbf{w}_{\Delta,c}^i|_{S^i}^2 + |\mathbf{w}_{\Delta,c}^j|_{S^j}^2) \\ & \leq C\frac{1}{\beta}(1+log(H/h))^2 \sum_{i=1}^N |\mathbf{w}_{\Delta,c}^i|_{S^i}^2 \\ & = C\frac{1}{\beta}(1+log(H/h))^2 |\mathbf{w}_\Delta|_{\tilde{S}}^2 \ , \end{array}$$

where $C > 0$ is independent of H and h.

□

We are now in the position to prove our main result.

THEOREM 1. *The condition number of the preconditioned augmented FETI-DP algorithm (20) satisfies,*

$$cond(M^{-1}F) \leq C\frac{1}{\beta}(1+log\frac{H}{h})^2 \ ,$$

where C is independent of H and h.

Proof: We will show that

$$4\lambda^T M\lambda \leq \lambda^T F\lambda \leq C\frac{1}{\beta}(1+log\frac{H}{h})^2\lambda^T M\lambda \ , \forall\lambda\in\Lambda \ .$$

Lower bound: From Klawonn et al. [8] or Mandel and Tezaur[10], we have

$$\lambda^T F \lambda = \max_{0 \neq \mathbf{v}_\Delta \in \widetilde{\mathbf{W}}_\Delta} \frac{|(\lambda, B_\Delta \mathbf{v}_\Delta)|^2}{|\mathbf{v}_\Delta|^2_{\tilde{S}}} .$$

From Lemma 2, we know that $B_\Delta^T B_\Delta \widetilde{\mathbf{W}}_\Delta \subset \widetilde{\mathbf{W}}_\Delta$, and from Lemma 4 we know that $|\mathbf{w}_\Delta|_{\tilde{S}} \leq |\mathbf{w}_\Delta|_{S_\Delta}$ for all $\mathbf{w}_\Delta \in B_\Delta^T B_\Delta \widetilde{\mathbf{W}}_\Delta$. Since $B_\Delta B_\Delta^T = 4I$, we have

$$\lambda^T F \lambda \geq \max_{0 \neq \mathbf{w}_\Delta \in \widetilde{\mathbf{W}}_\Delta} \frac{|(\lambda, B_\Delta B_\Delta^T B_\Delta \mathbf{w}_\Delta)|^2}{|B_\Delta^T B_\Delta \mathbf{w}_\Delta|^2_{\tilde{S}}} = 4 \max_{0 \neq \mathbf{w}_\Delta \in \widetilde{\mathbf{W}}_\Delta} \frac{|(\lambda, B_\Delta \mathbf{w}_\Delta)|^2}{|B_\Delta^T B_\Delta \mathbf{w}_\Delta|^2_{S_\Delta}} .$$

Since for any $\nu \in \Lambda$ there is a $\mathbf{w}_\Delta \in \widetilde{\mathbf{W}}_\Delta$ such that $\nu = B_\Delta \mathbf{w}_\Delta$, we have

$$\lambda^T F \lambda \geq 4 \frac{|(\lambda, \nu)|^2}{|B_\Delta^T \nu|^2_{S_\Delta}} .$$

Choosing $\nu = M\lambda$, we find

$$\lambda^T F \lambda \geq 4 \frac{|(\lambda, M\lambda)|^2}{|B_\Delta^T M\lambda|^2_{S_\Delta}} = 4 \frac{(\lambda, M\lambda)^2}{\lambda^T M^T B_\Delta S_\Delta B_\Delta^T M\lambda} = 4 \frac{(\lambda, M\lambda)^2}{\lambda^T M \lambda} = 4\lambda^T M \lambda .$$

Upper bound: Using Lemma 8, we have

$$\begin{aligned}
\lambda^T F \lambda &= \max_{0 \neq \mathbf{v}_\Delta \in \widetilde{\mathbf{W}}_\Delta} \frac{(\lambda, B_\Delta v_\Delta)^2}{|\mathbf{v}_\Delta|^2_{\tilde{S}}} \\
&\leq C \frac{1}{\beta}(1 + log\frac{H}{h})^2 \max_{0 \neq \mathbf{v}_\Delta \in \widetilde{\mathbf{W}}_\Delta} \frac{(\lambda, B_\Delta \mathbf{v}_\Delta)^2}{|B_\Delta^T B_\Delta \mathbf{v}_\Delta|^2_{S_\Delta}} \\
&= C \frac{1}{\beta}(1 + log\frac{H}{h})^2 \max_{0 \neq \mathbf{v}_\Delta \in \widetilde{\mathbf{W}}_\Delta} \frac{(\lambda, B_\Delta \mathbf{v}_\Delta)^2}{(M^{-1} B_\Delta \mathbf{v}_\Delta, B_\Delta \mathbf{v}_\Delta)} \\
&= C \frac{1}{\beta}(1 + log\frac{H}{h})^2 \max_{\nu \in \Lambda} \frac{(\lambda, \nu)^2}{(M^{-1}\nu, \nu)} \\
&= C \frac{1}{\beta}(1 + log\frac{H}{h})^2 (M\lambda, \lambda) .
\end{aligned}$$

□

5. Numerical results. We have tested our algorithm by solving a lid driven cavity problem on the domain $\Omega = [0,1] \times [0,1]$, with $\mathbf{f} = \mathbf{0}$, $g_x = 1, g_y = 0$ for $x \in [0,1], y = 1$, and $\mathbf{g} = \mathbf{0}$ elsewhere on the boundary. We have used both GMRES and CG to solve the preconditioned linear system (20), as well as the nonpreconditioned linear system (11). The initial guess is $\lambda = 0$ and the stopping criterion is $||r_k||_2/||r_0||_2 \leq 10^{-6}$, where r_k is the residual of the Lagrange multipliers at the k-th iteration.

Figure 1 gives the number of GMRES iterations for different number of subdomains with a fixed subdomain problem size $H/h = 8$, and for different subdomain problem size H/h with 4×4 subdomains. We see, from the left figure, that the convergence of the augmented FETI-DP method, with or without a preconditioner, is independent of the number of subdomains, while the preconditioned version needs less iterations. The right figure shows that the GMRES iteration count increases, in both the preconditioned and the nonpreconditioned cases, with the increase of the size of subdomain problem, but that it is growing much slower with the Dirichlet preconditioner than without.

Similar tests were also carried out with a conjugate gradient method, and the results are shown in Figure 2. It is interesting to see that for smaller problem, the nonpreconditioned algorithm behaves better, but for bigger problem the preconditioned version becomes advantageous. The reason is that the condition number of the preconditioned problem is bounded from above by the square of the logarithm of H/h, while the condition number of the nonpreconditioned problem is expected to be bounded only by a linear function of H/h.

In Figure 3, we demonstrate that the coarse saddle point problem in the preconditioner procedure is inf-sup stable, which means that the inf-sup constant of the coarse problem is bounded below from zero, with the increase of the size of the problem.

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FIG. 1. *GMRES iterations counts for the Stokes solver vs. number of subdomains for $H/h = 8$ (left) and vs. H/h for 4×4 subdomains (right)*

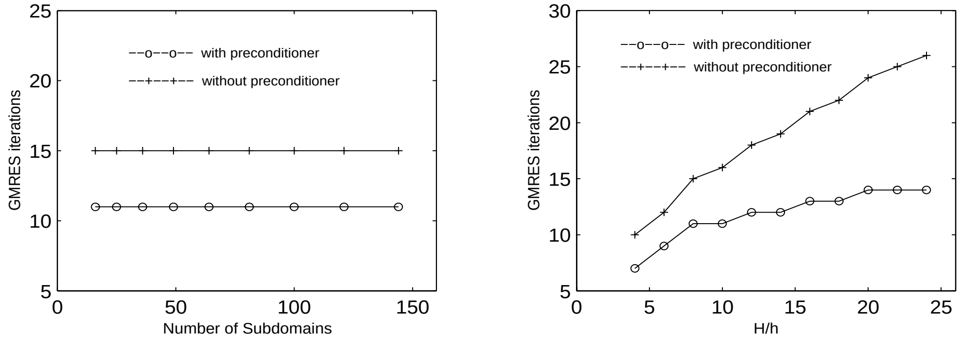

FIG. 2. *CG iterations counts for the Stokes solver vs. number of subdomains for $H/h = 8$ (left) and vs. H/h for 4×4 subdomains (right)*

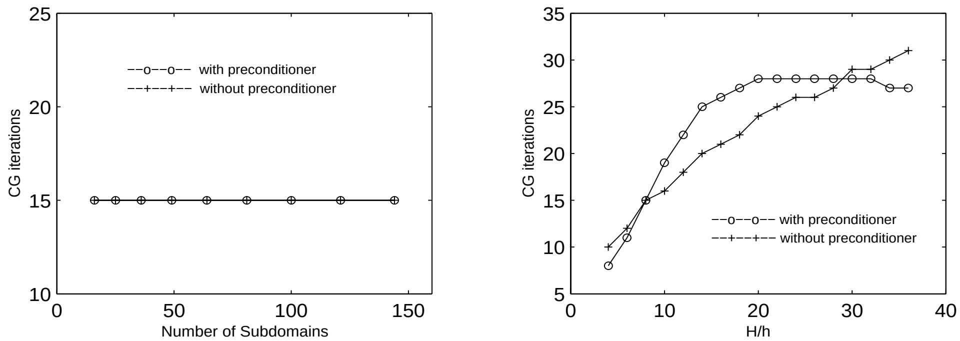

FIG. 3. *Inf-sup constant of the coarse saddle point problem vs. number of subdomains for $H/h = 8$ (left) and vs. H/h for 4×4 subdomains (right)*