Balancing Neumann-Neumann methods for incompressible Stokes equations

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Abstract

Balancing Neumann-Neumann methods are introduced and studied for incompressible Stokes equations discretized with mixed finite or spectral elements with discontinuous pressures. After decomposing the original domain of the problem into nonoverlapping subdomains, the interior unknowns, which are the interior velocity component and all except the constant pressure component, of each subdomain problem are implicitly eliminated. The resulting saddle point Schur complement is solved with a Krylov space method with a balancing Neumann-Neumann preconditioner based on the solution of a coarse Stokes problem with a few degrees of freedom per subdomain and on the solution of local Stokes problems with natural and essential boundary conditions on the subdomains. This preconditioner is of hybrid form in which the coarse problem is treated multiplicatively while the local problems are treated additively. The condition number of the preconditioned operator is independent of the number of subdomains and is bounded from above by the product of the square of the logarithm of the local number of unknowns in each subdomain and the inverse of the inf-sup constants of the discrete problem and of the coarse subproblem. Numerical results show that the method is quite fast; they are also fully consistent with the theory.

Keywords. Stokes equations, mixed finite elements, spectral elements, preconditioned iterations, domain decomposition, balancing Neumann-Neumann methods

AMS(MOS) subject classifications. 65N30, 65N35, 65N55

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1 Introduction

In this paper, we propose and study a balancing Neumann-Neumann domain decomposition method for incompressible Stokes equations. Neumann-Neumann type algorithms, together with FETI type algorithms and overlapping Schwarz methods, have emerged in the last decade among the most efficient and thoroughly tested preconditioned iterative solvers for the parallel solution of very large linear systems arising from finite or spectral element discretizations of elliptic partial differential equations; see, e.g., Gropp, Kaushik, Keyes, and Smith [16] and Bhardwaj, Day, Farhat, Lesoinne, Pierson, and Rixen [5]. We refer to Smith, Bjørstad, and Gropp [35] and Quarteroni and Valli [32] for a general introduction to domain decomposition methods.

Most of the theoretical and numerical work for Neumann-Neumann methods has been carried out for second order elliptic problems; see Mandel [25], Mandel and Brezina [26], Cowsar, Mandel and Wheeler [9], Dryja and Widlund [10], and Pavarino [29]. More recently, this family of methods has been extended to plate and shell problems, see Le Tallec, Mandel, and Vidrascu [22], to convection-diffusion problems, see Alart, Barboteu, Le Tallec, and Vidrascu [3] and Achdou, Le Tallec, Nataf, and Vidrascu [1], and to vector field problems, see Toselli [37]. We also note that the connection between Neumann-Neumann and FETI methods has been considered recently by Klawonn and Widlund [21]. In this paper, we extend the balancing Neumann-Neumann method to symmetric saddle point problems arising from incompressible Stokes equations.

Previous work on domain decomposition methods for incompressible Stokes equations has been based on other iterative substructuring methods, see Bramble and Pasciak [6], Pasciak [28], Quarteroni [31], Marini and Quarteroni [27], Fischer and Rønquist [12], Casarin [8], Rønquist [33], Le Tallec and Patra [23], Pavarino and Widlund [30], and Ainsworth and Sherwin [2], on overlapping Schwarz methods, see Gervasio [14], Fischer [11], Fischer, Miller, and Tufo [13], Klawonn and Pavarino [19], and Rønquist [34], and on block preconditioners, see Klawonn [18, 17], Klawonn and Pavarino [20] and the references therein.

Our balancing Neumann-Neumann preconditioner will be built from a number of components. They include solvers of Dirichlet problems on the subdomains Ω_i, into which the given region Ω has been subdivided. Here we use the fact that these are well-posed problems which are solvable for any boundary values provided that the constraints corresponding to the pressure components which are constant in each subdomain are dropped. We also use Neumann solvers for the subregions to construct local components of the preconditioner. Finally, we use a coarse space correction which maps any velocity field given on the interface Γ, i.e., the union of the interior part of the subdomain boundaries, onto a balanced velocity field, i.e., a vector field with values on Γ which allow a divergence free extension of the given boundary values into each subdomain. We must be careful and choose the coarse space so that it is stable, i.e., satisfies the appropriate inf–sup condition.

It is well known that a coarse space correction is necessary, in addition to many local corrections, in the design of successful, scalable domain decompo-

sition preconditioners; see, e.g., Widlund [38]. The design and study of such coarse spaces are in fact at the core of many contributions to domain decomposition theory. For balancing Neumann–Neumann methods, these issues have again come to the forefront in recent work by Achdou, Le Tallec, Nataf, and Vidrascu [1] and Alart, Barboteu, Le Tallec, and Vidrascu [3].

In the present study, a coarse space will also play a special and important role. Our coarse space correction will return the error vector of the iteration to the benign subspace of balanced velocities; the Stokes operator is symmetric, positive definite on this subspace and this allows us to use a preconditioned conjugate gradient method to achieve a very fast convergence rate.

We also note that we are quite fortunate in that we can use a discontinuous pressure approximation in our finite element approximations of Stokes equations. This makes it possible to regard all except the piece-wise constant pressure components as local variables, together with the interior velocity modes, resulting in a Schur complement system which involves only the interface velocities and one pressure parameter per subdomain. This sets the stage for the construction of a coarse space which includes the piecewise constant pressure components, one for each subdomain, and just a few velocity degrees of freedom for each of them. In our bounds of the condition number of the preconditioned system, the inf–sup constant of this coarse problem, as well as the inf–sup constant of the entire finite element model, will enter; naturally, we will build our preconditioner from stable components representing solvers on subdomains and a coarse, global problem.

The rest of this paper is organized as follows. In section 2, we briefly describe the Stokes system, certain finite and spectral element discretizations, and the resulting discrete system. The substructuring process, also known as static condensation, is described in section 3, both in matrix and variational form. The balancing Neumann-Neumann preconditioner is introduced in section 4, first in matrix form and then in the variational framework needed for the theoretical analysis. The main result of the paper and some auxiliary results are proven in section 5. Section 6 concludes the paper with some numerical results on a model Stokes problem in the plane.

2 The Stokes system: the continuous and discrete problems

Let $\Omega \subset \mathbf{R}^d, d = 2, 3$, be a polyhedral domain and $L^2_0(\Omega)$ be the subspace of $L^2(\Omega)$ of functions with zero mean value. The Stokes problem consists of finding the velocity $\mathbf{u} \in (H^1(\Omega))^d$ and the pressure $p \in L^2_0(\Omega)$ of an incompressible fluid

with viscosity ν by solving:

$$\begin{cases} \nu \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} dx - \int_\Omega \mathrm{div}\mathbf{v}\ p dx &= \int_\Omega \mathbf{f} \cdot \mathbf{v} dx \quad \forall \mathbf{v} \in (H_0^1(\Omega))^d, \\ -\int_\Omega \mathrm{div}\mathbf{u}\ q dx &= 0 \qquad\quad \forall q \in L_0^2(\Omega), \\ \mathbf{u}_{|\partial\Omega} = \mathbf{g}, \end{cases} \tag{1}$$

where $\mathbf{g} \in (H^{1/2}(\partial\Omega))^d$, $\mathbf{f} \in (H^{-1}(\Omega))^d$. We note that if $\int_{\partial\Omega} \mathbf{g} \cdot \mathbf{n} ds = 0$, i.e., $\mathbf{g}$ is balanced, then the solution $\mathbf{u}$ of this Dirichlet problem will be divergence free. For simplicity, we will set $\mathbf{g} = 0$ from now on. We also note that the techniques and results of this paper can be applied to the mixed formulation of the linear elasticity equations as well.

We will also need to consider Stokes problems with natural boundary conditions:

$$\nu \nabla u_i \cdot \mathbf{n} - p n_i = r_i \quad \text{on } \partial\Omega, \quad i = 1, \cdots, d, \tag{2}$$

derived by using Green's formula. In contrast with the case of Dirichlet boundary conditions, the pressure is now uniquely determined and the pressure space should now be taken to be $L^2(\Omega)$. In this case, as for the Laplace operator, each velocity component is determined only up to a constant and there is a compatibility condition between r_i and f_i, namely,

$$\int_\Omega \sum_{i=1}^d f_i\ dx + \int_{\partial\Omega} \sum_{i=1}^d r_i\ ds = 0.$$

We note that if the boundary conditions are mixed, part essential, part natural, then there is a unique solution without any compatibility conditions.

We will consider conforming discretizations of Stokes equations using finite as well as spectral finite elements, both with discontinuous pressures.

2.1 Finite element methods with discontinuous pressures

We assume that the domain Ω can be decomposed into N nonoverlapping subdomains Ω_i of characteristic size H forming a quadrilateral (hexahedral) finite element mesh τ_H. This coarse triangulation is further refined into a fine quadrilateral finite element triangulation τ_h of characteristic size h. Among the many choices of mixed finite elements available for Stokes equations, we consider the following:

a) $Q_2(h) - Q_0(h)$ mixed finite elements: the velocity space $\mathbf{V}$ is discretized by continuous, piecewise quadratic velocities:

$$\mathbf{V}^h = \{\mathbf{v} \in \mathbf{V} : v_{k\,|_T} \in Q_2(T),\ \forall T \in \tau_h,\ \ k = 1, \cdots, d\},$$

while the pressure space is discretized by discontinuous piecewise constant functions on τ_h

$$U^h = \{q \in U : q_{|_T} \in Q_0(T),\ \forall T \in \tau_h\}.$$

These elements satisfy the uniform inf-sup condition

$$\sup_{\mathbf{v}\in\mathbf{V}^h} \frac{(\mathrm{div}\mathbf{v}, q)}{\|\mathbf{v}\|_{H^1}} \geq \beta_h \|q\|_{L^2} \quad \forall q \in U^h, \tag{3}$$

with $\beta_h \geq c > 0$ independent of h, but they lead to nonoptimal error estimates; see Brezzi and Fortin [7, Ch. VI.4, p. 221].

b) $Q_2(h) - P_1(h)$ mixed finite elements: the velocity space is as before, while the pressure space consists of piecewise linear discontinuous pressures:

$$U^h = \{q \in U : q_{|_T} \in P_1(T),\ \forall T \in \tau_h\}.$$

These elements satisfy a uniform inf-sup condition (3) as well; there is also an optimal $O(h^2)$ error estimates for both velocities and pressures; see Brezzi and Fortin [7, Ch. VI, p. 216].

2.2 Spectral element methods: $Q_n - Q_{n-2}$

Let Ω_{ref} be the reference square or cube $(-1,1)^d, d = 2, 3$, and let $Q_n(\Omega_{\mathrm{ref}})$ be the set of polynomials on Ω_{ref} of degree n in each variable. We assume that the domain Ω can be decomposed into N nonoverlapping finite elements Ω_i, each of which is an image $\Omega_i = \phi_i(\Omega_{\mathrm{ref}})$, with ϕ_i an affine mapping. $\mathbf{V}$ is discretized, component by component, by continuous, piecewise polynomials of degree n:

$$\mathbf{V}^n = \{\mathbf{v} \in \mathbf{V} : v_{k|_{\Omega_i}} \circ \phi_i \in Q_n(\Omega_{\mathrm{ref}}),\ i = 1, \cdots, N,\ \ k = 1, \cdots, d\}.$$

The pressure space is discretized by piecewise polynomials of degree $n-2$, discontinuous across the boundaries of the elements Ω_i:

$$U^n = \{q \in U : q_{|_{\Omega_i}} \circ \phi_i \in Q_{n-2}(\Omega_{\mathrm{ref}}),\ i = 1, \cdots, N\}.$$

These mixed spectral elements are implemented using Gauss-Lobatto-Legendre (GLL(n)) quadrature, which also allows the construction of a very convenient nodal tensor-product basis for $\mathbf{V}^n$. Denote by $\{\xi_i\}_{i=0}^n$ the set of GLL(n) points of $[-1,1]$, and by σ_i the quadrature weight associated with ξ_i. Let $l_i(x)$ be the Lagrange interpolating polynomial of degree n which vanishes at all the GLL(n) nodes except at ξ_i, where it equals one. Each element of $Q_n(\Omega_{\mathrm{ref}})$ is expanded in the GLL(n) basis and each L^2-inner product of two scalar components u and v is replaced, in the three dimensional case, by

$$(u,v)_{n,\Omega} = \sum_{s=1}^{N} \sum_{i,j,k=0}^{n} (u \circ \phi_s)(\xi_i, \xi_j, \xi_k)(v \circ \phi_s)(\xi_i, \xi_j, \xi_k)|J_s|\sigma_i\sigma_j\sigma_k,$$

where $|J_s|$ is the determinant of the Jacobian of ϕ_s. The mass matrix based on these basis elements and GLL(n) quadrature are diagonal. Similarly, a very convenient basis for U^n consists of the tensor-product Lagrangian nodal basis functions associated with the internal GLL(n) nodes, i.e., the endpoints -1 and +1 are excluded. We will call these the pressure GLL(n) nodes.

The $Q_n - Q_{n-2}$ method satisfies a nonuniform inf-sup condition

$$\sup_{\mathbf{v}\in\widetilde{\mathbf{V}}^n} \frac{(\mathrm{div}\mathbf{v}, q)}{\|\mathbf{v}\|_{H^1}} \geq \beta_n \|q\|_{L^2} \quad \forall q \in U^n, \tag{4}$$

where $\beta_n = Cn^{-(\frac{d-1}{2})}$, $d = 2, 3$, and the constant C is independent of n and q; see Maday, Meiron, Patera, and Rønquist [24] and Stenberg and Suri [36]. However, numerical experiments, reported in [24], have also shown that for practical values of n, e.g., $n \leq 16$, the inf-sup constant β_n of the $Q_n - Q_{n-2}$ method decays much slower than what would be first expected from the theoretical bound.

An alternative, with a uniform bound on the inf-sup constant, is provided by the $Q_n - P_{n-1}$ method; see Bernardi and Maday [4]. However, this pressure space is less convenient than Q_{n-2} as far as implementation is concerned.

2.3 The discrete system

Let $\widetilde{\mathbf{V}}$ and $\widetilde{U}$ be the discrete velocity and pressure spaces. In the finite element case, we write $\widetilde{\mathbf{V}} \times \widetilde{U} = \mathbf{V}^h \times U^h$, while in the spectral element case we write $\widetilde{\mathbf{V}} \times \widetilde{U} = \mathbf{V}^n \times U^n$. Define the standard bilinear forms

$$a(\mathbf{u}, \mathbf{v}) = \nu \int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v} dx, \qquad b(\mathbf{v}, p) = -\int_\Omega \mathrm{div}\mathbf{v}\ p dx$$

and the linear functional

$$\mathbf{F}(\mathbf{v}) = \int_\Omega \mathbf{f} \cdot \mathbf{v} dx.$$

The discrete system obtained from (1) using finite or spectral elements is:

Find $\mathbf{u} \in \widetilde{\mathbf{V}}$ and $p \in \widetilde{U}$ such that

$$\left\{ \begin{array}{lcccll} a(\mathbf{u}, \mathbf{v}) & + & b(\mathbf{v}, p) & = & \mathbf{F}(\mathbf{v}) & \forall \mathbf{v} \in \widetilde{\mathbf{V}} \\ b(\mathbf{u}, q) & & & = & 0 & \forall q \in \widetilde{U}, \end{array} \right. \tag{5}$$

where we denote with the same letters the bilinear forms obtained using the appropriate quadrature rule described above. In matrix form, we have

$$K \left[\begin{array}{c} \mathbf{u} \\ p \end{array} \right] = \left[\begin{array}{cc} A & B^T \\ B & 0 \end{array} \right] \left[\begin{array}{c} \mathbf{u} \\ p \end{array} \right] = \left[\begin{array}{c} \mathbf{b} \\ 0 \end{array} \right].$$

On the subspace

$$\mathbf{V}_B = KerB = \{\mathbf{v} \in \widetilde{\mathbf{V}} : b(\mathbf{v}, q) = 0 \quad \forall q \in \widetilde{U}\},$$

the Stokes problem (5) is equivalent to the positive definite problem:

Find $\mathbf{u} \in \mathbf{V}_B$ such that

$$a(\mathbf{u}, \mathbf{v}) = \mathbf{F}(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}_B.$$

3 Substructuring for saddle point problems

The domain Ω is decomposed into open, nonoverlapping quadrilateral (hexahedral) subdomains Ω_i and the interface Γ, i.e.,

$$\Omega = \cup_{i=1}^{N} \Omega_i \cup \Gamma.$$

Here $\Gamma = \left(\bigcup_{i=1}^{N} \partial\Omega_i \right) \setminus \partial\Omega$. Each Ω_i typically consists of one, or a few, spectral elements of degree n or of many finite elements. We denote by Γ_h and $\partial\Omega_h$ the set of nodes belonging to the interface Γ and $\partial\Omega$, respectively. The starting point of our algorithm is the implicit elimination of the interior degrees of freedom, i.e., the velocity component that is supported in the open subdomains and what we will call the interior pressure component which has zero average over the individual subdomains. This process, also known as static condensation, is carried out by solving decoupled local Stokes problems on each subdomain Ω_i with Dirichlet boundary conditions for the velocities given on $\partial\Omega_i$. We then obtain a saddle point Schur complement problem for the interface velocities and a constant pressure in each subdomain. This reduced problem will be solved by a preconditioned Krylov space iteration normally the preconditioned conjugate gradient method.

For simplicity, we will use the same letters to denote both functions and their associated vector representations; the same convention will also be used for linear operators and their associated matrix forms.

3.1 Substructuring in matrix form

In order to eliminate the interior degrees of freedom, we reorder the vector of unknowns as

$$\begin{bmatrix} \mathbf{u}_I \\ p_I \\ \mathbf{u}_\Gamma \\ p_0 \end{bmatrix} \quad \begin{array}{l} \text{interior velocities} \\ \text{interior pressures with zero average} \\ \text{interface velocities} \\ \text{constant pressures in each } \Omega_i. \end{array}$$

Then, after using the same permutation, the discrete Stokes system matrix can be written as

$$\begin{bmatrix} K_{II} & K_{\Gamma I}^T \\ K_{\Gamma I} & K_{\Gamma\Gamma} \end{bmatrix} = \left[\begin{array}{cc|cc} A_{II} & B_{II}^T & A_{\Gamma I}^T & 0 \\ B_{II} & 0 & B_{I\Gamma} & 0 \\ \hline A_{\Gamma I} & B_{I\Gamma}^T & A_{\Gamma\Gamma} & B_0^T \\ 0 & 0 & B_0 & 0 \end{array}\right].$$

Eliminating the interior unknowns $\mathbf{u}_I$ and p_I by static condensation, we obtain the saddle point Schur complement system

$$S \begin{bmatrix} \mathbf{u}_\Gamma \\ p_0 \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{b}} \\ 0 \end{bmatrix}, \tag{6}$$

where

$$S = K_{\Gamma\Gamma} - K_{\Gamma I} K_{II}^{-1} K_{\Gamma I}^T =$$

$$= \begin{bmatrix} A_{\Gamma\Gamma} & B_0^T \\ B_0 & 0 \end{bmatrix} - \begin{bmatrix} A_{\Gamma I} & B_{I\Gamma}^T \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_{II} & B_{II}^T \\ B_{II} & 0 \end{bmatrix}^{-1} \begin{bmatrix} A_{\Gamma I}^T & 0 \\ B_{I\Gamma} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} S_\Gamma & B_0^T \\ B_0 & 0 \end{bmatrix},$$

and

$$\begin{bmatrix} \tilde{\mathbf{b}} \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{b}_\Gamma \\ 0 \end{bmatrix} - \begin{bmatrix} A_{\Gamma I} & B_{I\Gamma}^T \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_{II} & B_{II}^T \\ B_{II} & 0 \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{b}_I \\ 0 \end{bmatrix}.$$

By using a second permutation that reorders the interior velocities and pressures subdomain by subdomain, we note that K_{II}^{-1} represents the solution of N decoupled Stokes problems, one for each subdomain and all uniquely solvable, with Dirichlet data given on $\partial\Omega_i$

$$K_{II}^{-1} = \begin{bmatrix} {K_{II}^{(1)}}^{-1} & & 0 \\ & \ddots & \\ 0 & & {K_{II}^{(N)}}^{-1} \end{bmatrix}.$$

This is the matrix associated with the discrete Stokes extension operator $\mathcal{SH}$ described in the next section.

The Schur complement S does not need to be explicitly assembled since only its action Sv on a vector v is needed in a Krylov iteration. This operation essentially only requires the action of K_{II}^{-1} on a vector, i.e., the solution of N decoupled Stokes problems. In other words, Sv is computed by subassembling the actions of the subdomain Schur complements $S^{(i)}$ defined for Ω_i, by

$$S^{(i)} = K_{\Gamma\Gamma}^{(i)} - K_{\Gamma I}^{(i)} (K_{II}^{(i)})^{-1} {K_{\Gamma I}^{(i)}}^T =$$

$$= \begin{bmatrix} A_{\Gamma\Gamma}^{(i)} & {B_0^{(i)}}^T \\ B_0^{(i)} & 0 \end{bmatrix} - \begin{bmatrix} A_{\Gamma I}^{(i)} & {B_{I\Gamma}^{(i)}}^T \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_{II}^{(i)} & {B_{II}^{(i)}}^T \\ B_{II}^{(i)} & 0 \end{bmatrix}^{-1} \begin{bmatrix} {A_{\Gamma I}^{(i)}}^T & 0 \\ B_{I\Gamma}^{(i)} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} S_\Gamma^{(i)} & {B_0^{(i)}}^T \\ B_0^{(i)} & 0 \end{bmatrix}.$$

Once $\begin{bmatrix} \mathbf{u}_\Gamma \\ p_0 \end{bmatrix}$ is known, $\begin{bmatrix} \mathbf{u}_I \\ p_I \end{bmatrix}$ can be found by back-substitution,

$$\begin{bmatrix} \mathbf{u}_I \\ p_I \end{bmatrix} = \begin{bmatrix} A_{II} & B_{II}^T \\ B_{II} & 0 \end{bmatrix}^{-1} \left(\begin{bmatrix} \mathbf{b}_I \\ 0 \end{bmatrix} - \begin{bmatrix} A_{\Gamma I}^T & 0 \\ B_{I\Gamma} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_\Gamma \\ p_0 \end{bmatrix} \right).$$

We note that $\begin{bmatrix} \mathbf{u}_I \\ p_I \end{bmatrix}$ is independent of p_0.

3.2 Iterative substructuring in variational form

The substructuring procedure described in the previous section is associated with the space decomposition

$$\widetilde{\mathbf{V}} \times \widetilde{U} = \oplus_{i=1}^{N} \mathbf{V}_i \times U_i \oplus \mathbf{V}_\Gamma \times U_0,$$

where the interior spaces are defined as

$$\mathbf{V}_i = \widetilde{\mathbf{V}} \cap H_0^1(\Omega_i) \qquad\qquad U_i = \widetilde{U} \cap L_0^2(\Omega_i),$$

and the spaces of interface velocities and coarse pressures, constant in each subdomain, are defined as

$$\mathbf{V}_\Gamma = \mathcal{SH}(\widetilde{\mathbf{V}}) = \{\mathbf{v} \in \widetilde{\mathbf{V}} : \mathbf{v}_{|_{\Omega_i}} = \mathcal{SH}(\mathbf{v}_{|_{\partial\Omega_i}}),\ i = 1, \cdots, N\},$$

$$U_0 = \{q \in \widetilde{U} : q_{|_{\Omega_i}} = \text{ constant},\ i = 1, \cdots, N\}.$$

Here $\mathcal{SH} : \widetilde{\mathbf{V}}_{|\Gamma} \longrightarrow \widetilde{\mathbf{V}}$, is the velocity component of the discrete Stokes harmonic extension operator that maps an interface velocity $\mathbf{u}_\Gamma \in \widetilde{\mathbf{V}}_{|\Gamma}$ onto the solution $(\tilde{\mathbf{u}}, \tilde{p})$ of the following homogeneous Stokes problem, defined on each subdomain separately:

Find $\tilde{\mathbf{u}} \in \widetilde{\mathbf{V}}$ and $\tilde{p} \in \sum_{i=1}^{N} U_i$ such that on each Ω_i

$$\left\{ \begin{array}{lcccl} a(\tilde{\mathbf{u}}, \mathbf{v}) & + & b(\mathbf{v}, \tilde{p}) & = & 0 \quad \forall \mathbf{v} \in \mathbf{V}_i \\ b(\tilde{\mathbf{u}}, q) & & & = & 0 \quad \forall q \in U_i \\ \tilde{\mathbf{u}} = \mathbf{u}_\Gamma & \text{on} & \partial\Omega_i\ . & & \end{array} \right.$$

The following comparison of the energy of the discrete Stokes extensions $\mathcal{SH}$ and the discrete harmonic extensions $\mathcal{H}$ of each velocity component can be found in [15] and [6] for finite element discretizations and in [23] and [8] for spectral element discretizations. We note that the corresponding local bounds, for individual subdomains are equally valid and that the upper bound has an elementary proof.

Lemma 1

$$c\beta a(\mathcal{SH}\mathbf{u}_\Gamma, \mathcal{SH}\mathbf{u}_\Gamma) \leq a(\mathcal{H}\mathbf{u}_\Gamma, \mathcal{H}\mathbf{u}_\Gamma) \leq a(\mathcal{SH}\mathbf{u}_\Gamma, \mathcal{SH}\mathbf{u}_\Gamma) \qquad \forall \mathbf{u}_\Gamma \in \mathbf{V}_\Gamma,$$

where β is the inf-sup constant of the chosen mixed finite element space $\tilde{\mathbf{V}} \times \tilde{U}$.

If we define the interface inner product by

$$s(\mathbf{u}_\Gamma, \mathbf{v}_\Gamma) = a(\mathcal{SH}(\mathbf{u}_\Gamma), \mathcal{SH}(\mathbf{v}_\Gamma)) = \mathbf{u}_\Gamma^T S_\Gamma \mathbf{v}_\Gamma,$$

and by $b_0(\mathbf{u}_\Gamma, p_0)$ the restriction of the other bilinear form to the Stokes harmonic extensions, the variational formulation of the saddle point Schur complement problem (6) can be given by (see Pavarino and Widlund [30, Lemma 7.1]):

Find $\mathbf{u}_\Gamma \in \mathbf{V}_\Gamma$ and $p_0 \in U_0$ such that,

$$\left\{ \begin{array}{lcccl} s(\mathbf{u}_\Gamma, \mathbf{v}_\Gamma) & + & b_0(\mathbf{v}_\Gamma, p_0) & = & \widetilde{\mathbf{F}}(\mathbf{v}_\Gamma) \quad \forall \mathbf{v}_\Gamma \in \mathbf{V}_\Gamma \\ b_0(\mathbf{u}_\Gamma, q_0) & & & = & 0 \quad \forall q_0 \in U_0. \end{array} \right. \tag{7}$$

Problem (7) is equivalent to the positive definite problem:
Find $\mathbf{u}_\Gamma \in \mathbf{V}_{\Gamma,B}$ such that

$$s(\mathbf{u}_\Gamma, \mathbf{v}_\Gamma) = \widetilde{\mathbf{F}}(\mathbf{v}_\Gamma) \quad \forall \mathbf{v}_\Gamma \in \mathbf{V}_{\Gamma,B}, \tag{8}$$

where $\mathbf{V}_{\Gamma,B}$ is the subspace of *balanced* velocities defined by

$$\begin{aligned} \mathbf{V}_{\Gamma,B} &= KerB_0 = \{\mathbf{v}_\Gamma \in \mathbf{V}_\Gamma : B_0 \mathbf{v}_\Gamma = 0\} \\ &= \{\mathbf{v} \in \widetilde{\mathbf{V}} : \mathbf{v}_{|_{\Omega_i}} = \mathcal{SH}(\mathbf{v}_{|_{\partial\Omega_i}}) \text{ and } \int_{\partial\Omega_i} \mathbf{v} \cdot \mathbf{n} = 0,\ i = 1, \cdots, N\}. \end{aligned}$$

The equivalence of problems (7) and (8) follows from the fact that $\mathbf{V}_{\Gamma,B}$ consists of the functions of $\mathcal{SH}(\widetilde{\mathbf{V}})$ that satisfy the zero flux constraints $\{\int_{\partial\Omega_i} \mathbf{v} \cdot \mathbf{n} = 0,\ i = 1, \cdots, N\}$. We obtain (7) by using Lagrange multipliers $p_0 = (p_{0,i})_{i=1}^N$ to enforce the constraints given in the definition of the balanced subspace.

4 A Neumann-Neumann preconditioner

We will solve the saddle point Schur complement problem

$$S \left[\begin{array}{c} \mathbf{u}_\Gamma \\ p_0 \end{array} \right] = \left[\begin{array}{cc} S_\Gamma & B_0^T \\ B_0 & 0 \end{array} \right] \left[\begin{array}{c} \mathbf{u}_\Gamma \\ p_0 \end{array} \right] = \left[\begin{array}{c} \tilde{\mathbf{b}} \\ 0 \end{array} \right] \tag{9}$$

by a preconditioned Krylov space method such as GMRES or PCG. The latter can be applied to this indefinite problem because we will start and keep the iterates in the positive definite subspace $\mathbf{V}_{\Gamma,B}$.

4.1 Matrix form of the preconditioner

We propose a balancing Neumann-Neumann preconditioner based on the solution of a coarse Stokes problem with a few degrees of freedom per subdomain and of local Stokes problems with natural and essential boundary conditions on each subdomain. This preconditioner is of hybrid form in which the coarse problem is treated multiplicatively while the local problems are treated additively; cf. [35, p. 152]. It is closely analogous to the balancing Neumann-Neumann preconditioner for the positive definite case, except that the coarse and local problems are saddle point problems. The matrix form of the preconditioner is

$$Q = Q_H + (I - Q_H S) \sum_{i=1}^{N} Q_i (I - S Q_H),$$

where the coarse operator Q_H and local operators Q_i are defined below. The preconditioned operator is then

$$T = QS = T_0 + (I - T_0) \sum_{i=1}^{N} T_i (I - T_0),$$

where $T_0 = Q_H S$ and $T_i = Q_i S$. Q can be written as a three-step preconditioner similar to Smith, Bjørstad, and Gropp [35, p. 126]:

Given a residual vector r the preconditioned vector $\hat{r} = Qr$ is given by

$$\begin{aligned} t &\leftarrow Q_H r, \\ t &\leftarrow \sum_{i=1}^{N} Q_i (r - St), \\ \hat{r} &\leftarrow t + Q_H (r - St). \end{aligned}$$

We note that the first step can be left out if r is a residual in the range of $S(I - T_0)$; we will see that T_0 is a projection and that therefore $T_0(I - T_0) = 0$.

Coarse solver: Given a residual vector r, the coarse term $Q_H r$ is the solution of a coarse, global Stokes problem with a few velocity degrees of freedom and one constant pressure per subdomain Ω_i:

$$Q_H = R_H^T S_0^{-1} R_H,$$

where

$$R_H = \begin{bmatrix} L_0^T & 0 \\ 0 & I \end{bmatrix},$$

and

$$S_0 = R_H S R_H^T = \begin{bmatrix} L_0^T S_\Gamma L_0 & L_0^T B_0^T \\ B_0 L_0 & 0 \end{bmatrix}. \tag{10}$$

We will consider four choices for the matrix L_0. Some of the columns of L_0 are defined in terms of the Neumann-Neumann counting functions μ_i associated with each subdomain Ω_i: μ_i is zero at the interface nodes outside $\partial\Omega_i$ while its value at any node on $\partial\Omega_i$ equals the number of subdomains shared by that node. Its pseudo inverse $\mu_i^\dagger$ is the function $1/\mu_i(x)$ for all nodes where $\mu_i(x) \neq 0$, and it vanishes at all other points of $\Gamma_h \cup \partial\Omega_h$. We note that we use the function $\mu_i^\dagger$ in all or almost all of the subdomains and for each velocity component. Then the columns of L_0 are defined by one of the following four choices:

0) the inverse counting functions $\mu_i^\dagger$,

1) the $\mu_i^\dagger$ and the continuous coarse piecewise bi- or tri-linear functions,

2) the $\mu_i^\dagger$ and the continuous coarse piecewise bi- or tri-quadratic functions,

3) the $\mu_i^\dagger$ and the quadratic coarse edge/face bubble functions for the normal direction.

Choice 0) provides a quite minimal coarse velocity space and it turns out not to be inf-sup stable; it is included since it is the standard choice for the laplacian. Choices 1), 2), and 3) are enrichments of 0) that all turn out to be inf-sup stable uniformly in N. In particular, 2) is a very natural choice because it is based directly on a well known discretization of the Stokes problem on the coarse finite or spectral elements. We note that 1) and 2) might not be easy to adapt to unstructured subdomain meshes produced by automatic mesh partitioners. This is one of the reasons why we also have considered choice 3), which should be relatively easy to adapt for general unstructured subdomains.

In order to avoid linearly dependent $\mu_i^\dagger$ functions, and hence a singular coarse space problem, we might have to drop all of the components of these functions for one subdomain. This depends on the coarse triangulation: on a regular hexahedral mesh in three dimensions, we should, e.g., include at most $3(N-1)$ such functions.

We will use the same symbols μ_i and $\mu_i^\dagger$ for both the interface vectors and the functions obtained by extending them inside each subdomain by using the discrete Stokes harmonic extension operator $\mathcal{SH}$.

We can explicitly represent the velocity and pressure components of the coarse operator $Q_H S$ in matrix form.

Lemma 2 *Let* $\widetilde{S}_0 = L_0^T S_\Gamma L_0$, $\widetilde{S} = B_0 L_0 \widetilde{S}_0^{-1} L_0^T B_0^T$ *and consider* $\begin{bmatrix} \mathbf{v}_\Gamma \\ \eta \end{bmatrix} = (I - Q_H S) \begin{bmatrix} \mathbf{u}_\Gamma \\ \lambda \end{bmatrix}$. *Then,*

$$\begin{aligned} \mathbf{v}_\Gamma &= (I - \mathcal{P}_0 - \widetilde{\mathcal{P}})\mathbf{u}_\Gamma, \\ \eta &= G\mathbf{u}_\Gamma, \end{aligned}$$

where $\mathcal{P}_0$ *and* $\widetilde{\mathcal{P}}$ *are projections defined by*

$$\begin{aligned} \mathcal{P}_0 &= L_0 \widetilde{S}_0^{-1} L_0^T S_\Gamma - L_0 \widetilde{S}_0^{-1} L_0^T B_0^T \widetilde{S}^{-1} B_0 L_0 \widetilde{S}_0^{-1} L_0^T S_\Gamma, \\ \widetilde{\mathcal{P}} &= L_0 \widetilde{S}_0^{-1} L_0^T B_0^T \widetilde{S}^{-1} B_0, \end{aligned}$$

and

$$G = -\widetilde{S}^{-1} B_0 L_0 \widetilde{S}_0^{-1} L_0^T S_\Gamma + \widetilde{S}^{-1} B_0.$$

We note that $\widetilde{\mathcal{P}}\mathbf{u}_\Gamma = 0$ if $\mathbf{u}_\Gamma$ is balanced and that $(I - \mathcal{P}_0 - \widetilde{\mathcal{P}})\mathbf{u}_\Gamma$ is balanced because $B_0(I - \mathcal{P}_0 - \widetilde{\mathcal{P}}) = 0$. Moreover, we note that $\mathbf{v}_\Gamma$ and η are independent of λ, and that $\mathcal{P}_0^2 = \mathcal{P}_0, \widetilde{\mathcal{P}}^2 = \widetilde{\mathcal{P}}$, and $\widetilde{\mathcal{P}}\mathcal{P}_0 = \mathcal{P}_0\widetilde{\mathcal{P}} = 0$.

Proof. These formulas follow by explicitly computing the inverse of the coarse matrix (10) by using the following explicit formula, proven by a direct computation. Let A be an invertible matrix and let B have full row rank. Then,

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} - A^{-1}B^T S^{-1} B A^{-1} & A^{-1}B^T S^{-1} \\ S^{-1} B A^{-1} & -S^{-1} \end{bmatrix},$$

where $S = BA^{-1}B^T$.

□

Local solver: The local operators Q_i will only be applied to residuals of balanced velocity fields and thus the second residual component will vanish. We have also shown in Lemma 2 that the pressure components obtained in this step of the preconditioner plays no further role when we next apply the operator $(I - T_0)$. Each local operator Q_i is based on the solution of a local Stokes problem on Ω_i with natural boundary condition. This local problem is nonsingular for any subdomain Ω_i the boundary of which intersects $\partial\Omega$, but it is singular otherwise, i.e., for the *floating* subdomains. In the latter case any constant velocity belongs to the null space, while the pressure is now uniquely determined because a pressure term is present in the Neumann boundary condition; see (2). To avoid possible complications with singular problems, we modify the local Stokes problems on the floating subdomains, by adding ϵ times the velocity mass matrix to the local stiffness matrix $K^{(i)}$. We could also make these solutions unique by requiring that each velocity component has a zero average over Ω_i; the right hand sides will always be compatible.

Given a residual vector with a first component r_Γ and a zero second component, $Q_i r$ is the weighted solutions of a local Stokes problem on subdomain Ω_i with a natural boundary condition on $\partial\Omega_i \setminus \partial\Omega$:

$$Q_i r = \begin{bmatrix} R_i^T D_i^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} S_{\Gamma,\epsilon}^{(i)} & {B_0^{(i)}}^T \\ B_0^{(i)} & 0 \end{bmatrix}^{-1} \begin{bmatrix} D_i^{-1} R_i & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} r_\Gamma \\ 0 \end{bmatrix}. \qquad (11)$$

Here R_i are $0, 1$ restriction matrices mapping r_Γ into r_{Γ_i} and D_i are diagonal matrices representing multiplication by the counting functions μ_i. Moreover,

$$S_\epsilon^{(i)} = \begin{bmatrix} S_{\Gamma,\epsilon}^{(i)} & {B_0^{(i)}}^T \\ B_0^{(i)} & 0 \end{bmatrix}$$

is the local saddle point Schur complement, associated with subdomain Ω_i, of the regularized local stiffness matrix

$$K_\epsilon^{(i)} = \begin{bmatrix} A_{II,\epsilon}^{(i)} & {B_{II}^{(i)}}^T & {A_{\Gamma I,\epsilon}^{(i)}}^T & 0 \\ B_{II}^{(i)} & 0 & B_{I\Gamma}^{(i)} & 0 \\ A_{\Gamma I,\epsilon}^{(i)} & {B_{I\Gamma}^{(i)}}^T & A_{\Gamma\Gamma,\epsilon}^{(i)} & {B_0^{(i)}}^T \\ 0 & 0 & B_0^{(i)} & 0 \end{bmatrix},$$

where

$$A_\epsilon^{(i)} = A^{(i)} + \epsilon M^{(i)}.$$

ϵ is a positive parameter, and $M^{(i)}$ is the local velocity mass matrix.

4.2 Variational formulation of the preconditioner

The balancing Neumann-Neumann preconditioner introduced in the previous section is associated with further decomposing the interface space $\mathbf{V}_\Gamma \times U_0$ as

$$\mathbf{V}_\Gamma \times U_0 = \mathbf{V}_0 \times U_0 + \sum_{i=1}^{N} \mathbf{V}_{\Gamma_i} \times U_{0,i},$$

where the coarse and local spaces are defined below.

Coarse space: We will consider the following four choices for the coarse velocity space $\mathbf{V}_0$

$$\begin{aligned}
\mathbf{V}_0^0 &= \{\mathbf{v} \in \mathbf{V}_\Gamma : \mathbf{v} \in span\{\mu_i^\dagger\}\},\\
\mathbf{V}_0^1 &= \{\mathbf{v} \in \mathbf{V}_\Gamma : \mathbf{v} \in span\{\mu_i^\dagger\} \cup Q_1(\Gamma)\},\\
\mathbf{V}_0^2 &= \{\mathbf{v} \in \mathbf{V}_\Gamma : \mathbf{v} \in span\{\mu_i^\dagger\} \cup Q_2(\Gamma)\},\\
\mathbf{V}_0^3 &= \{\mathbf{v} \in \mathbf{V}_\Gamma : \mathbf{v} \in span\{\mu_i^\dagger\} \cup span\{normal\ direction\ edge/face\ bubble\ functions\}\}.
\end{aligned}$$

Here $Q_1(\Gamma)$ and $Q_2(\Gamma)$ are the subspaces of $\mathbf{V}_\Gamma$ of piecewise linear and quadratic polynomials on Γ, respectively. All these coarse spaces contain the kernel of the diagonally scaled local Stokes problems on the floating subdomains Ω_i given in formula (13) that are used in the definition of the local components of the preconditioner.

Coarse problem: Given $\begin{bmatrix} \mathbf{u}_\Gamma \\ \lambda \end{bmatrix} \in \mathbf{V}_\Gamma \times U_0$, define $\begin{bmatrix} \mathbf{u}_0 \\ \eta \end{bmatrix} = T_0 \begin{bmatrix} \mathbf{u}_\Gamma \\ \lambda \end{bmatrix} \in \mathbf{V}_0 \times U_0$ as the solution of the coarse Stokes problem

$$\left\{ \begin{array}{lcccll} s(\mathbf{u}_0 - \mathbf{u}_\Gamma, \mathbf{v}_0) & + & b_0(\mathbf{v}_0, \eta) & = & 0 & \forall \mathbf{v}_0 \in \mathbf{V}_0 \\ b_0(\mathbf{u}_0 - \mathbf{u}_\Gamma, q) & & & = & 0 & \forall q \in U_0. \end{array} \right. \tag{12}$$

We recall that the matrix form of this operator is $T_0 = Q_H S = R_H^T S_0^{-1} R_H S$ where S_0 is given by (10).

Local spaces: The local spaces are defined by

$$\mathbf{V}_{\Gamma,i} = \{\mathbf{v} \in \mathbf{V}_\Gamma : \quad \mathbf{v}(\mathbf{x}) = 0 \ \ \forall x \in \Gamma_h \setminus \partial\Omega_{i,h}\}, \qquad U_{0,i} = span\{q_{0,i}\}.$$

Local problems: We need to apply local Stokes solvers only to elements $\mathbf{w} = \begin{bmatrix} \mathbf{u}_\Gamma \\ \lambda \end{bmatrix}$ of $Range(I - T_0)$. Since $(I - T_0)$ is a projection on its range, we have $\mathbf{w} = \begin{bmatrix} \mathbf{u}_\Gamma \\ \lambda \end{bmatrix} = (I - T_0) \begin{bmatrix} \mathbf{u}_\Gamma \\ \lambda \end{bmatrix} = (I - T_0) \begin{bmatrix} \mathbf{u}_\Gamma \\ 0 \end{bmatrix}$; according to Lemma 2, $\mathbf{u}_\Gamma = (I - \mathcal{P}_0)\mathbf{u}_\Gamma$ is balanced and $\lambda = G\mathbf{u}_\Gamma$.

We define $T_i \mathbf{w} = T_i \mathbf{u}_\Gamma = \begin{bmatrix} T_i^u \mathbf{u}_\Gamma \\ T_i^p \mathbf{u}_\Gamma \end{bmatrix} \in \mathbf{V}_{\Gamma,i} \times U_{0,i}$ as the solution of a local Stokes problem with natural boundary conditions: $\forall \mathbf{v}_i \in \mathbf{V}_{\Gamma,i}, \forall q \in U_{0,i}$,

$$\left\{ \begin{array}{lcl} s_{i,\epsilon}(\mu_i T_i^u \mathbf{u}_\Gamma, \mu_i \mathbf{v}_i) + b_{0,i}(\mu_i \mathbf{v}_i, T_i^p \mathbf{u}_\Gamma) & = & s(\mathbf{u}_\Gamma, \mathbf{v}_i) + b_0(\mathbf{v}_i, G\mathbf{u}_\Gamma) \\ b_{0,i}(\mu_i T_i^u \mathbf{u}_\Gamma, q) & = & b_0(\mathbf{u}_\Gamma, q) \end{array} \right. \tag{13}$$

Here ϵ is a positive parameter,

$$s_{i,\epsilon}(\mathbf{u}_\Gamma, \mathbf{v}_\Gamma) = a_{i,\epsilon}(\mathcal{SH}_{i,\epsilon}(\mathbf{u}_\Gamma), \mathcal{SH}_{i,\epsilon}(\mathbf{v}_\Gamma)),$$

and

$$a_{i,\epsilon}(\mathbf{u}, \mathbf{v}) = \int_{\Omega_i} \nabla \mathbf{u} : \nabla \mathbf{v} dx + \epsilon \int_{\Omega_i} \mathbf{u} \cdot \mathbf{v} dx, \tag{14}$$

and $\mathcal{SH}_{i,\epsilon}$ is the velocity component of the discrete Stokes extension operator defined in terms of the regularized $a_{i,\epsilon}(\cdot,\cdot)$ velocity bilinear form instead of the standard $a(\cdot,\cdot)$ form. By $\mu_i T_i^u \mathbf{u}_\Gamma$ and $\mu_i \mathbf{v}_i$ we mean the finite or spectral element function with nodal values equal to the product of those of μ_i and the other function.

We recall that the matrix form of the operators T_i is given by $T_i = Q_i S$, with Q_i given in (11).

From the second equation in (13), we find that $\mu_i T_i^u \mathbf{u}_\Gamma$ is balanced over $\partial\Omega_i$ since $\mathbf{u}_\Gamma$ is balanced. If we also consider test functions in the same subspace, we see that the velocity component $T_i^u \mathbf{u}_\Gamma$ satisfies: $\forall \mathbf{v}_i \in \mathbf{V}_{\Gamma,i}$ such that $\mu_i \mathbf{v}_i$ is balanced over $\partial\Omega_i$:

$$s_{i,\epsilon}(\mu_i T_i^u \mathbf{u}_\Gamma, \mu_i \mathbf{v}_i) = s(\mathbf{u}_\Gamma, \mathbf{v}_i) + b_0(\mathbf{v}_i, G\mathbf{u}_\Gamma). \tag{15}$$

Given that the right-hand side of (13) is compatible, even for $\epsilon = 0$, we could alternatively choose to solve this singular system and enforce the constraints

$$\int_{\Omega_i} \mathcal{SH}(\mu_i T_i^u \mathbf{u}_\Gamma) dx = 0$$

for each floating subdomain.

5 Auxiliary and main results

We need a few more auxiliary results before we can formulate and prove our main result. We will work with the S_Γ-inner product

$$\Big\langle \mathbf{u}_\Gamma, \mathbf{v}_\Gamma \Big\rangle_{S_\Gamma} = s(\mathbf{u}_\Gamma, \mathbf{v}_\Gamma) = \mathbf{u}_\Gamma^T S_\Gamma \mathbf{v}_\Gamma.$$

On the balanced subspace $\mathbf{V}_{\Gamma,B} \times U_0$ this expression coincides with the bilinear form defined by S

$$\left\langle \begin{bmatrix} \mathbf{u}_\Gamma \\ \lambda \end{bmatrix}, \begin{bmatrix} \mathbf{v}_\Gamma \\ \eta \end{bmatrix} \right\rangle_S = \left\langle S \begin{bmatrix} \mathbf{u}_\Gamma \\ \lambda \end{bmatrix}, \begin{bmatrix} \mathbf{v}_\Gamma \\ \eta \end{bmatrix} \right\rangle,$$

since $B_0 \mathbf{u}_\Gamma = B_0 \mathbf{v}_\Gamma = 0$.

Lemma 3 *Let* $\begin{bmatrix} \mathbf{u}_\Gamma \\ \lambda \end{bmatrix} \in \mathit{Range}(I - T_0)$ *and let* $\begin{bmatrix} \mathbf{v}_\Gamma \\ \eta \end{bmatrix}$ *be arbitrary. Then,*

$$\begin{aligned}
a) \quad \left\langle S \begin{bmatrix} \mathbf{u}_\Gamma \\ \lambda \end{bmatrix}, \begin{bmatrix} \mathbf{v}_\Gamma \\ \eta \end{bmatrix} \right\rangle &= \left\langle S(I - T_0) \begin{bmatrix} \mathbf{u}_\Gamma \\ \lambda \end{bmatrix}, \begin{bmatrix} \mathbf{v}_\Gamma \\ \eta \end{bmatrix} \right\rangle \\
&= \left\langle S \begin{bmatrix} \mathbf{u}_\Gamma \\ \lambda \end{bmatrix}, (I - T_0) \begin{bmatrix} \mathbf{v}_\Gamma \\ \eta \end{bmatrix} \right\rangle; \\
b) \quad \left\langle S \begin{bmatrix} \mathbf{u}_\Gamma \\ \lambda \end{bmatrix}, \begin{bmatrix} \mathbf{v}_\Gamma \\ \eta \end{bmatrix} \right\rangle &= \left\langle S \begin{bmatrix} \mathbf{u}_\Gamma \\ \lambda \end{bmatrix}, \begin{bmatrix} \mathbf{v}_\Gamma \\ 0 \end{bmatrix} \right\rangle; \\
c) \quad \left\langle S \begin{bmatrix} \mathbf{u}_\Gamma \\ \lambda \end{bmatrix}, \begin{bmatrix} \mathbf{v}_\Gamma \\ \eta \end{bmatrix} \right\rangle &= \left\langle S_\Gamma \mathbf{u}_\Gamma, (I - \mathcal{P}_0 - \widetilde{\mathcal{P}}) \mathbf{v}_\Gamma \right\rangle \\
&= \left\langle S_\Gamma (I - \mathcal{P}_0 - \widetilde{\mathcal{P}}) \mathbf{u}_\Gamma, \mathbf{v}_\Gamma \right\rangle \\
&= \left\langle S_\Gamma (I - \mathcal{P}_0) \mathbf{u}_\Gamma, \mathbf{v}_\Gamma \right\rangle,
\end{aligned}$$

where $(I - \mathcal{P}_0 - \widetilde{\mathcal{P}})\mathbf{v}_\Gamma$ *is the velocity component of*

$$(I - T_0) \begin{bmatrix} \mathbf{v}_\Gamma \\ \eta \end{bmatrix} = (I - T_0) \begin{bmatrix} \mathbf{v}_\Gamma \\ 0 \end{bmatrix}.$$

Proof. a) Since $\begin{bmatrix} \mathbf{u}_\Gamma \\ \lambda \end{bmatrix} \in \mathit{Range}(I - T_0)$ and $I - T_0$ is a projection onto that subspace, we have $\begin{bmatrix} \mathbf{u}_\Gamma \\ \lambda \end{bmatrix} = (I - T_0) \begin{bmatrix} \mathbf{u}_\Gamma \\ \lambda \end{bmatrix}$. Since S and Q_H are symmetric and $T_0 = Q_H S$, we have

$$\begin{aligned}
\left\langle S(I - Q_H S) \begin{bmatrix} \mathbf{u}_\Gamma \\ \lambda \end{bmatrix}, \begin{bmatrix} \mathbf{v}_\Gamma \\ \eta \end{bmatrix} \right\rangle &= \left\langle (I - Q_H S) \begin{bmatrix} \mathbf{u}_\Gamma \\ \lambda \end{bmatrix}, S \begin{bmatrix} \mathbf{v}_\Gamma \\ \eta \end{bmatrix} \right\rangle \\
&= \left\langle S \begin{bmatrix} \mathbf{u}_\Gamma \\ \lambda \end{bmatrix}, (I - Q_H S) \begin{bmatrix} \mathbf{v}_\Gamma \\ \eta \end{bmatrix} \right\rangle.
\end{aligned}$$

b) Since $\begin{bmatrix} \mathbf{u}_\Gamma \\ \lambda \end{bmatrix} \in \mathit{Range}(I - T_0)$, $\mathbf{u}_\Gamma$ is balanced, and we find that

$$\begin{aligned}
\left\langle S \begin{bmatrix} \mathbf{u}_\Gamma \\ \lambda \end{bmatrix}, \begin{bmatrix} \mathbf{v}_\Gamma \\ \eta \end{bmatrix} \right\rangle &= \left\langle \begin{bmatrix} S_\Gamma \mathbf{u}_\Gamma + B_0^T \lambda \\ 0 \end{bmatrix}, \begin{bmatrix} \mathbf{v}_\Gamma \\ \eta \end{bmatrix} \right\rangle \\
= \left\langle \begin{bmatrix} S_\Gamma \mathbf{u}_\Gamma + B_0^T \lambda \\ 0 \end{bmatrix}, \begin{bmatrix} \mathbf{v}_\Gamma \\ 0 \end{bmatrix} \right\rangle &= \left\langle S \begin{bmatrix} \mathbf{u}_\Gamma \\ \lambda \end{bmatrix}, \begin{bmatrix} \mathbf{v}_\Gamma \\ 0 \end{bmatrix} \right\rangle.
\end{aligned}$$

c) The first formula is obtained by applying a) and b) and Lemma 2:

$$\begin{aligned}
\left\langle S \begin{bmatrix} \mathbf{u}_\Gamma \\ \lambda \end{bmatrix}, \begin{bmatrix} \mathbf{v}_\Gamma \\ \eta \end{bmatrix} \right\rangle &= \left\langle S \begin{bmatrix} \mathbf{u}_\Gamma \\ \lambda \end{bmatrix}, (I - T_0) \begin{bmatrix} \mathbf{v}_\Gamma \\ \eta \end{bmatrix} \right\rangle \\
= \left\langle \begin{bmatrix} S_\Gamma \mathbf{u}_\Gamma + B_0^T \lambda \\ 0 \end{bmatrix}, (I - T_0) \begin{bmatrix} \mathbf{v}_\Gamma \\ \eta \end{bmatrix} \right\rangle &= \mathbf{u}_\Gamma^T S_\Gamma (I - \mathcal{P}_0 - \widetilde{\mathcal{P}}) \mathbf{v}_\Gamma + \lambda^T B_0 (I - \mathcal{P}_0 - \widetilde{\mathcal{P}}) \mathbf{v}_\Gamma
\end{aligned}$$

$$= \mathbf{u}_\Gamma^T S_\Gamma (I - \mathcal{P}_0 - \widetilde{\mathcal{P}}) \mathbf{v}_\Gamma = \Big\langle S_\Gamma \mathbf{u}_\Gamma, (I - \mathcal{P}_0 - \widetilde{\mathcal{P}}) \mathbf{v}_\Gamma \Big\rangle.$$

The second formula follows similarly and the third follows from the second since $B_0 \mathbf{u}_\Gamma = 0$ and therefore $\widetilde{\mathcal{P}} \mathbf{u}_\Gamma = 0$.

□

Since $\mathbf{V}_0^2$ essentially is an enrichment of $Q_2(\Gamma)$, the following lemma follows from the stability of the well-known $Q_2 - Q_0$ finite elements and the stability of the discrete Stokes extension; see Pavarino and Widlund [30, Lemma 7.2] for a proof for the spectral element case and Casarin [8, Ch. 5] for related results.

Lemma 4 *The coarse space $\mathbf{V}_0^2 \times U_0$ satisfies the inf-sup condition*

$$\sup_{\mathbf{v}_\Gamma \in \mathbf{V}_0^2} \frac{(\mathrm{div} \mathcal{SH}(\mathbf{v}_\Gamma), q_0)^2}{a(\mathcal{SH}(\mathbf{v}_\Gamma), \mathcal{SH}(\mathbf{v}_\Gamma))} \geq \beta_0^2 \|q_0\|_{L^2}^2 \qquad \forall q_0 \in U_0,$$

where β_0 is independent of h, n, N.

Our numerical results, reported in section 6, indicate that a uniform inf-sup condition does not hold for the first coarse space $\mathbf{V}_0^0 \times U_0$. The results for all the others are quite satisfactory although we do not have a full theory. We note that the $Q_1 - Q_0$ elements by themselves are not inf-sup table but that we are using a richer velocity space which includes the $\mu_i^\dagger$ functions. We also work in the somewhat different context of Stokes extensions of traces on Γ.

Lemma 5

$$\|I - \mathcal{P}_0 - \widetilde{\mathcal{P}}\|_{S_\Gamma} \leq 2(1 + \frac{1}{\beta_0})$$

where β_0 is the inf-sup constant of the coarse space.

Proof. Any inf-sup stable problem satisfies a stability estimate given in Brezzi and Fortin [7, Th. 1.1, p. 42, eq. 1.29]):

$$\|\mathbf{u}_0\|_{S_\Gamma} \leq \sup_{\mathbf{v}_0 \in \mathbf{V}_0^2} \frac{s(\mathbf{u}_\Gamma, \mathbf{v}_0)}{\|\mathbf{v}_0\|_{S_\Gamma}} + \frac{2}{\beta_0} \sup_{q \in U_0} \frac{b_0(\mathbf{u}_\Gamma, q)}{\|q\|_{L^2}},$$

where $s(\mathbf{u}_\Gamma, \mathbf{v}_0)$ and $b_0(\mathbf{u}_\Gamma, q)$ are the two components of the right-hand side of the coarse problem (12). By the continuity of these two forms, it then follows

$$\|\mathbf{u}_0\|_{S_\Gamma} \leq (1 + \frac{2}{\beta_0}) \|\mathbf{u}_\Gamma\|_{S_\Gamma}.$$

By Lemma 2, $\mathbf{u}_0 = (\mathcal{P}_0 + \widetilde{\mathcal{P}}) \mathbf{u}_\Gamma$; hence

$$\|\mathbf{u}_0\|_{S_\Gamma} = \|(\mathcal{P}_0 + \widetilde{\mathcal{P}}) \mathbf{u}_\Gamma\|_{S_\Gamma} \leq (1 + \frac{2}{\beta_0}) \|\mathbf{u}_\Gamma\|_{S_\Gamma},$$

and therefore

$$\|I - \mathcal{P}_0 - \widetilde{\mathcal{P}}\|_{S_\Gamma} \leq \|I\|_{S_\Gamma} + \|\mathcal{P}_0 + \widetilde{\mathcal{P}}\|_{S_\Gamma} \leq 2(1 + \frac{1}{\beta_0}).$$

□

Theorem 1 *On the balanced subspace $\mathbf{V}_{\Gamma,B}\times U_0$ the balancing Neumann-Neumann operator T is symmetric positive definite with respect to the S bilinear form and*

$$cond(T) \leq C(1+\frac{1}{\beta_0})\frac{1}{\beta}\ \alpha,$$

where

$$\alpha = \begin{cases} (1+\log(H/h))^2 & \text{for finite elements} \\ \\ (1+\log n)^2 & \text{for spectral elements,} \end{cases}$$

β_0 and β are the inf-sup constants of the coarse problem and the original discrete Stokes problem, respectively.

Proof. Let $\mathbf{u}_\Gamma$ be balanced and let $\mathbf{w} = \begin{bmatrix} \mathbf{u}_\Gamma \\ \lambda \end{bmatrix}$. Then,

$$\begin{aligned} \Big\langle T\mathbf{w},\mathbf{w}\Big\rangle_S &= \Big\langle T_0\mathbf{w},\mathbf{w}\Big\rangle_S + \Big\langle (I-T_0)\sum_i T_i(I-T_0)\mathbf{w},\mathbf{w}\Big\rangle_S \\ &= \Big\langle T_0\mathbf{w},T_0\mathbf{w}\Big\rangle_S + \Big\langle \sum_i T_i(I-T_0)\mathbf{w},(I-T_0)\mathbf{w}\Big\rangle_S, \end{aligned} \tag{16}$$

because $\Big\langle T_0\mathbf{w},(I-T_0)\mathbf{w}\Big\rangle_S = 0$. Since T_0 is an orthogonal projection on the balanced subspace, we are left with providing a lower and upper bound for $\sum_i T_i$ on $Range(I-T_0)$; once such an inequality has been obtained we will simply add $\Big\langle T_0\mathbf{w},T_0\mathbf{w}\Big\rangle_S$ to each side of the inequalities. We can limit ourselves to proving lower and upper bounds for only the velocity component $\sum_i T_i^u$. Since $\mathbf{u}_\Gamma$ is balanced, we have $(I-T_0)\mathbf{w} = \begin{bmatrix} (I-\mathcal{P}_0)\mathbf{u}_\Gamma \\ G\mathbf{u}_\Gamma \end{bmatrix}$ and the velocity component of

$$\sum_i T_i(I-T_0)\mathbf{w} = \begin{bmatrix} \sum_i Q_i & 0 \\ 0 & 0 \end{bmatrix} S \begin{bmatrix} (I-\mathcal{P}_0)\mathbf{u}_\Gamma \\ G\mathbf{u}_\Gamma \end{bmatrix}$$

is

$$\sum_i Q_i[S_\Gamma(I-\mathcal{P}_0)\mathbf{u}_\Gamma + B_0^T G\mathbf{u}_\Gamma] = \sum_i T_i^u\mathbf{u}_\Gamma.$$

Therefore, by Lemma 3c,

$$\begin{aligned} \Big\langle \sum_i T_i(I-T_0)\mathbf{w},(I-T_0)\mathbf{w}\Big\rangle_S &= \Big\langle \sum_i Q_i[S_\Gamma(I-\mathcal{P}_0)\mathbf{u}_\Gamma + B_0^T G\mathbf{u}_\Gamma], S_\Gamma(I-\mathcal{P}_0)\mathbf{u}_\Gamma\Big\rangle \\ &= \Big\langle \sum_i T_i^u\mathbf{u}_\Gamma,(I-\mathcal{P}_0)\mathbf{u}_\Gamma\Big\rangle_{S_\Gamma} = \Big\langle (I-\mathcal{P}_0-\widetilde{\mathcal{P}})\sum_i T_i^u\mathbf{u}_\Gamma,\mathbf{u}_\Gamma\Big\rangle_{S_\Gamma}. \end{aligned} \tag{17}$$

Lower bound: Let $\begin{bmatrix} \mathbf{u}_\Gamma \\ \lambda \end{bmatrix}$ be an element in $Range(I - T_0)$; then $\mathbf{u}_\Gamma \in \mathbf{V}_{\Gamma,B}$. Since the pseudoinverses $\mu_i^\dagger$ of the counting functions define a partition of unity, we have $\mathbf{u}_\Gamma = \sum_{i=1}^N \mathbf{u}_i$ with $\mathbf{u}_i = \mu_i^\dagger \mathbf{u}_\Gamma \in \mathbf{V}_{\Gamma,i}$. Since $\mathbf{u}_\Gamma$ is balanced, we can use formula (15) and obtain

$$\Big\langle \mathbf{u}_\Gamma, \mathbf{u}_\Gamma \Big\rangle_{S_\Gamma} = s(\mathbf{u}_\Gamma, \mathbf{u}_\Gamma) = \sum_{i=1}^N s(\mathbf{u}_\Gamma, \mathbf{u}_i)$$

$$= \sum_{i=1}^N s_{i,\epsilon}(\mu_i T_i^u \mathbf{u}_\Gamma, \mu_i \mathbf{u}_i) - \sum_{i=1}^N b_0(\mathbf{u}_i, \lambda) = \sum_{i=1}^N s_{i,\epsilon}(\mu_i T_i^u \mathbf{u}_\Gamma, \mu_i \mathbf{u}_i), \tag{18}$$

because,

$$\sum_{i=1}^N b_0(\mathbf{u}_i, \lambda) = b_0(\sum_{i=1}^N \mathbf{u}_i, \lambda) = b_0(\mathbf{u}_\Gamma, \lambda) = 0.$$

By the Cauchy-Schwarz inequality and formula (15),

$$\sum_{i=1}^N s_{i,\epsilon}(\mu_i T_i^u \mathbf{u}_\Gamma, \mu_i \mathbf{u}_i) \le \Big(\sum_{i=1}^N s_{i,\epsilon}(\mu_i T_i^u \mathbf{u}_\Gamma, \mu_i T_i^u \mathbf{u}_\Gamma) \Big)^{1/2} \Big(\sum_{i=1}^N s_{i,\epsilon}(\mu_i \mathbf{u}_i, \mu_i \mathbf{u}_i) \Big)^{1/2}$$

$$= \Big(\sum_{i=1}^N s(\mathbf{u}_\Gamma, T_i^u \mathbf{u}_\Gamma) + \sum_{i=1}^N b_0(T_i^u \mathbf{u}_\Gamma, G\mathbf{u}_\Gamma) \Big)^{1/2} \Big(\sum_{i=1}^N s_{i,\epsilon}(\mathbf{u}_\Gamma, \mathbf{u}_\Gamma) \Big)^{1/2}$$

$$= \Big(s(\mathbf{u}_\Gamma, \sum_{i=1}^N T_i^u \mathbf{u}_\Gamma) + b_0(\sum_{i=1}^N T_i^u \mathbf{u}_\Gamma, G\mathbf{u}_\Gamma) \Big)^{1/2} s_\epsilon(\mathbf{u}_\Gamma, \mathbf{u}_\Gamma)^{1/2}. \tag{19}$$

The first factor in (19) is, by Lemma 3c,

$$s(\mathbf{u}_\Gamma, \sum_{i=1}^N T_i^u \mathbf{u}_\Gamma) + b_0(\sum_{i=1}^N T_i^u \mathbf{u}_\Gamma, G\mathbf{u}_\Gamma) = \Big\langle \begin{bmatrix} S_\Gamma \mathbf{u}_\Gamma + B_0^T G \mathbf{u}_\Gamma \\ 0 \end{bmatrix}, \begin{bmatrix} \sum_{i=1}^N T_i^u \mathbf{u}_\Gamma \\ \star \end{bmatrix} \Big\rangle$$

$$= \Big\langle \begin{bmatrix} \mathbf{u}_\Gamma \\ G\mathbf{u}_\Gamma \end{bmatrix}, \begin{bmatrix} \sum_{i=1}^N T_i^u \mathbf{u}_\Gamma \\ \star \end{bmatrix} \Big\rangle_S = \Big\langle \mathbf{u}_\Gamma, (I - \mathcal{P}_0 - \mathcal{P}) \sum_{i=1}^N T_i^u \mathbf{u}_\Gamma \Big\rangle_{S_\Gamma}. \tag{20}$$

Here, the value of $\star$ makes no difference. The second factor in (19) can be estimated as follows:

$$s_\epsilon(\mathbf{u}_\Gamma, \mathbf{u}_\Gamma) = a_\epsilon(\mathcal{SH}_\epsilon(\mathbf{u}_\Gamma), \mathcal{SH}_\epsilon(\mathbf{u}_\Gamma)) \le a_\epsilon(\mathcal{SH}(\mathbf{u}_\Gamma), \mathcal{SH}(\mathbf{u}_\Gamma))$$

$$= a(\mathcal{SH}(\mathbf{u}_\Gamma), \mathcal{SH}(\mathbf{u}_\Gamma)) + \epsilon \|\mathcal{SH}(\mathbf{u}_\Gamma)\|^2_{L^2(\Omega)} \le (1 + \frac{\epsilon}{\delta}) a(\mathcal{SH}(\mathbf{u}_\Gamma), \mathcal{SH}(\mathbf{u}_\Gamma))$$

$$= (1 + \frac{\epsilon}{\delta}) s(\mathbf{u}_\Gamma, \mathbf{u}_\Gamma), \tag{21}$$

where

$$\delta = \inf_{\mathbf{v}_\Gamma} \frac{a(\mathcal{SH}(\mathbf{v}_\Gamma), \mathcal{SH}(\mathbf{v}_\Gamma))}{\|\mathcal{SH}(\mathbf{v}_\Gamma)\|^2_{L^2(\Omega)}} > 0.$$

From (19), (20), and (21), we then have

$$\sum_{i=1}^{N} s_{i,\epsilon}(\mu_i T_i^u \mathbf{u}_\Gamma, \mu_i \mathbf{u}_\Gamma) \leq \Big\langle \mathbf{u}_\Gamma, (I - \mathcal{P}_0 - \mathcal{P}) \sum_{i=1}^{N} T_i^u \mathbf{u}_\Gamma \Big\rangle_{S_\Gamma}^{1/2} \Big((1 + \frac{\epsilon}{\delta}) s(\mathbf{u}_\Gamma, \mathbf{u}_\Gamma) \Big)^{1/2}.$$

Therefore, from this last estimate and (18), we find

$$\Big\langle \mathbf{u}_\Gamma, \mathbf{u}_\Gamma \Big\rangle_{S_\Gamma} \leq (1 + \frac{\epsilon}{\delta}) \Big\langle \mathbf{u}_\Gamma, (I - \mathcal{P}_0 - \mathcal{P}) \sum_{i=1}^{N} T_i^u \mathbf{u}_\Gamma \Big\rangle_{S_\Gamma},$$

and by (17)

$$\frac{1}{(1 + \frac{\epsilon}{\delta})} \Big\langle \mathbf{w}, \mathbf{w} \Big\rangle_S \leq \Big\langle \sum_i T_i (I - T_0) \mathbf{w}, (I - T_0) \mathbf{w} \Big\rangle_S. \tag{22}$$

Upper bound: We start by estimating the S_Γ-norm of an individual $T_i^u \mathbf{u}_\Gamma$. We note that

$$\Big\langle T_i^u \mathbf{u}_\Gamma, T_i^u \mathbf{u}_\Gamma \Big\rangle_{S_\Gamma} = s(T_i^u \mathbf{u}_\Gamma, T_i^u \mathbf{u}_\Gamma) = a(\mathcal{SH}(T_i^u \mathbf{u}_\Gamma), \mathcal{SH}(T_i^u \mathbf{u}_\Gamma)).$$

By comparing the energy of the discrete Stokes and harmonic extensions as in Lemma 1, we have

$$a(\mathcal{SH}(T_i^u \mathbf{u}_\Gamma), \mathcal{SH}(T_i^u \mathbf{u}_\Gamma)) \leq \frac{C}{\beta} a(\mathcal{H}(T_i^u \mathbf{u}_\Gamma), \mathcal{H}(T_i^u \mathbf{u}_\Gamma)).$$

Since $T_i^u \mathbf{u}_\Gamma \in \mathbf{V}_{\Gamma,i}$, we can now apply to each scalar component of $\mathcal{H}(T_i^u \mathbf{u}_\Gamma)$ a result proven in the scalar case in Dryja and Widlund [10, Lemma 4] for finite elements and in Pavarino [29, Lemma 6.2] for spectral elements, and obtain

$$a(\mathcal{H}(T_i^u \mathbf{u}_\Gamma), \mathcal{H}(T_i^u \mathbf{u}_\Gamma)) \leq C \ \alpha \ a_{i,\epsilon}(\mathcal{H}_{i,\epsilon}(\mu_i T_i^u \mathbf{u}_\Gamma), \mathcal{H}_{i,\epsilon}(\mu_i T_i^u \mathbf{u}_\Gamma)),$$

where $a_{i,\epsilon}(\cdot,\cdot)$ is given in formula (14) and

$$\alpha = \begin{cases} (1 + \log(H/h))^2 & \text{for finite elements} \\ \\ (1 + \log n)^2 & \text{for spectral elements.} \end{cases}$$

By using a local variant of Lemma 1 for each subdomain Ω_i, we can return to the discrete Stokes extensions:

$$\begin{aligned} a_{i,\epsilon}(\mathcal{H}_{i,\epsilon}(\mu_i T_i^u \mathbf{u}_\Gamma), \mathcal{H}_{i,\epsilon}(\mu_i T_i^u \mathbf{u}_\Gamma)) &\leq a_{i,\epsilon}(\mathcal{SH}_{i,\epsilon}(\mu_i T_i^u \mathbf{u}_\Gamma), \mathcal{SH}_{i,\epsilon}(\mu_i T_i^u \mathbf{u}_\Gamma)) \\ &= s_{i,\epsilon}(\mu_i T_i^u \mathbf{u}_\Gamma, \mu_i T_i^u \mathbf{u}_\Gamma). \end{aligned}$$

Therefore,

$$\Big\langle T_i^u \mathbf{u}_\Gamma, T_i^u \mathbf{u}_\Gamma \Big\rangle_{S_\Gamma} \leq C \frac{\alpha}{\beta} s_{i,\epsilon}(\mu_i T_i^u \mathbf{u}_\Gamma, \mu_i T_i^u \mathbf{u}_\Gamma).$$

By formula (15), Lemma 3c, and the Cauchy-Schwarz inequality, we have

$$s_{i,\epsilon}(\mu_i T_i^u \mathbf{u}_\Gamma, \mu_i T_i^u \mathbf{u}_\Gamma) = s(\mathbf{u}_\Gamma, T_i^u \mathbf{u}_\Gamma) + b(T_i^u \mathbf{u}_\Gamma, G\mathbf{u}_\Gamma)$$

$$= \Big\langle \begin{bmatrix} \mathbf{u}_\Gamma \\ G\mathbf{u}_\Gamma \end{bmatrix}, \begin{bmatrix} T_i^u \mathbf{u}_\Gamma \\ \star \end{bmatrix} \Big\rangle_S = \Big\langle \mathbf{u}_\Gamma, (I - \mathcal{P}_0 - \widetilde{\mathcal{P}}) T_i^u \mathbf{u}_\Gamma \Big\rangle_{S_\Gamma}$$

$$\leq \Big\langle \mathbf{u}_\Gamma, \mathbf{u}_\Gamma \Big\rangle_{S_\Gamma}^{1/2} \|I - \mathcal{P}_0 - \widetilde{\mathcal{P}}\|_{S_\Gamma} \Big\langle T_i^u \mathbf{u}_\Gamma, T_i^u \mathbf{u}_\Gamma \Big\rangle_{S_\Gamma}^{1/2}.$$

Therefore

$$\Big\langle T_i^u \mathbf{u}_\Gamma, T_i^u \mathbf{u}_\Gamma \Big\rangle_{S_\Gamma} \leq C \frac{\alpha^2}{\beta^2} \|I - \mathcal{P}_0 - \widetilde{\mathcal{P}}\|_{S_\Gamma}^2 \Big\langle \mathbf{u}_\Gamma, \mathbf{u}_\Gamma \Big\rangle_{S_\Gamma}.$$

By a standard coloring argument (see, e.g., Dryja and Widlund [10, Th. 1, p. 128]), we obtain

$$\Big\langle \sum_{i=1}^N T_i^u \mathbf{u}_\Gamma, \sum_{i=1}^N T_i^u \mathbf{u}_\Gamma \Big\rangle_{S_\Gamma} \leq C \frac{\alpha^2}{\beta^2} \|I - \mathcal{P}_0 - \widetilde{\mathcal{P}}\|_{S_\Gamma}^2 \Big\langle \mathbf{u}_\Gamma, \mathbf{u}_\Gamma \Big\rangle_{S_\Gamma}.$$

Hence by the Cauchy-Schwarz inequality,

$$\Big\langle \sum_{i=1}^N T_i^u \mathbf{u}_\Gamma, \mathbf{u}_\Gamma \Big\rangle_{S_\Gamma} \leq C \frac{\alpha}{\beta} \|(I - \mathcal{P}_0 - \widetilde{\mathcal{P}}\|_{S_\Gamma} \Big\langle \mathbf{u}_\Gamma, \mathbf{u}_\Gamma \Big\rangle_{S_\Gamma}.$$

Since $\mathbf{u}_\Gamma = (I - \mathcal{P}_0)\mathbf{u}_\Gamma$, we find by using Lemma 3c, that

$$\Big\langle (I - \mathcal{P}_0 - \widetilde{\mathcal{P}}) \sum_{i=1}^N T_i^u \mathbf{u}_\Gamma, \mathbf{u}_\Gamma \Big\rangle_{S_\Gamma} \leq C \frac{\alpha}{\beta} \|(I - \mathcal{P}_0 - \widetilde{\mathcal{P}}\|_{S_\Gamma} \Big\langle \mathbf{u}_\Gamma, \mathbf{u}_\Gamma \Big\rangle_{S_\Gamma},$$

and by (17)

$$\Big\langle \sum_i T_i (I - T_0)\mathbf{w}, (I - T_0)\mathbf{w} \Big\rangle_S \leq C \frac{\alpha}{\beta} \|(I - \mathcal{P}_0 - \widetilde{\mathcal{P}}\|_{S_\Gamma} \Big\langle \mathbf{w}, \mathbf{w} \Big\rangle_S. \tag{23}$$

In conclusion, we find by putting together the lower bound (22) and the upper bound (23) that

$$\frac{1}{(1 + \frac{\epsilon}{\delta})} \Big\langle \mathbf{w}, \mathbf{w} \Big\rangle_S \leq \Big\langle \sum_i T_i (I - T_0)\mathbf{w}, (I - T_0)\mathbf{w} \Big\rangle_S \leq C \frac{\|I - \mathcal{P}_0 - \widetilde{\mathcal{P}}\|_{S_\Gamma}}{\beta} \, \alpha \, \Big\langle \mathbf{w}, \mathbf{w} \Big\rangle_S,$$

and we can conclude our proof by using Lemma 5 and equation (16).

□

6 Numerical results

We report, in this last section, results of numerical experiments, carried out in Matlab 5.3 on Unix workstations, for a model Stokes problem on the unit square and with homogeneous Dirichlet boundary conditions. The problem was discretized with $Q_n - Q_{n-2}$ spectral elements and the domain Ω divided into $\sqrt{N} \times \sqrt{N}$ square subdomains. After the implicit elimination of the interior unknowns, the saddle point Schur complement (9) is solved iteratively by either GMRES with restart every 50 iterations or PCG. The initial guess is always zero, the right hand side is random and uniformly distributed, and the stopping criterion is $\|r_k\|_2/\|r_0\|_2 \leq 10^{-6}$, where r_k is the residual at the $k-$th iterate. As explained in section 4, the singularity of the local Neumann solves for the floating subdomains is avoided by shifting the diagonal of the local velocity stiffness matrices by $\epsilon = 10^{-5}$.

The Laplace operator. In order to have a reference to compare with, we first considered the Laplace instead of the Stokes operator. The results are reported in Table 1. In the upper half of the table the number of subdomains, $N = 3 \times 3$, is fixed, while the spectral degree n is increased from 2 to 12; in the lower half the spectral degree $n = 4$ is fixed and the number of subdomains N is increased from 2×2 to 12×12. The first three columns report the iteration count, and the maximum and minimum eigenvalues when CG is applied without any preconditioner. The next three columns report the same data when PCG is applied with a balancing Neumann-Neumann preconditioner with a coarse space spanned by only the $\mu_i^\dagger$ functions associated with floating subdomains. The final three columns report the same data when PCG is applied with a balancing Neumann-Neumann preconditioner with a coarse space spanned by all the $\mu_i^\dagger$ functions, except one in order to avoid a linearly dependent $\mu_i^\dagger$ functions in this particular geometry. These results clearly show that CG without preconditioning is neither optimal nor scalable, since the iteration counts grow with n and N, while PCG with a balancing Neumann-Neumann preconditioner is quasi-optimal and scalable; the iteration counts appear to grow at most logarithmically with n and are independent of N. The richer coarse space based on all $\mu_i^\dagger$ functions but one yields a better preconditioner (the condition number is almost half of that of the other balancing preconditioner), but the iteration counts are essentially the same. The growth of the maximum eigenvalue of the preconditioned operator with n is also plotted in the left panel of Figure 3.

The Stokes operator. In the next four tables, we report analogous sets of results for the Stokes operator. The saddle point Schur complement (9) for Stokes equations is solved by GMRES or PCG with and without our balancing Neumann-Neumann preconditioner. Each table corresponds to one of the four choices of coarse spaces described in section 4, $\mathbf{V}_0^0$ in Table 2, $\mathbf{V}_0^1$ in Table 3, $\mathbf{V}_0^2$ in Table 4, and $\mathbf{V}_0^3$ in Table 5. In the upper half of each table the number of subdomains, $N = 3 \times 3$, is fixed, while the spectral degree n is increased from 2 to 12; in the lower half the spectral degree $n = 4$ is fixed and the number of subdomains N is increased from 2×2 to 12×12. In each table, the first column reports the iteration counts of GMRES(50) without precondition-

ing, the second column the iteration counts of GMRES(50) with our balancing Neumann-Neumann preconditioner. The next three columns report the iteration counts and extreme eigenvalues of PCG with our balancing Neumann-Neumann preconditioner. The last column reports the square of the inf-sup constant β_0^2 of the coarse Stokes problem, computed as the minimum nonzero eigenvalue of $M_{p0}^{-1}(B_0L_0)(L_0^T S_\Gamma L_0)^{-1}(B_0L_0)^T$, where M_{p0} is the coarse pressure mass matrix and $(B_0L_0)(L_0^T S_\Gamma L_0)^{-1}(B_0L_0)^T$ is the pressure Schur complement for the coarse problem (10). The results of these tables are also plotted in Figure 1 (GMRES iteration counts), Figure 2 (PCG iteration counts), Figure 3 (maximum eigenvalue of the Laplacian and Stokes operators), Figure 4 (inverse coarse inf-sup constant $1/\beta_0$). These results show that GMRES without preconditioning is neither optimal nor scalable, while both GMRES and PCG with our balancing Neumann-Neumann preconditioner are quasi-optimal and scalable, except with the first choice of coarse space $\mathbf{V}_0^0$. In fact, the $\mathbf{V}_0^0$ coarse space does not seem to be inf-sup stable (last column of Table 2 and right panel of Figure 4) and the iteration counts of both GMRES and PCG seem to grow linearly with N in that case (right panels of Figures 1 and 2). The other three coarse spaces appear to be inf-sup stable uniformly in N and in fact the iteration counts of both GMRES and PCG are bounded from above independently of N. It is more difficult to detect numerically the dependence of the coarse inf-sup constant with respect to n; see right panel of Figure 4.

Overall, these numerical results seem to be in good agreement with the estimates of our main result: since we are considering here the two dimensional case and spectral elements, Theorem 1 states that the maximum eigenvalue of the preconditioner operator T grows as $O(\sqrt{n}(1+\log^2(n)))$, because $\beta_n = Cn^{-1/2}$ (see eq. (4)), while the minimum eigenvalue of T is always bounded from below by $1/(1+\frac{c}{\delta})$. We note that the condition numbers of T for the Stokes operator in Tables 3 and 4 are remarkably close to (almost always less than twice) the corresponding condition numbers of T for the Laplacian operator in Table 1. The situation would be even better if we had considered stable finite elements, since both the global and coarse inf-sup constants would be independent of H and h. Therefore the maximum eigenvalue of T would grow only as $O(1+\log^2(H/h))$.

Table 1: Laplace solver with balancing Neumann-Neumann preconditioner.

Fixed number of subdomains $N = 3 \times 3$									
	NO PREC.			NN PREC. FLOAT ONLY			NN PREC. ALL $\mu_i^\dagger - 1$		
n	it.	λ_{max}	λ_{min}	it.	λ_{max}	λ_{min}	it.	λ_{max}	λ_{min}
2	9	5.32	0.6667	6	2.05	1.0016	4	1.07	1.0001
3	14	5.65	0.3964	7	2.73	1.0008	6	1.43	1.0001
4	16	5.73	0.2799	8	3.44	1.0010	7	1.75	1.0003
5	19	5.77	0.2157	9	4.04	1.0009	8	2.11	1.0004
6	22	5.80	0.1752	9	4.59	0.9997	8	2.45	0.9998
7	24	5.83	0.1474	9	5.08	0.9997	9	2.77	0.9997
8	25	5.85	0.1271	10	5.54	0.9995	9	3.07	0.9998
9	28	5.86	0.1117	10	5.97	0.9991	9	3.36	0.9993
10	30	5.88	0.0996	10	6.36	0.9997	9	3.63	0.9999
11	31	5.89	0.0898	10	6.73	0.9989	10	3.89	0.9989
12	34	5.91	0.0818	10	7.08	0.9988	10	4.13	0.9990
Fixed spectral degree: $n = 4$									
	NO PREC.			NN PREC. FLOAT ONLY			NN PREC. ALL $\mu_i^\dagger - 1$		
N	it.	λ_{max}	λ_{min}	it.	λ_{max}	λ_{min}	it.	λ_{max}	λ_{min}
2×2	10	5.60	0.5439	3	2.25	1.0000	3	1.50	1.0000
3×3	16	5.73	0.2800	8	3.44	1.0010	7	1.75	1.0003
4×4	24	5.78	0.1655	10	3.07	1.0000	8	1.81	1.0005
5×5	31	5.80	0.1084	10	3.04	1.0005	8	1.84	1.0006
6×6	37	5.81	0.0762	10	2.98	0.9995	8	1.86	0.9998
7×7	42	5.82	0.0564	10	2.97	0.9998	8	1.87	1.0000
8×8	47	5.82	0.0434	10	2.98	0.9999	8	1.88	1.0002
9×9	52	5.83	0.0344	10	2.98	0.9998	8	1.88	0.9998
10×10	58	5.83	0.0279	10	2.98	0.9996	8	1.88	0.9999
11×11	64	5.83	0.0231	10	2.98	0.9999	8	1.88	1.0000
12×12	70	5.83	0.0194	10	2.98	1.0001	8	1.89	1.0002

Table 2: Stokes solver with balancing Neumann-Neumann preconditioner and first choice of coarse space $\mathbf{V}_0^0$.

Fixed number of subdomains $N = 3 \times 3$						
	GMRES	GMRES NN	PCG NN			β_0^2
n	it.	it.	it.	λ_{max}	λ_{min}	
2	22	6	6	1.99	1.0022	3.8945e-01
3	30	11	11	5.86	1.0039	1.3939e-01
4	38	12	13	7.83	1.0038	1.0445e-01
5	42	13	14	10.20	1.0022	7.9327e-02
6	46	15	15	11.65	1.0040	7.1022e-02
7	50+3	16	17	13.33	1.0112	6.1723e-02
8	50+4	17	17	14.67	1.0022	5.7662e-02
9	50+8	18	19	16.00	1.0112	5.2566e-02
10	50+19	18	19	17.27	1.0057	5.0166e-02
11	50+26	19	20	18.37	1.0033	4.6808e-02
12	50+43	19	21	19.61	1.0034	4.5234e-02
Fixed spectral degree: $n = 4$						
	GMRES	GMRES NN	PCG NN			β_0^2
N	it.	it.	it.	λ_{max}	λ_{min}	
2×2	16	9	10	2.84	1.0013	2.3819e-01
3×3	38	12	13	7.83	1.0038	1.0445e-01
4×4	50+26	14	15	10.53	1.0030	7.5698e-02
5×5	3*50+30	17	18	18.91	1.0020	4.0554e-02
6×6	5*50+19	20	20	22.92	1.0033	3.3473e-02
7×7	8*50+1	21	23	35.83	1.0038	2.0984e-02
8×8	10*50+41	23	25	40.27	1.0028	1.8777e-02
9×9	15*50+17	26	28	58.48	1.0028	1.2762e-02
10×10	19*50+40	27	28	62.59	1.0026	1.2003e-02
11×11	24*50+37	29	32	86.87	1.0024	8.5675e-03
12×12	34*50+49	30	33	89.89	1.0029	8.3314e-03

Table 3: Stokes solver with balancing Neumann-Neumann preconditioner and second choice of coarse space $\mathbf{V}_0^1$.

Fixed number of subdomains $N = 3 \times 3$						
	GMRES	GMRES NN	PCG NN			β_0^2
n	it.	it.	it.	λ_{max}	λ_{min}	
2	22	4	4	1.24	1.0000	4.9358e-01
3	30	8	9	2.39	1.0010	4.2758e-01
4	38	10	11	3.15	1.0008	3.8904e-01
5	42	11	12	3.88	1.0014	3.5665e-01
6	46	13	14	4.69	1.0008	3.3639e-01
7	50+3	13	14	5.24	1.0006	3.2215e-01
8	50+4	14	15	6.24	1.0015	3.1130e-01
9	50+8	15	16	6.44	1.0008	3.0323e-01
10	50+19	16	17	7.68	1.0013	2.9645e-01
11	50+26	16	17	7.58	1.0010	2.9120e-01
12	50+43	17	18	9.04	1.0009	2.8655e-01

Fixed spectral degree: $n = 4$						
	GMRES	GMRES NN	PCG NN			β_0^2
N	it.	it.	it.	λ_{max}	λ_{min}	
2×2	16	8	9	2.62	1.0012	4.5547e-01
3×3	38	10	11	3.15	1.0008	3.8903e-01
4×4	50+26	11	12	3.37	1.0006	3.7275e-01
5×5	3*50+30	11	12	3.49	1.0007	3.6303e-01
6×6	5*50+19	12	12	3.51	1.0008	3.5594e-01
7×7	8*50+1	12	12	3.60	1.0008	3.5090e-01
8×8	10*50+41	12	13	3.66	1.0007	3.4655e-01
9×9	15*50+17	12	13	3.80	1.0008	3.4306e-01
10×10	19*50+40	12	13	3.94	1.0008	3.4004e-01
11×11	24*50+37	12	13	4.06	1.0007	3.3748e-01
12×12	34*50+49	13	13	4.16	1.0007	3.3523e-01

Table 4: Stokes solver with balancing Neumann-Neumann preconditioner and third choice of coarse space $\mathbf{V}_0^2$.

Fixed number of subdomains $N = 3 \times 3$						
	GMRES	GMRES NN	PCG NN			β_0^2
n	it.	it.	it.	λ_{max}	λ_{min}	
2	22	(2)	(2)	(1.23)	(1.000)	-
3	30	7	8	2.28	1.0005	4.2215e-01
4	38	10	11	3.23	1.0010	3.1329e-01
5	42	11	12	4.05	1.0016	2.6010e-01
6	46	13	14	4.87	1.0005	2.1851e-01
7	50+3	14	15	5.62	1.0007	1.9749e-01
8	50+4	15	16	6.36	1.0008	1.7659e-01
9	50+8	16	16	7.06	1.0006	1.6541e-01
10	50+19	16	17	7.73	1.0003	1.5260e-01
11	50+26	17	17	8.37	1.0007	1.4573e-01
12	50+43	18	18	8.99	1.0006	1.3689e-01

Fixed spectral degree: $n = 4$						
	GMRES	GMRES NN	PCG NN			β_0^2
N	it.	it.	it.	λ_{max}	λ_{min}	
2×2	16	9	9	2.62	1.0031	3.0283e-01
3×3	38	10	11	3.23	1.0010	3.1329e-01
4×4	50+26	11	11	3.44	1.0010	2.8801e-01
5×5	3*50+30	11	12	3.54	1.0008	2.7240e-01
6×6	5*50+19	11	12	3.59	1.0007	2.6082e-01
7×7	8*50+1	11	12	3.63	1.0007	2.5391e-01
8×8	10*50+41	11	12	3.65	1.0008	2.4767e-01
9×9	15*50+17	11	12	3.67	1.0008	2.4372e-01
10×10	19*50+40	12	12	3.69	1.0010	2.3979e-01
11×11	24*50+37	12	12	3.70	1.0008	2.3715e-01
12×12	34*50+49	12	12	3.71	1.0008	2.3441e-01

Table 5: Stokes solver with balancing Neumann-Neumann preconditioner and fourth choice of coarse space $\mathbf{V}_0^3$.

Fixed number of subdomains $N = 3 \times 3$						
	GMRES	GMRES NN	PCG NN			β_0^2
n	it.	it.	it.	λ_{max}	λ_{min}	
2	22	5	5	1.35	1.0008	4.7442e-01
3	30	8	8	2.46	1.0007	3.5297e-01
4	38	11	11	3.29	1.0017	2.7204e-01
5	42	12	12	4.16	1.0025	2.2367e-01
6	46	13	14	5.00	1.0011	1.8980e-01
7	50+3	14	15	5.79	1.0026	1.7029e-01
8	50+4	15	16	6.78	1.0010	1.5276e-01
9	50+8	16	17	7.29	1.0025	1.4224e-01
10	50+19	17	17	8.52	1.0012	1.3141e-01
11	50+26	17	17	8.74	1.0011	1.2486e-01
12	50+43	18	18	10.16	1.0011	1.1737e-01
Fixed spectral degree: $n = 4$						
	GMRES	GMRES NN	PCG NN			β_0^2
N	it.	it.	it.	λ_{max}	λ_{min}	
2×2	16	9	10	2.84	1.0013	-
3×3	38	11	11	3.29	1.0017	2.7204e-01
4×4	50+26	11	12	3.52	1.0012	2.4066e-01
5×5	3*50+30	11	12	3.63	1.0014	2.0884e-01
6×6	5*50+19	12	12	3.68	1.0015	1.9034e-01
7×7	8*50+1	12	13	3.71	1.0012	1.7906e-01
8×8	10*50+41	12	13	3.74	1.0012	1.7116e-01
9×9	15*50+17	12	13	3.75	1.0012	1.6559e-01
10×10	19*50+40	12	13	3.76	1.0013	1.6127e-01
11×11	24*50+37	12	13	3.77	1.0010	1.5795e-01
12×12	34*50+49	12	13	3.78	1.0011	1.5520e-01

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Figure 1: GMRES iteration counts for the Stokes solver vs. spectral degree n when $N = 3 \times 3$ (left) and number of subdomains N when $n = 4$ (right)

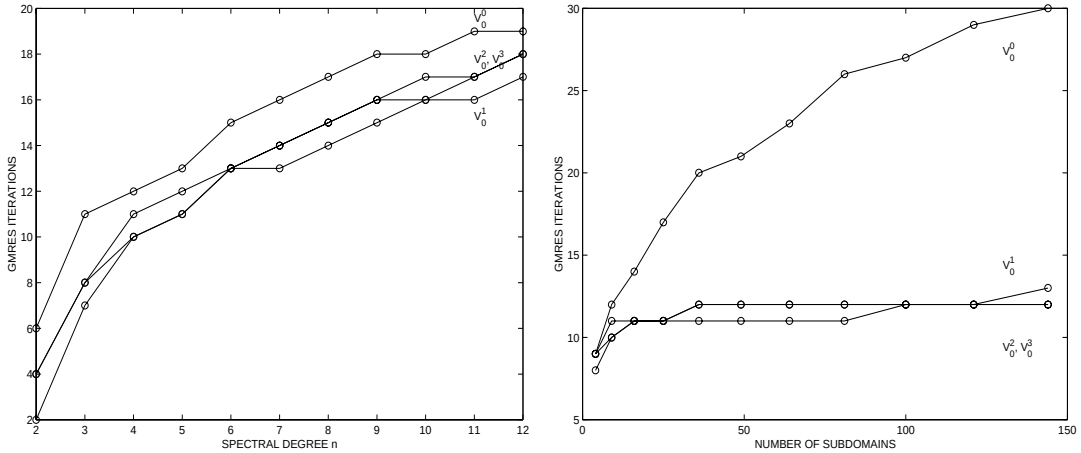

Figure 2: PCG iteration counts for the Stokes solver vs. spectral degree n when $N = 3 \times 3$ (left) and number of subdomains N when $n = 4$ (right)

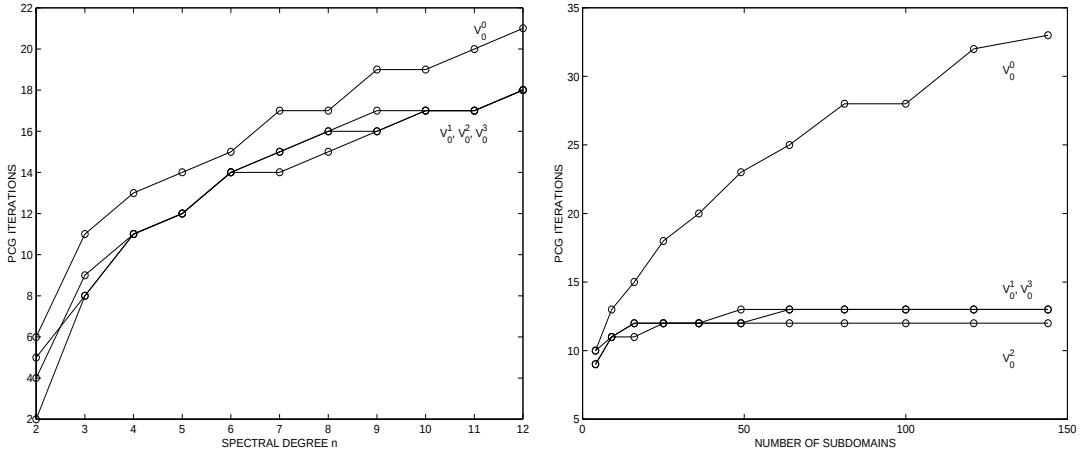

Figure 3: Maximum eigenvalue of the preconditioned operator vs. spectral degree n: Laplace solver (left) and Stokes solver (right)

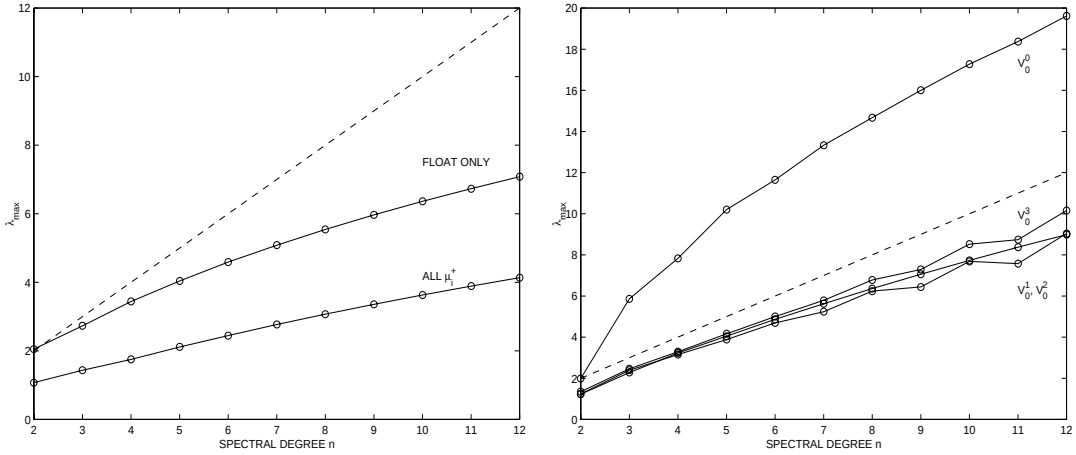

Figure 4: Inverse inf-sup constant $1/\beta_0$ of the coarse Stokes problem vs. spectral degree n when $N = 3 \times 3$ (left) and vs. number of subdomains N when $n = 4$ (right)