AN OVERLAPPING DOMAIN DECOMPOSITION PRECONDITIONER FOR A CLASS OF DISCONTINUOUS GALERKIN APPROXIMATIONS OF ADVECTION–DIFFUSION PROBLEMS

CAROLINE LASSER * AND ANDREA TOSELLI †

Abstract. We consider a scalar advection–diffusion problem and a recently proposed discontinuous Galerkin approximation, which employs discontinuous finite element spaces and suitable bilinear forms containing interface terms that ensure consistency. For the corresponding sparse, nonsymmetric linear system, we propose and study an additive, two–level overlapping Schwarz preconditioner, consisting of a coarse problem on a coarse triangulation and local solvers associated to suitable problems defined on a family of subdomains. This is a generalization of the corresponding overlapping method for approximations on continuous finite element spaces. Related to the lack of continuity of our approximation spaces, some interesting new features arise in our generalization, which have no analog in the conforming case. We prove an upper bound for the number of iterations obtained by using this preconditioner with GMRES, which is independent of the number of degrees of freedom of the original problem and the number of subdomains. The performance of the method is illustrated by several numerical experiments for different test problems, using linear finite elements in two dimensions.

Key words. advection-diffusion, domain decomposition, discontinuous Galerkin.

AMS subject classifications. 65F10, 65N22, 65N30, 65N55

1. Introduction.

We consider the following scalar advection-diffusion problem with Dirichlet conditions

$$\begin{aligned} \mathcal{L}u \;=\; -\nabla\cdot(a\nabla u) + b\cdot\nabla u + cu &\;=\; f, \quad \text{in } \Omega\,, \\ u &\;=\; 0, \quad \text{on } \Gamma\,, \end{aligned} \tag{1.1}$$

where Ω is a bounded open polyhedral domain in R^d, $d = 2, 3$, and Γ its boundary. Problem (1.1) describes a large class of diffusion-transport-reaction processes.

Discontinuous Galerkin (DG) approximations have been used since the early 1970s and are recently becoming more and more popular for the approximation of advection-diffusion problems; we refer to [5] for a comprehensive review of these methods. Here, we consider a discontinuous hp-finite element method proposed in [9]. As for many DG methods, the approximate solution belongs to a space of discontinuous finite element functions, i.e., it is piecewise polynomial of a certain degree on a given triangulation, being in general discontinuous across the elements. Increasing the polynomial degree as well as refining the triangulation results in better approximations of the desired solution. Suitable bilinear forms, which also contain interface contributions, are then employed, in order to ensure consistency. The corresponding systems of algebraic equations are sparse but often too large to be handled by direct solvers. In addition,

*Center for Mathematical Sciences, Technische Universität München, Arcisstr. 21, D-80290 München, Germany. E-mail: classer@mathematik.tu-muenchen.de. The work of this author was supported by the Studienstiftung des deutschen Volkes, while visiting the Courant Institute of Mathematical Sciences.

†Seminar for Applied Mathematics, ETH Zürich, Rämisstr. 101, CH-8092 Zürich, Switzerland. E-mail: toselli@sam.math.ethz.ch. Most of this work was carried out while the author was affiliated to the Courant Institute of Mathematical Sciences, New York, and supported in part by the Applied Mathematical Sciences Program of the U.S. Department of Energy under Contract DE-FGO288ER25053.

they are non-symmetric, since the bilinear forms contain advection- and interface-terms.

Fixing the polynomial degree $p \geq 1$, we construct and analyze a Schwarz-preconditioner for linear systems obtained from discontinuous hp–discretizations, to be used with a Krylov–type method, like GMRES. Our two–level Schwarz preconditioner is built from a coarse solver and a number of smaller local solvers, associated to a partition of the domain Ω. While the coarse level is designed to reduce the low-energy components of the error, the fine level splits the original problem into a number of smaller problems, not only to reduce the problem size but also to enable efficient parallel computing. We then generalize the additive Schwarz theory for non-symmetric problems, developed by Cai and Widlund in [2] and [3], to the class of DG approximations in question. Our main result is an upper bound for the convergence rate of the preconditioned system, which is independent of the number of degrees of freedom and the number of local problems.

We only know of one previous work on DD preconditioners for DG approximations. In [8], a two–level Schwarz preconditioner has been proposed and analyzed for a different type of DG approximations for the Poisson problem. As opposed to our approach, the method in [8] gives rise to a symmetric positive–definite problem and the Conjugate Gradient method can be employed. In [8] an explicit bound for the condition number for a non–overlapping preconditioner is obtained, which grows linearly with the number of degrees of freedom in each subdomain. The method that we present here is similar to that in [8], but we choose a different DG approximation, which we believe is more suited for advection–reaction–diffusion equations. The coarse space that we consider is also different, and we believe that it is more appropriate for the case of overlapping methods. We then use GMRES and prove an upper bound for the number of iterations obtained when a two–level overlapping preconditioner is employed. Due to the available error estimates for GMRES and the non–symmetry of our problem, bounds that are explicit in the relative overlap cannot be obtained in general, similarly to the case of conforming approximations; see [2, 3]. Our numerical results show however that, as expected, the rate of convergence improves when the the overlap increases.

The rest of the paper is organized as follows:
Section 2 introduces the model problem and the discontinuous finite element spaces. After defining the bilinear form and the corresponding discrete problem in section 3, we describe our overlapping Schwarz method in section 4. The technical tools used for the proof of the convergence result in section 6, are provided in section 5. We finally illustrate the performance of our algorithm in section 7 by several numerical experiments in the case of linear finite elements in two dimensions.

2. Model Problem and Finite Element Spaces. We consider problem (1.1) and make some further hypotheses. We assume that $a = \{a_{i,j}\}_{i,j=1}^{d}$ is a symmetric positive–definite matrix,

$$\xi^T a(x)\xi \geq \alpha_0 > 0, \quad \xi \in \mathrm{R}^d, \quad x \in \Omega,$$

b and c are a vector field in $W^{1,\infty}(\Omega)$ and a function in $L^\infty(\Omega)$, respectively, such that

$$(c - \frac{1}{2}\nabla \cdot b)(x) \;\geq\; \gamma_0 \;>\; 0\,, \quad x \in \Omega\,, \tag{2.1}$$

and the right-hand side f is a function in $L^2(\Omega)$. The existence of a unique solution of (1.1) is shown in [9]. We note that we have considered only the case of strongly–imposed homogeneous Dirichlet boundary conditions for simplicity, but that more general ones can be employed, such as Neumann, Robin, or weakly–imposed Dirichlet conditions. Our analysis remains valid in these cases.

In the following, the norm, seminorm, and inner product of a Hilbert space $\mathcal{H}$ are denoted by $\|\cdot\|_{\mathcal{H}}, |\cdot|_{\mathcal{H}}$, and $(\cdot,\cdot)_{\mathcal{H}}$, respectively.

In our analysis we will use some regularity properties for second order elliptic problems and tacitly assume that the domain Ω and the subdomains considered satisfy them. Such properties are certainly valid for general polygonal and polyhedral domains with angles between their edges (or faces) smaller than 2π. In particular we will assume that the Poisson problem on Ω (and consequently Problem (1.1) and its adjoint) with Dirichlet or Neumann conditions has $H^{\eta+3/2}$ regularity, for all $\eta < \eta_\Omega$, where $\eta_\Omega > 0$ depends on Ω and the particular type of boundary conditions considered; see, [6, Cor. 18.15 and Cor. 23.5].

We next introduce $\mathcal{T}_h$, a conforming, shape–regular triangulation of Ω consisting of open simplices κ with diameter $O(h)$. We denote by $\mathcal{P}_k(\kappa)$ the space of polynomials on $\bar\kappa$ of total degree $k \in \mathrm{N}_0$ and define the vector of local polynomial degrees $\mathbf{p} = (p_\kappa : \kappa \in \mathcal{T}_h)$. We consider the finite element space

$$S^{\mathbf{p}}(\Omega,\mathcal{T}_h) = \{u \in L^2(\Omega) : u|_{\bar\kappa} \in \mathcal{P}_{p_\kappa}(\kappa)\}\,.$$

Given $D \subseteq \Omega$, the union of some elements in $\mathcal{T}_h$, we define the product space

$$H^1(D,\mathcal{T}_h) = \{u \in L^2(D)|\; u|_\kappa \in H^1(\kappa)\,,\; \kappa \in \mathcal{T}_h\,,\; \kappa \subset D\}.$$

With an abuse of notation, we also denote by $H^1(D,\mathcal{T}_h)$ the subspace of $H^1(\Omega,\mathcal{T}_h)$ consisting of functions that vanish in $\Omega \setminus \bar D$. We equip $H^1(D,\mathcal{T}_h)$ with the broken Sobolev norm and seminorm, given by

$$\|u\|^2_{H^1(D,\mathcal{T}_h)} = \sum_{\substack{\kappa\in\mathcal{T}_h\\ \kappa\subset D}} \|u\|^2_{H^1(\kappa)}\,, \quad |u|^2_{H^1(D,\mathcal{T}_h)} = \sum_{\substack{\kappa\in\mathcal{T}_h\\ \kappa\subset D}} |u|^2_{H^1(\kappa)}\,,$$

and define $H^1_0(\Omega,\mathcal{T}_h)$ and $S^{\mathbf{p}}_0(\Omega,\mathcal{T}_h)$ as the subspaces of functions in $H^1(\Omega,\mathcal{T}_h)$ and $S^{\mathbf{p}}(\Omega,\mathcal{T}_h)$, respectively, vanishing on Γ. Our FE approximation space is chosen as

$$V^h = S^{\mathbf{p}}_0(\Omega,\mathcal{T}_h)\,.$$

We denote by $\mathcal{E}$ the set of all open $(d-1)$–dimensional faces (edges, for $d=2$) of the elements $\mathcal{T}_h$, and define the set of interior faces $\mathcal{E}_{int} = \{e \in \mathcal{E} : e \subset \Omega\}$ and the interior interface Γ_{int}, such that $\bar\Gamma_{int} = \cup_{e\in\mathcal{E}_{int}} \bar e$.

For $\kappa \in \mathcal{T}_h$, we denote the unit outward normal to $\partial\kappa$ at $x \in \partial\kappa$ by $\mu_\kappa(x)$ and partition the part of its boundary that is also contained in Γ_{int} into two sets:

$$\begin{aligned}
\partial_-\kappa &= \{x \in \partial\kappa \cap \Gamma_{int} : b(x)\cdot\mu_\kappa(x) < 0\} \qquad &&(\text{inflow part})\,,\\
\partial_+\kappa &= \{x \in \partial\kappa \cap \Gamma_{int} : b(x)\cdot\mu_\kappa(x) \geq 0\} \qquad &&(\text{outflow part})\,.
\end{aligned}$$

Given $v \in H^1(\Omega,\mathcal{T}_h)$, its restriction to $\bar D \subset \bar\Omega$ is denoted by $v_D = v|_{\bar D}$. Then, for $x \in \partial_-\kappa$ there exists a unique neighbor κ' with $x \in \partial\kappa'$ and set

$$v^+_\kappa(x) = v_\kappa(x)\,, \quad v^-_\kappa(x) = v_{\kappa'}(x)\,, \quad \lfloor v \rfloor_\kappa = v^+_\kappa - v^-_\kappa\,.$$

Given an interior face $e \in \mathcal{E}_{int}$, there are two elements κ_i, κ_j, with, e.g., $i > j$, that share this face. We define

$$[v]_e = v|_{\partial\kappa_i\cap e} - v|_{\partial\kappa_j\cap e}\,, \quad < v >_e = \frac{1}{2}(v|_{\partial\kappa_i\cap e} + v|_{\partial\kappa_j\cap e})\,,$$

and ν as the unit normal which points from κ_i to κ_j. We note, that μ and ν point in different directions in general and that $\lfloor\cdot\rfloor$ and $[\cdot]$ are distinct. While μ and $\lfloor\cdot\rfloor$ depend on the sign of the advective normal flux on an element boundary, ν and $[\cdot]$ depend on the element numbering. Similarly, for $e = \partial\kappa \cap \Gamma$, we set

$$[v]_e = v|_e.$$

Finally, we introduce a discontinuity-penalization function σ defined on Γ_{int}: for a face $e \in \mathcal{E}_{int}$, we denote the diameter of e by h_e and define

$$\sigma_e \;=\; \sigma_0 \cdot \frac{< \bar{a}p^2 >_e}{h_e}\,,$$

where $\bar{a} = ||a||$ and σ_0 is a suitably chosen positive constant.

3. Bilinear Form and Discrete Problem. For $u, v \in V^h$, we consider the bilinear form

$$\begin{aligned} B(u,v) = &\sum_{\kappa\in\mathcal{T}_h}\int_\kappa a\nabla u\cdot\nabla v dx + \sum_{\kappa\in\mathcal{T}_h}\int_\kappa (b\cdot\nabla u + cu)v dx \\ &- \sum_{\kappa\in\mathcal{T}_h}\int_{\partial_-\kappa\cap\Gamma_{int}} (b\cdot\mu)\lfloor u\rfloor v^+ ds + \int_{\Gamma_{int}} \sigma[u][v]ds \\ &+ \int_{\Gamma_{int}} \left([u] < (a\nabla v)\cdot\nu > - < (a\nabla u)\cdot\nu > [v]\right) ds\,, \end{aligned}$$

which has been proposed in [9]. Our DG approximation of (1.1) is then defined as the unique $u \in V^h$ such that

$$B(u,v) = (f,v)_{L^2(\Omega)}\,, \quad v \in V^h\,. \tag{3.1}$$

Problem (3.1) can be written in matrix form as

$$Bu = f, \tag{3.2}$$

where we have used the same notation for a function $u \in V^h$ and the corresponding vector of degrees of freedom, and a bilinear form, e.g., $B(\cdot,\cdot)$, and its matrix representation in the space V^h. Similarly, in the following we use the same notation for functional spaces and the corresponding spaces of vectors of degrees of freedom.

We next define some additional bilinear forms. It can be easily verified that

$$A(u,v) = \sum_{\kappa\in\mathcal{T}_h}\int_\kappa a\nabla u\cdot\nabla v dx + \int_{\Gamma_{int}} \sigma[u][v]ds\,,$$

defines a scalar product in $H^1_0(\Omega,\mathcal{T}_h)$ and a norm $\|\cdot\|_A = A(\cdot,\cdot)^{\frac{1}{2}}$.

Furthermore, let

$$\begin{aligned} D(u,v) &= \sum_{\kappa\in\mathcal{T}_h}\int_\kappa b\cdot\nabla u\, v\, dx - \sum_{\kappa\in\mathcal{T}_h}\int_{\partial_-\kappa\cap\Gamma_{int}} (b\cdot\mu)\lfloor u\rfloor v^+ ds\,, \\ S(u,v) &= \int_{\Gamma_{int}} \left([u] < (a\nabla v)\cdot\nu > - < (a\nabla u)\cdot\nu > [v]\right) ds\,, \\ C(u,v) &= (cu,v)_{L^2(\Omega)}\,. \end{aligned}$$

An important tool in the analysis of Schwarz methods is represented by some Poincaré and Friedrichs type inequalities valid for Sobolev spaces. The following lemma provides two generalizations to the discontinuous space $H^1(D, \mathcal{T}_h)$; see also [1, 8].

LEMMA 3.1 (Poincaré-Friedrichs). *Let $D \subseteq \Omega$ be a domain which is the union of some elements in $\mathcal{T}_h$. Then there exists a positive constant C depending only on the geometry of D but not on its size, and the shape-regularity constant of $\mathcal{T}_h$, such that, for all $u \in H^1(D, \mathcal{T}_h)$,*

$$\|u\|^2_{L^2(D)} \leq C H_D^2 \left(|u|^2_{H^1(D,\mathcal{T}_h)} + \sum_{\substack{e \in \mathcal{E} \\ e \subset \overline{D}}} \int_e h_e^{-1} [u]^2 ds \right), \tag{3.3}$$

where H_D is the diameter of D. If in addition $\int_D u dx = 0$, then

$$\|u\|^2_{L^2(D)} \leq C H_D^2 \left(|u|^2_{H^1(D,\mathcal{T}_h)} + \sum_{\substack{e \in \mathcal{E} \\ e \subset D}} \int_e h_e^{-1} [u]^2 ds \right). \tag{3.4}$$

Proof. Here, we only present a proof for the the Poincaré-type inequality (3.4). A proof for the Friedrichs inequality (3.3) can be found in [1] for the case of a convex D and can be easily generalized to our more general case.

We first suppose that D has unit diameter and proceed similarly to [1, Lem. 2.2]. Let $u \in H^1(D, \mathcal{T}_h)$ with $\int_D u dx = 0$ and $v \in H^{\eta+3/2}(D)$, for a $\eta > 0$, the solution of the following Neumann problem

$$-\Delta v = u, \text{ in } D, \quad \frac{\partial v}{\partial n} = 0, \text{ on } \partial D, \quad \int_D v dx = 0.$$

Then there exists a constant $C > 0$ such that

$$\|v\|_{H^{\eta+3/2}(D)} \leq C \|u\|_{L^2(D)}.$$

Integration by parts on each κ and summation over all the elements yields

$$\begin{aligned}
\|u\|^2_{L^2(D)} &= (u, -\Delta v)_{L^2(D)} \\
&= (\nabla u, \nabla v)_{L^2(D)} - \sum_{\kappa \subset D} \left(u, \frac{\partial v}{\partial n} \right)_{L^2(\partial\kappa \setminus \partial D)} \\
&\leq \left(|u|^2_{H^1(D,\mathcal{T}_h)} + \sum_{e \subset D} \int_e h_e^{-1} [u]^2 ds \right)^{\frac{1}{2}} \\
&\times \left(|v|^2_{H^1(D,\mathcal{T}_h)} + \sum_{\kappa \subset D} \int_{\partial\kappa \setminus \partial D} h_\kappa \left(\frac{\partial v}{\partial n} \right)^2 ds \right)^{\frac{1}{2}}
\end{aligned}$$

Using a trace inequality for $\partial v / \partial n$ as in [1] we obtain (3.4).

The corresponding inequalities for the case of a general D can be obtained employing a scaling argument. $\square$

We note that (3.3) is the generalization of the corresponding estimate for a function in $H^1(\Omega)$ with support contained in $\bar{D}$ to a discontinuous function in $H^1(D, \mathcal{T}_h)$.

In particular, (3.3) remains valid for a function that is constant in D and vanishes in $\Omega\setminus\bar{D}$, due to the contributions on the edges on ∂D. On the other hand, (3.4) requires additional restrictions on u, since it is not valid for a constant function on D.

The following inverse inequalities are proven in [13, Sect. 4.6.1].

LEMMA 3.2 (Local Inverse Inequalities). *There exists a positive constant C depending only on the shape-regularity constant of $\mathcal{T}_h$ such that for all $u \in \mathcal{P}_{p_\kappa}(\kappa)$ and for all $\kappa \in \mathcal{T}_h$*

$$\|u\|^2_{L^2(\partial\kappa)} \leq C\,\frac{p_\kappa^2}{h_\kappa}\,\|u\|^2_{L^2(\kappa)}\,, \tag{3.5}$$

$$|u|^2_{H^1(\kappa)} \leq C\,\frac{p_\kappa^4}{h_\kappa^2}\,\|u\|^2_{L^2(\kappa)}\,. \tag{3.6}$$

Using these tools, we obtain the following Lemmata.

LEMMA 3.3 (Continuity). *There exists $C>0$ such that*

$$|B(u,v)| \;\leq\; C\,\|u\|_A\|v\|_A\,, \quad u,v \in V^h\,.$$

Proof. The bilinear form B consists of five contributions I, II, III, IV, and V, all of which can be bounded by $C\|u\|_A\|v\|_A$:
We easily find

$$|I| = |\sum_{\kappa\in\mathcal{T}_h}\int_\kappa a\nabla u\cdot\nabla v\,dx| \;\leq\; C\,\|u\|_A\|v\|_A\,,$$

$$|IV| = |\int_{\Gamma_{int}}\sigma[u][v]\,ds| \;\leq\; C\,\|u\|_A\|v\|_A\,.$$

The Cauchy-Schwarz inequality and Lemma 3.1 with $D=\Omega$ yield

$$\begin{aligned}|II| = |\sum_{\kappa\in\mathcal{T}_h}\int_\kappa(b\cdot\nabla u + cu)v\,dx| \;&\leq\; C\sum_{\kappa\in\mathcal{T}_h}\left(|u|_{H^1(\kappa)}\|v\|_{L^2(\kappa)} + \|u\|_{L^2(\kappa)}\|v\|_{L^2(\kappa)}\right)\\ &\leq C\,\|u\|_A\|v\|_A\,.\end{aligned}$$

Applying the inverse inequality (3.5), Lemma 3.1, and the definition of σ, we find

$$\begin{aligned}|III| &= |\sum_{\kappa\in\mathcal{T}_h}\int_{\partial_-\kappa\cap\Gamma_{int}}(b\cdot\mu)\lfloor u\rfloor v^+ ds|\\ &\leq C\left(\sum_{\kappa\in\mathcal{T}_h}h_\kappa^{-1}\|\lfloor u\rfloor\|^2_{L^2(\partial_-\kappa\cap\Gamma_{int})}\right)^{\frac{1}{2}}\left(\sum_{\kappa\in\mathcal{T}_h}h_\kappa\|v^+\|^2_{L^2(\partial_-\kappa\cap\Gamma_{int})}\right)^{\frac{1}{2}}\\ &\leq C\left(\int_{\Gamma_{int}}\sigma[u]^2 ds\right)^{\frac{1}{2}}\|v\|_{L^2(\Omega)} \;\leq\; C\,\|u\|_A\|v\|_A\,.\end{aligned}$$

Using (3.5), we finally obtain

$$|V| = |\int_{\Gamma_{int}}([u] < (a\nabla v)\cdot\nu > - < (a\nabla u)\cdot\nu > [v])\,ds|$$

$$
\le C \left(\sum_{e\in\mathcal{E}_{int}} h_e^{-1} \|[u]\|^2_{L^2(e)} \cdot \sum_{\substack{\kappa\in\mathcal{T}_h \\ \partial\kappa\subset\Gamma_{int}}} h_\kappa \| < a\nabla v > \|^2_{L^2(\partial\kappa)} \right)^{\frac{1}{2}}
$$

$$
+ \, C \left(\sum_{\substack{\kappa\in\mathcal{T}_h \\ \partial\kappa\subset\Gamma_{int}}} h_\kappa \| < a\nabla u > \|^2_{L^2(\partial\kappa)} \cdot \sum_{e\in\mathcal{E}_{int}} h_e^{-1} \|[v]\|^2_{L^2(e)} \right)^{\frac{1}{2}}
$$

$$
\le C \left(\int_{\Gamma_{int}} \sigma[u]^2 ds \cdot \sum_{\kappa\in\mathcal{T}_h} \|a\nabla v\|^2_{L^2(\kappa)} \right)^{\frac{1}{2}} + C \left(\sum_{\kappa\in\mathcal{T}_h} \|a\nabla u\|^2_{L^2(\kappa)} \cdot \int_{\Gamma_{int}} \sigma[v]^2 ds \right)^{\frac{1}{2}}
$$

$$
\le C \, \|u\|_A \|v\|_A \, . \qquad \square
$$

LEMMA 3.4 (Coercivity). *We have*

$$
B(u,u) \ge \|u\|_A^2 \, , \quad u \in H_0^1(\Omega, \mathcal{T}_h) \, .
$$

Proof.

$$
\begin{aligned}
B(u,u) \;&=\; \sum_{\kappa\in\mathcal{T}_h} \|\sqrt{a}\nabla u\|^2_{L^2(\kappa)} + \int_{\Gamma_{int}} \sigma[u]^2 ds \\
&+\; \sum_{\kappa\in\mathcal{T}_h} \int_\kappa (b\cdot\nabla u + cu) u dx - \sum_{\kappa\in\mathcal{T}_h} \int_{\partial_-\kappa\cap\Gamma_{int}} (b\cdot\mu) \lfloor u \rfloor u^+ ds \\
&=: \|u\|_A^2 + R(u,u)
\end{aligned}
$$

Therefore, we just have to make sure that $R(u,u) \ge 0$. Integration by parts yields

$$
\begin{aligned}
R(u,u) = &\sum_{\kappa\in\mathcal{T}_h} \int_\kappa \left(-\frac{1}{2}(\nabla\cdot b) + c \right) u^2 dx \\
&+ \sum_{\kappa\in\mathcal{T}_h} \left(\int_{\partial\kappa\cap\Gamma_{int}} \frac{1}{2}(b\cdot\mu)(u^+)^2 ds - \int_{\partial_-\kappa\cap\Gamma_{int}} (b\cdot\mu)\lfloor u \rfloor u^+ ds \right) \, .
\end{aligned}
$$

Condition (2.1) ensures that the first sum is positive. To deal with the second sum, we consider an interior face $e \subset \mathcal{E}_{int}$ which is common to the elements κ and κ'. Let e be an inflow edge of, e.g., κ'. Then the second sum can be written as

$$
\begin{aligned}
&\sum_{e\subset\mathcal{E}_{int}} \int_e \left(\frac{1}{2}(b\cdot\mu_\kappa)(u_\kappa)^2 + \frac{1}{2}(b\cdot\mu_{\kappa'})(u_{\kappa'})^2 - (b\cdot\mu_{\kappa'})(u_{\kappa'} - u_\kappa)u_{\kappa'} \right) \, ds \\
= &\sum_{e\subset\mathcal{E}_{int}} \int_e \frac{1}{2} |b\cdot\mu_{\kappa'}|(u_{\kappa'} - u_\kappa)^2 \, ds \;=\; \int_{\Gamma_{int}} \frac{1}{2} |b\cdot\mu| [u]^2 ds \;\ge\; 0 \, ,
\end{aligned}
$$

where we have used the fact that $e \subset \partial_-\kappa'$ also belongs to $\partial_+\kappa$. $\square$

Using similar arguments as in the proofs of Lemmata 3.3 and 3.4, we can prove the following Lemma:

LEMMA 3.5. *There exists a constant $C > 0$ such that for all $u, v \in V^h$*

$$
\begin{aligned}
|D(u,v)| &\le C\|u\|_{L^2(\Omega)} \|v\|_A \, , \\
|D(u,v)| &\le C\|u\|_A \|v\|_{L^2(\Omega)} \, .
\end{aligned}
$$

Finally, we are able to control the interface penalization contribution by requiring that the penalization coefficient is sufficiently large:

LEMMA 3.6. *Let $H > 0$ and $\sigma_0 \geq c_0/H$ for some constant $c_0 > 0$. Then there exists $C > 0$, such that for all $u, v \in V^h$*

$$|S(u,v)| \leq C\sqrt{H}\,\|u\|_A\|v\|_A\,.$$

Proof. Since $\sigma^{-1} \leq CHh$, using the inverse inequality (3.5), we obtain

$$\begin{aligned}|S(u,v)| &\leq \left(\sum_{\kappa\in\mathcal{T}_h}\sigma\|[u]\|^2_{L^2(\partial\kappa)}\right)^{\frac{1}{2}}\left(\sum_{\kappa\in\mathcal{T}_h}\sigma^{-1}\| < a\nabla v > \|^2_{L^2(\partial\kappa)}\right)^{\frac{1}{2}}\\ &\leq C\,\|u\|_A\sqrt{H}\left(\sum_{\kappa\in\mathcal{T}_h}h\,\| < a\nabla v > \|^2_{L^2(\partial\kappa)}\right)^{\frac{1}{2}} \leq C\sqrt{H}\,\|u\|_A\|v\|_A\,. \qquad \square\end{aligned}$$

We remark that the restriction imposed by the previous lemma on σ does not appear to be required in practice; see Section 7.

4. An overlapping Schwarz Method. In this section, we introduce our two–level algorithm. It is the generalization of the classical overlapping method with a standard coarse space. We refer to [15] and [14] for further details and some implementation issues.

We first introduce a shape–regular coarse triangulation of Ω

$$\mathcal{T}_H = \{\Omega_i\}_{1\leq i\leq N},$$

of diameter $H > h$ and suppose that $\mathcal{T}_h$ is obtained by refining $\mathcal{T}_H$. We next extend each Ω_i to a larger region $\Omega_i' \subset \Omega$, in such a way that Ω_i' is the union of some elements in $\mathcal{T}_h$. Concerning the overlap of the extended subregions, we assume that there exists a constant $\alpha > 0$ such that

$$dist(\partial\Omega_i'\cap\Omega, \partial\Omega_i) \geq \alpha H\,, \quad 1 \leq i \leq N\,. \tag{4.1}$$

Before proceeding, we remark that more general partitions and coarse meshes can be employed in overlapping methods. In particular, the coarse mesh does not need to be related to the fine one, and the non–overlapping partition $\{\Omega_i\}$ does not need to be related to the coarse mesh $\mathcal{T}_H$. Indeed, one only needs to assume that the diameter of $\mathcal{T}_H$ and the diameters of the $\{\Omega_i\}$ are of the same size H; see, e.g., [4]. Our results and proofs remain valid in this more general case.

The first problem we need to address is the choice of the local solvers associated to the $\{\Omega_i'\}$. Our FE spaces are discontinuous and at a first glance there are no traces to match! We then proceed in a pure algebraic way, by first defining some local spaces (or, equivalently, by extracting some blocks from B) and identify the corresponding problems, if any, that they represent.

Our local spaces are defined by

$$V_i = \{\, u \in V^h : u(x) = 0 \text{ for } x \in \Omega\backslash\overline{\Omega_i'}\,\}\,, \quad 1 \leq i \leq N\,. \tag{4.2}$$

We note that a function in V_i is discontinuous and, as opposed to the case of conforming approximations, in general does not vanish on $\partial\Omega_i'$. Let $R_i^T :\ V_i \to V^h$ be the natural interpolation operator from the subspace V_i into V_h. We recall that the restriction operator $R_i : V^h \to V_i$, defined as the transpose of R_i^T with respect to the Euclidean scalar product, puts to zero the degrees of freedom outside $\bar{\Omega}_i'$. The matrix block corresponding to the space V_i is obtained by extracting *all* the degrees of freedom relative to the elements contained in Ω_i' and is equal to

$$B_i = R_i B R_i^T : V_i \longrightarrow V_i.$$

It can easily be verified that the matrix B_i is the representation of the following local bilinear form:

$$\begin{aligned} B_i(u,v) &= \sum_{\substack{\kappa\in\mathcal{T}_h\\ \kappa\subset\Omega_i'}} \int_\kappa (a\nabla u\cdot\nabla v + b\cdot\nabla u v + cuv)dx \\ &- \sum_{\substack{\kappa\in\mathcal{T}_h\\ \kappa\subset\Omega_i'}} \int_{\partial_-\kappa\cap\Omega_i'} (b\cdot\mu)\lfloor u\rfloor v^+ ds + \int_{\Gamma_{int}\cap\Omega_i'} \sigma[u][v]ds \\ &+ \int_{\Gamma_{int\cap\Omega_i'}} ([u] < (a\nabla v)\cdot\nu > - < (a\nabla u)\cdot\nu > [v])\, ds \end{aligned}$$
$$\left.\begin{aligned} &- \sum_{\substack{\kappa\in\mathcal{T}_h\\ \kappa\subset\Omega_i'}} \int_{\partial_-\kappa\cap\partial\Omega_i'} (b\cdot\mu)u^+v^+ds \\ &+ \frac{1}{2}\int_{\Gamma_{int\cap\partial\Omega_i'}} (u((a\nabla v)\cdot\nu) - (a\nabla u)\cdot\nu)v)\, ds + \int_{\Gamma_{int}\cap\partial\Omega_i'} \sigma uvds \end{aligned}\right\}$$

for $u, v \in V_i$. The contributions in the first three lines come from the DG approximation of the operator $\mathcal{L}$ on Ω_i', while the remaining contributions are boundary contributions on $\partial\Omega_i'$, which appear since we have kept the boundary degrees of freedom in the definition of V_i. We first consider the pure hyperbolic case $a = 0$. Following [9], we see that B_i is the approximation of a Dirichlet problem with *weakly* imposed boundary conditions on the inflow part of the boundary $\partial\Omega_i'$ and it is therefore well–posed. This is opposed to the standard overlapping method for conforming approximations, where, by extracting local blocks, strongly imposed Dirichlet conditions on *all* $\partial\Omega_i'$ and thus potentially ill–posed local problems are obtained. In the pure diffusive case $b = 0$, we note the presence of the term $1/2$ in the skew–symmetric boundary contribution, arising from the average of the fluxes. Without this multiplicative factor, B_i would still be the approximation of a Dirichlet problem with weakly imposed boundary conditions on $\partial\Omega_i'$; see [9]. Despite the presence of the term $1/2$, we note however that B_i is positive–definite thanks to the presence of the penalization contribution and the local problem on Ω_i' is well–posed. In the general transport–diffusion case, the local matrices are still positive–definite, even if they do not in general represent Dirichlet local problems and we will prove that our choice of local problems leads to an optimal and scalable method.

We also note that, thanks to the choice of the local spaces, the case of zero overlap,

$$\Omega_i' = \Omega_i, \quad 1 \le i \le N,$$

can be considered, as was already noted in [8]. This has no analog in the conforming case and is due to the fact that we work with discontinuous FE spaces. Most of our numerical results show that the number of iterations obtained in this case is comparable, even if larger, to that for the overlapping case.

We now introduce our coarse solver. It is defined on $\mathcal{T}_H$ and is the FE approximation of our original problem on the *continuous, piecewise linear* FE space

$$V_0 = S^1(\Omega, \mathcal{T}_H) \cap H_0^1(\Omega) \subset V^h.$$

If $R_0^T : V_0 \to V^h$ is the natural interpolation operator from the subspace V_0 into V_h, then our coarse solver is

$$B_0 = R_0 B R_0^T,$$

and it can be easily shown to be positive–definite. We are now ready to define our Schwarz preconditioner

$$\hat{B}^{-1} = \sum_{i=0}^{N} R_i^T B_i^{-1} R_i.$$

In order to analyze the spectral properties of the corresponding preconditioned system $\hat{B}^{-1}B$, we write the latter using some projections; see [14]. As is standard practice in Schwarz methods, for $0 \le i \le N$ we define the B-projections $P_i : V^h \to V_i$ by

$$B(P_i u, v) \; = \; B(u, v)\,, \quad v \in V_i\,.$$

It can be easily shown (see [14]) that

$$P_i = (R_i^T B_i^{-1} R_i)\, B,$$

and consequently that the preconditioned matrix $\hat{B}^{-1}B$ is equal to the additive Schwarz operator:

$$P = \sum_{i=0}^{N} P_i\,.$$

In Theorem 6.1, we will show that P is invertible.

We consider the generalized minimum residual method (GMRES) applied to the preconditioned system

$$Pu = g, \tag{4.3}$$

where $g = \hat{B}^{-1} f$. Some convergence bounds for GMRES are proven in [7], to which we refer for a description of the algorithm. We denote by

$$c_P = \inf_{u \neq 0} \frac{A(u, Pu)}{A(u,u)} \quad \text{and} \quad C_P = \sup_{u \neq 0} \frac{\|Pu\|_A}{\|u\|_A}$$

the smallest eigenvalue of the symmetric part and the operator norm of P, respectively, Then, if $c_p > 0$, GMRES applied to (4.3) converges in a finite number of steps, and after m steps the norm of the residual is bounded by

$$\|r_m\|_A \; \le \; \left(1 - \frac{c_p^2}{C_P^2}\right)^{\frac{m}{2}} \|r_0\|_A\,.$$

5. Technical Tools. In this section, we provide all the technical tools needed for the proof of our convergence result contained Theorem 6.1.

Let $\widetilde{B_y}$ be a ball of radius H centered at the point $y \in \Omega$, and set $B_y = \widetilde{B_y} \cap \Omega$. The following definition of the quasi-interpolant as well as the proof of Lemma 5.1 are given for $d = 2$. Our definitions and analysis can easily be adapted to the case $d = 3$.

We define an interpolation operator

$$Q_H : L^2(\Omega) \to V_0,$$

by assigning a nodal value to every vertex a, b, c of every coarse element $K \in \mathcal{T}_H$. We set

$$(Q_H u)(y) = meas(B_y)^{-1} \int_{B_y} u(x)\, dx, \quad y \in \{a, b, c\}.$$

The following lemma ensures that Q_H is stable and provides an error bound.

LEMMA 5.1 (Coarse Mesh Quasi-Interpolant). *There exists $C > 0$, independent of h and H, such that, for all $u \in H^1(\Omega, \mathcal{T}_h)$*

$$\|Q_H u - u\|^2_{L^2(\Omega)} \le CH^2 \|u\|^2_A, \tag{5.1}$$

$$\|Q_H u\|^2_A \le C \|u\|^2_A. \tag{5.2}$$

Proof. We consider a coarse element $K \in \mathcal{T}_H$ with vertices a, b, c and denote by $\widetilde{K}$ the smallest convex neighborhood of K that also contains B_a, B_b, and B_c. We clearly have,

$$\|Q_H u\|_{L^2(K)} \le C \|u\|_{L^2(\widetilde{K})}, \quad u \in L^2(\Omega).$$

Since $\widetilde{K}$ has a diameter of order H, inequality (3.4) yields a positive constant C independent of h and H, such that for $v \in H^1(\Omega, \mathcal{T}_h)$ with $\int_{\widetilde{K}} v\, dx = 0$

$$\|v\|^2_{L^2(\widetilde{K})} \le CH^2 \left(|v|^2_{H^1(\widetilde{K}, \mathcal{T}_h)} + \int_{\Gamma_{int} \cap \widetilde{K}} \sigma[v]^2 ds \right).$$

Let now $u \in H^1(\Omega, \mathcal{T}_h)$ and $\bar{u} := u - meas(\widetilde{K})^{-1} \int_{\widetilde{K}} u dx$. Since Q_H reproduces constant functions on K, we obtain

$$\begin{aligned} \|Q_H u - u\|^2_{L^2(K)} = \|Q_H \bar{u} - \bar{u}\|^2_{L^2(K)} &\le C \|\bar{u}\|^2_{L^2(\widetilde{K})} \\ &\le CH^2 \left(|u|^2_{H^1(\widetilde{K}, \mathcal{T}_h)} + \int_{\Gamma_{int} \cap \widetilde{K}} \sigma[u]^2 ds \right). \end{aligned}$$

Summing over all $K \in \mathcal{T}_H$ and taking into account that for each $x \in \Omega$ the number of extended elements $\widetilde{K}$ to which it belongs is uniformly bounded, we have, for $u \in H^1(\Omega, \mathcal{T}_h)$,

$$\begin{aligned} \|Q_H u - u\|^2_{L^2(\Omega)} &\le C \sum_{\widetilde{K} \in \mathcal{T}_H} \|Q_H u - u\|^2_{L^2(\widetilde{K})} \\ &\le CH^2 \sum_{\widetilde{K} \in \mathcal{T}_H} \left(|u|^2_{H^1(\widetilde{K}, \mathcal{T}_h)} + \int_{\Gamma_{int} \cap \widetilde{K}} \sigma[u]^2 ds \right) \\ &\le CH^2 \|u\|^2_A, \end{aligned}$$

which concludes the proof of (5.1).

Using the inverse inequality (3.6) for an element $K \in \mathcal{T}_H$ and (3.4), we find

$$\begin{aligned}
|Q_H u|^2_{H^1(K)} = |Q_H \bar{u}|^2_{H^1(K)} &\leq C\, H^{-2}\, \|Q_H \bar{u}\|^2_{L^2(K)} \\
&\leq CH^{-2} \left(\|Q_H \bar{u} - \bar{u}\|^2_{L^2(K)} + \|\bar{u}\|^2_{L^2(\widetilde{K})} \right) \\
&\leq C \left(|u|^2_{H^1(\widetilde{K}, \mathcal{T}_h)} + \int_{\Gamma_{int} \cap \widetilde{K}} \sigma[u]^2 ds \right) .
\end{aligned}$$

Since $Q_H u$ is continuous in Ω, $\|Q_h u\|_A$ is equal to the broken H^1–seminorm, and summing over all $K \in \mathcal{T}_H$ concludes the proof of inequality (5.2). □

We note that we have used the interpolant Q_H instead of the L^2 orthogonal projection, in order to make our analysis valid in the case of a coarse mesh that is not quasi–uniform; see, e.g., [4].

The following lemma ensures that, for every function in the discontinuous space V^h, a stable decomposition can be found for the family of subspaces $\{V_i\}$.

Lemma 5.2 (Decomposition). *There exists a constant $C_0 > 0$, independent of h and H, such that for all $u \in V^h$ there exists $\{u_i \in V_i\}_{0 \leq i \leq N}$ with $u = \sum_{i=0}^N u_i$ and*

$$\sum_{i=0}^{N} \|u_i\|^2_A \;\leq\; C_0^2\, \|u\|^2_A \, .$$

Proof. We denote by $C(\Omega, \mathcal{T}_h) = \{u \in L^2(\Omega) : u|_{\bar{\kappa}} \in C(\bar{\kappa})\, , \kappa \in \mathcal{T}_h\}$ the space of piecewise continuous functions. We define the operator

$$I^h : C(\Omega, \mathcal{T}_h) \rightarrow V^h\, ,$$

where for each element $\kappa \in \mathcal{T}_h$, the restriction $I^h|_{\bar{\kappa}}$ to $\bar{\kappa}$ is equal to the nodal interpolation operator onto $\mathcal{P}_{p_\kappa}(\kappa)$.

For $u \in V^h$, we define

$$\begin{cases} u_0 = Q_H u, \\ u_i = I^h(\theta_i(u - u_0))\, , \quad 1 \leq i \leq N\, , \end{cases}$$

where $\{\theta_i\}_{1 \leq i \leq N}$ is a piecewise linear partition of unity relative to the family $\{\Omega'_i\}_{1 \leq i \leq N}$; see, e.g., [14]. We recall, in particular, that $\theta_i \in [0,1]$, $supp(\theta_i) \subset \bar{\Omega}'_i$, for $1 \leq i \leq N$, and $\sum_{i=1}^N \theta_i(x) = 1$ for all $x \in \Omega$. Furthermore, our assumption (4.1) on the overlap of the extended subdomains ensures that $\|\nabla \theta_i\|_{L^\infty(\Omega)} \leq CH^{-1}$, where C depends on α. By construction, $u_i \in V_i$ for $0 \leq i \leq N$, and $u = \sum_{i=0}^N u_i$.

Let $w = u - u_0$. The same arguments used in the proof of the decomposition lemma for standard conforming finite elements [14, Chapter 5.3], yield, for $\kappa \in \mathcal{T}_h$ and $1 \leq i \leq N$,

$$|u_i|^2_{H^1(\kappa)} \;\leq\; 2\, |w|^2_{H^1(\kappa)} + CH^{-2}\, \|w\|^2_{L^2(\kappa)}\, .$$

Since for each $x \in \Omega$ the number of $u_i(x)$, which differ from zero, is uniformly bounded (finite covering), summing over i yields

$$\sum_{i=1}^{N} |u_i|^2_{H^1(\kappa)} \;\leq\; C\, |w|^2_{H^1(\kappa)} + CH^{-2}\, \|w\|^2_{L^2(\kappa)}\, .$$

We next sum over all the elements κ and obtain

$$\sum_{i=1}^{N} |u_i|^2_{H^1(\Omega,\mathcal{T}_h)} \ \leq\ C\,|w|^2_{H^1(\Omega,\mathcal{T}_h)} + CH^{-2}\,\|w\|^2_{L^2(\Omega)}\,.$$

Furthermore, we have, for all $1 \leq i \leq N$,

$$\|[\theta_i w]\|_{L^\infty(\Gamma_{int})} \ \leq\ \|[w]\|_{L^\infty(\Gamma_{int})}\,,$$

where we have used the fact that θ_i is continuous and that $\|\theta_i\|_{L^\infty(\Omega)} \leq 1$. Since $w \in V^h$, we obtain

$$\int_{\Gamma_{int}} \sigma[u_i]^2 ds \leq \int_{\Gamma_{int}} \sigma[w]^2 ds\,.$$

The finite covering of the subdomains yields

$$\sum_{i=1}^{N} \int_{\Gamma_{int}} \sigma[u_i]^2 ds \ \leq\ C \int_{\Gamma_{int}} \sigma[w]^2 ds\,.$$

Summing the H^1-seminorms and jump terms, we obtain

$$\sum_{i=1}^{N} \|u_i\|^2_A \ \leq\ C\,\|w\|^2_A + CH^{-2}\,\|w\|_{L^2(\Omega)}\,,$$

and the proof is concluded by applying Lemma 5.1. $\square$

REMARK 1. *The proof of the previous lemma can be carried out also in the case of zero overlap: $\Omega_i' = \Omega_i$. In this case the partition of unity $\{\theta_i\}$ consists of the (discontinuous) characteristic functions of the subdomains $\{\Omega_i\}$. However, C_0^2 grows linearly with H/h in this case; see also [8] for a similar algorithm.*

The following lemma contains some bounds for the B-projections $\{P_i\}$.

LEMMA 5.3 (B-Projections). *There exists $C > 0$, such that for all $u \in V^h$,*

$$\begin{aligned}
&\|P_0 u\|_A \leq C\,\|u\|_A,\\
&\|P_0 u - u\|_{L^2(\Omega)} \leq C\,H^\gamma\,\|u\|_A\\
&\|P_i u\|_{L^2(\Omega)} \leq C\,H\,\|P_i u\|_A\,, \quad 1 \leq i \leq N\,,
\end{aligned}$$

where $\gamma > 1/2$ is related to the regularity constant of the adjoint problem with Dirichlet boundary conditions.

Proof. The coercivity and continuity of B, and the definition of P_0 yield

$$\|P_0 u\|^2_A \ \leq\ B(P_0 u, P_0 u) \ =\ B(u, P_0 u) \ \leq\ C\,\|u\|_A \|P_0 u\|_A\,,$$

which gives the first inequality.

In order to obtain a bound for the error $u - P_0 u$, we consider the auxiliary problem

$$\mathcal{L}^* w = P_0 u - u \quad \text{in } \Omega, \quad w = 0 \quad \text{on } \Gamma\,,$$

where $\mathcal{L}^*$ is the adjoint of $\mathcal{L}$. We have for any $w_0 \in V_0$

$$\begin{aligned}
\|P_0 u - u\|^2_{L^2(\Omega)} &= (P_0 u - u, \mathcal{L}^* w)_{L^2(\Omega)} = B(P_0 u - u, w)\\
&= B(P_0 u - u, w - w_0) \leq C\,\|P_0 u - u\|_A \|w - w_0\|_A\,.
\end{aligned}$$

Since $P_0u - u \in L^2(\Omega)$, then $w \in H^{\eta+3/2}(\Omega)$ for a $\eta > 0$, and the Sobolev embedding theorem implies $H^{\eta+3/2}(\Omega) \subset C(\overline{\Omega})$. Therefore, $w - w_0$ is continuous, and $\|w - w_0\|_A$ is equal to the broken H^1–seminorm. Standard approximation estimates yield the existence of $w_0 \in V_0$ such that

$$\|w - w_0\|_{H^1(\Omega)} \ \leq \ C\, H^\gamma \, \|w\|_{H^{1+\gamma}(\Omega)} \ ,$$

with $\gamma = \eta + 1/2$; see, e.g., [12]. Therefore,

$$\|P_0u - u\|^2_{L^2(\Omega)} \ \leq \ C\, H^\gamma \, \|P_0u - u\|_A \|P_0u - u\|_{L^2(\Omega)} \ ,$$

which gives the L^2-bound.

The inequalities for $i > 0$ result from the observation that P_iu vanishes outside a region of diameter $O(H)$ and the Friedrichs inequality in Lemma 3.1. ☐

As for the analogous algorithm in the conforming case ([2, 14]), we need to control the lower–order and skew–symmetric terms of the bilinear form B. Lemmata 3.1, 3.5, 3.6 and 5.3 set the stage for the proof of the following bounds, which can be carried out as in [14, Lem. 16, Ch. 5.4].

LEMMA 5.4. *There exists a constant $C > 0$, independent of h and H, such that for all $u \in V^h$ and $0 \leq i \leq N$*

$$\begin{aligned}
|C(P_iu - u, P_iu)| &\leq C\, H^{\beta_i} \left(\|u\|_A^2 + \|P_iu\|_A^2 \right) \ , \\
|D(P_iu - u, P_iu)| &\leq C\, H^{\beta_i} \left(\|u\|_A^2 + \|P_iu\|_A^2 \right) \ , \\
|S(P_iu - u, P_iu)| &\leq C\, \sqrt{H} \left(\|u\|_A^2 + \|P_iu\|_A^2 \right) \ ,
\end{aligned}$$

where $\beta_0 = \gamma$ and $\beta_i = 1$ for $i > 0$.

6. The convergence result. We have now completed all the preparations required to obtain a lower bound for c_P and an upper bound for C_P. We remark that the following proof is similar to those in [2], [3], and [14, Ch. 5.4].

THEOREM 6.1. *There exist constants $C > 0$, $H_0 > 0$, $c(H_0) > 0$, such that, for all $u \in V^h$,*

$$\begin{aligned}
A(Pu, Pu) \ &\leq \ C\, A(u, u)\, , \\
c(H_0) A(u, u) \ &\leq \ A(u, Pu), \quad H \leq H_0.
\end{aligned}$$

Proof. First we observe, that the finite covering property implies

$$\|Pu\|^2_A \ = \ \left\| \sum_{i=0}^N P_iu \right\|^2_A \ \leq \ C \, \sum_{i=0}^N \|P_iu\|^2_A \, . \tag{6.1}$$

Since B is coercive and continuous, we find

$$\begin{aligned}
\sum_{i=0}^N \|P_iu\|^2_A &\leq \sum_{i=0}^N B(P_iu, P_iu) = \sum_{i=0}^N B(u, P_iu) = B(u, \sum_{i=0}^N P_iu) \\
&\leq C\, \|u\|_A \left\| \sum_{i=0}^N P_iu \right\|_A \ \leq \ C\, \|u\|_A \left(\sum_{i=0}^N \|P_iu\|^2_A \right)^{\frac{1}{2}} \ .
\end{aligned} \tag{6.2}$$

Combining (6.1) and (6.2), we obtain $\|Pu\|^2_A \ \leq \ C\, \|u\|^2_A$, which proves our upper bound.

Since $A(u,Pu) = \sum_{i=0}^{N} A(u,P_iu)$, we need to consider the term $A(u,P_iu)$ for $0 \le i \le N$. Using the definition of P_i and B, we have

$$
\begin{aligned}
0 &= B(P_iu - u, P_iu) \\
&= A(P_iu - u, P_iu) + C(P_iu - u, P_iu) + D(P_iu - u, P_iu) + S(P_iu - u, P_iu)\,,
\end{aligned}
$$

and consequently, using Lemma 5.4,

$$
\begin{aligned}
A(u,P_iu) &\ge A(P_iu,P_iu) - |C(P_iu - u, P_iu)| - |D(P_iu - u, P_iu)| - |S(P_iu - u, P_iu)| \\
&\ge \left(1 - C\max(H^{\beta_i}, \sqrt{H}, H)\right) \|P_iu\|_A^2 - C\,\max(H^{\beta_i}, \sqrt{H}, H)\,\|u\|_A^2\,.
\end{aligned}
$$

If we choose H small enough such that

$$
\omega = \min_{0\le i\le N} \left(1 - C\max(H^{\beta_i}, \sqrt{H}, H)\right),
$$

is positive, we have

$$
A(u,P_iu) \ge \omega\|P_iu\|_A^2 - \delta_i\|u\|_A^2\,,
$$

where $\delta_i = \max(H^{\beta_i}, \sqrt{H}, H)$. Again, the finite covering implies

$$
A(u,Pu) \;\ge\; \omega \sum_{i=0}^{N} \|P_iu\|_A^2 - C\,\|u\|_A^2\,. \tag{6.3}
$$

The coercivity and continuity of B, Lemma 5.2, and the Cauchy-Schwarz inequality yield

$$
\begin{aligned}
\|u\|_A^2 \le B(u,u) \;&=\; \sum_{i=0}^{N} B(u,u_i) \;=\; \sum_{i=0}^{N} B(P_iu,u_i) \\
&\le C \sum_{i=0}^{N} \|P_iu\|_A\|u_i\|_A \;\le\; C\,\left(\sum_{i=0}^{N}\|P_iu\|_A^2\right)^{\frac{1}{2}} \cdot \left(\sum_{i=0}^{N}\|u_i\|_A^2\right)^{\frac{1}{2}} \\
&\le C\,\left(\sum_{i=0}^{N}\|P_iu\|_A^2\right)^{\frac{1}{2}} \cdot C_0\,\|u\|_A\,,
\end{aligned}
$$

and therefore $\sum_{i=0}^{N} \|P_iu\|_A^2 \;\ge\; C\,\|u\|_A^2$, which, combined with (6.3), gives the desired lower bound for H sufficiently small. □

REMARK 2. *We note that our analysis is valid for FE spaces of arbitrary polynomial degree on each element, but the constants C, H_0, and c in Theorem 6.1 depend on $p := \max\{p_\kappa |\ \kappa \in \mathcal{T}_h\}$ in general.*

7. Numerical results. We present some numerical results to illustrate the performance of our overlapping Schwarz algorithm for piecewise linear finite elements in two dimensions. We have tested the two–level preconditioner introduced in the previous sections, as well as the one–level preconditioner built on the same partitions, and we are interested in the performance of the two methods when varying h, H, and the overlap. We consider Problem (1.1) in $\Omega = (0,1)^2$ with weakly–imposed Dirichlet boundary conditions; see, e.g., [9]. Our test cases are for a Poisson problem, an advection-diffusion equation with constant coefficients, and an advection-diffusion equation with a rotating flow field.

We use a two-level subdivision of Ω, consisting of a fine triangulation $\mathcal{T}_h$, obtained by dividing Ω into h^{-2} squares that are then cut into two triangles, and a coarse triangulation consisting of H^{-2} squares Ω_i, which are possibly extended in order to form a partition $\{\widetilde{\Omega_i}\}$ by adding $q \in \mathrm{N}_0$ layers of h-level triangles in all directions. We set $\Omega_i' = \widetilde{\Omega_i} \cap \Omega$. The overlap is $\delta = qh$, $\delta \geq 0$.

Though our theory requires the penalization parameter σ_0 to be of order H^{-1}, our experiments show that in practice this restriction is not required. We have chosen $\sigma_0 = 1$ and solved the coarse and local problems exactly by using Gaussian elimination.

We remark that all our theoretical estimates employ the A–induced scalar product, but that our GMRES implementation employs the standard Euclidean product. Our theoretical results are still valid in this case:
The inverse estimates (3.5) and (3.6) yield positive constants d_0, d_1 independent of h, such that

$$d_0 h^d \|x\|_2^2 \;\leq\; \|x\|_A^2 \;\leq\; d_1 h^{d-2} \|x\|_2^2, \quad x \in \mathrm{R}^n \;;$$

see for example [10, Sect. 7.7]. Therefore, the use of the Euclidean norm increases the iteration counts only by an additive term of order $\log_{10}(h)$, which is hard to observe in our computational experiments; see also [11, Sect. 5].

In our experiments we stop GMRES as soon as $\|r_i\|_2 \leq 10^{-6}\|r_0\|_2$ or after 100 iterations. Our numerical results have been obtained with *Matlab 5.3*.

7.1. Poisson equation. We first consider the Poisson equation with inhomogeneous Dirichlet conditions:

$$-\Delta u = xe^y \quad \text{in } \Omega\,, \quad u = -xe^y \quad \text{on } \Gamma\,.$$

and partitions into $N \times N$ squares ($H = 1/N$), with $N = 2, 4, 8, 16, 32$.

Tables 7.1 show the iteration counts for the one– and two–level algorithms, as functions of h and the inverse of the relative overlap. We have also considered the case of zero overlap, denoted by $H/\delta = \infty$. We note that both methods appear to be rather insensitive to the size of the original problem when H is fixed, but that, as expected, the iterations for the one–level preconditioner (table on the left hand-side) grow with the number of subdomains. The two–level algorithm (table on the right hand-side), on the other hand, appears to be scalable and this confirms our analysis. We also note that the iteration numbers decrease when the relative overlap increases. Since our convergence bound for the two–level preconditioner is not explicit in the overlap, we can only give the heuristic explanation that the subproblems capture more and more of the entire problem when the overlap is increased. Finally, we remark that the restriction on the penalization term $\sigma_0 > C/H$ does not appear to be required in practice. This is essential, since if this coefficient is too high, the accuracy of the FE solution deteriorates.

The case of zero overlap requires a special discussion. Our results show that the number of iterations obtained are generally comparable to, but slightly higher than, those obtained in the case of $\delta > 0$ for both algorithms. The iterations are considerably higher only for the case $h = 1/128$ and $H = 1/8$. From our numerical results, we are unable to deduce whether the two–level method is optimal or non–optimal with the number of iterations growing as a power of H/h. We refer to the following tables for a clearer behavior of the convergence rate in this case, and to [8] for a method with the

		H/δ				
h^{-1}	H^{-1}	∞	16	8	4	2
16	2	17	-	16	14	12
16	4	24	-	-	22	17
32	2	22	21	17	14	12
32	4	33	-	30	23	18
32	8	44	-	-	38	29
64	2	30	27	22	17	14
64	4	45	40	32	24	18
64	8	60	-	53	41	30
64	16	84	-	-	73	54
128	4	60	54	44	33	25
128	8	82	72	57	43	31
128	16	100	-	100	78	57
128	32	100	-	-	100	100

		H/δ				
h^{-1}	H^{-1}	∞	16	8	4	2
16	2	13	-	11	11	11
16	4	13	-	-	13	14
32	2	16	13	12	11	10
32	4	15	-	13	12	13
32	8	13	-	-	13	15
64	2	21	16	14	12	11
64	4	19	15	14	13	13
64	8	16	-	13	13	14
64	16	13	-	-	13	15
128	4	25	18	16	14	13
128	8	35	15	14	13	14
128	16	15	-	13	13	15
128	32	12	-	-	13	15

TABLE 7.1

Poisson's equation: Iteration counts for GMRES and the one–level and two–level preconditioners, respectively, versus h and the relative overlap.

same local solvers but a different coarse space, which exhibits a rate of convergence that appears to grow linearly with H/h. However, we believe that due to the minimal communication between the subdomains and the relatively small iteration counts that we have obtained, the two-level algorithm with zero overlap might be competitive in practice.

7.2. Advection–diffusion problem with constant coefficients. We next consider the advection-diffusion equation

$$-\Delta u + b \cdot \nabla u = f \quad \text{in } \Omega\,, \quad u = 0 \quad \text{on } \Gamma\,,$$

with constant coefficients and zero Dirichlet boundary conditions. We consider the two cases

$$b \in \{-(k\pi, k\pi) : k = 3, 300\}\,.$$

The right-hand side f is always chosen such that the exact solution is $u = xe^{xy}\sin(\pi x)\sin(\pi y)$.

Tables 7.2 present the results for $k = 3$, for the one– and two–level algorithms, respectively. As for the Poisson problem with non–vanishing overlap, the iteration counts decrease when the overlap increases and are independent of the number of subdomains for the two–level method. The use of a coarse solver improves the convergence properties.

In this case, the behavior for zero overlap appears to be more regular. As expected, the iteration counts increase when the number of subdomains increases for the one–level algorithm. On the other hand, if a coarse solver is employed, the number of iterations appears to grow like H/h, *when h is fixed*. For a fixed value of H/h, slower convergence rates are obtained for h larger. We can then conclude that, for the case of zero overlap, the iteration counts are indeed bounded by a $C(H/h)$, with C a suitable constant; see also [8]. However, we believe that in this case as well the two-level algorithm with zero overlap might be competitive in practice.

		H/δ				
h^{-1}	H^{-1}	∞	16	8	4	2
16	4	25	-	-	15	17
32	4	33	-	21	16	17
32	8	45	-	-	25	22
64	4	49	28	22	16	17
64	8	59	-	36	27	24
64	16	84	-	-	47	39
128	4	43	28	22	16	17
128	8	59	36	27	24	24
128	16	84	-	47	39	39

		H/δ				
h^{-1}	H^{-1}	∞	16	8	4	2
16	4	15	-	-	14	16
32	4	16	-	15	14	15
32	8	12	-	-	14	16
64	4	20	16	16	15	15
64	8	14	-	13	13	16
64	16	10	-	-	12	16
128	4	20	16	16	15	15
128	8	14	13	13	16	16
128	16	10	-	12	16	16

TABLE 7.2

Case of $b = -(3\pi, 3\pi)$: iteration counts for GMRES with the one–level and two–level preconditioners, respectively, versus h and the relative overlap.

		H/δ				
h^{-1}	H^{-1}	∞	16	8	4	2
16	4	13	-	-	12	16
32	4	14	-	13	13	16
32	8	22	-	-	16	21
64	4	15	13	13	13	16
64	8	23	-	21	17	20
64	16	38	-	-	26	27
128	4	15	13	13	14	16
128	8	23	21	17	20	20
128	16	38	-	26	27	27

		H/δ				
h^{-1}	H^{-1}	∞	16	8	4	2
16	4	32	-	-	21	19
32	4	32	-	28	21	18
32	8	74	-	-	32	23
64	4	32	30	27	21	18
64	8	73	-	47	32	23
64	16	100	-	-	36	27
128	4	33	31	27	21	18
128	8	73	47	32	23	23
128	16	100	-	36	27	28

TABLE 7.3

Case of $b = -(300\pi, 300\pi)$: iteration counts for GMRES with the one– and two–level preconditioners, versus h and the relative overlap.

Our second set of results is for $k = 300$ and is shown in Tables 7.3. All the remarks made for Tables 7.2 remain valid in this case, but the iteration counts for the two–level method are considerably higher. This is a case with very strong convection (the Reynolds number is approximately 1000), and the one–level method performs fairly well. A coarse space not only does not seem necessary, but can slow down the convergence considerably. We believe that such behavior is partly due to our coarse solver, which, in this case, comes from a non–stabilized approximation of an advection–diffusion problem on a continuous FE space and a different type of coarse solver needs to be devised for some kinds of convection–dominated problems. Note also that the iterations for the one–level method appear to depend only on H, and grow linearly with $1/H$. For the case of zero overlap, the same remarks made before remain valid.

7.3. Advection–diffusion problem with a rotating flow field and boundary layers. Finally, we consider an advection-diffusion equation with a rotating wind $b = 0.5\,(y+1, -x-1)$, a constant $c = 10^{-4}$, the right-hand side $f = 0$, and discon-

		H/δ				
h^{-1}	H^{-1}	∞	16	8	4	2
16	4	22	-	-	14	16
32	4	30	-	19	15	17
32	8	39	-	-	23	22
64	4	40	26	20	16	18
64	8	53	-	33	25	24
64	16	72	-	-	42	37
128	4	54	28	21	16	18
128	8	53	33	25	24	26
128	16	72	-	42	37	42

		H/δ				
h^{-1}	H^{-1}	∞	16	8	4	2
16	4	13	-	-	13	14
32	4	15	-	13	13	13
32	8	14	-	-	13	15
64	4	19	15	14	13	14
64	8	16	-	14	13	14
64	16	13	-	-	13	15
128	4	24	18	14	13	14
128	8	16	14	13	13	14
128	16	13	-	13	15	14

TABLE 7.4

Rotating flow field, case of $\nu = 1$: iteration counts for GMRES with the one– and two–level preconditioners, versus h and the relative overlap.

		H/δ				
h^{-1}	H^{-1}	∞	16	8	4	2
16	4	13	-	-	10	13
32	4	16	-	11	10	14
32	8	23	-	-	15	17
64	4	19	13	10	10	14
64	8	28	-	18	15	18
64	16	43	-	-	25	25
128	4	25	13	11	10	14
128	8	28	18	15	18	19
128	16	43	-	25	25	27

		H/δ				
h^{-1}	H^{-1}	∞	16	8	4	2
16	4	27	-	-	19	16
32	4	28	-	22	19	16
32	8	33	-	-	20	18
64	4	31	26	23	19	17
64	8	36	-	24	20	17
64	16	23	-	-	17	19
128	4	35	26	23	19	17
128	8	36	24	20	17	17
128	16	23	-	17	19	18

TABLE 7.5

Rotating flow field, case of $\nu = 0.01$: iteration counts for GMRES with the one– and two–level preconditioners, versus h and the relative overlap.

tinuous Dirichlet boundary data:

$$\begin{aligned} -\nu\Delta u + b\cdot\nabla u + cu = f, &\quad \text{in } \Omega\,, \\ u = 1 &\quad \text{if} \quad (x,y) \in]0.5,1] \times \{-1,1\} \cup \{1\} \times [0,1]\,, \\ u = 0 &\quad \text{elsewhere on } \Gamma\,. \end{aligned}$$

We note that for small values of ν there are internal layers and boundary layers along the four sides of Ω.

Tables 7.4 show the results for the two methods for a case of small Reynolds number ($\nu = 1$). We note that the same remarks made for Tables 7.2 apply in this case for both algorithms. We then consider a convection–dominated case. Tables 7.5 show the results for a case of a much smaller diffusion ($\nu = 0.01$). As for a parallel constant flow, the results for the one–level method are better than those with a coarse space, even though, due to the smaller Reynolds number (100) the difference is not as large as in Tables 7.3.

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