A FETI PRECONDITIONER FOR TWO DIMENSIONAL EDGE ELEMENT APPROXIMATIONS OF MAXWELL'S EQUATIONS ON NON-MATCHING GRIDS

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Abstract. A class of FETI methods for the mortar approximation of a vector field problem in two dimensions is introduced and analyzed. Edge element discretizations of lowest degree are considered. The method proposed can be employed with geometrically conforming and non–conforming partitions. Our numerical results show that its condition number increases only with the number of unknowns in each subdomains, and is independent of the number of subdomains and the size of the problem.

Key words. Edge elements, Maxwell's equations, domain decomposition, FETI, preconditioners, non-matching grids

AMS subject classifications. 65F10, 65N22, 65N30, 65N55

1. Introduction. In this paper, we consider the boundary value problem

$$\begin{array}{rcll} L\mathbf{u} := \mathbf{curl}\,(a\,\mathrm{curl}\,\mathbf{u}) + A\,\mathbf{u} & = & \mathbf{f} & \text{in } \Omega, \\ \mathbf{u}\cdot\mathbf{t} & = & 0 & \text{on } \partial\Omega, \end{array} \tag{1}$$

with Ω a bounded polygonal domain in $\mathbb{R}^2$. Here

$$\mathbf{curl}\, v := \begin{bmatrix} \dfrac{\partial v}{\partial x_2} \\ -\dfrac{\partial v}{\partial x_1} \end{bmatrix}, \quad \mathrm{curl}\,\mathbf{u} := \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2};$$

see, e.g., [17]. The coefficient matrix A is a symmetric, uniformly positive definite matrix–valued function with entries $A_{ij} \in L^\infty(\Omega)$, $1 \le i, j \le 2$, and $a \in L^\infty(\Omega)$ is a positive function bounded away from zero. The domain Ω has unit diameter and $\mathbf{t}$ is the unit tangent to its boundary.

The weak formulation of problem (1) requires the introduction of the Hilbert space $H(\mathrm{curl};\Omega)$, defined by

$$H(\mathrm{curl};\Omega) := \left\{\mathbf{v} \in (L^2(\Omega))^2 | \quad \mathrm{curl}\,\mathbf{v} \in L^2(\Omega)\right\}.$$

The space $H(\mathrm{curl};\Omega)$ is equipped with the following inner product and graph norm,

$$(\mathbf{u},\mathbf{v})_{\mathrm{curl}} := \int_\Omega \mathbf{u}\cdot\mathbf{v}\, d\mathbf{x} + \int_\Omega \mathrm{curl}\,\mathbf{u}\,\mathrm{curl}\,\mathbf{v}\, d\mathbf{x}, \qquad \|\mathbf{u}\|^2_{\mathrm{curl}} := (\mathbf{u},\mathbf{u})_{\mathrm{curl}}.$$

The tangential component $\mathbf{u}\cdot\mathbf{t}$, of a vector $\mathbf{u} \in H(\mathrm{curl};\Omega)$ on the boundary $\partial\Omega$, belongs to the space $H^{-\frac{1}{2}}(\partial\Omega)$; see [17, 8]. The subspace of vectors in $H(\mathrm{curl};\Omega)$ with vanishing tangential component on $\partial\Omega$ is denoted by $H_0(\mathrm{curl};\Omega)$.

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For any $\mathcal{D} \subset \Omega$, we define the bilinear form

$$a_{\mathcal{D}}(\mathbf{u},\mathbf{v}) := \int_{\mathcal{D}} (a \operatorname{curl}\mathbf{u} \operatorname{curl}\mathbf{v} + A\, \mathbf{u}\cdot\mathbf{v})\, d\mathbf{x}, \quad \mathbf{u},\, \mathbf{v} \in H(\operatorname{curl};\Omega). \tag{2}$$

The variational formulation of equation (1) is:
Find $\mathbf{u} \in H_0(\operatorname{curl};\Omega)$ such that

$$a_{\Omega}(\mathbf{u},\mathbf{v}) = \int_{\Omega} \mathbf{f}\cdot\mathbf{v}\, d\mathbf{x}, \quad \mathbf{v} \in H_0(\operatorname{curl};\Omega). \tag{3}$$

We discretize this problem using edge elements, also known as Nédélec elements; see [24]. These are vector–valued finite elements that only ensure the continuity of the tangential component across the common side of adjacent mesh triangles, as is physically required for the electric and magnetic fields, which are solutions of Maxwell's equations.

In this paper, we consider a mortar approximation of this problem. The computational domain is partitioned into a family of non–overlapping subdomains and independent triangulations are introduced in each subdomain. The weak continuity of the tangential component of the solution is then enforced by using suitable integral conditions that require that the jumps across the subdomain inner boundaries are perpendicular to suitable finite element spaces defined on the edges of the partition. We note that the mortar method was originally introduced in [10] for finite element approximations in H^1. Mortar approximations for edge element approximations in two and three dimensions have been studied in [3] and [4], respectively. There has also been additional recent work for the case of sliding meshes for the study of electromagnetic fields in electrical engines; see, e.g., [25].

The applications that we have in mind are mainly problems arising from static and quasi–static Maxwell's equations (eddy current problems); see, e.g., [6, 5]. In this paper, we only consider the model problem (3), where the dependency on the time variable or on the frequency has been eliminated, and we generically refer to it as Maxwell's equations. A good preconditioner for this model problem is the first step for the efficient solution of linear systems arising from the edge element approximation of static problems, and of time– or frequency–dependent problems arising from the quasi–static approximation of Maxwell's equations.

The aim of this paper is to build an iterative method of Finite Element Tearing and Interconnecting (FETI) type for a mortar edge element approximation of problem (1). FETI methods were first introduced for the solution of conforming approximations of elasticity problems in [15]. In this approach, the original domain Ω is decomposed into non-overlapping subdomains Ω_i, $i = 1,\ldots,N$. On each subdomain Ω_i a local stiffness matrix is obtained from the finite element discretization of $a_{\Omega_i}(\cdot,\cdot)$. Analogously, a set of right hand sides is built. The continuity of the solution corresponding to the primal variables is then enforced, by using Lagrange multipliers, across the interface defined by the subdomain inner boundaries. In the original FETI algorithm, the primal variables are then eliminated by solving local Neumann problems, and an equation in the Lagrange multipliers is obtained. Several preconditioners have been proposed and studied for its solution; see, e.g., [14, 16, 23, 13, 29, 26, 19, 32].

Many iterative methods for the solution of linear systems arising from mortar approximations have been proposed. We cite, in particular, [22, 1, 20, 2, 9, 11, 7, 12, 21, 33] and refer to the references therein for a more detailed discussion.

To our knowledge, the application of FETI preconditioners to mortar approximations was first explored in [18] and then tested more systematically in [28]. The idea is fairly simple and relies on the observation that mortar approximations with Lagrange multipliers and FETI formulations, where the pointwise continuity across the substructures is enforced by using Lagrange multipliers, give rise to indefinite linear systems that have the same form. FETI preconditioners can then be devised for mortar approximations in a straightforward way; see [28]. In this paper, we apply the FETI preconditioner introduced in [19] for the case of non–redundant Lagrange multipliers, to the mortar approximation originally studied in [3]. Our work generalizes that for FETI preconditioners for two–dimensional conforming edge–element approximations in [32]. As opposed to the H^1 case, the generalization of FETI preconditioners to mortar approximations requires some modifications in $H(\mathrm{curl}\,)$. More precisely, the coarse components of the preconditioners need to be modified here in order to obtain a scalable method, and a suitable scaling matrix Q has to be introduced; see section 5. As shown in [28], no modification appears to be necessary for nodal finite elements in H^1. Finally, we note that in this paper we only consider problems without jumps of the coefficients. For conforming approximations, FETI methods that are robust with respect to large variations of jumps of the coefficients have been developed and studied, see [26, 19], but the case of nodal or edge element approximations on non–matching grids still needs to be explored and is left for a forthcoming paper.

The outline of the remainder of this paper is as follows. In section 2, we introduce a partition of the domain Ω and local finite element spaces. In section 3, we consider the mortar condition and in section 4, we present our FETI method, in terms of a projection onto a low–dimensional subspace and a local preconditioner. The expressions for the projection and the preconditioner are then given in section 5 and some numerical results for geometrically conforming and non–conforming partitions are presented in section 6.

2. Finite element spaces. We first consider a non–overlapping partition of the domain Ω,

$$\mathcal{F}_H = \left\{ \Omega_i,\ i = 1, \cdots, N \mid \quad \bigcup_{i=1}^{N} \overline{\Omega_i} = \overline{\Omega} \quad ; \quad \Omega_k \cap \Omega_l = \emptyset \quad , \quad 1 \le k < l \le N \right\},$$

such that each subdomain Ω_i is a connected polygonal open set in $\mathbb{R}^2$. We remark that $\mathcal{F}_H$ does not need to be geometrically conforming. We denote the diameter of Ω_i by H_i and the maximum of the diameters of the subdomains by H:

$$H := \max_{1 \le i \le N} \{H_i\}.$$

The elements of $\mathcal{F}_H$ are also called *substructures*. Let $\mathbf{t}_i$ be the unit tangent to $\partial\Omega_i$, chosen so that following the direction of $\mathbf{t}_i$, Ω_i is on the left.

For every subdomain Ω_i, we define the set of its open edges that do not lie on $\partial\Omega$ by $\{\Gamma^{i,j} \mid j \in \mathcal{I}_i\}$. We then define the interface Γ, also called the "skeleton" of the decomposition, as the union of the edges of $\mathcal{F}_H$ that do not lie on $\partial\Omega$:

$$\Gamma := \bigcup_{i=1}^{N} \partial\Omega_i \setminus \partial\Omega = \bigcup_{i=1}^{N} \bigcup_{j \in \mathcal{I}_i} \Gamma^{i,j}.$$

We also define the local spaces of restrictions of vectors in $H_0(\mathrm{curl}\,;\Omega)$ to Ω_i:

$$H_\star(\mathrm{curl}\,;\Omega_i) := \{\mathbf{u}_i \in H(\mathrm{curl}\,;\Omega_i)|\ \mathbf{u}_i\cdot\mathbf{t} = 0 \text{ on } \partial\Omega\cap\partial\Omega_i\}.$$

For every substructure Ω_i, we consider a triangulation $\mathcal{T}_{i,h}$, made of triangles or rectangles. Let $\mathcal{E}_{i,h}$ be the set of edges of $\mathcal{T}_{i,h}$. For every edge $e\in\mathcal{E}_{i,h}$, we fix a direction, given by a unit vector $\mathbf{t}_e$. The length of the edge e is denoted by $|e|$. The local triangulations are assumed shape–regular and quasi–uniform, and they do not need to match across the inner boundaries of the subdomains. We define h as the maximum of the mesh–sizes of the triangulations.

We next consider the lowest–order Nédélec finite element (FE) spaces, originally introduced in [24], defined on each subdomain Ω_i as

$$X_h(\Omega_i) = X_i := \{\mathbf{u}_i\in H_\star(\mathrm{curl}\,;\Omega_i)|\ \mathbf{u}_{i_{|t}} \in \mathcal{R}(t),\ t\in\mathcal{T}_{i,h}\},$$

where, in the case of triangular meshes, we have

$$\mathcal{R}(t) := \left\{ \begin{bmatrix} \alpha_1+\alpha_3x_2 \\ \alpha_2-\alpha_3x_1 \end{bmatrix} \mid \alpha_k\in\mathbb{R}\right\}.$$

We recall that the tangential component of a vector $\mathbf{u}_i\in X_i$ is constant on the edges of the triangulation $\mathcal{T}_{i,h}$, and that the degrees of freedom can be chosen as the values of the tangential component on the edges

$$\lambda_{e_k}(\mathbf{u}_i) = u_k^{(i)} := \mathbf{u}_i\cdot\mathbf{t}_{e_k}{}_{|e_k}, \quad e_k\in\mathcal{E}_{i,h}. \tag{4}$$

We next introduce the product space

$$X_h(\Omega) = X := \prod_{i=1}^N X_i \subset \prod_{i=1}^N H_\star(\mathrm{curl}\,;\Omega_i),$$

the spaces of tangential vectors

$$W_h(\partial\Omega_i) = W_i := \{(\mathbf{u}_i\cdot\mathbf{t}_i)\,\mathbf{t}_i \text{ restricted to } \partial\Omega_i\setminus\partial\Omega\ \mid \mathbf{u}_i\in X_i\},$$

and the product space

$$W_h(\Gamma) = W := \prod_{i=1}^N W_i.$$

We note that we have chosen a different definition of the trace spaces than that employed in [32]. Here, the spaces W_i consist of piecewise constant tangential *vectors* on $\partial\Omega_i\setminus\partial\Omega$.

Throughout this paper, we will use the following conventions. We will use the same notation for the vectors in X_i and tangential vectors in W_i. We denote a generic vector function in X_i using a bold letter with the subscript i, e.g., $\mathbf{u}_i$, and the column vector of its degrees of freedom, defined in (4), using the same letter with the superscript (i), e.g., $u^{(i)}$. Its k–th degree of freedom corresponding to the edge e_k, defined in (4), is $u_k^{(i)}$. A generic vector in the product space X (or W) is also denoted by a bold letter, e.g., $\mathbf{u}$, and the corresponding vector of degrees of freedom by the same letter, e.g., u. We will use the same notation for the spaces of functions X_i and W_i and the corresponding spaces of degrees of freedom.

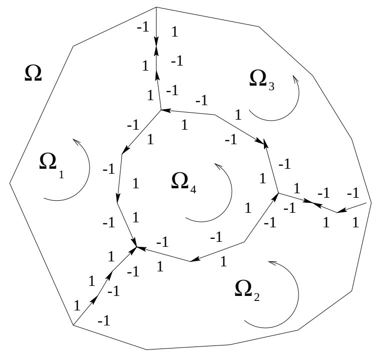

FIG. 1. *Example of partition of the domain Ω. We show the directions of the subdomain boundaries, given by the unit vectors $\{\mathbf{t}_i\}$, those of the fine edges on the interface Γ, and the corresponding values of the degrees of freedom $t^{(i)}$.*

Given the unit vectors $\mathbf{t}_i$, the column vectors $t^{(i)}$ are defined by

$$t_k^{(i)} := \mathbf{t}_i \cdot \mathbf{t}_{e_k}, \quad e_k \subset \partial\Omega_i \setminus \partial\Omega, \quad e_k \in \mathcal{E}_{i,h}.$$

We will need these tangential vectors in the definition of our FETI method; see section 5. We remark that, in case that all the edges e_k on $\partial\Omega_i$ have the same direction of the boundary $\partial\Omega_i$, the entries of the vector $t^{(i)}$ are equal to one. Figure 1 shows an example of a partition, with the directions of the subdomain boundaries and of the fine edges on the interface Γ, and the corresponding degrees of freedom $t^{(i)}$.

Finally, for $i = 1, \cdots, N$, we define the discrete harmonic extensions with respect to the bilinear forms $a_{\Omega_i}(\cdot,\cdot)$ into the interior of Ω_i

$$\mathcal{H}_i : \; W_i \longrightarrow X_i.$$

We recall that $\mathcal{H}_i \mathbf{u}_i$ minimizes the energy $a_{\Omega_i}(\mathcal{H}_i \mathbf{u}_i, \mathcal{H}_i \mathbf{u}_i)$ among all the vectors of X_i with tangential component equal to $\mathbf{u}_i$ on $\partial\Omega_i \setminus \partial\Omega$.

3. A mortar condition. The mortar method presented in this section was originally developed and studied in [3]. We consider the skeleton Γ and choose a splitting of Γ as the disjoint union of some edges $\{\overline{\Gamma}^{k,j}\}$, that we call *mortars*. We note that this partition is not in general unique; see Figure 2 (left) for an example of decomposition.

A unique set of indices corresponds to this choice and we denote it by

$$\mathcal{I}_M := \{m = (k,j) \text{ such that } \Gamma^{k,j} \text{ is a mortar }\}.$$

To simplify the notation, we denote the mortars by $\{\Gamma^m | \; m \in \mathcal{I}_M\}$. We have

$$\overline{\Gamma} := \bigcup_{i=1}^{M} \overline{\Gamma}^m, \quad \Gamma^m \cap \Gamma^n = \emptyset, \quad \text{ if } m \neq n \text{ and } n, m \in \mathcal{I}_M.$$

For any m in $\mathcal{I}_M$, we denote by W^m the space $W^{k,j}$ $(m = (k,j) \in \mathcal{I}_M)$ given by

$$W^{k,j} := \{(\mathbf{u}_k \cdot \mathbf{t}_k)\, \mathbf{t}_k \text{ restricted to } \Gamma^{k,j} \;\; | \; \mathbf{u}_k \in X_k\}.$$

We note that the vectors in $W^{k,j}$ are also the restrictions of vectors in W_k to $\Gamma^{k,j}$. Before introducing the mortar space, we need to fix a last point. Let Γ^m be a mortar

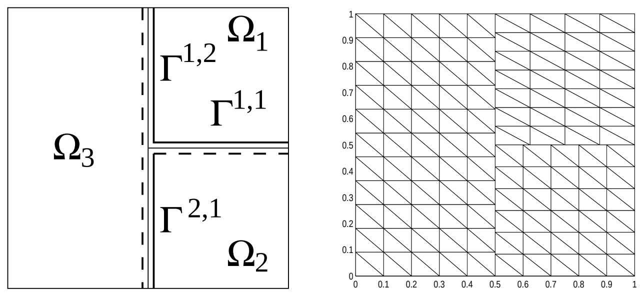

FIG. 2. *An example where the domain is decomposed into three (rectangular) non-overlapping subdomains. The skeleton has two partitions (figure on the left): one in terms of mortars (solid dark line) and the other in terms of non–mortars (long dashed lines). On the right, an example of discretization of the subdomains by means of triangular grids that do not match at the interfaces between adjacent subdomains.*

edge with $m = (k, j)$ and $\mathbf{u} \in X_h$: for almost every $\mathbf{x} \in \Gamma^m$, there exists an index l $(1 \leq l \leq N)$, $l \neq k$ such that $\mathbf{x} \in \Gamma^m \cap \partial\Omega_l$. At this point $\mathbf{x}$ we have two fields, namely $\mathbf{u}_k$ and $\mathbf{u}_l$. Since the domain decomposition is in general non–conforming, the value of l depends on $\mathbf{x}$ and we denote by $\mathcal{J}_l$ the set of indices l $(1 \leq l \leq N)$ such that $\Gamma^m \cap \partial\Omega_l \neq \emptyset$. We then define

$$\mathbf{u}_{-k}(\mathbf{x}) := \mathbf{u}_l(\mathbf{x}) \quad , \quad \mathbf{x} \in \Gamma^m \cap \partial\Omega_l, \quad l \in \mathcal{J}_l.$$

The function $\mathbf{u}_{-k}$ is defined for almost all $\mathbf{x} \in \Gamma^m$. In general, it is not the tangential component at Γ^m of a field $\mathbf{u} \in H_\star(\mathrm{curl}\,; \Omega_i)$: it can indeed correspond to tangential components from different subdomains which share a subset of Γ^m and live on different grids.

The equality between $\mathbf{u}_{-k} \cdot \mathbf{t}_k$ and $\mathbf{u}_k \cdot \mathbf{t}_k$ at Γ^m becomes a too stringent condition since the two fields are in general defined on different and non–matching grids. As is usually done in non–conforming mortar domain decomposition methods, we impose these constraints in a weak form by means of suitable Lagrange multipliers. Here, the Lagrange multiplier space consists of the tangential components of the shape functions at the mortar edges; see [3].

REMARK 3.1. *The definition of the mortar space for the edge elements is simpler than for nodal finite elements. In the nodal case, the space of Lagrange multipliers cannot be chosen as a space of traces on the mortar edges, but only as a suitable subspace of it. In the edge case, it is not necessary to decrease the dimension of the multiplier space since the information is associated to edges and not to nodes; see [10] for more details.*

The Lagrange multiplier (mortar) space is now defined by

$$\{\mathbf{v} \in L^2(\Gamma) \mid \mathbf{v}_{|\Gamma^m} \in W^m, \quad m \in \mathcal{I}_M\}.$$

We remark that this is a space of tangential vectors on Γ. The transmission conditions at the interface between adjacent subdomains are then weakly imposed by means of these Lagrange multipliers. A solution $\mathbf{u} \in W$ is required to satisfy the constraints

$$\int_{\Gamma^m} (\mathbf{u}_k \cdot \mathbf{t}_k - \mathbf{u}_{-k} \cdot \mathbf{t}_k\,)\, \mathbf{v} \cdot \mathbf{t}_k \, ds = 0, \quad \mathbf{v} \in W^m, \quad m = (k, j) \in \mathcal{I}_M. \tag{5}$$

The set of transmission conditions can be expressed in matrix form in the following way:

Let $\mathbf{w}_i^k$ be the basis function associated to the k-th mesh edge of $\partial\Omega_i \setminus \partial\Omega$. We introduce two matrices C and D by

$$C_{kl} := \int_{\Gamma^m} (\mathbf{w}_i^k \cdot \mathbf{t}_i)(\mathbf{w}_i^l \cdot \mathbf{t}_i)\, ds = \delta_{kl}\, |e_k|, \quad e_k, e_l \subset \Gamma^m,$$

$$D_{kn} := \int_{\Gamma^m} (\mathbf{w}_i^k \cdot \mathbf{t}_i)(\mathbf{w}_r^n \cdot \mathbf{t}_i)\, ds, \quad e_k \subset \Gamma^m, \quad e_n \subset \partial\Omega_r, \quad r \in \mathcal{J}_m,$$

where $\Gamma^m = \Gamma^{i,j}$. Then the matching conditions (5) have the form

$$Bu = 0 \quad \text{where} \quad B = C - D.$$

We remark that the entries of C and D depend on the particular choice of degrees of freedom defined in (4).

The matrix B can also be written as

$$B = \left[B^{(1)}\; B^{(2)}\; \cdots\; B^{(N)} \right],$$

where the local matrices $B^{(i)}$ act on vectors in W_i. The entries of B do not belong in general to $\{0, 1, -1\}$ as in the conforming case described in [32] but, since we are working with non–matching grids, they take into account the edge intersections at the interfaces.

4. A FETI method. In this section, we introduce a FETI method for the solution of the linear system arising from the mortar edge element approximation of problem (3).

We first assemble the local stiffness matrices, relative to the bilinear forms $a_{\Omega_i}(\cdot,\cdot)$, and the local load vectors. The degrees of freedom that are not on the interface Γ only belong to one substructure and can be eliminated in parallel by block Gaussian elimination. Let $f^{(i)}$ be the resulting right hand sides and let $S^{(i)}$ be the Schur complement matrices

$$S^{(i)} :\; W_i \longrightarrow W_i,$$

relative to the degrees of freedom on $\partial\Omega_i \setminus \partial\Omega$.

We recall that the local Schur complements satisfy the following property

$$u^{(i)^t} S^{(i)} u^{(i)} = a_{\Omega_i}(\mathcal{H}_i \mathbf{u}_i, \mathcal{H}_i \mathbf{u}_i); \tag{6}$$

see, e.g., [27, 31].

Following [19, 32], we can then write our mortar problem as

$$\begin{array}{lclcl} Su & + & B^t\lambda & = & f \\ Bu & & & = & 0 \end{array} \tag{7}$$

where

$$u := \begin{bmatrix} u^{(1)} \\ \vdots \\ u^{(N)} \end{bmatrix} \in W, \quad S := \mathrm{diag}\{\mathrm{S}^{(1)}, \cdots, \mathrm{S}^{(\mathrm{N})}\}, \quad \mathrm{f} := \begin{bmatrix} f^{(1)} \\ \vdots \\ f^{(N)} \end{bmatrix}.$$

The vector λ is a Lagrange multiplier relative to the weak continuity constraint $Bu = 0$.

We remark that the $S^{(i)}$ are always invertible and, consequently, there is no natural coarse space associated to the substructures; we are in a similar case as the one considered in [13]. We first find u from the first equation in (7), and substitute its value in the second equation. We obtain the system

$$F\lambda = d, \tag{8}$$

where

$$F := B\,S^{-1}\,B^t, \quad d := B\,S^{-1}\,f.$$

Following [19, 32, 28], we now define a preconditioner. Since we assume that the coefficients do not have any jump, we do not need to introduce a set of scaling matrices as is required for problems with coefficient jumps; see, e.g., [30, 31, 19, 32].

We introduce the matrices,

$$R := \left[R^{(1)}\; R^{(2)}\; \cdots\; R^{(M)}\right], \quad G := Q\,B\,R, \tag{9}$$

where $R^{(i)}$ are vectors in W, related to the substructures $\{\Omega_i\}$ and Q is a suitable invertible matrix that we will specify in the next section. More precisely, we suppose that $R^{(i)}$ is obtained from a local vector $\mathbf{r}_i \in W_i$ on $\partial\Omega_i \setminus \partial\Omega$, by extending it by zero on the boundaries of the other substructures. We will make a particular choice of R for problem (3), in section 5, and specify the dimension M.

Following [13, 32], we define the projection

$$P := I - G(G^tFG)^{-1}G^tF,$$

onto the complement of $Range(G)$. This projection is orthogonal with respect to the scalar product induced by F. Following [19, 32], we next define the preconditioner

$$\widehat{M}^{-1} := (BB^t)^{-1}\,BSB^t\,(BB^t)^{-1}.$$

It can be easily seen that BB^t is invertible and is block–diagonal only if the partition $\mathcal{F}_H$ is geometrically conforming.

Now, we consider a projected conjugate gradient method, as in [13, 32].

1. Initialize

$$\begin{aligned} \lambda^0 &= G(G^tFG)^{-1}G^td \\ q^0 &= d - F\lambda^0 \end{aligned}$$

2. Iterate $k = 1, 2, \cdots$ until convergence

$$\begin{array}{rcl} \text{Project: } w^{k-1} & = & P^tq^{k-1} \\ \text{Precondition: } z^{k-1} & = & \widehat{M}^{-1}w^{k-1} \\ \text{Project: } y^{k-1} & = & Pz^{k-1} \\ \beta^k & = & \langle y^{k-1}, w^{k-1}\rangle / \langle y^{k-2}, w^{k-2}\rangle \quad [\beta^1 = 0] \\ p^k & = & y^{k-1} + \beta^k p^{k-1} \quad [p^1 = y^0] \\ \alpha^k & = & \langle y^{k-1}, w^{k-1}\rangle / \langle p^k, Fp^k\rangle \\ \lambda^k & = & \lambda^{k-1} + \alpha^k p^k \\ q^k & = & q^{k-1} - \alpha^k Fp^k \end{array}$$

The first projection can be omitted; because of the choice of the initial vector λ^0, we have $w^{k-1} = q^{k-1}$ after the first projection step. Here, we have denoted the residual at the k–th step by q^k. In practice, partial or full re-orthogonalization may be required; cf. [16].

The method presented here is equivalent to using the conjugate gradient method for solving the following preconditioned system

$$P\widehat{M}^{-1}P^tF\lambda = P\widehat{M}^{-1}P^td, \quad \lambda \in \lambda_0 + V, \tag{10}$$

with

$$V := Range(P). \tag{11}$$

We remark that the matrices S and S^{-1} do not need to be calculated in practice. The action of S on a vector requires the solution of a Dirichlet problem on each substructure, while the action of S^{-1} requires the solution of a Neumann problem on each substructure; see [27, Ch. 4].

5. A particular choice of the matrices R and Q.

In this section, we consider a particular choice of the matrices R and Q in the definition of the FETI algorithm for problem (3).

We proceed in a similar way as in [32, Sect. 5], but we will need to introduce a suitable matrix Q, different from the identity.

The definition of R is the same as in the conforming case, see [32, Sect. 5], and is given in terms of local vectors.

DEFINITION 5.1. *The local vectors* $\{\mathbf{r}_i,\ i = 1,\cdots,N\}$, *with the corresponding vectors of degrees of freedom* $\{r^{(i)}\}$, *are the unique vectors that satisfy*

$$r^{(i)^t}v^{(i)} = \sum_{\substack{e_k\subset\partial\Omega_i\setminus\partial\Omega \\ e_k\in\mathcal{E}_{i,h}}} r_k^{(i)}v_k^{(i)} = \int\limits_{\partial\Omega_i\setminus\partial\Omega} \mathbf{v}_i\cdot\mathbf{t}_i\,ds, \quad \mathbf{v}_i\in W_i.$$

The global vectors $R^{(i)}$ *are obtained by extending the local vectors* $r^{(i)}$ *by zero outside* $\partial\Omega_i$.

We can easily find that

$$r_k^{(i)} = |e_k|\,t_k^{(i)}, \quad e_k\subset\partial\Omega_i\setminus\partial\Omega.$$

The vectors $\mathbf{r}_i$ have then the same direction as the $\mathbf{t}_i$ and are scaled using the lengths of the edges of the triangulations $\mathcal{T}_{i,h}$.

We then define the matrix Q as

$$Q := (BB^t)^{-1}. \tag{12}$$

We remark that in the case of a conforming triangulation the matrix Q is a multiple of the identity; see [32]. For matching grids, we then obtain the same preconditioner as introduced in [32] for conforming approximations. Here, our choice of Q does not require any additional calculation, since $(BB^t)^{-1}$ is also needed for the application of the preconditioner $\widehat{M}^{-1}$.

It remains to decide how many of the local vectors $R^{(i)}$ need to be considered in the definition of the matrix R. We introduce $\mathcal{G}_H$ as the dual graph of the partition $\mathcal{F}_H$. Thus, $\mathcal{G}_H$ has a vertex for each substructure of $\mathcal{F}_H$ and there is an edge in

$\mathcal{G}_H$ between two vertices if the intersection of the boundaries of the corresponding substructures has positive measure. As in [32], we define the matrix R by

$$R := \begin{cases} \left[R^{(1)}\ R^{(2)}\ \cdots\ R^{(N-1)}\right], & \text{if } \mathcal{G}_H \text{ is two–colorable,} \\ \left[R^{(1)}\ R^{(2)}\ \cdots\ R^{(N)}\right], & \text{otherwise .} \end{cases} \tag{13}$$

The following result can be proven using [32, Lem. 5.2 and Th. 5.1].

LEMMA 5.1. *Let R be defined in (13). Then the matrix G has full rank.*

REMARK 5.1. *An analogous FETI method can also be devised for problems involving the bilinear form*

$$\int_\Omega (a \operatorname{div} \mathbf{u} \operatorname{div} \mathbf{v} + A\, \mathbf{u} \cdot \mathbf{v})\, dx, \quad \mathbf{u}, \mathbf{v} \in H(\operatorname{div}; \Omega),$$

discretized with the lowest-order Raviart–Thomas spaces. Here, $H(\operatorname{div}; \Omega)$ is the space of vectors in $(L^2)^2$, with divergence in L^2. Since, in two dimensions, vectors in the Raviart–Thomas spaces can be obtained from those in the Nédélec spaces by a rotation of ninety degrees, the unit outward normal vectors $\mathbf{n}_i$ to the boundaries $\partial\Omega_i$, instead of the unit tangent vectors $\mathbf{t}_i$, have to be employed in the construction of the local functions $\mathbf{r}_i$. All the definitions in this paper remain valid in this case. For Raviart-Thomas discretizations in three dimensions, an analogous method can also be defined and all our definitions remain valid.

6. Numerical results. The purpose of this section is to show that, for problems without jumps, the FETI method proposed here performs similarly to the corresponding method for conforming approximations; see [32, Sect. 6]. In particular, our method appears to be scalable, its condition number only depends on the number of degrees of freedom per subdomain, and it is quite insensitive to variations of the ratios of the coefficients. In addition, it appears to be robust when the meshes of adjacent subdomains are very different.

In many iterative substructuring methods, an important role is played by the ratio H/h that measures the number of degrees of freedom per subdomain. In particular, the condition number of these methods grows only quadratically with the logarithm of H/h; see, e.g., [27]. This ratio is regarded as a local quantity and can vary greatly from one subdomain to another. In our numerical results, we always report the maximum value of this ratio taken over the subdomains.

We consider the domain $\Omega = (0,1)^2$ and assume that the coefficient matrix A is diagonal and equal to

$$A = \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix}.$$

In our first set of results, we consider a family of geometrically conforming partitions of Ω, into $2^d \times 2^d$ substructures of equal size, with $d = 1, 2, 3, 4$. For a fixed partition, we consider two kinds of uniform triangulations for the substructures, in such a way that on the interface between two adjacent substructures the meshes do not match. The ratio between the mesh–sizes of the two triangulations is $h_1/h_2 = 4/3$. Figure 3 shows an example of this checkerboard–type discretization for $d = 2$.

In Table 1, we show the estimated condition number and the number of iterations to obtain a relative residual $\|q_k\|/\|f\|$ less than 10^{-6}, as a function of the diameter of the finer mesh and the partition. Here, q_k is the $k-$th residual as defined in the

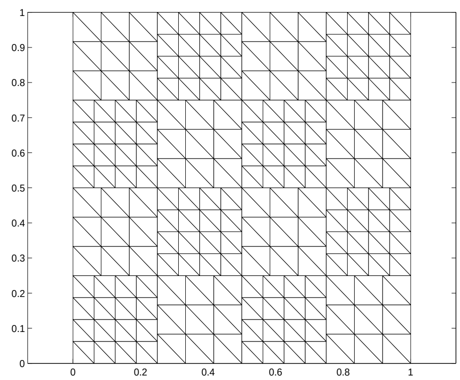

FIG. 3. *A conforming partition and a checkerboard–type discretization.*

TABLE 1

The FETI method for conforming partitions with $h_1/h_2 = 4/3$. Estimated condition number and number of CG iterations necessary to obtain a relative residual $\|q_k\|/\|f\|$ less than 10^{-6} (in parentheses), versus H/h and h. Case of $a = 1$, $b = 1$. The asterisks denote the cases for which we have not enough memory to run the corresponding algorithm.

H/h	32	16	8	4
$1/h = 32$ (1600 el.)	-	1.805 (5)	2.941 (9)	2.179 (8)
$1/h = 64$ (6400 el.)	2.151 (6)	4.045 (11)	3.035 (10)	2.165 (7)
$1/h = 128$ (25600 el.)	5.314 (12)	4.175 (12)	3.013 (9)	*

algorithmic description given in section 4. For a fixed ratio H/h, the condition number and the number of iterations are quite insensitive to the dimension of the fine meshes. In addition, even for non–matching grids, the ratio H/h appears to play an important role.

TABLE 2

The FETI method for conforming partitions with $h_1/h_2 = 4/3$. Estimated condition number and number of CG iterations necessary to obtain relative preconditioned residual ($\|q_k\|/\|f\|$) less than 10^{-6} (in parentheses), versus H/h and b. Case of $1/h = 128$ and $a = 1$.

H/h	8	16	32
b=0.0001	3.091 (16)	4.216 (20)	5.364 (18)
b=0.001	3.078 (14)	4.21 (18)	5.358 (16)
b=0.01	3.069 (13)	4.203 (16)	5.353 (15)
b= 0.1	3.044 (11)	4.192 (14)	5.346 (14)
b= 1	3.013 (9)	4.175 (12)	5.314 (12)
b= 10	2.992 (8)	4.114 (11)	5.154 (11)
b= 100	2.939 (9)	3.829 (11)	4.379 (11)
b= 1000	2.501 (7)	2.746 (8)	2.486 (7)
b=1e+04	1.418 (4)	1.493 (4)	1.533 (4)
b=1e+05	1.037 (2)	1.042 (2)	1.044 (2)
b=1e+06	1.06 (2)	1.046 (2)	1.044 (2)

In Table 2, we show some results when the ratio of the coefficients b and a changes. For a fixed value of $1/h = 128$ and $a = 1$, and for the partitions into 2^d by 2^d

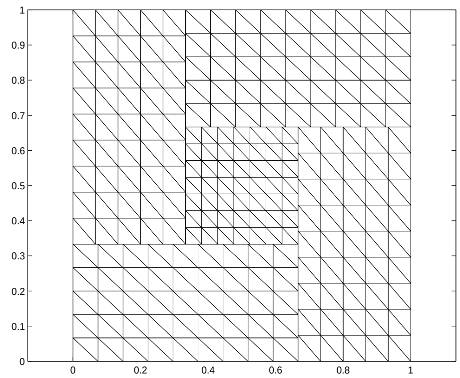

FIG. 4. *A block consisting of five subdomains, employed for building a non–conforming partition.*

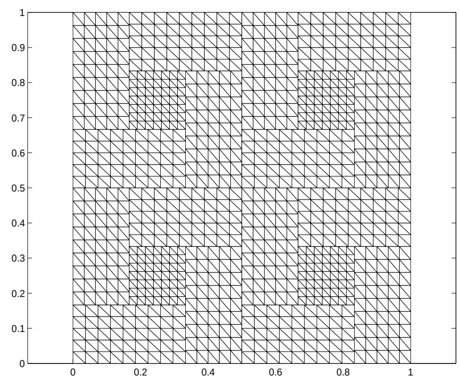

FIG. 5. *A non–conforming partition consisting of four blocks.*

substructures, with $d = 2, 3, 4$, the estimated condition number and the number of iterations are shown as a function of H/h and b. The number of iterations and the condition number appear to be bounded independently of the ratio of the coefficients.

We then consider some test cases relative to geometrically non–conforming partitions of the domain $(0, 1)^2$. We consider partitions consisting of $2^d \times 2^d$ equal blocks, $d = 0, 1, 2, 3$. A block is made of five non-conforming subdomains and is shown in Figure 4 together with a possible triangulation. Figure 5 shows a partition for the case $d = 1$ (four blocks and twenty subdomains). The number of subdomains is thus five times the number of blocks. We then consider uniform triangulations for the subdomains in each block. The rectangular subdomains have the same mesh.

We first consider a case where the ratio between the mesh sizes of the rectangular and square subdomains is $h_1/h_2 = 7/5$; see Figures 4 and 5 for two examples. In Table 3, we show the estimated condition number and the number of iterations to obtain a relative residual $\|q_k\|/\|f\|$ less than 10^{-6}, as a function of the diameter of the finer mesh and the ratio H/h. The condition number appears to increase slowly with H/h and to be quite insensitive to the size of the fine meshes.

In Table 4, we show some results when the ratio of the coefficients b and a changes. For a fixed value of $1/h = 84$ (29312 elements) and $a = 1$, and for the partitions into 2^d by 2^d blocks, with $d = 1, 2, 3$, the estimated condition number and the number of

TABLE 3

The FETI method for non–conforming partitions with $h_1/h_2 = 7/5$. Estimated condition number and number of CG iterations necessary to obtain a relative residual $\|q_k\|/\|f\|$ less than 10^{-6} (in parentheses), versus H/h and h. Case of $a = 1$, $b = 1$.

H/h	20	10	5
$1/h = 21$ (1832 el.)	-	2.728 (8)	2.63 (10)
$1/h = 42$ (7328 el.)	3.459 (8)	3.23 (11)	2.876 (10)
$1/h = 84$ (29312 el.)	4.034 (13)	3.619 (12)	2.901 (10)
$1/h = 168$ (117248 el.)	4.552 (14)	3.619 (12)	*

iterations are shown as a function of H/h and b.

TABLE 4

The FETI method for non–conforming partitions with $h_1/h_2 = 7/5$. Estimated condition number and number of CG iterations necessary to obtain relative preconditioned residual ($\|q_k\|/\|f\|$) less than 10^{-6} (in parentheses), versus H/h and b. Case of $1/h = 84$ and $a = 1$.

H/h	5	10	20
b=0.0001	2.966 (17)	3.667 (20)	4.068 (21)
b=0.001	2.955 (15)	3.663 (18)	4.064 (19)
b=0.01	2.951 (14)	3.658 (16)	4.06 (17
b= 0.1	2.933 (12)	3.651 (14)	4.054 (15)
b= 1	2.901 (10)	3.619 (12)	4.034 (13)
b= 10	2.882 (9)	3.561 (11)	3.93 (12)
b= 100	2.769 (9)	3.214 (10)	3.284 (10)
b= 1000	2.305 (7)	2.197 (7)	2.229 (7)
b=1e+04	1.656 (5)	1.523 (4)	1.54 (4)
b=1e+05	1.173 (3)	1.178 (3)	1.086 (2)
b=1e+06	1.135 (2)	1.115 (2)	1.089 (2)

For the same non–conforming partitions, we finally consider a case where the ratio between the diameters of the meshes of the rectangular and square subdomains is larger. We choose $h_1/h_2 = 2.8$. In Table 5, we show some results when the ratio of the coefficients b and a changes. For a fixed value of $1/h = 168$ (48128 elements) and $a = 1$, and for the partitions into 2^d by 2^d blocks, with $d = 1, 2, 3$, the estimated condition number and the number of iterations are shown as a function of H/h and b. In this case, the meshes of adjacent substructures are very different but the condition numbers and the number of iterations are still quite satisfactory.

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REFERENCES

[1] Yves Achdou, Yuri A. Kuznetsov, and Olivier Pironneau. Substructuring preconditioners for the Q_1 mortar element method. *Numer. Math.*, 71(4):419–449, 1995.

[2] Yves Achdou, Yvon Maday, and Olof B. Widlund. Iterative substructuring preconditioners for mortar element methods in two dimensions. *SIAM J. Numer. Anal.*, 36(2):551–580, 1999.

[3] Adnene Ben Abdallah, Faker Ben Belgacem, and Yvon Maday. Mortaring the two-dimensional "Nédélec" finite elements for the discretization of the Maxwell equations. M^2AN, 2000. To appear.

TABLE 5

The FETI method for non–conforming partitions with $h_1/h_2 = 2.8$. Estimated condition number and number of CG iterations necessary to obtain relative preconditioned residual ($\|q_k\|/\|f\|$) less than 10^{-6} (in parentheses), versus H/h and b. Case of $1/h = 168$ and $a = 1$.

H/h	14	28	56
b=0.0001	5.058 (23)	5.062 (24)	6.275 (24)
b=0.001	5.054 (21)	5.056 (22)	6.27 (23)
b=0.01	5.045 (19)	5.043 (19)	6.258 (21)
b= 0.1	5.026 (16)	5.032 (17)	6.241 (18)
b= 1	4.994 (14)	5.008 (15)	6.205 (16)
b= 10	4.922 (13)	4.977 (14)	6.094 (15)
b= 100	4.833 (12)	4.761 (13)	5.48 (13)
b= 1000	4.448 (11)	3.938 (10)	4.078 (9)
b=1e+04	3.381 (8)	2.669 (6)	2.668 (6)
b=1e+05	2.257 (5)	1.901 (4)	1.906 (3)
b=1e+06	1.947 (3)	1.786 (3)	1.436 (2)

[4] Faker Ben Belgacem, Annalisa Buffa, and Yvon Maday. The mortar element method for 3D Maxwell equations: first results. Technical report, Laboratoire d'Analyse Numérique, Université Pierre et Marie Curie, 1999. Submitted to SIAM J. Numer. Anal.

[5] Ozkár Bíró. Edge element formulations of eddy current problems. *Comput. Meth. in Appl. Mech. and Eng.*, 169:391–405, 1999.

[6] Alain Bossavit. *Electromagnétisme en vue de la modélisation.* Mathèmatiques & Applications. Springer-Verlag France, Paris, 1993.

[7] Dietrich Braess and Wolfgang Dahmen. Stability estimates of the mortar finite element method for 3–dimensional problems. *East-West J. Numer. Math.*, 6:249–263, 1998.

[8] Franco Brezzi and Michel Fortin. *Mixed and hybrid finite element methods.* Springer-Verlag, New York, 1991.

[9] Mario A. Casarin and Olof B. Widlund. A hierarchical preconditioner for the mortar finite element method. *ETNA*, 4:75–88, June 1996.

[10] Yvon Maday Christine Bernardi and Anthony T. Patera. A new nonconforming approach to domain decomposition: The mortar elements method. In H. Brezis and J.L. Lions, editors, *Nonlinear partial differential equations and their applications*, pages 13–51. Pitman, 1994.

[11] Maksymilian Dryja. An additive Schwarz method for elliptic mortar finite element problems in three dimensions. In P. E. Bjørstad, M. Espedal, and D. E. Keyes, editors, *Ninth International Conference of Domain Decomposition Methods.* John Wiley & Sons, Ltd, Strasbourg, France, 1997. Held in Ullensvang at the Hardanger Fjord, Norway, June 4–7, 1996.

[12] Bernd Engelmann, Ronald H.W. Hoppe, Yuri Iliash, Yuri A. Kuznetsov, Yuri Vassilevski, and Barbara I. Wohlmuth. Adaptive macro–hybrid finite element methods. In H. Bock, F. Brezzi, R. Glowinski, G. Kanschat, Y. A. Kuznetsov, J. Périaux, and R. Rannacher, editors, *Proceedings of the 2nd European Conference on Numerical Methods*, pages 294–302. World Scientific, Singapore, 1998.

[13] Charbel Farhat, Po-Shu Chen, and Jan Mandel. A scalable Lagrange multiplier based domain decomposition method for time-dependent problems. *Int. J. Numer. Meth. Eng.*, 38:3831–3853, 1995.

[14] Charbel Farhat, Jan Mandel, and François-Xavier Roux. Optimal convergence properties of the FETI domain decomposition method. *Comp. Methods Appl. Mech. Eng.*, 115:367–388, 1994.

[15] Charbel Farhat and François-Xavier Roux. A method of Finite Element Tearing and Interconnecting and its parallel solution algorithm. *Int. J. Numer. Meth. Eng.*, 32:1205–1227, 1991.

[16] Charbel Farhat and François-Xavier Roux. Implicit parallel processing in structural mechanics. In J. Tinsley Oden, editor, *Computational Mechanics Advances*, volume 2 (1), pages 1–124. North-Holland, 1994.

[17] Vivette Girault and Pierre-Arnaud Raviart. *Finite Element Methods for Navier-Stokes Equa-*

tions. Springer-Verlag, New York, 1986.

[18] Axel Klawonn and Dan Stefanica. A numerical study of a class of FETI preconditioners for mortar finite elements in two dimensions. Technical Report 773, Department of Computer Science, Courant Institute, November 1998.

[19] Axel Klawonn and Olof B. Widlund. FETI and Neumann–Neumann iterative substructuring methods: connections and new results. Technical Report 796, Department of Computer Science, Courant Institute, December 1999.

[20] Yuri A. Kuznetsov. Efficient iterative solvers for elliptic problems on nonmatching grids. *Russ. J. Numer. Anal. Math. Modelling*, 10(3):187–211, 1995.

[21] Catherine Lacour. Iterative substructuring preconditioners for the mortar finite element method. In P. E. Bjørstad, M. Espedal, and D. E. Keyes, editors, *Ninth International Conference on Domain Decomposition*, pages 406–412. John Wiley & Sons, Ltd, Strasbourg, France, 1997. Held in Ullensvang at the Hardanger Fjord, Norway, June 4–7, 1996.

[22] Patrick Le Tallec and Taoufik Sassi. Domain decomposition with nonmatching grids: Augmented Lagrangian approach. *Math. Comp.*, 64(212):1367–1396, 1995.

[23] Jan Mandel and Radek Tezaur. Convergence of a substructuring method with Lagrange multipliers. *Numer. Math.*, 73:473–487, 1996.

[24] Jean-Claude Nédélec. Mixed finite elements in R^3. *Numer. Math.*, 35:315–341, 1980.

[25] Francesca Rapetti. The mortar edge element method on non–matching grids for eddy current calculations in moving structures. *Int. J. Num. Meth.*, 1999. Submitted.

[26] Daniel Rixen and Charbel Farhat. A simple and efficient extension of a class of substructure based preconditioners to heterogeneous structural mechanics problems. *Int. J. Numer. Meth. Eng.*, 44:489–516, 1999.

[27] Barry F. Smith, Petter E. Bjørstad, and William D. Gropp. *Domain Decomposition: Parallel Multilevel Methods for Elliptic Partial Differential Equations*. Cambridge University Press, 1996.

[28] Dan Stefanica. *Domain decomposition methods for mortar finite elements*. PhD thesis, Courant Institute of Mathematical Sciences, September 1999.

[29] Radek Tezaur. *Analysis of Lagrange multiplier based domain decomposition*. PhD thesis, University of Colorado at Denver, 1998.

[30] Andrea Toselli. *Domain decomposition methods for vector field problems*. PhD thesis, Courant Institute of Mathematical Sciences, May 1999. Technical Report 785, Department of Computer Science, Courant Institute of Mathematical Sciences, New York University.

[31] Andrea Toselli. Neumann–Neumann methods for vector field problems. Technical Report 786, Department of Computer Science, Courant Institute, June 1999. Submitted to ETNA.

[32] Andrea Toselli and Axel Klawonn. A FETI domain decomposition method for Maxwell's equations with discontinuous coefficients in two dimensions. Technical Report 788, Department of Computer Science, Courant Institute, 1999. Submitted to SIAM J. Numer. Anal.

[33] Barbara I. Wohlmuth. Discretization methods and iterative solvers based on domain decomposition, November 1999. Mathematisch–Naturwissenschaftliche Fakultät, Universität Augsburg, Habilitationsschrift, 169pp.