A FETI DOMAIN DECOMPOSITION METHOD FOR MAXWELL'S EQUATIONS WITH DISCONTINUOUS COEFFICIENTS IN TWO DIMENSIONS

ANDREA TOSELLI * AND AXEL KLAWONN†

Abstract. A class of FETI methods for the edge element approximation of vector field problems in two dimensions is introduced and analyzed. First, an abstract framework is presented for the analysis of a class of FETI methods where a natural coarse problem, associated with the substructures, is lacking. Then, a family of FETI methods for edge element approximations is proposed. It is shown that the condition number of the corresponding method is independent of the number of substructures and grows only polylogarithmically with the number of unknowns associated with individual substructures. The estimate is also independent of the jumps of both of the coefficients of the original problem. Numerical results validating the theoretical bounds are given. The method and its analysis can be easily generalized to Raviart–Thomas element approximations in two and three dimensions.

Key words. Edge elements, Maxwell's equations, domain decomposition, FETI, preconditioners, heterogeneous coefficients

AMS subject classifications. 65F10, 65N22, 65N30, 65N55

1. Introduction. In this paper, we consider the boundary value problem

$$\begin{aligned} L\mathbf{u} := \mathbf{curl}\,(a\,\mathrm{curl}\,\mathbf{u}) + A\,\mathbf{u} &= \mathbf{f} \quad \text{in } \Omega, \\ \mathbf{u}\cdot\mathbf{t} &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

with Ω a bounded polygonal domain in $\mathbb{R}^2$. The domain Ω has unit diameter and $\mathbf{t}$ is its unit tangent. We have

$$\mathbf{curl}\,v := \begin{bmatrix} \dfrac{\partial v}{\partial x_2} \\ -\dfrac{\partial v}{\partial x_1} \end{bmatrix}, \quad \mathrm{curl}\,\mathbf{u} := \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2};$$

see, e.g., [16]. The coefficient matrix A is a symmetric uniformly positive definite matrix–valued function with entries $A_{ij} \in L^\infty(\Omega)$, $1 \le i,j \le 2$, and $a \in L^\infty(\Omega)$ is a positive function bounded away from zero.

The weak formulation of problem (1) requires the introduction of the Hilbert space $H(\mathrm{curl}\,;\Omega)$, defined by

$$H(\mathrm{curl}\,;\Omega) := \left\{\mathbf{v} \in (L^2(\Omega))^2 | \quad \mathrm{curl}\,\mathbf{v} \in L^2(\Omega)\right\}.$$

The space $H(\mathrm{curl}\,;\Omega)$ is equipped with the following inner product and graph norm,

$$(\mathbf{u},\mathbf{v})_{\mathrm{curl}} := \int_\Omega \mathbf{u}\cdot\mathbf{v}\,dx + \int_\Omega \mathrm{curl}\,\mathbf{u}\,\mathrm{curl}\,\mathbf{v}\,dx, \qquad \|\mathbf{u}\|^2_{\mathrm{curl}} := (\mathbf{u},\mathbf{u})_{\mathrm{curl}},$$

*Courant Institute of Mathematical Sciences, 251 Mercer Street, New York, N.Y. 10012, USA. E–mail: toselli@cims.nyu.edu. URL: http://www.math.nyu.edu/~toselli. This work was supported in part by the Applied Mathematical Sciences Program of the U.S. Department of Energy under Contract DEFGO288ER25053.

†Institut für Numerische und instrumentelle Mathematik, Westfälische Wilhelms–Universität Münster, Einsteinstr. 62, 48149 Münster, Germany. E–mail: klawonn@math.uni-muenster.de. URL: http://wwwmath.uni-muenster.de/math/u/klawonn This work was supported in part by the National Science Foundation under Grants NSF-CCR-9732208 and in part by the US Department of Energy under Contract DE-FG02-92ER25127.

and the tangential component $\mathbf{u}\cdot\mathbf{t}$, of a vector $\mathbf{u}\in H(\mathrm{curl}\,;\Omega)$ on the boundary $\partial\Omega$, belongs to the space $H^{-\frac{1}{2}}(\partial\Omega)$; see [16, 7]. The subspace of vectors in $H(\mathrm{curl}\,;\Omega)$ with vanishing tangential component on $\partial\Omega$ is denoted by $H_0(\mathrm{curl}\,;\Omega)$.

For any $\mathcal{D}\subset\Omega$, we define the bilinear form

$$a_{\mathcal{D}}(\mathbf{u},\mathbf{v}) := \int\limits_{\mathcal{D}} (a\,\mathrm{curl}\,\mathbf{u}\,\mathrm{curl}\,\mathbf{v} + A\,\mathbf{u}\cdot\mathbf{v})\,dx, \quad \mathbf{u},\,\mathbf{v}\in H(\mathrm{curl}\,;\Omega). \tag{2}$$

The variational formulation of Equation (1) is:
Find $\mathbf{u}\in H_0(\mathrm{curl}\,;\Omega)$ such that

$$a_{\Omega}(\mathbf{u},\mathbf{v}) = \int\limits_{\Omega} \mathbf{f}\cdot\mathbf{v}\,dx, \quad \mathbf{v}\in H_0(\mathrm{curl}\,;\Omega). \tag{3}$$

We discretize this problem using edge elements, also known as Nédélec elements; see [25]. These are vector finite elements that only ensure the continuity of the tangential component across the elements, as is physically required for the electric and magnetic fields, solutions of Maxwell's equations.

The applications that we have in mind are mainly problems arising from static and quasi–static Maxwell's equations (eddy current problems); see, e.g., [6, 5]. In this paper, we only consider the model problem (3), where the dependency on the time or the frequency has been eliminated, and we will generically refer to it as Maxwell's equations. A good preconditioner for this model problem is the first step for the efficient solution of linear systems arising from the edge element approximation of static problems, and time– or frequency–dependent problems arising from the quasi–static approximations of Maxwell's equations.

The aim of this paper is to build and analyze a domain decomposition method with Lagrange multipliers for the solution of linear systems arising from the edge element approximations. Our algorithm belongs to the family of Finite Element Tearing and Interconnecting (FETI) methods which have been first introduced for the solution of elasticity problems in [14]. In this approach, the original domain Ω is decomposed into nonoverlapping subdomains Ω_i, $i=1,\ldots,N$. On each subdomain Ω_i a local stiffness matrix is obtained from the finite element discretization of $a_{\Omega_i}(\cdot,\cdot)$. Analogously, a set of right hand sides is built. The continuity of the solution corresponding to the primal variables is then enforced by using Lagrange multipliers across the interface defined by the subdomain boundaries. In the original FETI algorithm, the primal variables are then eliminated by solving local Neumann problems, and an equation in the Lagrange multipliers is obtained.

When considering the Poisson equation or stationary elasticity problems, the local matrices are in general singular, with a small null space consisting of constants or rigid body motions, respectively. Thus, auxiliary local problems need to be solved, in order to obtain the components of the solution in the local null spaces. These auxiliary problems play the role of a coarse problem and result in a condition number of the method independent of the number of subdomains; see [13, 15]. In a variant of the FETI method, introduced in [13], an additional Dirichlet problem is solved exactly on each subdomain, in each iteration. This variant is also known as the FETI method with Dirichlet preconditioner.

In [23, 32], it was first shown for scalar, second–order elliptic equations that the condition number of the Dirichlet variant of the FETI algorithm is bounded by $C\,(1+\log(H/h))^3$, where H and h are a typical diameter of a subdomain and of a finite element, respectively, and C is a constant independent of H and h. Since $(H/h)^d, d = 2, 3$ is a measure for the number of unknowns per subdomain, this gives a convergence bound for the FETI algorithm which only grows polylogarithmically with the number of unknowns associated with an individual substructure. In [21], this result was extended to the elliptic system of linear elasticity and an algorithm with inexact subdomain solvers based on FETI was proposed and analyzed. An improved condition number estimate of the order of $(1 + \log(H/h))^2$ will be given in [22]. For a different FETI algorithm developed by Park et al., see, e.g., Park, Justino, and Felippa [26], a condition number estimate of the same polylogarithmic order was given in the thesis of Tezaur [32].

Some variants, for which the condition number is also independent of possibly large jumps of the coefficients, were later proposed in [29, 30] based on mechanical arguments. A mathematical analysis of these and of some extended FETI algorithms will be given in [22].

The family of FETI methods has also been extended to problems that lack natural auxiliary coarse problems, e.g., time–dependent problems from elastodynamics, see [9, 12], and acoustic scattering problems, see [11].

Furthermore, FETI algorithms have also been developed for plate and shell problems, see, e.g., [12, 10] for algorithmic descriptions and numerical results and [24, 32] for a mathematical analysis.

In this paper, we consider a FETI method for the edge element approximation of Problem (3). Here, the local problems are not singular and, as in the case of time–dependent problems, there is no natural coarse problem associated with the subdomains. We will proceed as in [9], and propose a set of local functions that will allow us to build a coarse space for the Lagrange multipliers. In addition, following [30, 22], we also propose a family of preconditioners, built from the values of the coefficient A in (2). An important feature of our method is that the condition number is independent of the jumps of *both* coefficients in (2), as it is also the case for the Neumann–Neumann method considered in [33, 35]. We know of no previous work on a FETI method for the vector problem (1), nor of any previous theoretical study of a FETI method for the case where more than one coefficient has jumps.

In order to analyze our preconditioner and prove a polylogarithmic bound, we will first introduce an abstract framework for a family of FETI methods without a natural coarse problem; see [9]. We generalize the analysis in [22] to this class of problems and introduce a new assumption on the space of coarse functions. Our theory then leads to the choice of a suitable coarse space for Problem (3). The method proposed can be easily generalized to Raviart–Thomas approximations in two and three dimensions. To our knowledge, the case of Problem (3) in three dimensions still remains to be treated.

The study and analysis of preconditioners for Nédélec and Raviart–Thomas approximations is very recent; extensive work to extend classical Schwarz preconditioners to these vector problems has started only in the past three years. We note that two–level overlapping Schwarz preconditioners were initially developed for two dimensions, in [3], and then extended to three dimensions, in [34, 20]. Multigrid and multilevel methods were considered in [3, 2, 19, 18, 4, 20], and iterative substructuring methods in [1, 28, 36, 37, 33, 35].

The outline of the remainder of this paper is as follows. In section 2, we introduce appropriate Sobolev and finite element spaces, and in section 3, we formulate a FETI algorithm for the solution of (3). Section 4 is devoted to the development of an abstract framework needed for the analysis of our method, and in section 5, we apply these results and study the convergence properties of a particular class of FETI methods for the solution of (3). In section 6, we present some numerical results to validate our analysis. In an appendix, we give the quite technical proof of Lemma 5.3 stated in section 5.

2. Continuous and discrete spaces. In addition to $H(\mathrm{curl}\,;\Omega)$, we also use some standard Sobolev spaces. Given a bounded open Lipschitz domain $\mathcal{D} \subset \mathbb{R}^2$, with boundary $\partial\mathcal{D}$, let $|\cdot|_{1/2;\partial\mathcal{D}}$ denote the semi–norm of the Sobolev space $H^{1/2}(\partial\mathcal{D})$. Throughout, we work with scaled norms for the space $H^{1/2}(\partial\mathcal{D})$ obtained by dilation from the standard definition of the Sobolev norm on a region with unit diameter. Thus, with the L_2–norm $\|\cdot\|_{0;\mathcal{D}}$, we have

$$\|\phi\|^2_{\frac{1}{2};\partial\mathcal{D}} := |\phi|^2_{\frac{1}{2};\partial\mathcal{D}} + \frac{1}{H_{\mathcal{D}}}\|\phi\|^2_{0;\partial\mathcal{D}}.$$

Here and in the following, given a subset $\mathcal{D} \subset \mathbb{R}^2$, we denote its diameter by $H_{\mathcal{D}}$.

As already mentioned, the tangential component of any vector field $\mathbf{u} \in H(\mathrm{curl}\,;\mathcal{D})$ belongs to $H^{-\frac{1}{2}}(\partial\mathcal{D})$, and the corresponding trace operator is continuous and surjective; see [16, 7]. Here, $H^{-\frac{1}{2}}(\partial\mathcal{D})$ is equipped with the norm

$$||u||_{-\frac{1}{2};\partial\mathcal{D}} := \sup_{\substack{\phi \in H^{\frac{1}{2}}(\partial\mathcal{D}) \\ \phi \neq 0}} \frac{\langle u, \phi\rangle}{||\phi||_{\frac{1}{2};\partial\mathcal{D}}}, \tag{4}$$

where $\langle\cdot,\cdot\rangle$ represents the duality pairing between $H^{-\frac{1}{2}}(\partial\mathcal{D})$ and $H^{\frac{1}{2}}(\partial\mathcal{D})$. If $\mathbf{t}_{\mathcal{D}}$ is the unit tangent vector to $\partial\mathcal{D}$, the following inequality holds

$$||\mathbf{u}\cdot\mathbf{t}_{\mathcal{D}}||^2_{-\frac{1}{2};\partial\mathcal{D}} \leq C\left(||\mathbf{u}||^2_{0;\mathcal{D}} + H^2_{\mathcal{D}}||\mathrm{curl}\,\mathbf{u}||^2_{0;\mathcal{D}}\right), \tag{5}$$

with a constant C that is independent of $H_{\mathcal{D}}$. The scaling factor is obtained by dilation from a region of unit diameter. From now on, we denote by C a positive generic constant, uniformly bounded from above, and by c a positive generic constant uniformly bounded away from zero.

We next consider a triangulation $\mathcal{T}_h$ of the domain Ω, made of triangles or rectangles. Let $\mathcal{E}_h$ be the set of edges of $\mathcal{T}_h$. For every edge $e_k \in \mathcal{E}_h$, we fix a direction, given by a unit vector $\mathbf{t}_{e_k}$, tangent to e_k. The length of the edge e_k is denoted by $|e_k|$.

We also consider a non overlapping partition of the domain Ω,

$$\mathcal{F}_H = \left\{ \Omega_i \mid \quad 1 \leq i \leq N, \quad \bigcup_{i=1}^{N} \overline{\Omega_i} = \overline{\Omega} \right\},$$

such that each Ω_i is connected and is the union of some elements in $\mathcal{T}_h$. We suppose that $\mathcal{F}_H$ has at least two elements. We denote the diameter of Ω_i by H_i and define H as the maximum of the diameters of the subdomains:

$$H := \max_{1\leq i\leq N}\{H_i\}.$$

The elements of $\mathcal{F}_H$ are also called *substructures*. In the following, we always assume that the substructures are images of a reference square under sufficiently regular maps, which effectively means that their aspect ratios remain uniformly bounded. In addition, we assume that the ratio of the diameters of two adjacent subregions is bounded away from zero and infinity.

Let $\mathbf{t}_i$ be the unit tangent to $\partial\Omega_i$, such that, when going along $\partial\Omega_i$ following the direction of $\mathbf{t}_i$, Ω_i is on the left.

We define the *edges* of the partition as the intersections E_{ij}

$$\overline{E}_{ij} := \partial\Omega_i \cap \partial\Omega_j, \quad i \neq j, \quad |E_{ij}| > 0,$$

where $|E_{ij}|$ denotes the measure of E_{ij} and $\overline{E}_{ij}$ its closure. Let $\mathcal{E}_H$ be the set of edges of $\mathcal{F}_H$, and let the interface Γ be the union of the edges of $\mathcal{F}_H$, or, equivalently the parts of the subdomain boundaries that do not belong to $\partial\Omega$:

$$\Gamma := \bigcup_{i=1}^{N} \partial\Omega_i \setminus \partial\Omega.$$

For every subdomain Ω_i, let $\mathcal{I}_i$ be the set of indices j, such that E_{ij} is an edge of Ω_i:

$$\mathcal{I}_i := \{j \mid E_{ij} \subset \partial\Omega_i,\ E_{ij} \in \mathcal{E}_H\}.$$

Our assumptions on the partition $\mathcal{F}_H$ ensure that the the number of edges $|\mathcal{I}_i|$ is uniformly bounded.

We also define a *vertex* of the partition $\mathcal{F}_H$ as a non–empty intersection of the closure of two different edges in $\mathcal{E}_H$. Let $\mathcal{V}_H$ be the set of vertices of $\mathcal{F}_H$.

We assume that the coefficients a and A are constant in each substructure Ω_i and denote them by a_i and A_i, respectively. We also assume that

$$0 < \beta_i \|\mathbf{x}\|^2 \leq \mathbf{x}^t A_i \mathbf{x} \leq \gamma_i \|\mathbf{x}\|^2, \quad \mathbf{x} \in \mathbb{R}^2, \tag{6}$$

for $i = 1, \cdots, N$. Here, $\|\cdot\|$ denotes the standard Euclidean norm.

We define the local spaces

$$\mathrm{H}_\star(\mathrm{curl}\,;\Omega_i) := \{\mathbf{u}_i \in \mathrm{H}(\mathrm{curl}\,;\Omega_i) |\ \mathbf{u}_i \cdot \mathbf{t} = 0 \text{ on } \partial\Omega \cap \partial\Omega_i\}.$$

We consider the lowest–order Nédélec finite element (FE) spaces, originally introduced in [25], defined on each subdomain Ω_i

$$X_h(\Omega_i) = X_i := \{\mathbf{u}_i \in \mathrm{H}_\star(\mathrm{curl}\,;\Omega_i) |\ \mathbf{u}_{i|_t} \in \mathcal{R}(t),\ t \in \mathcal{T}_h,\ t \subset \Omega_i\},$$

where, in the case of triangular meshes, we have

$$\mathcal{R}(t) := \left\{ \begin{bmatrix} \alpha_1 + \alpha_3 x_2 \\ \alpha_2 - \alpha_3 x_1 \end{bmatrix} \mid \alpha_i \in \mathbb{R} \right\}.$$

We recall that the tangential component of a vector $\mathbf{u}_i \in X_i$ on the edges of the triangulation $\mathcal{T}_h$ is constant, and that the degrees of freedom can be taken as the values of the tangential component on the edges

$$\lambda_{e_k}(\mathbf{u}_i) = \mathbf{u}_k^{(i)} := \mathbf{u}_i \cdot \mathbf{t}_{e_k|_{e_k}}, \quad e_k \in \mathcal{E}_h, \quad e_k \subset \overline{\Omega_i}. \tag{7}$$

As in the case of nodal elements, the L^2–norm of a vector $\mathbf{u}_i \in \mathcal{R}(t)$ can be bounded from above and below by means of its degrees of freedom

$$c \sum_{e \subset \partial t} (|e|\, \lambda_e(\mathbf{u}_i))^2 \leq \|\mathbf{u}_i\|_{0;t}^2 \ \leq\ C \sum_{e \subset \partial t} (|e|\, \lambda_e(\mathbf{u}_i))^2\,, \tag{8}$$

where the constants c and C only depend on the aspect ratio of the element t. The proof given for nodal elements in [27, Prop. 6.3.1] can easily be adapted to the present case.

We next consider the product space

$$X_h(\Omega) = X := \prod_{i=1}^{N} X_i \subset \prod_{i=1}^{N} \mathrm{H}_\star(\mathrm{curl}\,;\Omega_i),$$

the trace spaces

$$W_h(\partial\Omega_i) = W_i := \{\mathbf{u}_i \cdot \mathbf{t}_i \text{ restricted to } \partial\Omega_i \setminus \partial\Omega \ \mid \mathbf{u}_i \in X_i\},$$

and the product space

$$W_h(\Gamma) = W := \prod_{i=1}^{N} W_i.$$

The local trace spaces W_i consist of piecewise constant functions on $\partial\Omega_i \setminus \partial\Omega$.

Throughout this paper, we will use the following notations. We denote a generic vector function in X_i using a bold letter with the subscript i, e.g., $\mathbf{u}_i$, and the column vector of its degrees of freedom (7) using the same bold letter with the superscript (i), e.g., $\mathbf{u}^{(i)}$. Its k–th degree of freedom corresponding to the edge e_k, defined in (7), is $\mathbf{u}_k^{(i)}$. We define its tangential component on $\partial\Omega_i \setminus \partial\Omega$ by

$$u_i := \mathbf{u}_i \cdot \mathbf{t}_i, \quad \text{on } \partial\Omega_i \setminus \partial\Omega,$$

which is a piecewise constant function that is uniquely determined by the degrees of freedom (7) on $\partial\Omega_i \setminus \partial\Omega$. Let $u^{(i)}$ be the column vector of these degrees of freedom defined componentwise by

$$u_k^{(i)} := \mathbf{u}_k^{(i)}, \quad e_k \subset \partial\Omega_i \setminus \partial\Omega.$$

We will also use the same notation for the spaces of functions X_i and W_i and the corresponding spaces of degrees of freedom.

We say that a given vector $\mathbf{u} \in X$ is continuous if its tangential component is continuous across the edges $E_{ij} \in \mathcal{E}_H$. In this case, with our notations, we have

$$\begin{aligned} u_{i|_{E_{ij}}} &= -u_{j|_{E_{ij}}}, \quad E_{ij} \in \mathcal{E}_H, \\ u_k^{(i)} &= u_k^{(j)}, \quad e_k \subset E_{ij}, \quad E_{ij} \in \mathcal{E}_H. \end{aligned}$$

We remark that, given an edge $E_{ij} \subset \mathcal{E}_H$, the vectors $\mathbf{t}_i$ and $\mathbf{t}_j$ have opposite direction on E_{ij}, but the direction of a fine edge $e_k \subset E_{ij}$ is the same on $\partial\Omega_i$ and on $\partial\Omega_j$.

Given the unit vectors $\mathbf{t}_i$, we define the column vectors $t^{(i)}$ of degrees of freedom

$$t_k^{(i)} := \mathbf{t}_i \cdot \mathbf{t}_{e_k}, \quad e_k \subset \partial\Omega_i \setminus \partial\Omega, \quad e_k \in \mathcal{E}_h.$$

We remark that, in case all the edges e_k on $\partial\Omega_i$ have the same direction of the boundary $\partial\Omega_i$, the entries of the vector $t^{(i)}$ are equal to one. Figure 1 shows an example of a partition, with the directions of the subdomain boundaries and of the fine edges on the interface Γ, and the corresponding values of the degrees of freedom $t^{(i)}$.

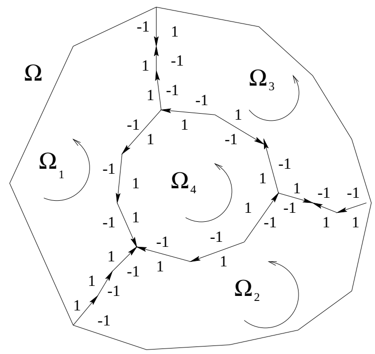

FIG. 1. *Example of partition of the domain Ω. We show the directions of the subdomain boundaries, given by the unit vectors $\{\mathbf{t}_i\}$, those of the fine edges on the interface Γ, and the corresponding values of the degrees of freedom $t^{(i)}$.*

Finally, for $i = 1, \cdots, N$, we define the extensions into the interior of Ω_i

$$\mathcal{H}_i : W_i \longrightarrow X_i,$$

that are discrete harmonic with respect to the bilinear forms $a_{\Omega_i}(\cdot,\cdot)$. We recall that $\mathbf{u}_i = \mathcal{H}_i u_i$ minimizes the energy $a_{\Omega_i}(\mathbf{u}_i, \mathbf{u}_i)$ among all the vectors of X_i with tangential component equal to u_i on $\partial\Omega_i \setminus \partial\Omega$.

3. A FETI method. In this section, we introduce a FETI method for the solution of the linear system arising from the edge element discretization of problem (3). Throughout the paper, we denote the Euclidean scalar product in l^2 by $\langle\cdot,\cdot\rangle$. We first assemble the local stiffness matrices, relative to the bilinear forms $a_{\Omega_i}(\cdot,\cdot)$, and the local load vectors. The degrees of freedom that are not on the interface Γ only belong to one substructure and can be eliminated in parallel by block Gaussian elimination. Let $f^{(i)}$ be the resulting right hand sides and $S^{(i)}$ the Schur complement matrices

$$S^{(i)} : W_i \longrightarrow W_i,$$

relative to the degrees of freedom on $\partial\Omega_i \setminus \partial\Omega$.

We recall that the local Schur complements satisfy the following property

$$|u^{(i)}|^2_{S^{(i)}} := \langle u^{(i)}, S^{(i)} u^{(i)}\rangle = a_{\Omega_i}(\mathcal{H}_i u_i, \mathcal{H}_i u_i); \tag{9}$$

see, e.g., [31, 35].

Following [22], we can then reformulate our problem as

$$\begin{array}{rcrcl} Su & + & B^t\lambda & = & f \\ Bu & & & = & 0 \end{array} \tag{10}$$

where

$$u := \begin{bmatrix} u^{(1)} \\ \vdots \\ u^{(N)} \end{bmatrix} \in W, \quad S := \mathrm{diag}\{\mathrm{S}^{(1)}, \cdots, \mathrm{S}^{(\mathrm{N})}\}, \quad \mathrm{f} := \begin{bmatrix} f^{(1)} \\ \vdots \\ f^{(N)} \end{bmatrix}.$$

The matrix B, the entries of which belong to $\{0, 1, -1\}$, evaluates the difference of the corresponding degrees of freedom on the interface Γ and can be written as

$$B = \left[B^{(1)} \, B^{(2)} \, \cdots \, B^{(N)} \right],$$

where the local matrices $B^{(i)}$ act on vectors in W_i. We note that, with our terminology, a vector $w \in W$ is continuous if and only if $Bw = 0$. The vector λ is a Lagrange multiplier relative to the pointwise continuity constraint $Bu = 0$.

We remark that the $S^{(i)}$ are always invertible and, consequently, there is no natural coarse space associated to the substructures; we are in a similar case as the one considered in [9].

We first find u from the first equation in (10), and substitute its value in the second equation. We obtain the system

$$F\lambda = d, \tag{11}$$

where

$$F := B\,S^{-1}\,B^t, \quad d := B\,S^{-1}\,f.$$

In order to build a preconditioner for (11), we first introduce a set of scaling functions on the boundaries $\partial\Omega_i$, using only the coefficient A; see [33, 35] for a Neumann–Neumann method. Our family of scaling functions will depend on a parameter

$$\delta \geq 1/2. \tag{12}$$

Let Ω_i be a substructure. We define a piecewise constant function $\mu_i^\dagger \in W_i$ by

$$\mu_i^\dagger{}_{|E_{ij}} \equiv \frac{\gamma_i^\delta}{\gamma_i^\delta + \gamma_j^\delta}, \quad j \in \mathcal{I}_i, \tag{13}$$

where γ_i and γ_j are the largest eigenvalues of the coefficient matrices A_i and A_j, respectively. We remark that $\mu_i^\dagger$ is constant on each coarse edge E_{ij} and satisfies

$$(\mu_i^\dagger + \mu_j^\dagger)_{|E_{ij}} = 1, \quad E_{ij} \in \mathcal{E}_H. \tag{14}$$

We consider the matrices,

$$R := \left[R^{(1)} \, R^{(2)} \, \cdots \, R^{(M)} \right], \quad G := B\,R, \tag{15}$$

where $R^{(i)}$ are vectors in W, related to the substructures $\{\Omega_i\}$. More precisely, we suppose that the generic $R^{(i)}$ is obtained from a local vector $r^{(i)} \in W_i$ on $\partial\Omega_i$, by extending it by zero on the boundaries of the other substructure. We will make a particular choice of R for Problem (3), in section 5, and specify the dimension M.

Following [9], we then define the projection

$$P := I - G(G^t F G)^{-1} G^t F,$$

onto the complement of $Range(G)$, orthogonal with respect to the scalar product induced by F. Following [22, Sect. 4], we define the preconditioner

$$\widehat{M}^{-1} := (BD^{-1}B^t)^{-1}\,BD^{-1}SD^{-1}B^t\,(BD^{-1}B^t)^{-1},$$

where

$$D := \begin{bmatrix} D^{(1)} & O & \cdots & O \\ O & D^{(2)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ O & \cdots & O & D^{(N)} \end{bmatrix}.$$

Here, the local matrices $D^{(i)}$ are diagonal and represent the multiplication by the local scaling functions $\mu_i^\dagger$:

$$D_{kk}^{(i)} := \mu^\dagger_{i\,|_{e_k}}, \quad e_k \subset \partial\Omega_i \setminus \partial\Omega.$$

It can be easily seen that $BD^{-1}B^t$ is a diagonal matrix, since there are no crosspoints in our discretizations. Therefore, $\widehat{M}^{-1}$ results in the same preconditioner as the one for redundant Lagrange multipliers derived by mechanical arguments in [30] and analyzed in [22].

Now, we consider a projected conjugate gradient method, as in [9].

1. Initialize

$$\begin{aligned} \lambda^0 &= G(G^tFG)^{-1}G^td \\ q^0 &= d - F\lambda^0 \end{aligned}$$

2. Iterate $k = 1, 2, \cdots$ until convergence

$$\begin{aligned} \text{Project: } w^{k-1} &= P^t q^{k-1} \\ \text{Precondition: } z^{k-1} &= \widehat{M}^{-1} w^{k-1} \\ \text{Project: } y^{k-1} &= P z^{k-1} \\ \beta^k &= \langle y^{k-1}, w^{k-1}\rangle / \langle y^{k-2}, w^{k-2}\rangle \quad [\beta^1 = 0] \\ p^k &= y^{k-1} + \beta^k p^{k-1} \quad [p^1 = y^0] \\ \alpha^k &= \langle y^{k-1}, w^{k-1}\rangle / \langle p^k, Fp^k\rangle \\ \lambda^k &= \lambda^{k-1} + \alpha^k p^k \\ q^k &= q^{k-1} - \alpha^k F p^k \end{aligned}$$

The first projection can be omitted; because of the choice of the initial vector λ^0, we have $w^{k-1} = q^{k-1}$ after the first projection step. Here, we have denoted the residual at the k–th step by q^k. In practice, partial or full reorthogonalization may be required; cf. [15].

The method presented here is equivalent to using the conjugate gradient method for solving the following preconditioned system

$$P\widehat{M}^{-1}P^tF\lambda = P\widehat{M}^{-1}P^td, \quad \lambda \in \lambda_0 + V, \tag{16}$$

with

$$V := Range(P). \tag{17}$$

We remark that the matrices S and S^{-1} do not need to be calculated in practice. The action of S on a generic vector requires the solution of a Dirichlet problem on each substructure, while the action of S^{-1} requires the solution of a Neumann problem on each substructure; see [31, Ch. 4].

4. Analysis of the FETI preconditioner. In this section, we develop an abstract framework in order to analyze the preconditioned system (16). Our analysis also applies to the method described in [9] for time–dependent elasticity problems and is similar to the one in [22] for the original FETI method. It has been modified in order to treat the case where the local matrices $S^{(i)}$ are invertible and the projection P is orthogonal with respect to the scalar product induced by F.

We note that, in the following, we use the same notation for operators and their matrix representations. Let $U := Range(B)$ be the space of the Lagrange multipliers. Since S is invertible, we have

$$Ker(S) = \{0\}, \quad Range(S) = W.$$

In addition to the subspace $V \subset U$, defined in (17), we also need the space $V' \subset U$, defined by

$$V' := Range(P^t) = Ker(P)^{\perp}. \tag{18}$$

Since P is not symmetric, V and V' are in general different. Note that it can also be easily shown that V' is isomorphic to the *dual* space of V using $\langle \cdot, \cdot \rangle$ as dual pairing.

We now consider a subspace of *coarse* functions in W, which has a similar function as the kernel of the matrix S in the original FETI method; see, e.g., [22]. We define

$$Z := Range(R) = span\{R^{(i)} \,|\, i = 1, \cdots, M\},$$

where R has been defined in (15). We will make the following assumption on R; see [22].

ASSUMPTION 4.1. *We have*

$$Z \cap Ker(B) = Range(R) \cap Ker(B) = \{0\}.$$

This assumption means that Z does not contain any continuous vector and consequently, that the matrix $G = BR$ has full rank. It is immediate to check that the kernel of P is equal to the range of the matrix G and that

$$V = \{\lambda \in U | \quad \langle \lambda, Bz \rangle_F = 0, \; z \in Z\}, \tag{19}$$

$$V' = \{\lambda \in U | \quad \langle \lambda, Bz \rangle = 0, \; z \in Z\}, \tag{20}$$

where $\langle \cdot, \cdot \rangle_F$ denotes the scalar product induced by F,

$$\langle \lambda, \mu \rangle_F := \langle \lambda, F\mu \rangle, \quad \lambda, \mu \in U.$$

We will also need the space Z', defined by

$$Z' := \left\{ w \in Ker(B)^{\perp} | \; Bw = FBz, \text{ with } z \in Z \right\}. \tag{21}$$

As for V and V', it can easily be shown that Z' is isomorphic to the dual space of Z. The following lemma characterizes the space Z'.

LEMMA 4.1. *A vector $z' \in W$ belongs to Z' if and only if there exists a $z \in Z$, such that*

$$z' = B^t(BB^t)^{-1}FBz. \tag{22}$$

Proof. We first suppose that $z' \in Z'$. For a suitable $z \in Z$, we then have

$$Bz' = FBz = BB^t(BB^t)^{-1}FBz,$$

and, consequently,

$$z' - B^t(BB^t)^{-1}FBz = 0,$$

since $z' \in Ker(B)^{\perp}$. The rest of the proof is trivial. □

From (19) and Lemma 4.1, we also have the following characterization of V

$$V = \{\lambda \in U | \quad \langle \lambda, Bz' \rangle = 0,\ z' \in Z'\}. \tag{23}$$

The following lemma generalizes Lemma 4 in [22].

LEMMA 4.2. *For any $w \in W$, there exists a unique $z'_w \in Z'$, such that*

$$B(w + z'_w) \in V'.$$

Proof. Let $z' \in Z'$, i.e.,

$$z' = B^t(BB^t)^{-1}FBz,$$

for a suitable $z \in Z$. From (20), we have that $B(w + z') \in V'$ if and only if

$$\tilde{z}^t B^t B(w + z') = 0, \quad \tilde{z} \in Z,$$

and, using the definition of z', if and only if

$$\tilde{z}^t B^t FBz = -\tilde{z}^t B^t Bw, \quad \tilde{z} \in Z. \tag{24}$$

We need to show that (24) has a unique solution $z = z_w \in Z$, for all $w \in W$. It is enough to show uniqueness. If the solution of (24) is not unique, there exists a non–vanishing vector $z \in Z$, such that

$$z^t B^t FBz = 0,$$

and, consequently, $z \in Ker(B)$. Using Assumption 4.1, we deduce that $z = 0$, and this is a contradiction. □

Given the preconditioned system (16), we define the operator

$$\widehat{C}^{-1} : V' \longrightarrow V,$$

as the restriction of $P\widehat{M}^{-1}$ to $V' = Range(P^t)$. We have the following Lemma.

LEMMA 4.3. *The operator $\widehat{C}^{-1}$ is selfadjoint and positive definite.*

Proof. The first property can be easily checked. The proof of the second one is similar to [22, Lem. 3]. Since S is positive definite, we have, for every $\lambda \in V'$,

$$\langle \widehat{C}^{-1}\lambda, \lambda \rangle = \langle \widehat{M}^{-1}\lambda, \lambda \rangle = \left\| S^{\frac{1}{2}} D^{-1} B^t (BD^{-1}B^t)^{-1} \lambda \right\|^2.$$

The right hand side is always non negative and vanishes if and only if $(BD^{-1}B^t)^{-1}\lambda \in Ker(B^t)$. Since B has full rank, this condition implies $(BD^{-1}B^t)^{-1}\lambda = 0$ and, consequently, $\lambda = 0$. □

The inverse of $\widehat{C}^{-1}$

$$\widehat{C} : V \longrightarrow V',$$

is therefore well defined.

In order to bound the smallest and largest eigenvalues of the preconditioned operator $P\widehat{M}^{-1}P^tF$, it is enough to find two positive constants C_1 and C_2 such that

$$C_1 \langle \widehat{C}\lambda, \lambda\rangle \leq \langle F\lambda, \lambda\rangle \leq C_2 \langle \widehat{C}\lambda, \lambda\rangle, \quad \lambda \in V.$$

In order to find these bounds, we define two norms. In V', we define

$$\|\lambda\|_{V'} := \langle \widehat{C}^{-1}\lambda, \lambda\rangle = \langle \widehat{M}^{-1}\lambda, \lambda\rangle = |D^{-1}B^t(BD^{-1}B^t)^{-1}\lambda|_S^2, \quad \lambda \in V', \tag{25}$$

where

$$|w|_S^2 := \langle w, Sw\rangle, \quad w \in W,$$

while in V, we define

$$\|\lambda\|_V := \sup_{\mu \in V'} \frac{\langle \lambda, \mu\rangle}{\|\mu\|_{V'}}, \quad \lambda \in V. \tag{26}$$

It can easily be checked that

$$\|\lambda\|_V^2 = \langle \widehat{C}\lambda, \lambda\rangle, \quad \lambda \in V.$$

We finally consider a jump operator, originally defined in [22]

$$P_D := D^{-1}B^t(BD^{-1}B^t)^{-1}B, \tag{27}$$

which will be essential to prove our bounds. It is immediate that P_D is a projection which is orthogonal to the scalar product induced by D, and preserves jumps, since $BP_D = B$; see [22, Lem. 2].

The following lemma can be found in [22, Lem. 1].

LEMMA 4.4. *For any $\lambda \in U$, there exists a $\widehat{w} \in Range(P_D)$, such that $\lambda = B\widehat{w}$.*

Proof. Since $U = Range(B)$, there exists a $\widetilde{w} \in W$, such that $\lambda = B\widetilde{w}$. We can then choose $\widehat{w} := P_D\widetilde{w}$, since $B\widehat{w} = BP_D\widetilde{w} = B\widetilde{w}$. □

We will make the following assumption on P_D; see Lemmas 6 and 7, and the proof of Theorem 1 in [22].

ASSUMPTION 4.2. *There exists a parameter ω, such that, for any $w \in W$ with $Bw \in V'$,*

$$|P_Dw|_S^2 \leq \omega\, |w|_S^2.$$

Before proving our main result, we need the following lemma, which introduces a representation formula for the matrix F. The proofs in [23, 22] can easily be adapted to our case.

LEMMA 4.5. *For every $\lambda \in U$, we have*

$$\langle F\lambda, \lambda\rangle = \sup_{w\in W} \frac{\langle \lambda, Bw\rangle^2}{|w|_S^2}.$$

We are now ready to prove our main abstract theorem.

THEOREM 4.1. *If Assumptions 4.1 and 4.2 hold, we have*

$$\langle \widehat{C}\lambda, \lambda\rangle \leq \langle F\lambda, \lambda\rangle \leq \omega\, \langle \widehat{C}\lambda, \lambda\rangle, \quad \lambda \in V,$$

where ω is the parameter in Assumption 4.2. Then,

$$\kappa(P\widehat{M}^{-1}P^tF) \leq \omega,$$

where κ denotes the condition number of the preconditioned system.

Proof. Our proof follows that of Theorem 1 in [22]. Let $\lambda \in V$. We first prove the lower bound. We consider an arbitrary but fixed vector $\mu \in V'$. Lemma 4.4 ensures that there exists $\widehat{w} \in Range(P_D)$ such that $\mu = B\widehat{w}$. Using Lemma 4.5, we have

$$\langle F\lambda, \lambda\rangle \geq \frac{\langle \lambda, B\widehat{w}\rangle^2}{|\widehat{w}|_S^2} = \frac{\langle \lambda, B\widehat{w}\rangle^2}{|P_D\widehat{w}|_S^2} = \frac{\langle \lambda, B\widehat{w}\rangle^2}{|D^{-1}B^t(BD^{-1}B^t)^{-1}B\widehat{w}|_S^2}. \tag{28}$$

Using the definitions of $\widehat{M}^{-1}$ and $\|\mu\|_{V'}$, the last expression in (28) is equal to

$$\frac{\langle \lambda, \mu\rangle^2}{|D^{-1}B^t(BD^{-1}B^t)^{-1}\mu|_S^2} = \frac{\langle \lambda, \mu\rangle^2}{\|\mu\|_{V'}^2}, \quad \mu \in V'.$$

Taking the supremum over $\mu \in V'$ and using the definition of the norm $\|\lambda\|_V$ gives the lower bound.

We now consider the upper bound. Using the definitions of Z' and V, we can write

$$\sup_{w\in W} \frac{\langle \lambda, Bw\rangle^2}{|w|_S^2} = \sup_{w\in W} \frac{\langle \lambda, B(w+z')\rangle^2}{|w|_S^2} = \sup_{w\in W} \frac{\langle \lambda, Bw\rangle^2}{|w+z'|_S^2}, \quad z' \in Z'. \tag{29}$$

According to Lemma 4.2, we can find a unique $z'_w \in Z'$ such that $B(w + z'_w) \in V'$. Choosing $z' = z'_w$ in (29) and using Assumption 4.2, we have

$$\begin{aligned}\sup_{w\in W} \frac{\langle \lambda, Bw\rangle^2}{|w|_S^2} &\leq \omega \sup_{w\in W} \frac{\langle \lambda, Bw\rangle^2}{|P_D(w+z'_w)|_S^2} = \omega \sup_{\substack{\tilde{w}=w+z'_w\\ w\in W}} \frac{\langle \lambda, B\tilde{w}\rangle^2}{|P_D\tilde{w}|_S^2}\\ &= \omega \sup_{\substack{\tilde{w}\in W\\ B\tilde{w}\in V'}} \frac{\langle \lambda, B\tilde{w}\rangle^2}{|P_D\tilde{w}|_S^2} = \omega \sup_{\mu\in V'} \frac{\langle \lambda, \mu\rangle^2}{\|\mu\|_{V'}^2}.\end{aligned}$$

The upper bound is proven by using the definition of the norm $\|\lambda\|_V$. ☐

5. A particular choice of the matrix R for Maxwell's equations. In this section, we consider a particular choice of the matrix R in the definition of the FETI algorithm for Problem (3); see (15). Our aim is to find a suitable R such that Assumptions 4.1 and 4.2 hold with an ω that does not depend on the diameter of the substructures and the jumps of the coefficients a and A in (3).

We consider the jump operator P_D and recall some of its properties; see [22]. Given $w \in W$, we first define the vector

$$v := E_D w := w - P_D w.$$

Using the definition of P_D, we can easily find that the $E_D w$ is continuous. In addition, $E_D w$ is equal to the D–weighted average of w on the interface Γ; see [22, Lem. 5]. This can be seen by considering a continuous vector $x \in W$, for every edge $e_k \in \mathcal{E}_h$ on Γ, that satisfies

$$x_k^{(i)} = x_k^{(j)} = 1, \quad \text{if } e_k \in E_{ij},$$

and is equally zero elsewhere. Since $Bx = 0$, we have $x^t D P_D w = 0$. Using the fact that v is continuous, we then find

$$v_k^{(i)} = v_k^{(j)} = x^t D w = D_{kk}^{(i)} w_k^{(i)} + D_{kk}^{(j)} w_k^{(j)}, \quad e_k \subset E_{ij}, \quad j \in \mathcal{I}_i, \tag{30}$$

or, equivalently, considering the corresponding functions,

$$\mathbf{v}_i \cdot \mathbf{t}_{i|E_{ij}} = \mathbf{v}_j \cdot \mathbf{t}_{i|E_{ij}} = \left[\mu_i^\dagger(\mathbf{w}_i \cdot \mathbf{t}_i) + \mu_j^\dagger(\mathbf{w}_j \cdot \mathbf{t}_i)\right]_{|E_{ij}}, \quad j \in \mathcal{I}_i.$$

Using (30) and (14), we can find a formula for $P_D w$. If we define

$$u := P_D w = w - v, \tag{31}$$

we find that

$$u_k^{(i)} = D_{kk}^{(j)}(w_k^{(i)} - w_k^{(j)}), \quad e_k \subset E_{ij}, \quad j \in \mathcal{I}_i, \tag{32}$$

or, equivalently,

$$\begin{aligned} u_{i|E_{ij}} = \mathbf{u}_i \cdot \mathbf{t}_{i|E_{ij}} &= \left[\mu_j^\dagger(\mathbf{w}_i \cdot \mathbf{t}_i - \mathbf{w}_j \cdot \mathbf{t}_i)\right]_{|E_{ij}} = \left[\mu_j^\dagger(\mathbf{w}_i \cdot \mathbf{t}_i + \mathbf{w}_j \cdot \mathbf{t}_j)\right]_{|E_{ij}} \\ &= \left[\mu_j^\dagger\,(w_i + w_j)\right]_{|E_{ij}}, \quad j \in \mathcal{I}_i. \end{aligned} \tag{33}$$

We note that $\mathbf{t}_i$ and $\mathbf{t}_j$ have opposite directions along E_{ij}. Using (9), we see that, in order to bound $|u|_S^2$, we need to bound the energy of the discrete harmonic extensions $\{\mathcal{H}_i u_i\}$.

The discrete harmonic extension $\mathcal{H}_i$ satisfies the following stability estimate

$$H_i^2 \|\mathrm{curl}\, \mathcal{H}_i u_i\|_{0;\Omega_i}^2 + \|\mathcal{H}_i u_i\|_{0;\Omega_i}^2 \leq C\, \|u_i\|_{-\frac{1}{2};\partial\Omega_i}^2,$$

with a constant C that does not depend on the diameter of Ω_i. This estimate is obtained by considering the corresponding bound for a substructure of unit diameter and by using dilation. This shows that, in general, a bound for the energy

$$a_{\Omega_i}(\mathcal{H}_i u_i, \mathcal{H}_i u_i),$$

depends on the diameter of Ω_i.

For an arbitrary substructure Ω_i, we define the subset of W_i of tangential traces with mean value zero:

$$W_i^0 := \{u_i \in W_i \mid \int_{\partial\Omega_i} u_i\, ds = 0\}$$

The following lemma ensures that a curl–free extension can be found of the traces in W_i^0. The same argument as in [37, Lem. 4.3] can be employed for its proof; see also [33, Sect. 5.5].

LEMMA 5.1. *Let Ω_i be a substructure. Then, there exists an extension operator $\widetilde{\mathcal{H}}_i : W_i^0 \to X_i$, such that, for any $u_i \in W_i^0$,*

$$\operatorname{curl} \widetilde{\mathcal{H}}_i u_i = 0,$$

and

$$||\widetilde{\mathcal{H}}_i u_i||_{0;\Omega_i} \le C ||u_i||_{-\frac{1}{2};\partial\Omega_i}. \tag{34}$$

Here C is independent of h, H_i, and u_i.

We note that, if $u_i \in W_i^0$, Lemma 5.1 ensures that

$$a_{\Omega_i}(\mathcal{H}_i u_i, \mathcal{H}_i u_i) \le a_{\Omega_i}(\widetilde{\mathcal{H}}_i u_i, \widetilde{\mathcal{H}}_i u_i) \le C\, \gamma_i\, ||u_i||^2_{-\frac{1}{2};\partial\Omega_i}, \tag{35}$$

with a constant C that is independent of H_i. Next, we try to find a subspace of functions w for which the tangential traces u_i of $u = P_D w$ belong to W_i^0. More precisely, we require that

$$\begin{aligned}\int_{\partial\Omega_i} u_i\, ds \quad &= \sum_{\substack{e_k \subset \partial\Omega_i \\ e_k \in \mathcal{E}_h}} u_k^{(i)} t_k^{(i)} |e_k| = \sum_{j \in \mathcal{I}_i} \sum_{\substack{e_k \subset E_{ij} \\ e_k \in \mathcal{E}_h}} u_k^{(i)} t_k^{(i)} |e_k| \\ &= \sum_{j \in \mathcal{I}_i} \sum_{\substack{e_k \subset E_{ij} \\ e_k \in \mathcal{E}_h}} t_k^{(i)} |e_k|\, D_{kk}^{(j)}\, (w_k^{(i)} - w_k^{(j)}) = 0, \quad i = 1, \cdots, N.\end{aligned} \tag{36}$$

In Assumption 4.2, we consider functions $w \in W$, such that $Bw \in V'$. Using the definition of V', we deduce that $Bw \in V'$ if and only if

$$z^t B^t B w = 0, \quad z \in Z,$$

or, equivalently,

$$\sum_{j=1}^{N} {r^{(i)}}^t {B^{(i)}}^t B^{(j)} w^{(j)} = 0, \quad i = 1, \cdots, N, \tag{37}$$

where the local vectors $r^{(i)}$ are those introduced in section 3 in order to build the matrix R. Using the definition of the local matrices $B^{(i)}$, (37) can be written as

$$\sum_{j \in \mathcal{I}_i} \sum_{\substack{e_k \subset E_{ij} \\ e_k \in \mathcal{E}_h}} r_k^{(i)} (w_k^{(i)} - w_k^{(j)}) = 0 \quad i = 1, \cdots, N. \tag{38}$$

We can then find the local vectors $r^{(i)}$, by making (38) equivalent to (36). We find, for $i = 1, \cdots, N$,

$$r_k^{(i)} := D_{kk}^{(j)}\, t_k^{(i)}\, |e_k|, \quad e_k \subset E_{ij}, \quad j \in \mathcal{I}_i, \tag{39}$$

or, equivalently, by considering the corresponding functions,

$$r_{i|_{e_k}} := |e_k|\, \mu^\dagger_{j|_{e_k}} = |e_k|\, [1 - \mu_i^\dagger]_{|_{e_k}}, \quad e_k \subset E_{ij}, \quad j \in \mathcal{I}_i.$$

We remark that the local function r_i is always positive and thus is the tangential component of a vector that has the same direction as $\mathbf{t}_i$; it is scaled using the functions $\mu_j^\dagger$, relative to the substructures adjacent to Ω_i, and the diameters of the fine edges on $\partial\Omega_i$. The corresponding expression for the vector of degrees of freedom $r^{(i)}$ must take into account that the direction of $\partial\Omega_i$ and that of the edges $\{e_k\}$ may differ.

We can formalize the definition of the vectors $\{r^{(i)}\}$ in the following way:

DEFINITION 5.1. *The local vectors $\{r^{(i)},\ i = 1, \cdots, N\}$ are the unique vectors that satisfy*

$$\sum_{\substack{e_k \subset \partial\Omega_i \\ e_k \in \mathcal{E}_h}} {r_k^{(i)}}^t v_k^{(i)} = \sum_{j \in \mathcal{I}} \int_{E_{ij}} \mu_j^\dagger\, v_i\, ds = \int_{\partial\Omega_i} (1 - \mu_i^\dagger)\, v_i\, ds, \quad v_i \in W_i.$$

The global vectors $R^{(i)}$ are obtained by extending the local vectors $r^{(i)}$ by zero outside $\partial\Omega_i$.

We need to build the matrix R; cf. (15). If all the vectors $R^{(i)}$ are used to build R, then there are cases in which the space $span\{R^{(i)} |\ i = 1, \cdots, N\}$ contains a continuous function, thus failing to satisfy Assumption 4.1. On the other hand, we need (36) to be valid for *every* subdomain Ω_i; see the proof of Theorem 5.2. The following lemma helps us determine how many of the vectors $R^{(i)}$ need to be employed for constructing the matrix R and ensures that, in case any of the $R^{(i)}$ is not used, (36) remains valid for every Ω_i.

We first define $\mathcal{G}_H$ as the dual graph of the partition $\mathcal{F}_H$. We recall that $\mathcal{G}_H$ is obtained by considering a vertex for each substructure of $\mathcal{F}_H$ and an edge in $\mathcal{G}_H$ between two vertices if the corresponding substructures have a common edge. It will be useful to identify the vertices of $\mathcal{G}_H$ with the centers of mass of the subdomains of $\mathcal{F}_H$.

LEMMA 5.2.

1) *The graph $\mathcal{G}_H$ is two–colorable if and only if each vertex in $\mathcal{V}_H$ that does not belong to $\partial\Omega$ belongs to an even number of substructures.*
2) *The space $span\{R^{(i)} |\ i = 1, \cdots, N\}$ contains a non–vanishing continuous function if and only if $\mathcal{G}_H$ is two–colorable.*
3) *If one of the vectors $R^{(i)}$ is removed, say $R^{(N)}$, then the space $span\{R^{(i)} |\ i = 1, \cdots, N-1\}$ does not contain any continuous function.*
4) *If $\mathcal{G}_H$ is two–colorable, then the condition*

$$z^t B^t B w = 0, \quad z \in span\{R^{(i)} |\ i = 1, \cdots, N-1\},$$

implies that, for every $w \in W$,

$$\int_{\partial\Omega_i} u_i\, ds = 0, \quad i = 1, \cdots, N,$$

where $u = P_D w$.

Proof. Since Ω is connected and $\mathcal{F}_H$ consists of more than one subdomain, $\mathcal{G}_H$ has at least one edge. Then, $\mathcal{G}_H$ is two–colorable if and only if all its cycles have an even number of vertices; see, e.g., [17, Th. 2.1.6]. In addition, we can associate a cycle of $\mathcal{G}_H$ to each vertex of the partition $\mathcal{F}_H$ that does not lie on $\partial\Omega$, by considering the shortest cycle that connects the centers of mass of the subdomains that have this vertex. We remark that the number of vertices of this cycle is equal to the number of

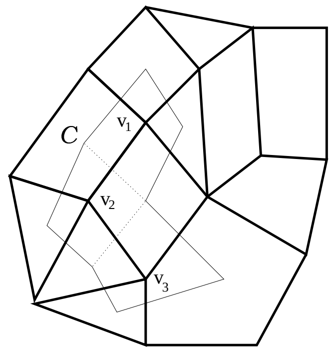

FIG. 2. *Example of cycle $\mathcal{C}$ in $\mathcal{G}_H$ for a particular partition $\mathcal{F}_H$.*

subdomains that share this vertex.
Part 1: We first suppose that $\mathcal{G}_H$ is two–colorable. Consider any vertex of $\mathcal{F}_H$ that does not lie on $\partial\Omega$. Then the cycle associated to this vertex has an even number of vertices of $\mathcal{G}_H$, and, consequently, it belongs to an even number of subdomains.

We suppose that every vertex of $\mathcal{F}_H$, which does not lie on $\partial\Omega$, belongs to an even number of subdomains. Consider any cycle $\mathcal{C}$ of $\mathcal{G}_H$. We can suppose, without loss of generality, that $\mathcal{C}$ has no repeated vertices, except the initial and the final. Let $\{v_1, \cdots, v_K\}$ be the vertices in $\mathcal{V}_H$ inside $\mathcal{C}$, and let $\{\mathcal{C}_1, \cdots, \mathcal{C}_K\}$ be the cycles associated to $\{v_1, \cdots, v_K\}$. It is immediate to see that $\mathcal{C}$ can be obtained by considering the cycles $\{\mathcal{C}_1, \cdots, \mathcal{C}_K\}$ and deleting the common edges. We refer to Figure 2 for an example. Since $\{\mathcal{C}_1, \cdots, \mathcal{C}_K\}$ all have an even number of edges, and the common edges are removed two by two, then $\mathcal{C}$ must consist of an even number of edges and, since $\mathcal{C}$ is arbitrary, $\mathcal{G}_H$ is two–colorable.
Part 2: We first suppose that $\mathcal{G}_H$ is two–colorable, with colors $\{1, -1\}$. Let c_i be the color of Ω_i. It is then immediate to see that the vector

$$\sum_{i=1}^{N} c_i \gamma_i^\delta R^{(i)},$$

does not vanish and that it is continuous, since, on the generic edge $E_{ij} \in \mathcal{E}_H$, $\gamma_i^\delta R^{(i)}$ and $\gamma_j^\delta R^{(j)}$ have the same absolute value and opposite sign.

We next suppose that there is a non–vanishing continuous vector

$$\sum_{i=1}^{N} \alpha_i R^{(i)}, \tag{40}$$

with at least one coefficient, say α_k, different from zero.

We first prove that all coefficients $\{\alpha_i\}$ are non–vanishing. Suppose, by contradiction, that there is a coefficient, say α_N, that vanishes. Since Ω is connected, we can find a sequence of subdomains

$$\{\Omega_{i_1}, \Omega_{i_2}, \cdots, \Omega_{i_M}\} \subset \mathcal{F}_H,$$

such that

$$\Omega_{i_1} = \Omega_N, \quad \Omega_{i_M} = \Omega_k,$$

and such that any two consecutive subdomains in the sequence have a common edge:

$$\overline{\Omega}_{i_j} \cap \overline{\Omega}_{i_{j+1}} = E_j \subset \mathcal{E}_H, \quad j = 1, \cdots, M-1,$$

Since $\alpha_N = 0$ and the sum in (40) is continuous on the edge E_1, we have $\alpha_{i_2} = 0$. In the same way, we can prove that $\alpha_{i_j} = 0$, for $j = 3, \cdots, M$, and finally that $\alpha_k = 0$, which is a contradiction.

Consider then the integers $\{c_i := sign(\alpha_i)\}$. We now prove that the set $\{c_i\}$ provides a two–coloring of $\mathcal{G}_H$. Consider two adjacent subdomains Ω_i and Ω_j that share the edge E_{ij}. We have to prove that c_i and c_j have opposite sign. We first remark that, since the unit vectors $\mathbf{t}_i$ and $\mathbf{t}_j$ have opposite direction along E_{ij}, the entries of the vectors $R^{(i)}$ and $R^{(j)}$ have opposite sign on E_{ij}; see Definition 5.1 and (39). As the sum in (40) is continuous along the edge E_{ij} and the only contribution on E_{ij} comes from $\alpha_i R^{(i)}$ and $\alpha_j R^{(j)}$, the coefficients α_i and α_j, and therefore c_i and c_j, must be of opposite sign.
Part 3: The proof employs an argument similar to the previous one, used for proving that, if the sum (40) is continuous, then the coefficients α_i are all non–vanishing.
Part 4: By construction of the vectors $R^{(i)}$ (see (36), (37), and (39)), we have

$$\int_{\partial\Omega_i} u_i \, ds = 0, \quad i = 1, \cdots, N-1.$$

Let $\{c_i \in \{1, -1\}\}$ be a two–coloring of $\mathcal{G}_H$. Using (33), we deduce that, along the generic edge $E_{ij} \in \mathcal{E}_H$, the tangential traces u_i and u_j have the same sign. Using (13), we then have

$$\sum_{i=1}^{N} c_i \gamma_i^{\delta} \int_{\partial\Omega_i} u_i \, ds = 0,$$

since the contributions on the generic edge E_{ij} cancel. We finally find that

$$\int_{\partial\Omega_N} u_N \, ds = -c_N \gamma_N^{-\delta} \sum_{i=1}^{N-1} c_i \gamma_i^{\delta} \int_{\partial\Omega_i} u_i \, ds = 0.$$

□

We define the matrix R by

$$R := \begin{cases} \left[R^{(1)}\; R^{(2)}\; \cdots\; R^{(N-1)}\right], & \text{if } \mathcal{G}_H \text{ is two–colorable,} \\ \left[R^{(1)}\; R^{(2)}\; \cdots\; R^{(N)}\right], & \text{otherwise .} \end{cases} \tag{41}$$

Lemma 5.2 ensures that Assumption 4.1 holds

THEOREM 5.1. *Let R be defined in (41). Then Assumption 4.1 holds.*

Before showing that Assumption 4.2 also holds, we need a decomposition lemma for the tangential traces on the boundary of a substructure. The proof can be carried out using a similar argument as that for the stronger result contained in [37, Lem. 4.4], and for completeness is given in Appendix A.

LEMMA 5.3. *Let Ω_i be a substructure and let $\{w_{ij},\ j \in \mathcal{I}_i\}$ be functions in W_i, which vanish on $\partial\Omega_i \setminus E_{ij}$. Let*

$$w_i := \sum_{j \in \mathcal{I}_i} w_{ij}.$$

Then there exists a constant C, independent of h and H, such that

$$\|w_{ij}\|^2_{-\frac{1}{2};\partial\Omega_i} \le C(1+\log(H/h))^2 \|w_i\|^2_{-\frac{1}{2};\partial\Omega_i}. \tag{42}$$

We are now ready to prove the following theorem.

THEOREM 5.2. *Let the matrix R be defined as in (41). Then Assumption 4.2 holds with*

$$\omega = C\,\eta\,\left(1+\log\left(\frac{H}{h}\right)\right)^2,$$

where

$$\eta := \max_{i=1,\cdots,N} \max\left\{\frac{\gamma_i}{\beta_i}, \frac{\gamma_i H_i^2}{a_i}\right\},$$

and where C is independent of h, H, the coefficients a and A, and the parameter δ.

Proof. Let $Z = Range(R)$. Consider an arbitrary function $w \in W$, such that

$$z^t B^t B w = 0, \quad z \in Z.$$

We need to bound the quantity

$$|u|_S^2 = \sum_{i=1}^N |u^{(i)}|^2_{S^{(i)}} = \sum_{i=1}^N a_{\Omega_i}(\mathcal{H}_i u_i, \mathcal{H}_i u_i),$$

where $u := P_D w$. We consider an arbitrary substructure Ω_i. Lemma 5.2 and Definition 5.1 ensure that $u_i \in W_i^0$, for all $i = 1,\cdots,N$. Using (35) and (33), we find

$$\begin{aligned} a_{\Omega_i}(\mathcal{H}_i u_i, \mathcal{H}_i u_i) &\le C\,\gamma_i\,\|u_i\|^2_{-\frac{1}{2};\partial\Omega_i} \le C\,\gamma_i \left\| \sum_{j\in\mathcal{I}_i} \mu_j^\dagger (w_i + w_j)\vartheta_{ij} \right\|^2_{-\frac{1}{2};\partial\Omega_i} \\ &\le C \sum_{j\in\mathcal{I}_i} \frac{\gamma_j^{2\delta}\gamma_i}{(\gamma_i^\delta+\gamma_j^\delta)^2} \|\vartheta_{ij} w_i\|^2_{-\frac{1}{2};\partial\Omega_i} + C \sum_{j\in\mathcal{I}_i} \frac{\gamma_j^{2\delta}\gamma_i}{(\gamma_i^\delta+\gamma_j^\delta)^2} \|\vartheta_{ij} w_j\|^2_{-\frac{1}{2};\partial\Omega_i}, \end{aligned} \tag{43}$$

where C is independent of H_i, and ϑ_{ij} is the characteristic function of E_{ij} in $\partial\Omega_i$. We first consider the first term on the right of (43). Using Lemma 5.3, we find

$$\begin{aligned} &\sum_{j\in\mathcal{I}_i} \frac{\gamma_j^{2\delta}\gamma_i}{(\gamma_i^\delta+\gamma_j^\delta)^2} \|\vartheta_{ij} w_i\|^2_{-\frac{1}{2};\partial\Omega_i} \\ &\quad \le C\,(1+\log(H/h))^2\,\gamma_i\,\|w_i\|^2_{-\frac{1}{2};\partial\Omega_i} \sum_{j\in\mathcal{I}_i} \frac{\gamma_j^{2\delta}}{(\gamma_i^\delta+\gamma_j^\delta)^2}. \end{aligned} \tag{44}$$

In order to bound the the second term on the right of (43), we use Lemma 5.3 and [33, Lem. 5.5.2]. We can then write

$$\begin{aligned} \sum_{j\in\mathcal{I}_i} \frac{\gamma_j^{2\delta}\gamma_i}{(\gamma_i^\delta+\gamma_j^\delta)^2} \|\vartheta_{ij} w_j\|^2_{-\frac{1}{2};\partial\Omega_i} &\le C \sum_{j\in\mathcal{I}_i} \frac{\gamma_j^{2\delta}\gamma_i}{(\gamma_i^\delta+\gamma_j^\delta)^2} \|\vartheta_{ij} w_j\|^2_{-\frac{1}{2};\partial\Omega_j} \\ &\le C\,(1+\log(H/h))^2 \sum_{j\in\mathcal{I}_i} \frac{\gamma_j^{2\delta}\gamma_i}{(\gamma_i^\delta+\gamma_j^\delta)^2\gamma_j}\,\gamma_j \|w_j\|^2_{-\frac{1}{2};\partial\Omega_j}, \end{aligned} \tag{45}$$

where, with an abuse of notation, we have also denoted by ϑ_{ij} the characteristic function of E_{ij} on $\partial\Omega_j$. We combine (43), (44), and (45), and sum over the substructures Ω_j. By noting that, for a generic substructure Ω_i, exactly $(1 + |\mathcal{I}_i|)$ terms in $\|w_i\|^2_{-\frac{1}{2};\partial\Omega_i}$ contribute to this sum, we obtain

$$|u|_S^2 \leq C\,(1+\log(H/h))^2 \sum_{i=1}^{N} \left(\gamma_i \, \|w_i\|^2_{-\frac{1}{2};\partial\Omega_i} \sum_{j\in\mathcal{I}_i} \frac{\gamma_j^{2\delta}\gamma_i + \gamma_i^{2\delta}\gamma_j}{(\gamma_i^\delta + \gamma_j^\delta)^2\gamma_i} \right).$$

It can be easily checked that the generic term

$$\frac{\gamma_j^{2\delta}\gamma_i + \gamma_i^{2\delta}\gamma_j}{(\gamma_i^\delta + \gamma_j^\delta)^2\gamma_i},$$

is a homogeneous function of degree zero of γ_i and γ_j, that can be bounded by 2, uniformly in $\gamma_i > 0$, $\gamma_j > 0$, and $\delta \geq 1/2$. We then obtain

$$|u|_S^2 \leq C\,(1+\log(H/h))^2 \sum_{i=1}^{N} \gamma_i \, \|w_i\|^2_{-\frac{1}{2};\partial\Omega_i}.$$

The proof is completed by employing the trace estimate (4). $\square$

We remark that the same constant η also appears in the estimates for other substructuring methods for the same problem; see [36, 37, 33, 35].

The estimate given in Theorem 5.2 remains uniformly valid when the coefficient matrix A tends to zero, but becomes unbounded when a becomes small. This situation occurs when time–dependent Maxwell's equations are considered. It corresponds to the case when the time step approaches zero. The following lemma ensures that in the limit case $a = 0$, the condition number of the FETI preconditioned system is bounded independently of H/h and the jumps of the coefficient A.

LEMMA 5.4. *In the limit case $a = 0$, there is a constant C, independent of h, H, the coefficient matrix A, and the parameter δ, such that Assumption 4.2 holds with*

$$\omega = C\xi,$$

where

$$\xi := \max_{i=1,\cdots,N} \left\{ \frac{\gamma_i}{\beta_i} \right\}.$$

Proof. The proof can be carried out as in Theorem 5.2, by noting that, in this case, the bilinear forms $a_i(\cdot,\cdot)$ are just weighted L^2–scalar products and that (8) can then be employed. We refer to [33, Lem. 5.6.4] and [35, Lem. 6.4] for a proof of a similar result. $\square$

REMARK 5.1. *An analogous FETI method can be also devised for problems involving the bilinear form*

$$\int_\Omega (a \operatorname{div}\mathbf{u} \operatorname{div}\mathbf{v} + A\,\mathbf{u}\cdot\mathbf{v})\,dx, \qquad \mathbf{u},\mathbf{v} \in H(\operatorname{div};\Omega),$$

discretized with the lowest order Raviart–Thomas spaces. Here, $H(\operatorname{div};\Omega)$ is the space of vectors in L^2, with divergence in L^2. Since, in two dimensions, vectors in the

TABLE 1

FETI method. Estimated condition number and number of CG iterations necessary to obtain a relative preconditioned residual $\|z_k\|/\|f\|$ less than 10^{-6} (in parentheses), versus H/h and n. Case of $a = 1$, $b = 1$. The asterisks denote the cases for which we have not enough memory to run the corresponding algorithm.

H/h	32	16	8	4	2
n=32	-	1.529 (3)	2.399 (7)	1.804 (7)	1.299 (5)
n=64	1.801 (4)	3.228 (8)	2.485 (8)	1.807 (7)	1.299 (5)
n=128	4.215 (9)	3.332 (10)	2.487 (8)	1.784 (6)	*
n=192	-	3.348 (10)	2.476 (8)	*	*
n=256	4.341 (11)	3.319 (9)	*	*	*

Raviart–Thomas spaces can be obtained from those in the Nédélec spaces by a rotation of ninety degrees, the unit outward normal vectors $\mathbf{n}_i$ to the boundaries $\partial\Omega_i$, instead of the unit tangent vectors $\mathbf{t}_i$, have to be employed in the construction of the local functions r_i. All the results in this paper remain valid in this case. For Raviart-Thomas discretizations in three dimensions, an analogous method can be defined and all our results, except Part 1 of Lemma 5.2, remain valid in this case as well.

6. Numerical results. We consider the domain $\Omega = (0,1)^2$ and two uniform triangulations $\mathcal{T}_h$ and $\mathcal{T}_H$. The fine triangulation is made of triangles, and the coarse one of squares that are unions of fine triangles. The substructures Ω_i are the elements of the coarse triangulation $\mathcal{T}_H$. The fine triangulation $\mathcal{T}_h$ consists of $2 * n^2$ triangles, with $h = 1/n$. We assume that the coefficient matrix A is diagonal and equal to

$$A = \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix}.$$

We use the value $\delta = 1/2$. In addition, since we have observed no difference in the number of iterations with or without full reorthogonalization, in all our tests, we use the standard two–term recurrence implementation for the conjugate gradient method, as described in section 3.

In Table 1, we show the estimated condition number and the number of iterations to obtain a relative preconditioned residual $\|z_k\|/\|f\|$ less than 10^{-6}, as a function of the dimensions of the fine and coarse meshes. Here, z_k is the $k-$th preconditioned residual as defined in the algorithmic description given in section 3. For a fixed ratio H/h, the condition number and the number of iterations are quite insensitive to the dimension of the fine mesh.

In Table 2, we show some results when the ratio of the coefficients b and a change. For a fixed value of $n = 128$ and $a = 1$, the estimated condition number and the number of iterations are shown as a function of H/h and b. In accordance with Theorem 5.2, the condition number is independent of the ratio b/a, when $b/a \leq 1$. Table 2 also shows that, in practice, this holds for $b/a \geq 1$ as well, and that, when b/a is very large, the condition number tends to be independent of H/h. Our numerical results thus confirm our analysis of the limit case $a = 0$, in Lemma 5.4.

We finally consider a case where the coefficients have jumps. In Table 3, we show some results when the coefficient b has jumps across the interface. We consider the checkerboard distribution shown in Figure 3, where b is equal to b_1 in the shaded area and to b_2 elsewhere. For a fixed value of $n = 128$, $b_1 = 100$, and $a = 1$, the estimated condition number and the number of iterations are shown as a function of H/h and b_2.

TABLE 2

FETI method. Estimated condition number and number of CG iterations necessary to obtain relative preconditioned residual ($||z_k||/||f||$) less than 10^{-6} (in parentheses), versus H/h and b. Case of $n = 128$ and $a = 1$.

H/h	4	8	16
b=0.0001	1.782 (6)	2.49 (8)	3.337 (10)
b=0.001	1.782 (6)	2.49 (8)	3.337 (10)
b=0.01	1.782 (6)	2.49 (8)	3.336 (10)
b= 0.1	1.782 (6)	2.49 (8)	3.336 (10)
b= 1	1.784 (6)	2.487 (8)	3.332 (10)
b= 10	1.788 (7)	2.47 (8)	3.307 (10)
b= 100	1.764 (7)	2.407 (8)	3.103 (10)
b= 1000	1.701 (6)	2.081 (7)	2.232 (7)
b=1e+04	1.356 (5)	1.382 (4)	1.387 (4)
b=1e+05	1.012 (2)	1.015 (2)	1.015 (2)
b=1e+06	1.04 (3)	1.037 (2)	1.037 (2)

TABLE 3

FETI method. Checkerboard distribution for b: (b_1, b_2). Estimated condition number and number of CG iterations to obtain a relative preconditioned residual ($||z_k||/||f||$) less than 10^{-6} (in parentheses), versus H/h and b_2. Case of $n = 128$, $a = 1$, and $b_1 = 100$.

H/h	4	8	16
b2=0.0001	4.116 (17)	5.987 (22)	8.416 (26)
b2=0.001	4.095 (16)	5.96 (20)	8.374 (25)
b2= 0.01	4.04 (15)	5.882 (19)	8.249 (23)
b2= 0.1	3.876 (13)	5.648 (17)	7.909 (21)
b2= 1	3.445 (12)	5.018 (15)	6.994 (18)
b2= 10	2.577 (9)	3.733 (12)	5.158 (14)
b2= 100	1.764 (7)	2.407 (8)	3.103 (10)
b2= 1000	2.506 (9)	3.37 (11)	3.988 (12)
b2=1e+04	2.737 (10)	3.094 (11)	3.515 (11)
b2=1e+05	2.196 (9)	2.73 (10)	3.355 (11)
b2=1e+06	2.089 (9)	2.653 (10)	3.336 (12)

For $b_2 = 100$, the coefficient b has a uniform distribution, and this corresponds to a minimum for the condition number and the number of iterations. When b_2 decreases or increases, the condition number and the number of iterations also increase, but they can still be bounded independently of b_2.

In Table 4, we show some results when the coefficient a has jumps. We consider the checkerboard distribution shown in Figure 3, where a is equal to a_1 in the shaded area and to a_2 elsewhere. For a fixed value of $n = 128$, $a_1 = 0.01$, and $b = 1$, the estimated condition number and the number of iterations are shown as a function of H/h and a_2. We remark that for $a_2 = 0.01$, the coefficient a has a uniform distribution. A slight increase in the number of iterations and the condition number is observed, when a_2 is decreased or increased and when H/h is large.

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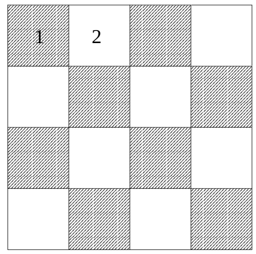

FIG. 3. *Checkerboard distribution of the coefficients in the unit square.*

TABLE 4
Checkerboard distribution for a: (a_1, a_2). Estimated condition number and number of CG iterations to obtain a relative preconditioned residual ($\|z_k\|/\|f\|$) less than 10^{-6} (in parentheses), versus H/h and a_2. Case of $n = 128$, $b = 1$, and $a_1 = 0.01$.

H/h	4	8	16
a2=1.e-07	2.799 (8)	4.492 (12)	7.286 (15)
a2=1.e-06	2.409 (8)	3.812 (11)	6.208 (14)
a2=1.e-05	1.817 (7)	2.651 (9)	4.048 (11)
a2=1.e-04	1.794 (7)	2.448 (8)	3.229 (10)
a2=1.e-03	1.784 (7)	2.419 (8)	3.072 (9)
a2=1.e-02	1.764 (7)	2.4 (8)	3.247 (10)
a2=1.e-01	1.772 (7)	2.407 (8)	3.103 (10)
a2=1.e+00	1.774 (7)	2.458 (8)	3.265 (10)
a2=1.e+01	1.774 (7)	2.458 (8)	3.265 (10)
a2=1.e+02	1.774 (7)	2.458 (8)	3.265 (10)
a2=1.e+03	1.774 (7)	2.458 (8)	3.265 (10)

Appendix. Proof of Lemma 5.3.

We will use the same argument as that for the stronger result contained in [37, Lem. 4.4]. We will first recall some technical tools; they were originally developed for the three dimensional case in [37], and then adapted to two dimensions in [33, Sect. 5.5].

We consider an arbitrary substructure Ω_i. We will first replace the $H^{-1/2}$–norm in W_i by an equivalent one, by considering the supremum over a finite dimensional space of $H^{1/2}$. We define $V_h(\partial\Omega_i) \subset H^{1/2}(\partial\Omega_i)$ as

$$V_h(\partial\Omega_i) := Q_h(\partial\Omega_i) + B_h(\partial\Omega_i).$$

Here $Q_h(\partial\Omega_i)$ is the space of all continuous piecewise linear functions and $B_h(\partial\Omega_i)$ a space of bubble functions vanishing on the edges $e \in \mathcal{E}_h$, $e \subset \partial\Omega_i$:

$$\begin{aligned} Q_h(\partial\Omega_i) &:= \{\phi \in C^0(\partial\Omega_i),\ \phi_{|_e} \in \mathbb{P}_1(e), e \subset \partial\Omega_i,\ e \in \mathcal{E}_h\}, \\ B_h(\partial\Omega_i) &:= \{\psi \in C^0(\partial\Omega_i),\ \psi_{|_e} = \alpha_e\phi_1\phi_2, e \subset \partial\Omega_i,\ e \in \mathcal{E}_h, \alpha_e \in \mathbb{R}\}, \end{aligned}$$

where ϕ_1 and ϕ_2 are the nodal basis functions that span $\mathbb{P}_1(e)$ on the edge e.

The space $V_h(\partial\Omega_i)$ satisfies the following property; we refer to [37, Lem. 4.2] for a proof.

LEMMA A.1. *There exists a constant C, which depends only on the aspect ratio of Ω_i, such that for each $\phi \in Q_h(\partial\Omega_i)$ and $\psi \in B_h(\partial\Omega_i)$, the following equivalence* *holds*

$$\|\phi + \psi\|_{\frac{1}{2};\partial\Omega_i} \leq \|\phi\|_{\frac{1}{2};\partial\Omega_i} + \|\psi\|_{\frac{1}{2};\partial\Omega_i} \leq C\|\phi + \psi\|_{\frac{1}{2};\partial\Omega_i}. \tag{46}$$

The following lemma introduces an equivalent norm in W_i and can be proved similarly to [37, Lem. 4.2].

LEMMA A.2. *There exist constants, c and C, such that, for all $u \in W_i$, we have*

$$c \sup_{\substack{\phi \in V_h(\partial\Omega_i) \\ \phi \neq 0}} \frac{\langle u, \phi \rangle}{\|\phi\|_{\frac{1}{2};\partial\Omega_i}} \leq \|u\|_{-\frac{1}{2};\partial\Omega_i} \leq C \sup_{\substack{\phi \in V_h(\partial\Omega_i) \\ \phi \neq 0}} \frac{\langle u, \phi \rangle}{\|\phi\|_{\frac{1}{2};\partial\Omega_i}}.$$

We are now ready to prove Lemma 5.3. Consider a generic edge $E_{ij} \in \mathcal{E}_H$, with $j \in \mathcal{I}_i$. Using Lemmas A.2 and A.1, we have

$$||w_{ij}||_{-\frac{1}{2};\partial\Omega_i} \leq C \left(\sup_{\substack{\phi \in Q_h(\partial\Omega_i) \\ \phi \neq 0}} \frac{\langle w_{ij}, \phi \rangle}{\|\phi\|_{\frac{1}{2};\partial T}} + \sup_{\substack{\psi \in B_h(\partial\Omega_i) \\ \psi \neq 0}} \frac{\langle w_{ij}, \psi \rangle}{\|\psi\|_{\frac{1}{2};\partial\Omega_i}} \right). \tag{47}$$

We then decompose ψ into the sum of terms ψ_{ij} supported on individual edges $E_{ij} \subset \partial\Omega_i$

$$\psi = \sum_{j \in \mathcal{I}_i} \psi_{ij}. \tag{48}$$

Similarly, we decompose ϕ into the sum of contributions supported on individual edges. Let χ_{ij} be a continuous piecewise linear function on $\partial\Omega_i$ that vanishes outside E_{ij} and is identically one at every interior mesh point in E_{ij}. We define

$$\phi_{ij} := \chi_{ij}\phi, \quad j \in \mathcal{I}_i,$$

and the remainder ϕ_w

$$\phi_w := \phi - \sum_{j \in \mathcal{I}_i} \phi_{ij}. \tag{49}$$

We remark that ϕ_w is the sum of different contributions, one for each vertex of $\partial\Omega_i$, and has support contained in the union of the two fine edges in $\mathcal{E}_h$ that end at that vertex.

Local inverse estimates combined with interpolation arguments easily give

$$\|\psi_{ij}\|^2_{\frac{1}{2};\partial\Omega_i} \leq C\|\psi\|^2_{\frac{1}{2};\partial\Omega_i}. \tag{50}$$

Similar arguments give

$$\|\phi_w\|^2_{\frac{1}{2};\partial\Omega_i} \leq C\frac{1}{h}\|\phi_w\|^2_{0;\partial\Omega_i} \leq C\|\phi_w\|^2_{L^\infty(\partial\Omega_i)}. \tag{51}$$

Using [8, Lem. 3.3], we have,

$$\|\phi_w\|^2_{L^\infty(\partial\Omega_i)} \leq C(1 + \log(H/h))\ \|\phi\|^2_{\frac{1}{2};\partial\Omega_i}. \tag{52}$$

Combining (51) and (52), we find

$$\|\phi_w\|^2_{\frac{1}{2};\partial\Omega_i} \leq C(1 + \log(H/h))\ \|\phi\|^2_{\frac{1}{2};\partial\Omega_i}. \tag{53}$$

Using the inequality $\|q\|_{1;\Omega_i} \le C\|q\|^2_{1/2;\partial\Omega_i}$, which is valid for discrete harmonic functions, and the same argument as in [31, p. 172, Th. 3], we have,

$$\|\phi_{ij}\|^2_{\frac{1}{2};\partial\Omega_i} \le C(1+\log(H/h))^2\|\phi\|^2_{\frac{1}{2};\partial\Omega_i}. \tag{54}$$

By using the splitting (49), we find

$$\begin{aligned} \langle w_{ij},\phi\rangle &= \sum_{k\in\mathcal{I}_i}\langle w_{ij},\phi_{ik}\rangle + \langle w_{ij},\phi_w\rangle \\ &= \langle w_i,\phi_{ij}\rangle + \langle w_{ij},\phi_w\rangle. \end{aligned} \tag{55}$$

The first term on the right side of (55) can be bounded by means of (54)

$$|\langle w_i,\phi_{ij}\rangle| \le C(1+\log(H/h))\,\|\phi\|_{\frac{1}{2};\partial\Omega_i}\,\|w_i\|_{-\frac{1}{2};\partial\Omega_i}. \tag{56}$$

The second term on the right side of (55) can be bounded using the following argument: For each ϕ_w there is a unique $\tilde{\psi}_{ij} \in B_h(\partial\Omega_i)$ that vanishes outside E_{ij}, such that

$$\int_e \phi_w\ ds = \int_e \tilde{\psi}_{ij}\ ds, \quad e\in\mathcal{E}_h,\ e\subset E_{ij}.$$

Moreover, this mapping is continuous

$$\|\tilde{\psi}_{ij}\|^2_{\frac{1}{2};\partial\Omega_i} \le C\frac{1}{h}\|\tilde{\psi}_{ij}\|^2_{0;\partial\Omega_i} \le C\frac{1}{h}\|\phi_w\|^2_{0;\partial\Omega_i} \le C\|\phi_w\|^2_{\frac{1}{2};\partial\Omega_i}.$$

By means of this bound and (53), we finally obtain

$$\begin{aligned} |\langle w_{ij},\phi_w\rangle| = \left|\langle w_{ij},\tilde{\psi}_{ij}\rangle\right| = \left|\langle w_i,\tilde{\psi}_{ij}\rangle\right| &\le C\|w_i\|_{-\frac{1}{2};\partial\Omega_i}\|\phi_w\|_{\frac{1}{2};\partial\Omega_i} \\ &\le C(1+\log(H/h))^{1/2}\|w_i\|_{-\frac{1}{2};\partial\Omega_i}\|\phi\|_{\frac{1}{2};\partial\Omega_i}. \end{aligned} \tag{57}$$

Using (50), we find for the second term on the right hand side of (47)

$$\begin{aligned} \frac{|\langle w_{ij},\psi\rangle|}{\|\psi\|_{\frac{1}{2};\partial\Omega_i}} = \frac{|\langle w_i,\psi_{ij}\rangle|}{\|\psi\|_{\frac{1}{2};\partial\Omega_i}} &\le \frac{\|w_i\|_{-\frac{1}{2};\partial\Omega_i}\ \|\psi_{ij}\|_{\frac{1}{2};\partial\Omega_i}}{\|\psi\|_{\frac{1}{2};\partial\Omega_i}} \\ \le C\frac{\|w_i\|_{-\frac{1}{2};\partial\Omega_i}\ |\psi_{ij}|_{\frac{1}{2};\partial\Omega_i}}{|\psi|_{\frac{1}{2};\partial\Omega_i}} &\le C\|w_i\|_{-\frac{1}{2};\partial\Omega_i}. \end{aligned} \tag{58}$$

The proof is completed by combining (47), (49), (56), (57), and (58).

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