ITERATIVE SUBSTRUCTURING PRECONDITIONERS FOR MORTAR ELEMENT METHODS IN TWO DIMENSIONS

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Abstract. The mortar methods are based on domain decomposition and they allow for the coupling of different variational approximations in different subdomains. The resulting methods are nonconforming but still yield optimal approximations. In this paper, we will discuss iterative substructuring algorithms for the algebraic systems arising from the discretization of symmetric, second order, elliptic equations in two dimensions. Both spectral and finite element methods, for geometrically conforming as well as nonconforming domain decompositions, are studied. In each case, we obtain a polylogarithmic bound on the condition number of the preconditioned matrix.

Key words. domain decomposition, iterative substructuring, mortar finite element method

AMS(MOS) subject classifications. 65F30, 65N22, 65N30, 65N55

1. Introduction. Since the late nineteen eighties, interest has developed in non-overlapping domain decomposition methods coupling different variational approximations in different subdomains. The *mortar element methods*, see [10], have been designed for this purpose and they allow us to combine different discretizations in an optimal way. Optimality means that the error is bounded by the sum of the subregion-by-subregion approximation errors without any constraints on the choice of the different discretizations. One can, for example, couple spectral methods of different polynomial degrees, or spectral methods with finite elements, or different finite element methods with different meshes. Also, the domain partitioning need not be geometrically conforming, *i.e.* the intersection of the closures of two neighboring subdomains may only be parts of certain edges of these subdomains.

The basic ideas of the mortar method can be outlined as follows: the *skeleton* of the decomposition (*i.e.* the union of the subdomains interfaces) is itself partitioned into *mortars*. Each mortar is an entire edge of one of the subdomains; the mortars are disjoint open sets. The chosen local discretizations may force the method to be nonconforming and we only impose a type of weak continuity. For each subdomain Ω_k and for each nonmortar side Γ_k^j of $\partial\Omega_k$, we introduce a carefully chosen discrete space $\tilde{W}_{kh}^j$ of functions supported on Γ_k^j. Weak continuity, in this context, then means that the $L^2(\Gamma_k^j)-$projection of the jump across Γ_k^j into the space $\tilde{W}_{kh}^j$ vanishes. In the first version of the mortar method, strong continuity constraints were also imposed at the vertices of the subdomains but this turned out not to be necessary. A second version of the mortar method, developed and analyzed by Ben Belgacem and Maday [6],[7], does not require such constraints. In particular for problems in three dimensions, the second version offers important advantages over the first and in what follows, we shall exclusively work with this more recently developed method. We note that, in a finite element context, similar nonconforming methods have been studied by Le Tallec et al

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[17],[18],[19] .

Mortar element methods offer many advantages:

• They increase the portability of spectral methods.

• In the context of finite elements, they provide flexibility in the construction of the mesh. For example, they may be used in some cases to avoid updating the finite element mesh (sliding meshes [5]) or, on the contrary, to simplify the adaption of the meshes ([9]).

• They are well suited for parallel computing.

There has already been several implementations of the mortar methods, among them [5] with sliding meshes, [23] for spectral element methods, [18] for a nonconforming finite element method for elasticity problems, and [4] for the Navier Stokes equation.

In the present paper, we propose algorithms for solving the algebraic linear systems arising from the mortar methods. After the elimination, in parallel, of the degrees of freedom internal to the subdomains, there remains to find the traces of the solution on the subdomain boundaries, *i.e.* to solve the *Schur complement system*. In our methods, we work only with the true unknowns of the Schur complement systems, *i.e.* the unknowns associated with the mortars and the vertices of the subdomains. The method presented here can be viewed as a generalization of an iterative substructuring algorithm first introduced by Bramble, Pasciak, and Schatz [12], for two-dimensional conforming discretizations and which was reinterpreted in terms of block-Jacobi methods in [14]. The algorithm consists essentially of decomposing suitably the discrete space into a direct sum of subspaces in such a way that the related block-Jacobi preconditioned conjugate gradient method has a satisfactory rate of convergence. Each mortar can be associated in a natural way with a subspace but, in addition, a global *coarse space* must be included to deal with the low frequency error. We obtain a polylogarithmic bound, in terms of the local number of unknowns, on the number of iterations required for a given accuracy. Therefore, the proposed algorithm can be considered as almost perfectly scalable.

Other algorithms have also been proposed. A Neumann-Neumann preconditioner is studied and tested in [18]. In [2], a saddle point formulation of the system, as in [6], is considered, and iterative methods based on a certain class of preconditioners is suggested. A saddle point algorithm for which the internal degrees of freedom need not be eliminated is proposed in [16]. In [13], a method based on a hierarchical basis representation, cf. [24], is developed and tested for low order mortar finite elements and geometrically conforming decompositions of the regions.

In three dimensions, the preconditioner of this paper is not satisfactory, and another iterative substructuring method has been proposed; see [22]. In addition, an extension of the theory for two-level Schwarz algorithms, using overlapping subregions, has been completed for the mortar finite element case; see [26].

The paper is organized as follows. In Section 2, a brief review is given of the mortar finite element method in the geometrically conforming case. An iterative substructuring preconditioner for that case is proposed and studied in Section 3. The geometrically nonconforming mortar finite element method is discussed in Section 4. Finally, a generalization to the spectral element method in the geometrically conforming and geometrically nonconforming cases is carried out in Section 5.

2. Mortar Element Methods in the Geometrically Conforming Case. Let Ω be a bounded polygonal domain of $\mathbb{R}^2$, and let $\{\Omega_k\}_{k=1}^{K}$ be a partition of Ω

into K non-overlapping open quadrilaterals:

$$\overline{\Omega} = \cup_{k=1}^{K} \overline{\Omega}_k \qquad \text{where} \qquad \Omega_k \cap \Omega_\ell = \emptyset \quad \text{if } k \neq \ell.$$

We make this restriction to polygonal domains and subdomains only to simplify the presentation. The domain decomposition is called *geometrically conforming* if the intersection of the closure of two subdomains is either empty, a vertex, or an entire common edge of the two subdomains. For any $1 \leq k \neq \ell \leq K$, let $\Gamma_{k\ell}$ be the closed straight segment, possibly degenerate, given by $\Gamma_{k\ell} = \overline{\Omega}_k \cap \overline{\Omega}_\ell$. Let us also introduce $\mathcal{V}$ as the set of crosspoints of the domain decomposition which are not on $\partial\Omega$, and the *skeleton*, defined by $\Gamma = \cup_{1 \leq k < \ell \leq K} \Gamma_{k\ell}$.

We assume that the subdomains have uniformly bounded aspect ratios but there is no need to assume that the subdomains form a quasiuniform coarse triangulation.

All what follows concerns the Dirichlet problem for Poisson's equation

$$(1) \qquad \begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

but our results hold for any self-adjoint, elliptic, second order operator.

Families of finite element triangulations $\mathcal{T}_{k,h}$ are associated with the $\Omega_k, 1 \leq k \leq K$, which we assume satisfy the classical shape regularity assumption on the elements. We denote by h_k the maximum diameter of the elements of $\mathcal{T}_{k,h}$. To simplify our analysis, we also assume that the meshes are quasiuniform for each subregion Ω_k. We recall that quasiuniformity for a triangular mesh means that there exist two positive constants τ and σ such that for all triangles T of $\mathcal{T}_{k,h}$, $\tau h_k \leq h_T \leq \sigma \rho_T$. Here h_T is the diameter of T, and ρ_T the diameter of the circle inscribed in T. Let X_{kh} be the related space of piecewise linear continuous finite element functions which vanish on $\partial\Omega$. Denoting by Tr_k the trace operator from Ω_k onto $\partial\Omega_k$, we set $\mathcal{X}_{kh} \equiv Tr_k X_{kh}$. The product spaces X_h and $\mathcal{X}_h$ are defined by:

$$X_h \equiv \prod_{1 \leq k \leq K} X_{kh} \qquad \mathcal{X}_h \equiv \prod_{1 \leq k \leq K} \mathcal{X}_{kh}.$$

If $|\Gamma_{k\ell}| \neq 0$, we also introduce $\mathcal{X}_{k,\ell,h}$ by

$$\mathcal{X}_{k,\ell,h} \equiv \left\{ (Tr_k \mathbf{v}_{kh})_{|\Gamma_{k\ell}}, \quad \mathbf{v}_{kh} \in X_{kh} \right\}.$$

Clearly, $\mathcal{X}_{k,\ell,h}$ is a subspace of the space $\bar{\mathcal{X}}_{k,\ell,h}$ of the piecewise linear continuous functions on the corresponding mesh of $\Gamma_{k\ell}$ ($\mathcal{X}_{k,\ell,h} = \bar{\mathcal{X}}_{k,\ell,h}$ if $\bar{\Gamma}_{k\ell} \cap \partial\Omega = \emptyset$). The dimension of $\bar{\mathcal{X}}_{k,\ell,h}$, denoted by $N_{k\ell} + 2$, equals the number of nodes of $\mathcal{T}_{k,h}$ on $\Gamma_{k\ell}$.

We note that the meshes need not match at the interface between two subdomains. Thus, in order to discretize the space $H_0^1(\Omega)$, we have to introduce, for each $1 \leq k < \ell \leq K$ with $|\Gamma_{k\ell}| > 0$, a space $\tilde{W}_{k,\ell,h}$ of Lagrange multipliers used to impose a weak continuity constraint across $\Gamma_{k\ell}$. A choice has to be made since this space of Lagrange multipliers can be associated with either $\bar{\mathcal{X}}_{k,\ell,h}$ or $\bar{\mathcal{X}}_{\ell,k,h}$. One strategy is always to choose the one of largest dimension, but we emphasize that any other choice can also be supported by existing theory, and that the same asymptotical error bound results in all cases.

In the case where the Lagrange multiplier space $\tilde{W}_{k,\ell,h}$ is based on $\bar{\mathcal{X}}_{k,\ell,h}$, let $\{\phi_i\}, 0 \leq i \leq N_{k\ell} + 1$, be the shape functions of $\mathcal{X}_{k,\ell,h}$ associated with the nodes of $\mathcal{T}_{k,h}$ on $\Gamma_{k\ell}$, with ϕ_0 and $\phi_{N_{k\ell}+1}$ associated with the endpoints of $\Gamma_{k\ell}$. Then, $\tilde{W}_{k,\ell,h}$ is

FIG. 1. *Shape functions of the spaces $\bar{\mathcal{X}}_{k,\ell,h}$ (left) and $\tilde{W}_{k,\ell,h}$ (right).*

chosen as the space spanned by $(\phi_0 + \phi_1, \phi_2, ..., \phi_i, ..., \phi_{N_{k\ell}-1}, \phi_{N_{k\ell}} + \phi_{N_{k\ell}+1})$; it is a subspace of $\bar{\mathcal{X}}_{k,\ell,h}$ of codimension two. Figure 1 illustrates the construction of $\tilde{W}_{k,\ell,h}$ from $\bar{\mathcal{X}}_{k,\ell,h}$:

It is now possible to define the subspace Y_h of X_h:

$$Y_h \equiv \left\{ \begin{array}{l} \mathbf{v}_h = (\mathbf{v}_{kh})_{1\le k\le K} \in X_h : \\ \forall 1 \le k < \ell \le K, \forall \mu_{k\ell h} \in \tilde{W}_{k,\ell,h}, \quad \int_{\Gamma_{k\ell}} (Tr_k \mathbf{v}_{kh} - Tr_\ell \mathbf{v}_{\ell h}) \mu_{k\ell h} = 0 \end{array} \right\},$$

and the subspace $\mathcal{Y}_h$ of $\mathcal{X}_h$:

$$\mathcal{Y}_h \equiv \left\{ v_h \in \mathcal{X}_h : \forall 1 \le k < \ell \le K, \forall \mu_{k\ell h} \in \tilde{W}_{k,\ell,h}, \int_{\Gamma_{k\ell}} (v_{kh} - v_{\ell h}) \mu_{k\ell h} = 0 \right\}.$$

Consider an edge $\Gamma_{k\ell}$, $|\Gamma_{k\ell}| > 0$. Assuming that the Lagrange multiplier space $\tilde{W}_{k,\ell,h}$ is built from the mesh $\mathcal{T}_{kh}$, then the nodes of $\mathcal{T}_{kh} \cap \Gamma_{k\ell} \backslash \mathcal{V}$ and $\mathcal{T}_{\ell h} \cap \Gamma_{k\ell} \backslash \mathcal{V}$ are called *slave* and *master* nodes, respectively, because the value of $v_h \in \mathcal{Y}_h$ at any slave node is completely determined by the values at the master nodes and crosspoints. Assuming that $|\Gamma_{k\ell}| > 0$, then the edge $\partial\Omega_k \cap \Gamma_{k\ell}$ is said to be a slave, or nonmortar, and master, or mortar, edge of Ω_k, respectively, if the space $\tilde{W}_{k,\ell,h}$ is based on the mesh $\mathcal{T}_{kh}$ and $\mathcal{T}_{\ell h}$, respectively. We denote by $N_{\mathcal{V}}$ the number of degrees of freedom at the crosspoints of the domain decomposition, and by N_m and N_s the number of master and slave nodes, respectively. Then, the dimension of $\mathcal{X}_h$ and $\mathcal{Y}_h$ are $N_{\mathcal{V}} + N_m + N_s$ and $N_{\mathcal{V}} + N_m$, respectively.

Let $a(\cdot,\cdot) : X_h \times X_h \to \mathbb{R}$, be the bilinear form:

$$a(\mathbf{u}_h, \mathbf{v}_h) \equiv \sum_{k=1}^{K} \int_{\Omega_k} \nabla \mathbf{u}_{kh} \cdot \nabla \mathbf{v}_{kh}.$$

The discretized problem corresponding to (1) is: Find $\mathbf{u}_h \in Y_h$ such that

$$a(\mathbf{u}_h, \mathbf{v}_h) = \sum_{k=1}^{K} \int_{\Omega_k} f\, \mathbf{v}_{kh}, \quad \forall \mathbf{v}_h \in Y_h.$$

It is natural to introduce two subspaces which are orthogonal in the sense of this energy inner product. The first, $X_h^\circ \subset X_h$, consists of functions which vanish on the interfaces, *i.e.* $X_h^\circ \equiv \{\mathbf{v}_h \in X_h : \forall k, 1 \le k \le K, Tr_k \mathbf{v}_{kh} = 0\}$. The other is the subspace of the discrete harmonic extensions $\tilde{\mathbf{u}}_h \in X_h$ of $u_h \in \mathcal{X}_h$, *i.e.* the unique solution $\tilde{\mathbf{u}}_h$ of

$$\begin{aligned} a(\tilde{\mathbf{u}}_h, \mathbf{v}_h) &= 0, \qquad \forall \mathbf{v}_h \in X_h^\circ, \\ \tilde{u}_{kh} &= u_{kh}. \end{aligned} \tag{2}$$

In order to reduce the size of the problem, it is possible to solve, in parallel, a discrete Dirichlet problem for each subdomain, *i.e.* to find $\mathbf{u}_h^\circ \in X_h^\circ$ such that

$$\forall \mathbf{v}_h \in X_h^\circ, \quad a(\mathbf{u}_h^\circ, \mathbf{v}_h) = \sum_{k=1}^{K} \int_{\Omega_k} f\, \mathbf{v}_{kh}.$$

Defining the bilinear form $s_h : \mathcal{X}_h \times \mathcal{X}_h \to I\!R$, corresponding to a discrete Poincaré-Steklov operator S_h:

$$s_h(u_h, v_h) \equiv a(\tilde{\mathbf{u}}_h, \tilde{\mathbf{v}}_h), \tag{3}$$

there remains to find $u_h \in \mathcal{Y}_h$ such that

$$\forall v_h \in \mathcal{Y}_h, \quad s_h(u_h, v_h) = (f, \tilde{\mathbf{v}}_h) - a(\mathbf{u}_h^\circ, \tilde{\mathbf{v}}_h). \tag{4}$$

The solution of (2) is then given by $\mathbf{u}_h = \mathbf{u}_h^\circ + \tilde{\mathbf{u}}_h$.

The goal of the next section is to find a basis of $\mathcal{Y}_h$ for which a block diagonal preconditioner for S_h yields a condition number almost independent of the mesh parameters.

3. Preconditioners for the Geometrically Conforming Mortar Element Method. In the following c and C will denote positive constants uniformly bounded away from 0 and ∞, respectively. They are, in particular, independent of the H_k and h_k, the diameters of the subdomain Ω_k and its elements, and in the spectral case, of the degree of the polynomials.

3.1. Decomposition of the space $\mathcal{Y}_h$. The purpose of this section is to decompose the vector space $\mathcal{Y}_h$ into the direct sum of a coarse space $\mathcal{Y}_H$ of dimension $N_{\mathcal{V}}$ (the number of degrees of freedom associated with the crosspoints) and of a fine space $\mathcal{Y}_h^H$ of dimension N_m (the number of master nodes):

$$\mathcal{Y}_h = \mathcal{Y}_h^H \oplus \mathcal{Y}_H.$$

Here

$$\mathcal{Y}_h^H \equiv \{v_h \in \mathcal{Y}_h : \quad v_h = 0 \quad \text{on } \mathcal{V}\}.$$

A few notations will be needed in order to specify the coarse space $\mathcal{Y}_H$. Let A be a crosspoint and let $\mathcal{K}_A$ denote the set

$$\mathcal{K}_A \equiv \{1 \le k \le K : \quad A \in \overline{\Omega_k}\}.$$

It is clear that $N_{\mathcal{V}} = \sum_{A \in \mathcal{V}}$ cardinal $(\mathcal{K}_A)$. For each crosspoint A, and for each $k \in \mathcal{K}_A$, we define a basis vector $e^{A,k} \in \mathcal{Y}_h$, such that,

1. $e_k^{A,k}(A) = 1$;
2. for all vertices $B \neq A$ of Ω_k, $e_k^{A,k}(B) = 0$;
3. for all $\ell \neq k$, and for all vertices B of Ω_ℓ, $e_\ell^{A,k}(B) = 0$;
4. $e^{A,k}$ is linear on master edges.

For a given crosspoint $A \in \bar{\Omega}_k$, $e^{A,k}$ clearly vanishes on all edges except those which have A as an endpoint. Let A and B be the endpoints of $\Gamma_{k\ell}, |\Gamma_{k\ell}| > 0$. In the case where $\Gamma_{k\ell}$ is a master side of $\partial\Omega_k$, the restriction of $e_k^{A,k}$ to $\Gamma_{k\ell}$ is the linear function $\tilde{e}^{k,\ell,A}$ with the value 1 at A and 0 at B. The restriction of $e_l^{A,k}$ to $\Gamma_{k\ell}$ is the unique function $\tilde{e}^{\ell,k,A} \in \mathcal{X}_{\ell,k,h}$ such that

$$\begin{array}{rl} & \tilde{e}^{\ell,k,A}(A) = \tilde{e}^{\ell,k,A}(B) = 0, \\ \forall \mu_{\ell k h} \in \tilde{W}_{\ell,k,h}, & \int_{\Gamma_{k\ell}} (\tilde{e}^{\ell,k,A} - \tilde{e}^{k,\ell,A})\mu_{\ell k h} = 0 \end{array} \quad . \tag{5}$$

If conversely, $\Gamma_{k\ell}$ is a slave side of $\partial\Omega_k$, the restriction of $e_l^{A,k}$ to $\Gamma_{k\ell}$ is 0, while the restriction of $e_k^{A,k}$ to $\Gamma_{k\ell}$ is the unique function $\tilde{e}^{k,\ell,A}$ in $\mathcal{X}_{k,\ell,h}$ such that,

$$\tilde{e}^{k,\ell,A}(A) = 1, \quad \tilde{e}^{k,\ell,A}(B) = 0, \quad \forall \mu_{k\ell h} \in \tilde{W}_{k,\ell,h}, \quad \int_{\Gamma_{k\ell}} \tilde{e}^{k,\ell,A} \mu_{k\ell h} = 0, \tag{6}$$

The coarse space $\mathcal{Y}_H$ is defined by

$$\mathcal{Y}_H \equiv \{e^{A,k}; \quad A \in \mathcal{V}, k \in \mathcal{K}_A\}^{span}. \tag{7}$$

It is clear that the dimension of $\mathcal{Y}_H$ is $N_{\mathcal{V}}$.

In what follows, it will be necessary to have accurate estimates of certain Sobolev norms of the basis vectors of the coarse space $\mathcal{Y}_H$.

LEMMA 1. *Let A be a crosspoint and $k, \ell \in \mathcal{K}_A$, with $|\Gamma_{k\ell}| > 0$. Assume that $\partial\Omega_k \cap \Gamma_{k\ell}$ is a slave side of Ω_k and let $\tilde{e}^{k,\ell,A}$ be the unique function in $\mathcal{X}_{k,\ell,h}$ defined by (6). Then,*

$$||\tilde{e}^{k,\ell,A}||_{L^2(\Gamma_{k\ell})} \leq C\sqrt{h_k}, \tag{8}$$

$$||\tilde{e}^{k,\ell,A}||_{H^1(\Gamma_{k\ell})} \leq C\frac{1}{\sqrt{h_k}}, \tag{9}$$

$$||\tilde{e}^{k,\ell,A}||_{L^\infty(\Gamma_{k\ell})} \leq C. \tag{10}$$

Proof. Let $\tilde{E}$ be the vector of coordinates of $\tilde{e}^{k,\ell,A}$ in the previously described basis $\{\phi_i\}_{0\leq i\leq N_{k\ell}+1}$ of shape functions of $\mathcal{X}_{k,\ell,h}$ associated to the nodes of $\mathcal{T}_{k,h}$ which lie on $\Gamma_{k\ell}$. Using the boundary values for $\tilde{e}^{k,\ell,A}$, we find that the vector $\tilde{E} \equiv (E_1, \ldots, E_{N_{k\ell}})^T$ satisfies:

$$\tilde{B}\tilde{E} = F,$$

where $F = (-1/2l_0, 0, \ldots, 0)^T$ and

$$\tilde{B} \equiv \begin{pmatrix} \frac{1}{2}l_0 + \frac{1}{3}l_1 & \frac{1}{6}l_1 & & & & \\ \frac{1}{6}l_1 & \frac{1}{3}(l_1+l_2) & \frac{1}{6}l_2 & & & \\ \ddots & \ddots & \ddots & & & \\ & \frac{1}{6}l_i & \frac{1}{3}(l_i+l_{i+1}) & \frac{1}{6}l_{i+1} & & \\ & & \ddots & \ddots & \ddots & \\ & & & \frac{1}{6}l_{N_{kl}-2} & \frac{1}{3}(l_{N_{kl}-2}+l_{N_{kl}-1}) & \frac{1}{6}l_{N_{kl}-1} \\ & & & & \frac{1}{6}l_{N_{kl}-1} & \frac{1}{3}l_{N_{kl}-1} + \frac{1}{2}l_{N_{kl}} \end{pmatrix}.$$

Here l_i is the length of the i-th mesh interval of $\Gamma_{k\ell}$. Since the mesh $\mathcal{T}_{kh}$ is quasiuniform, $\tilde{B}$ is spectrally equivalent to the diagonal matrix $D \equiv h_k I$ and therefore the Euclidean norm of the vector $\tilde{E}$ is of order 1, since the Euclidean norm of F is of order h_k. Therefore, (8) is proved. The next inequality, (9), now follows by using a well known inverse inequality for quasiuniform meshes. Finally, (10) is obtained from (8) and (9) and the Gagliardo-Nirenberg interpolation inequality. □

REMARK 1. *In the same way, we can also prove the same estimates for the function $\tilde{e}^{\ell,k,A} - \tilde{e}^{k,\ell,A}$ when the side $\partial\Omega_k \cap \Gamma_{k\ell}$ is a master side of Ω_k.*

REMARK 2. *From (10) and Remark 1, it follows immediately that $\forall A \in \mathcal{V}, \forall k \in \mathcal{K}_A$,*

$$\forall \ell \in \{1, \dots K\}, \quad \|e^{A,k}\|_{L^\infty(\partial\Omega_\ell)} \leq C. \tag{11}$$

Denoting by $|.|_{1/2,*}$ the product semi-norm on $\prod_{1\leq k\leq K} H^{1/2}(\partial\Omega_k)$, it will prove useful to have bounds of $|e^{A,k}|_{1/2,*}$ for any $A \in \mathcal{V}$ and $k \in \mathcal{K}_A$. Three cases can be distinguished:

1. Both sides of Ω_k adjacent to A are slave sides.
2. Both sides of Ω_k adjacent to A are master sides.
3. One side of Ω_k adjacent to A is a slave side, the other a master side.

In the third case, we have the following result.

LEMMA 2. *Let A be a crosspoint and let $k, \ell, m \in \mathcal{K}_A$, with $|\Gamma_{k\ell}| > 0$ and $|\Gamma_{km}| > 0$. Assume that $\partial\Omega_k \cap \Gamma_{k\ell}$ is a slave side of Ω_k and that $\partial\Omega_k \cap \Gamma_{km}$ is a master side of Ω_k. Then,*

$$|e_k^{A,k}|^2_{H^{1/2}(\partial\Omega_k)} \leq C(1 + \log(\frac{H_k}{h_k})).$$

Proof. From the quasiuniformity of the mesh $\mathcal{T}_{kh}$, there exists a constant C such that for all $\epsilon \in (0, 1/2)$,

$$|e_k^{A,k}|^2_{H^{1/2}(\partial\Omega_k)} \leq C h_k^{-2\epsilon} |e_k^{A,k}|^2_{H^{1/2-\epsilon}(\partial\Omega_k)}. \tag{12}$$

Let $f^{k,m,A}$ and $f^{k,\ell,A}$ be the functions on $\partial\Omega_k$, which coincide with $e_k^{A,k}$ on Γ_{km} and $\Gamma_{k\ell}$, respectively, and with 0 on $\partial\Omega_k \backslash \Gamma_{km}$ and and $\partial\Omega_k \backslash \Gamma_{k\ell}$, respectively. It is then clear that at all mesh points which are not vertices,

$$e_k^{A,k} = f^{k,m,A} + f^{k,\ell,A}.$$

The semi-norm $|f^{k,m,A}|^2_{H^{1/2-\epsilon}(\partial\Omega_k)}$ can be computed explicitly, because $f^{k,m,A}$ is piecewise linear, and the following bound is obtained:

$$|f^{k,m,A}|^2_{H^{1/2-\epsilon}(\partial\Omega_k)} \leq C \frac{1}{\epsilon} H_k^{2\epsilon}. \tag{13}$$

It now follows from (8) and an inverse inequality that

$$|f^{k,\ell,A}|^2_{H^{1/2-\epsilon}(\partial\Omega_k)} \leq C h_k^{2\epsilon}. \tag{14}$$

Choosing $\epsilon = (1 + \log(\frac{H_k}{h_k}))^{-1}$ and combining (13) and (14), we obtain the desired result by using (12). $\square$

The next lemma is proved in the same way as Lemma 2.

LEMMA 3. *Let A be a crosspoint and let $k, \ell \in \mathcal{K}_A$, with $|\Gamma_{k\ell}| > 0$. Assume that $\partial\Omega_k \cap \Gamma_{k\ell}$ is a master side of Ω_k. Then,*

$$|e_\ell^{A,k}|^2_{H^{1/2}(\partial\Omega_\ell)} \leq C(1 + \log(\frac{H_\ell}{h_\ell})).$$

It is also possible to prove the following result for the first and second cases:

LEMMA 4. *Let A be a crosspoint and let $k, \ell, m \in \mathcal{K}_A$, with $|\Gamma_{k\ell}| > 0$ and $|\Gamma_{km}| > 0$. Assume that the sides $\partial\Omega_k \cap \Gamma_{k\ell}$ and $\partial\Omega_k \cap \Gamma_{km}$ are either both master or both slave sides of Ω_k. Then,*

$$|e_k^{A,k}|^2_{H^{1/2}(\partial\Omega_k)} \leq C.$$

Proof. The result is very easy when both sides are master sides, because $e_k^{A,k}$ is then continuous and piecewise linear on $\partial\Omega_k$. When both sides are slave sides, it follows from Lemma 1 that

$$\|e_k^{A,k}\|^2_{L^2(\partial\Omega_k)} \leq C h_k,$$

and the proof is completed by using an inverse inequality. □

Lemmas 2-4 can be summarized in the following corollary:

COROLLARY 1. *Let A be a crosspoint and let $k \in \mathcal{K}_A$. Then,*

$$|e^{A,k}|^2_{1/2,*} \leq C \max_{\ell \in \mathcal{K}_A}(1 + \log(\frac{H_\ell}{h_\ell})).$$

3.2. A block-Jacobi preconditioner.

Let $\tilde{S}$ be the matrix of S_h in the new basis described above. The matrix $\tilde{S}$ can be written as

$$\tilde{S} = \begin{pmatrix} \tilde{S}_{hh} & \tilde{S}_{hH} \\ \tilde{S}_{hH}^T & \tilde{S}_{HH} \end{pmatrix}. \tag{15}$$

In order to design a preconditioner for $\tilde{S}$, we replace the block $\tilde{S}_{hH}$ by 0, and the block $\tilde{S}_{hh}$ by its block diagonal part with one block for each mortar. The resulting preconditioner is $\hat{S}$ with

$$\hat{S} = \begin{pmatrix} \hat{S}_{hh} & 0 \\ 0 & \tilde{S}_{HH} \end{pmatrix}. \tag{16}$$

In this section, we will develop bounds for the condition number of the preconditioned matrix $\hat{S}^{-1}\tilde{S}$. For that purpose, the following well known result will prove useful: There exist two constants c and C such that

$$\forall v_h \in \mathcal{Y}_h, \quad c|v_h|^2_{1/2,*} \leq s_h(v_h, v_h) \leq C|v_h|^2_{1/2,*};$$

see, e.g., [11], in particular the discussion of an extension theorem for finite element spaces. Let $\hat{s}_h$ be the bilinear form corresponding to the matrix $\hat{S}$. The following lemma gives an upper bound for the eigenvalues of $\hat{S}^{-1}\tilde{S}$:

LEMMA 5. *There exists a positive constant C such that*

$$\forall v_h \in \mathcal{Y}_h, \quad s_h(v_h, v_h) \leq C\hat{s}_h(v_h, v_h).$$

Proof. Consider an element $v_h \in \mathcal{Y}_h$. There then exists a unique pair $(v_h^H, v_H) \in \mathcal{Y}_h^H \times \mathcal{Y}_H$ such that

$$v_h = v_h^H + v_H. \tag{17}$$

Obviously,

$$s_h(v_h, v_h) \le 2(s_h(v_H, v_H) + s_h(v_h^H, v_h^H)). \tag{18}$$

Observing that for any $x \in \Gamma$, there is a uniform bound on the number of subspaces with elements which do not all vanish at x, we deduce that there exists a constant C such that

$$s_h(v_h^H, v_h^H)) \le C \sum_{1 \le k < \ell \le K; |\Gamma_{k\ell}|>0} s_h(v_h^H|_{\Gamma_{k\ell}}, v_h^H|_{\Gamma_{k\ell}})) = C\hat{s}_h(v_h^H, v_h^H).$$

In addition,

$$\hat{s}_h(v_H, v_H) = s_h(v_H, v_H).$$

□

To find a lower bound for the eigenvalues of $\hat{S}^{-1}\tilde{S}$, the following lemma is needed:

LEMMA 6. *There exists a constant C such that*

$$\forall v_h \in \mathcal{Y}_h, \quad \hat{s}_h(v_H, v_H) \le C \max_{\ell \in \mathcal{K}_A} (1 + \log(\frac{H_\ell}{h_\ell}))^2 s_h(v_h, v_h),$$

where $v_H \in \mathcal{Y}_H$ is the coarse space component of v_h.

Proof. Consider a vector $v_h \in \mathcal{Y}_h$, and let $(v_h^H, v_H) \in \mathcal{Y}_h^H \times \mathcal{Y}_H$ be given as in (17). Then,

$$\hat{s}_h(v_H, v_H) \le C|v_H|^2_{1/2,*} = C \sum_{1 \le k \le K} |v_{kH}|^2_{H^{1/2}(\partial\Omega_k)}.$$

Consider specifically the subdomain Ω_ℓ and denote by $\{\mathcal{V}_i\}_{1 \le i \le N_{\mathcal{V},\ell}}$ the vertices of $\partial\Omega_\ell$ and by $\{\Omega_{k_i}\}_{1 \le i \le N_{\mathcal{V},\ell}}$ the subdomains adjacent to Ω_ℓ. We choose a numbering such that $\forall i < N_{\mathcal{V},\ell}$, Γ_{ℓ,k_i} joins the crosspoints $\mathcal{V}_i$ and $\mathcal{V}_{i+1}$.

It is clear that

$$v_{H|\partial\Omega_\ell} = \sum_{1 \le i \le N_{\mathcal{V},\ell}} v_{\ell h}(\mathcal{V}_i) e_\ell^{\mathcal{V}_i,\ell} + \sum_{1 \le i \le N_{\mathcal{V},\ell}} \sum_{k \in \mathcal{K}(\mathcal{V}_i); |\Gamma_{k\ell}|>0} v_{kh}(\mathcal{V}_i) e_\ell^{\mathcal{V}_i,k}.$$

Denote by $w_{\ell H}$ the continuous piecewise linear function on $\partial\Omega_\ell$ which interpolates $v_{\ell h}$ at the $\mathcal{V}_i$. We can then write $v_{\ell H}$ as

$$v_{\ell H} = w_{\ell H} + \sum_{1 \le i \le N_{\mathcal{V},\ell}} \sum_{\substack{k \ne \ell \in \mathcal{K}(\mathcal{V}_i); \\ \Gamma_{k\ell} \text{ is a slave side of } \partial\Omega_\ell}} (v_{kh}(\mathcal{V}_i) - v_{\ell h}(\mathcal{V}_i)) \tilde{e}^{\ell,k,\mathcal{V}_i}. \tag{19}$$

Here $\tilde{e}^{\ell,k,\mathcal{V}_i}$ is defined by (5). Proceeding exactly as in [12], we can prove that

$$|w_{\ell H}|^2_{H^{1/2}(\partial\Omega_\ell)} \le C(1 + \log(\frac{H_\ell}{h_\ell})) \ |v_{\ell h}|^2_{H^{1/2}(\partial\Omega_\ell)}.$$

In addition, since $v_h \in \mathcal{Y}_h$, for $|\Gamma_{k\ell}| > 0$,

$$< v_{\ell h} >_{\Gamma_{k\ell}} = < v_{kh} >_{\Gamma_{k\ell}},$$

where $< v_{kh} >_{\Gamma_{k\ell}}$ denotes the mean value of v_{kh} over the edge $\Gamma_{k\ell}$. Therefore, $\forall i \in \{1, \ldots, N_{\mathcal{V},\ell}\}$, $\forall k \neq \ell \in \mathcal{K}(\mathcal{V}_i)$ such that $\Gamma_{k\ell}$ is a slave side of $\partial\Omega_\ell$,

$$v_{kh}(\mathcal{V}_i) - v_{\ell h}(\mathcal{V}_i) = (v_{kh}(\mathcal{V}_i) - < v_{kh} >_{\Gamma_{k\ell}}) + (< v_{\ell h} >_{\Gamma_{k\ell}} - v_{\ell h}(\mathcal{V}_i)).$$

But, see, e.g., [15],

$$|v_{kh}(\mathcal{V}_i) - < v_{kh} >_{\Gamma_{k\ell}} |^2 \leq C(1 + \log(\frac{H_k}{h_k})) \ \ |v_{kh}|^2_{H^{1/2}(\partial\Omega_k)}, \tag{20}$$

and

$$|v_{\ell h}(\mathcal{V}_i) - < v_{\ell h} >_{\Gamma_{k\ell}} |^2 \leq C(1 + \log(\frac{H_\ell}{h_\ell})) \ \ |v_{\ell h}|^2_{H^{1/2}(\partial\Omega_\ell)}.$$

In addition, from Lemma 3,

$$|\tilde{e}^{\ell,k,\mathcal{V}_i}|^2_{H^{1/2}_{00}(\Gamma_{k\ell})} \leq C(1 + \log(\frac{H_\ell}{h_\ell})),$$

which gives the desired result since Ω_ℓ has a uniformly bounded number of neighbors. □

We can now prove a lower bound for the eigenvalues of $\hat{S}^{-1}\tilde{S}$:

THEOREM 1. *There exists a constant C such that*

$$\forall v_h \in \mathcal{Y}_h, \quad \hat{s}_h(v_h, v_h) \leq C \max_k (1 + \log(\frac{H_k}{h_k}))^2 s_h(v_h, v_h).$$

Proof. Consider a vector $v_h \in \mathcal{Y}_h$, and let $(v_h^H, v_H) \in \mathcal{Y}_h^H \times \mathcal{Y}_H$ be given by (17). It is clear that

$$\begin{array}{rcl} \hat{s}_h(v_h^H, v_h^H) & = & \hat{s}_h(v_h - v_H, v_h - v_H) \\ & \leq & C \displaystyle\sum_{1 \leq k < \ell \leq K; |\Gamma_{k\ell}| > 0} |v_{kh} - v_{kH}|^2_{H^{1/2}_{00}(\Gamma_{k\ell})} + |v_{\ell h} - v_{\ell H}|^2_{H^{1/2}_{00}(\Gamma_{k\ell})}. \end{array}$$

We now focus on the term $|v_{kh} - v_{kH}|^2_{H^{1/2}_{00}(\Gamma_{k\ell})}$. Using exactly the same arguments as in [12],[15], it is possible to bound this expression by

$$C(1 + \log(\frac{H_k}{h_k}))^2 |v_{kh}|^2_{H^{1/2}(\Gamma_{k\ell})} + C(1 + \log(\frac{H_\ell}{h_\ell}))^2 |v_{\ell h}|^2_{H^{1/2}(\Gamma_{k\ell})},$$

which completes the proof of the theorem. To make our paper more self contained, we will outline a proof of this result.

Assume that $\Gamma_{k\ell}$ is the segment $(0, H)$. By the definition of the $H^{1/2}_{00}(\Gamma_{k\ell})$-norm,

$$|v_{kh} - v_{kH}|^2_{H^{1/2}_{00}(\Gamma_{k\ell})} = |v_{kh} - v_{kH}|^2_{H^{1/2}(\Gamma_{k\ell})} +$$

$$\int_{x=0}^{H} \frac{|v_{kh}(x) - v_{kH}(x)|^2}{x} dx + \int_{x=0}^{H} \frac{|v_{kh}(x) - v_{kH}(x)|^2}{H - x} dx. \tag{21}$$

Clearly,

$$|v_{kh} - v_{kH}|^2_{H^{1/2}(\Gamma_{k\ell})} \leq |v_{kh} - v_{kH}|^2_{H^{1/2}(\partial\Omega_k)} \leq 2|v_{kh}|^2_{H^{1/2}(\partial\Omega_k)} + 2|v_{kH}|^2_{H^{1/2}(\partial\Omega_k)},$$

and it is possible to use Lemma 6.

Since the last two terms of (21) are very similar, we concentrate on the first. As in [15], this integral is split into two, over $(0, h_k)$, (h_k, H), respectively. It is easily seen that

$$\int_{h_k}^{H} \frac{|v_{kh}(x) - v_{kH}(x)|^2}{x} dx \leq C(1 + \log(\frac{H}{h_k}))\|v_{kh} - v_{kH}\|^2_{L^\infty(\Gamma_{k\ell})}, \tag{22}$$

and that

$$\int_{0}^{h_k} \frac{|v_{kh}(x) - v_{kH}(x)|^2}{x} dx \leq C\|v_{kh} - v_{kH}\|^2_{L^\infty(\Gamma_{k\ell})}. \tag{23}$$

From (11), it follows that

$$\|v_{kH}\|^2_{L^\infty(\Gamma_{k\ell})} \leq C(\|v_{kh}\|^2_{L^\infty(\Gamma_{k\ell})} + \|v_{\ell h}\|^2_{L^\infty(\Gamma_{k\ell})}). \tag{24}$$

Thus, from (22),(23), and (24),

$$\begin{aligned} \|v_{kh} - v_{kH}\|^2_{H^{1/2}_{00}(\Gamma_{k\ell})} &\leq C(1 + \log(\tfrac{H_k}{h_k}))\|v_{kh}\|^2_{L^\infty(\Gamma_{k\ell})} \\ &+ C(1 + \log(\tfrac{H_\ell}{h_\ell}))\|v_{\ell h}\|^2_{L^\infty(\Gamma_{k\ell})}. \end{aligned}$$

We now use the following very important property of the projection into the coarse space: the component $v_{kH}|_{\Gamma_{k\ell}}$ depends only on $v_{kh}|_{\Gamma_{k\ell}}$ and $v_{lh}|_{\Gamma_{k\ell}}$, and, for any $c \in \mathbb{R}$, $v_{kH}|_{\Gamma_{k\ell}} - c$ is associated through this mapping to $v_{kh}|_{\Gamma_{k\ell}} - c$ and $v_{\ell h}|_{\Gamma_{k\ell}} - c$. Recalling that $< v_{kh} >_{\Gamma_{k\ell}} = < v_{\ell h} >_{\Gamma_{k\ell}}$, and choosing $c = < v_{kh} >_{\Gamma_{k\ell}}$, we find,

$$\begin{aligned} \|v_{kh} - v_{kH}\|^2_{H^{1/2}_{00}(\Gamma_{k\ell})} &= \|v_{kh} - c - (v_{kH} - c)\|^2_{H^{1/2}_{00}(\Gamma_{k\ell})} \\ \leq C(1 + \log(\tfrac{H_k}{h_k}))\|v_{kh} - c\|^2_{L^\infty(\Gamma_{k\ell})} &+ C(1 + \log(\tfrac{H_\ell}{h_\ell}))\|v_{\ell h} - c\|^2_{L^\infty(\Gamma_{k\ell})} \\ \leq C(1 + \log(\tfrac{H_k}{h_k}))^2 |v_{kh}|^2_{H^{1/2}(\Gamma_{k\ell})} &+ C(1 + \log(\tfrac{H_\ell}{h_\ell}))^2 |v_{\ell h}|^2_{H^{1/2}(\Gamma_{k\ell})} \end{aligned}$$

as in (20). □

We can now obtain a bound on the condition number of $\hat{S}^{-1}\tilde{S}$:

THEOREM 2. *There exists a constant C such that*

$$cond(\hat{S}^{-1}\tilde{S}) \leq C \max_{1 \leq k \leq K} (1 + \log(\frac{H_k}{h_k}))^2. \tag{25}$$

REMARK 3. *In order to design a convenient and inexpensive preconditioner, we should replace the blocks of $\hat{S}_{hh}$ in a suitable way. The preconditioners defined above can be simplified in two ways: first the fine space blocks can be replaced by more convenient matrices by using for instance hierarchical bases as described in [24] and [13]. Another possible simplification is crucial for parallelism: it makes sense to replace the block $\hat{S}_{hh}$ of the preconditioner, related to the fine space, by a matrix $\check{S}_h$ corresponding to a bilinear form $\check{s}_h : \mathcal{Y}_h^H \times \mathcal{Y}_h^H \to \mathbb{R}$, constructed as follows: Each $v_h \in \mathcal{Y}_h^H$ is mapped to $\check{v}_h \in \mathcal{X}_h$ given by*

$$\begin{aligned} \check{v}_h = v_h \quad &\textit{on the mortar sides,} \\ \check{v}_h = 0 \quad &\textit{on the nonmortar sides,} \end{aligned} \tag{26}$$

and

$$\check{s}_h(u_h, v_h) \equiv s_h(\check{u}_h, \check{v}_h). \tag{27}$$

For the resulting preconditioner, it is easy to prove, by using the stability result of Ben Belgacem [6], Lemma 1, that the condition number estimate (25) remains valid in the geometrically conforming case. A full discussion will be given, in Subsection 4.3, of the geometrically nonconforming case.

4. Preconditioners for the Geometrically Nonconforming Mortar Element Methods.

4.1. The geometrically nonconforming mortar element method. In this section, we turn to the mortar element method in the case when the decomposition is no longer geometrically conforming. We will still assume that the aspect ratios of the subdomains are bounded by a positive constant, and we recall that H_k is the diameter of the subdomain Ω_k. We also assume that there exists a constant c such that if $|\partial\Omega_k \cap \partial\Omega_\ell| > 0$ then

$$\frac{H_k}{H_\ell} > c. \tag{28}$$

Before formulating the discrete problem, we will adapt some of our previous notations and introduce some new ones. For $k \in \{1, \dots, K\}$, let $\{\Gamma_k^j\}_{1\le j\le j(k)}$ denote the edges of $\partial\Omega_k$. Among the set of all edges $\{\Gamma_k^j;\ 1 \le k \le K,\ 1 \le j \le j(k)\}$, we select a family of mortars $\{\gamma_m\}_{1\le m\le M}$, satisfying the following three conditions:

1. $\underset{1\le n\le M}{\cup} \bar{\gamma}_m = \Gamma$;
2. $\forall (m,n) \in \{1, \dots, M\}^2$, $m \ne n$, $\gamma_m \cap \gamma_n = \emptyset$;
3. $\forall m \in \{1, \dots, M\}$ there exists $k(m), j(m)$ such that $\gamma_m = \Gamma_{k(m)}^{j(m)}$.

Denoting by $\mathcal{X}_{kh}^j$ the vector space of the traces on Γ_k^j of the functions of X_{kh}, we introduce the vector space W_h

$$W_h \equiv \prod_{1\le m\le M} \mathcal{X}_{k(m)h}^{j(m)}.$$

As in the geometrically conforming case, let us introduce the space $\bar{\mathcal{X}}_{kh}^j$ of the piecewise linear continuous functions on the corresponding mesh of Γ_k^j. Then $\tilde{W}_{kh}^j$ denotes the subspace of $\bar{\mathcal{X}}_{kh}^j$ of the functions which are constant in the two end segments of $\mathcal{T}_{kh} \cap \Gamma_k^j$.

The nonconforming approximation of $H_0^1(\Omega)$ is given by the space

$$Y_h \equiv \left\{ \begin{array}{l} \mathbf{v}_h \in X_h; \exists \chi_h \in W_h \text{ such that } \forall k \in \{1, \dots, K\},\ \forall j \in \{1, \dots, j(k)\}, \\ \text{ if } \exists m \text{ such that } (k,j) = (k(m), j(m)), \quad Tr_k v_{kh}|_{\Gamma_k^j} = \chi_h|_{\Gamma_k^j}, \\ \text{ else } \quad \int_{\Gamma_k^j} (Tr_k v_{kh}|_{\Gamma_k^j} - \chi_h)\mu_{kh}^j = 0, \quad \forall \mu_{kh}^j \in \tilde{W}_{kh}^j. \end{array} \right\}.$$

We can also introduce the trace space $\mathcal{Y}_h$

$$\mathcal{Y}_h \equiv Tr(Y_h).$$

As in Section 2, the edge Γ_k^j is called a mortar or master side of Ω_k if there exists $m \in \{1, \dots, M\}$ such that $(k,j) = (k(m), j(m))$, and a nonmortar or slave side of Ω_k otherwise.

Again, the unknowns interior to each subdomain can be eliminated by solving, in parallel, one Dirichlet problem for each subdomain, and we are led to the problem of solving (4). As in the previous section, the goal is to find a basis of $\mathcal{Y}_h$ for which a block-Jacobi preconditioner leads to condition numbers which are almost independent of the size of the subdomains and elements. As in Section 3, the preconditioner will consist of a coarse space block and a block for each mortar.

4.2. Decomposition of the vector space $\mathcal{Y}_h$. As in Subsection 3.1, we decompose the vector space $\mathcal{Y}_h$ into the direct sum of a coarse space $\mathcal{Y}_H$ of dimension $N_{\mathcal{V}}$ (the number of degrees of freedom associated with the crosspoints) and a fine space $\mathcal{Y}_h^H$ of dimension N_m (the number of master nodes). Thus,

$$\mathcal{Y}_h = \mathcal{Y}_h^H \oplus \mathcal{Y}_H, \tag{29}$$

where

$$\mathcal{Y}_h^H \equiv \{v_h \in \mathcal{Y}_h : \quad v_h = 0 \quad \text{on } \mathcal{V}\}.$$

A basis of $\mathcal{Y}_H$ is defined as follows. For each vertex A, and for each $k \in \mathcal{K}_A$, the basis vector $e^{A,k} \in \mathcal{Y}_H$ is fully determined by the following four conditions:

1. $e_k^{A,k}(A) = 1$;
2. for all vertices $B \neq A$ of Ω_k, $e_k^{A,k}(B) = 0$;
3. for all $\ell \neq k$, for all vertices B of Ω_ℓ, $e_\ell^{A,k}(B) = 0$;
4. $e^{A,k}$ is linear on the master edges.

As in Subsection 3.1, the coarse space $\mathcal{Y}_H$ is defined by

$$\mathcal{Y}_H \equiv \{e^{A,k}; \quad A \in \mathcal{V}, k \in \mathcal{K}_A\}^{span}.$$

Consider first a vertex $A \in \Gamma_k^j$ where Γ_k^j is a slave side of Ω_k and let B be the other end point of Γ_k^j. Then,

$$\begin{aligned}
&e_\ell^{A,k}|_{\Gamma_\ell^i} = 0, \quad \forall \ell \neq k, \forall i \in j(\ell) \text{ such that } |\Gamma_\ell^i \cap \Gamma_k^j| > 0;\\
&e_k^{A,k}(A) = 1; \quad e_k^{A,k}(B) = 0;\\
&\textstyle\int_{\Gamma_k^j} e_k^{A,k} \mu_{kh}^j = 0, \quad \forall \mu_{kh}^j \in \tilde{W}_{kh}^j.
\end{aligned}$$

Exactly as in Lemma 1, we can prove that

$$\|e_k^{A,k}\|_{L^2(\Gamma_k^j)} \leq C\sqrt{h_k}, \qquad \|e_k^{A,k}\|_{H^1(\Gamma_k^j)} \leq C\frac{1}{\sqrt{h_k}}, \qquad \|e_k^{A,k}\|_{L^\infty(\Gamma_k^j)} \leq C.$$

Assume now that Γ_k^j is a master side of Ω_k. Let $\ell \neq k$, and let $i \in j(\ell)$ satisfy $|\Gamma_\ell^i \cap \Gamma_k^j| > 0$. We will estimate $\|e_\ell^{A,k} - e_k^{A,k}|_{\Gamma_\ell^i}\|_{L^2(\Gamma_\ell^i)}$.

Let $\tilde{e}_\ell^{A,k}|_{\Gamma_\ell^i}$ be the trivial extension of $e_k^{A,k}$ onto $\Gamma_\ell^i \backslash \Gamma_k^j$. Just as in Lemma 1, and Remark 1, we can prove that

$$\|e_\ell^{A,k} - \tilde{e}_\ell^{A,k}\|_{L^2(\Gamma_\ell^i)} \leq C\sqrt{h_\ell}. \tag{30}$$

To give a flavor of the proof, let us consider the case depicted in Figure 2:

Let C_0 and C_1 be the endpoints of Γ_ℓ^i and let D_0 and D_1 be the endpoints of the mesh segment containing the crosspoint A. We introduce the continuous function $\tilde{e}^{A,k,\ell}$ defined on Γ_ℓ^i, which is piecewise linear on the mesh of Γ_ℓ^i, and satisfies

$$\tilde{e}^{A,k,\ell} = e_k^{A,k} \text{ on } (D_1, C_1), \quad \tilde{e}^{A,k,\ell} = 0 \text{ on } (C_0, D_0), \quad \tilde{e}^{A,k,\ell} \text{ linear on } (D_0, D_1).$$

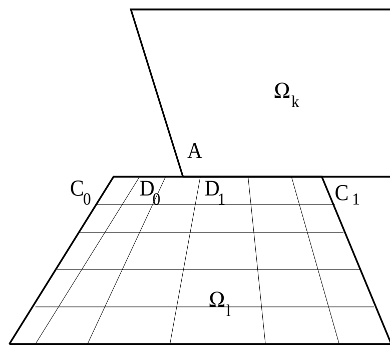

FIG. 2.

In turn, $\tilde{e}^{A,k,\ell}$ is split into the sum of two piecewise linear functions $\tilde{e}_1^{A,k,\ell}$ and $\tilde{e}_2^{A,k,\ell}$ such that

$$\tilde{e}^{A,k,\ell} = \tilde{e}_1^{A,k,\ell} + \tilde{e}_2^{A,k,\ell},$$

where

$$\tilde{e}_1^{A,k,\ell}(C_1) = \tilde{e}^{A,k,\ell}(C_1), \quad \tilde{e}_1^{A,k,\ell} = 0 \text{ on } (C_0, D_1), \quad \tilde{e}_1^{A,k,\ell} \text{ linear on } (D_1, C_1).$$

Let π_h be the L^2-projection onto $\mathcal{X}_{\ell h}^i \cap H_0^1(\Gamma_\ell^i)$ along $\tilde{W}_{\ell h}^i$. It is clear that

$$e_\ell^{A,k} = \tilde{e}_2^{A,k,\ell} + \pi_h(\tilde{e}_1^{A,k,\ell}) + \pi_h(\tilde{e}_\ell^{A,k} - \tilde{e}^{A,k,\ell}).$$

Therefore,

$$\|e_\ell^{A,k} - \tilde{e}_\ell^{A,k}\|_{L^2(\Gamma_\ell^i)} \le \|(I-\pi_h)(\tilde{e}_\ell^{A,k} - \tilde{e}^{A,k,\ell})\|_{L^2(\Gamma_\ell^i)} + \|(I-\pi_h)(\tilde{e}_1^{A,k,\ell})\|_{L^2(\Gamma_\ell^i)}.$$

It is clear that $\|\tilde{e}_\ell^{A,k} - \tilde{e}^{A,k,\ell}\|_{L^2(\Gamma_\ell^i)} \le C\sqrt{h_\ell}$ and from the L^2 stability of π_h, the first term of the right hand side of the inequality above is bounded by $C\sqrt{h_\ell}$. Then, an argument as in Remark 1 yields the same bound for the second term $\|(I-\pi_h)\tilde{e}_1^{A,k,\ell}\|_{L^2(\Gamma_\ell^i)}$.

From the observation (30), it is possible to prove the following lemma in the same way as Lemma 3.

LEMMA 7. *Let A be a crosspoint and let Γ_k^j be a master side of Ω_k with an end point A. Let $\ell \neq k$, and let $i \in j(\ell)$ satisfy $|\Gamma_\ell^i \cap \Gamma_k^j| > 0$. Then,*

$$|e_\ell^{A,k}|^2_{H^{1/2}(\partial\Omega_\ell)} \le C(1 + \log(\frac{H_\ell}{h_\ell})). \tag{31}$$

As in Subsection 3.2, let $\tilde{S}$ be the matrix of S_h in the new basis described above. Again, the matrix $\tilde{S}$ can be described by formula (15) and it is possible to define a block diagonal preconditioner $\hat{S}$ by (16). The bilinear form related to $\hat{S}$ is called $\hat{s}_h$. An upper bound for the eigenvalues of $\hat{S}^{-1}\tilde{S}$ is given by a counterpart of Lemma 5, which is proved exactly as for the geometrically conforming case. To find a lower bound, we have to prove an analogue of Lemma 6:

LEMMA 8. *There exists a positive constant C such that*

$$\forall v_h \in \mathcal{Y}_h, \quad \hat{s}_h(v_H, v_H) \le C \max_k (1 + \log(\frac{H_k}{h_k}))^2 s_h(v_h, v_h),$$

where v_H is the projection of v_h on $\mathcal{Y}_H$ along $\mathcal{Y}_h^H$.

Proof. Consider a vector $v_h \in \mathcal{Y}_h$, and let $(v_h^H, v_H) \in \mathcal{Y}_h^H \times \mathcal{Y}_H$ be given by (17). One can check that

$$\hat{s}_h(v_H, v_H) \leq C|v_H|^2_{1/2,*} = C \sum_{1\leq k\leq K} |v_{kH}|^2_{H^{1/2}(\partial\Omega_k)}.$$

We focus on one subdomain denoted by Ω_0, with sides $(\Gamma_0^j)_{1\leq j\leq j(0)}$. Let w_H be the continuous piecewise linear function on $\partial\Omega_0$ taking the same values as v_{0h} at the vertices of $\partial\Omega_0$. v_{0H} can be rewritten as

$$v_{0H} = w_H + \sum_{\substack{1\leq j\leq j(0),\\ \Gamma_0^j \text{ is a slave side of } \partial\Omega_0}} \tilde{w}_H^j,$$

where the functions $\tilde{w}_H^j$ will be specified below.

Let us focus on a single slave side Γ_0^j of $\partial\Omega_0$, which we call $\gamma = [A, B]$ for convenience. The related space of Lagrange multipliers $\tilde{W}_{0h}^j$ is denoted $\tilde{W}_{\gamma h}$. Let $(\Omega_k)_{1\leq k\leq k(\gamma)}$ be the subdomains such that $|\gamma \cap \partial\Omega_k| > 0$, ordered in such a way that Ω_k and Ω_{k+1} are adjacent. In the rest of the proof, we assume that $k(\gamma) > 1$, but the results also hold for $k(\gamma) = 1$ with a slight modification, which will not be discussed here. Also denote by $\gamma_k = [C_{k-1}, C_k]$ the side of $\partial\Omega_k$ adjacent to γ, as shown in Figure 3:

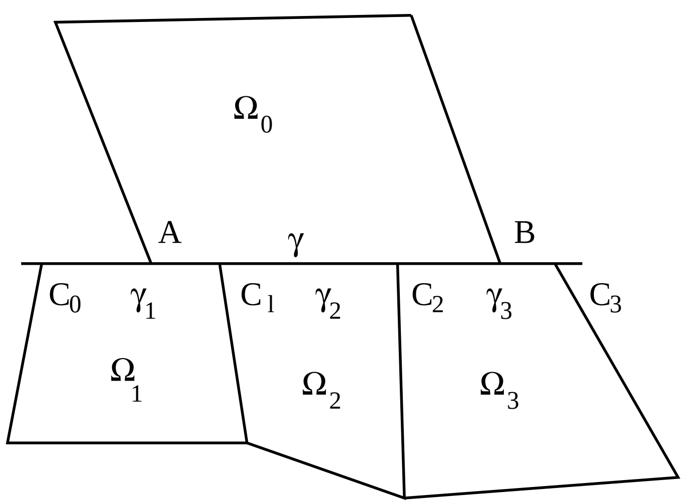

FIG. 3. *The side γ of $\partial\Omega_0$ and adjacent subdomains*

The length of γ and γ_k are called d_γ and d_{γ_k}, respectively.

A calculation shows that

$$\tilde{w}_H^j = \sum_{k=1}^{k(\gamma)} (\alpha_k e_0^{C_{k-1},k} + \beta_k e_0^{C_k,k}),$$

where

$$\alpha_k = v_{kh}(C_{k-1}) - w_H(C_{k-1}),$$

and

$$\beta_k = v_{kh}(C_k) - w_H(C_k),$$

where the function w_H has been extended linearly outside γ. Let us further focus on the term $\beta_k e_0^{C_k,k}$; the term $\alpha_k e_0^{C_{k-1},k}$ can be estimated in the same way. For $0 \le k \le k(\gamma)$, β_k can be rewritten as

$$\beta_k = \frac{d(A,C_k)}{d_\gamma}(v_{kh}(C_k) - v_{0h}(B)) + (1 - \frac{d(A,C_k)}{d_\gamma})(v_{kh}(C_k) - v_{0h}(A)).$$

For simplicity only, we assume that for $0 \le k \le k(\gamma)$, the intersection of γ and γ_k contains the support of at least one basis function of the Lagrange multiplier space related to γ; this hypothesis can be eliminated by decomposing the quantities $v_{kh}(C_k) - v_{0h}(A)$ and $v_{kh}(C_k) - v_{0h}(B)$ further. Then, for $0 \le k \le k(\gamma)$, it is possible to choose a nonnegative $\mu_h \in \tilde{W}_{\gamma h}$ supported in γ_k, such that

$$v_{kh}(C_k) - v_{0h}(A) = \left(v_{kh}(C_k) - \frac{\int_\gamma v_{kh}\mu_h}{\int_\gamma \mu_h} \right) + \left(\frac{\int_\gamma v_{0h}\mu_h}{\int_\gamma \mu_h} - v_{0h}(A) \right),$$

$$v_{kh}(C_k) - v_{0h}(B) = \left(v_{kh}(C_k) - \frac{\int_\gamma v_{kh}\mu_h}{\int_\gamma \mu_h} \right) + \left(\frac{\int_\gamma v_{0h}\mu_h}{\int_\gamma \mu_h} - v_{0h}(B) \right).$$

Therefore, as in the proof of Lemma 6,

$$(32)\quad |(v_{kh} - w_H)(C_k)|^2 \le C(1+\log(\frac{H_k}{h_k}))|v_{kh}|^2_{H^{1/2}(\gamma_k)} + C(1+\log(\frac{H_0}{h_0}))|v_{0h}|^2_{H^{1/2}(\gamma)}.$$

Additionally, because of (28), we can prove the following estimate:

$$(33)\qquad \sum_{k=1}^{k(\gamma)} |e_0^{C_k,k}|^2_{H^{1/2}(\partial\Omega_0)} \le C(1+\log(\frac{H_0}{h_0})),$$

which is slightly stronger than (31).

For $k < k(\gamma)$, the ratio $\frac{d(A,C_k)}{d_\gamma}$ is smaller than one and the estimate

$$|\sum_{k=1}^{k(\gamma)-1} \beta_k e_0^{C_k,k}|^2_{H^{1/2}(\partial\Omega_0)} \le C(1+\log(\frac{H_0}{h_0})) \max_{\ell=0,k}(1+\log(\frac{H_\ell}{h_\ell}))$$

follows from (32) and (33). For $k = k(\gamma)$, the situation is somewhat more difficult because $\frac{d(A,C_k)}{d_\gamma}$ may be large; however, in this case $|e_0^{C_{k(\gamma)},k(\gamma)}|^2_{H^{1/2}(\partial\Omega_0)}$ is bounded by $C\left(\frac{d_\gamma}{d(A,C_{k(\gamma)})}\right)^2(1+\log(\frac{H_0}{h_0}))$. Therefore, we find that

$$|\beta_{k(\gamma)} e_0^{C_{k(\gamma)},k(\gamma)}|^2_{H^{1/2}(\partial\Omega_0)} \le C(1+\log(\frac{H_0}{h_0})) \max_{\ell=0,k}(1+\log(\frac{H_\ell}{h_\ell})).$$

□

Exactly as for the geometrically conforming case, it is now possible to prove the following result:

THEOREM 3. *There exists a constant C such that*

$$cond(\hat{S}^{-1}\tilde{S}) \le C \max_{1\le k\le K}(1+\log(\frac{H_k}{h_k}))^2.$$

4.3. The fine space block $\hat{S}_{hh}$. To simplify the implementation, it makes sense to replace the block $\hat{S}_{hh}$ of the preconditioner, related to the fine space, by a matrix $\check{S}_h$ corresponding to a bilinear form $\check{s}_h : \mathcal{Y}_h^H \times \mathcal{Y}_h^H \to I\!R$, constructed as follows: Each $v_h \in \mathcal{Y}_h^H$ is mapped to $\check{v}_h \in \mathcal{X}_h$ given by

$$\begin{aligned} \check{v}_h &= v_h \quad \text{on the mortar sides,} \\ \check{v}_h &= 0 \quad \text{on the nonmortar sides,} \end{aligned}$$

and

$$\check{s}_h(u_h, v_h) \equiv s_h(\check{u}_h, \check{v}_h).$$

The related preconditioner is called $\check{S}$, and it is easy to prove that

$$\check{s}_h(v_h, v_h) \leq \hat{s}_h(v_h, v_h). \tag{34}$$

To obtain a condition number estimate, we first state the following lemma.

LEMMA 9. *Let Ω_0 be a subdomain and let γ be a nonmortar side of $\partial\Omega_0$. Let $\{\gamma_k\}_{1\leq k\leq k(\gamma)}$ be the mortars adjacent to γ; here γ_k is a whole side of $\partial\Omega_k$. Then, there exists a constant C such that $\forall v_h \in \mathcal{Y}_h^H$,*

$$|v_{0h}|^2_{H^{1/2}_{00}(\gamma)} \leq C \max_{0\leq k\leq k(\gamma)} (1 + \log(\frac{H_k}{h_k}))^2 \sum_{k=1}^{k(\gamma)} |v_{kh}|^2_{H^{1/2}_{00}(\gamma_k)}.$$

Proof. Let the function $\tilde{v}_h$ be defined on $\cup_{1\leq k\leq k(\gamma)}\bar{\gamma}_k$ by the restrictions of v_{kh} to γ_k. Let π be the mortar projection which maps $\tilde{v}_h$ to $v_{0h}|_\gamma$. As in the proof of Lemma 8, we set $\gamma = (A, B)$ and $\gamma_k = (C_{k-1}, C_k)$. Let w be the piecewise linear continuous function which interpolates $\tilde{v}_h$ at $(C_k)_{0\leq k\leq k(\gamma)}$, A and B. Of course, w vanishes at all the C_k and thus depends linearly only on the two parameters $v_{1h}(A)$ and $v_{k(\gamma)h}(B)$. When necessary, we shall use the notation $w(\cdot, \cdot)$.

It is clear that

$$v_{0h}|_\gamma = \pi w + \pi(\tilde{v}_h - w).$$

Let us first bound πw. Using the same arguments as in Lemma 7, we find that the square of the $H^{1/2}_{00}(\gamma)$-norm of $\pi w(0, 1)$ and $\pi w(1, 0)$ are bounded by $C(1 + \log(\frac{H_0}{h_0}))$. Additionally,

$$|v_{1h}(A)|^2 \leq C(1 + \log(\frac{H_1}{h_1}))\|v_{1h}\|^2_{H^{1/2}_{00}(\gamma_1)},$$

$$|v_{k(\gamma)h}(B)|^2 \leq C(1 + \log(\frac{H_{k(\gamma)}}{h_{k(\gamma)}}))\|v_{k(\gamma)h}\|^2_{H^{1/2}_{00}(\gamma_{k(\gamma)})}.$$

Therefore,

$$|\pi w|^2_{H^{1/2}_{00}(\gamma)} \leq C \max_k (1 + \log(\frac{H_k}{h_k}))^2 \sum_{k=1}^{k(\gamma)} \|v_{kh}\|^2_{H^{1/2}_{00}(\gamma_k)}. \tag{35}$$

There remains to estimate $\pi(\tilde{v}_h - w)$. From the stability of the operator π, given in Lemma 1 of [6], we have

$$\|\pi(\tilde{v}_h - w)\|^2_{H^{1/2}_{00}(\gamma)} \leq C \sum_{k=1}^{k_\gamma} \|\tilde{v}_h - w\|^2_{H^{1/2}_{00}(\gamma\cap\gamma_k)}. \tag{36}$$

The right hand side of (36) can be bounded using the same argument as in the proof of Theorem 1, and we obtain

$$\|\tilde{v}_h - w\|^2_{H^{1/2}_{00}(\gamma\cap\gamma_k)} \leq C(1+\log(\frac{H_k}{h_k}))^2 \|v_{kh}\|^2_{H^{1/2}_{00}(\gamma_k)}. \tag{37}$$

The proof of the lemma follows by using (35), (36), and (37). □

The following corollary is an analogue of Lemma 5:

COROLLARY 4.1. *There exists a positive constant C such that*

$$\forall v_h \in \mathcal{Y}_h, \quad s_h(v_h, v_h) \leq C \max_{1\leq k\leq K} (1+\log(\frac{H_k}{h_k}))^2 \check{s}_h(v_h, v_h).$$

The next result follows from Corollary 4.1, the analogue of Theorem 1 for the geometrically nonconforming case, and (34):

THEOREM 4. *There exists a constant C such that*

$$cond(\check{S}^{-1}\tilde{S}) \leq C \max_{1\leq k\leq K} (1+\log(\frac{H_k}{h_k}))^4. \tag{38}$$

5. Preconditioners for the Mortar Spectral Method.

5.1. The geometrically conforming case. For simplicity only, we shall now assume that the subdomains Ω_k are rectangles with sides parallel to the coordinate axes. We first give a brief review of the mortar spectral method; see [7] for more details. For a review on the analysis of the Legendre spectral methods, see, e.g., [8]. Consider a family of integers $\{N_k\}_{k\in\{1\ldots K\}}$, all greater than two. Let X_{kN} be the space of polynomials on Ω_k of degree N_k in each space variables, which vanish on $\partial\Omega_k \cap \partial\Omega$, and let $\mathcal{X}_{kN}$ be the space of traces of functions of X_{kN} on $\partial\Omega_k$. A product space is defined by

$$X_N \equiv \prod_{k=1}^{K} X_{kN}.$$

For each subdomain Ω_k, a quadrature formula is given, by the Gauss-Lobatto-Legendre formula on $(-1,1)$, an affine transformation, and tensorization. We denote the corresponding nodes and weights by $\xi_{i,j,k}$ and $\omega_{i,j,k}$, and by $\sum_{GL,k}$ the quadrature formula

$$\sum_{GL,k} f \equiv \sum_{i,j=0}^{N_k} \omega_{i,j,k} f(\xi_{i,j,k}).$$

The nodes $\xi_{i,j,k}$ define a grid $\mathcal{T}_{kN}$ on Ω_k. A discrete bilinear form is given by

$$a_N(\mathbf{u},\mathbf{v}) \equiv \sum_{k=1}^{N} \sum_{GL,k} \nabla \mathbf{u}_{kN} \cdot \nabla \mathbf{v}_{kN}.$$

Consider two adjacent subdomains Ω_k and Ω_ℓ and let $L_{k\ell}$ be equal to N_k or N_ℓ. Let $N_{k\ell} \equiv L_{k\ell} - 2$ and denote by $\tilde{W}_{k,\ell,N}$ the space of polynomials of degree $N_{k\ell}$. The subspace Y_N of X_N is now introduced by

$$Y_N \equiv \left\{ \begin{array}{l} \mathbf{v}_N \in X_N : \\ \forall 1\leq k<\ell\leq K, \quad \forall \mu_{k\ell N} \in \tilde{W}_{k,\ell,N}, \quad \int_{\Gamma_{k\ell}} (Tr_k \mathbf{v}_{kN} - Tr_\ell \mathbf{v}_{\ell N})\mu_{k\ell N} = 0 \end{array} \right\},$$

and the subspace $\mathcal{Y}_N$ of $\mathcal{X}_N$ by

$$\mathcal{Y}_N \equiv \left\{ \begin{array}{l} v_N \in \mathcal{X}_N : \\ \forall 1 \leq k < \ell \leq K, \quad \forall \mu_{k\ell N} \in \tilde{W}_{k,\ell,N}, \quad \int_{\Gamma_{k\ell}} (v_{kN} - v_{\ell N}) \mu_{k\ell N} = 0 \end{array} \right\}.$$

Slave and mortar sides are introduced exactly as in Section 2, and the Poincaré-Steklov bilinear form $s_N : \mathcal{Y}_N \times \mathcal{Y}_N \to I\!R$, can be introduced in terms of the problem of finding $u_N \in \mathcal{Y}_N$ such that

$$\forall v_N \in \mathcal{Y}_N, \quad s_N(u_N, v_N) = \sum_{k=1}^{N} \sum_{GL,k} f \tilde{\mathbf{v}}_{kN} - a_N(\mathbf{u}_N^\circ, \tilde{\mathbf{v}}_N).$$

Here $\mathbf{u}_N^\circ$ is obtained by solving the analogue of (3) and $\tilde{\mathbf{v}}_N$ is the discrete harmonic extension of v_N.

As before, we look for a decomposition of the space $\mathcal{Y}_N$ into the direct sum of a coarse space $\mathcal{Y}_H$, of dimension $N_{\mathcal{V}}$, and a fine space $\mathcal{Y}_N^H$. For each crosspoint A and for each $k \in \mathcal{K}_A$, the basis vector $e^{A,k}$ of $\mathcal{Y}_H$ is fully determined by the following conditions:

1. $e_k^{A,k}(A) = 1$;
2. for all vertices $B \neq A$ of Ω_k, $e_k^{A,k}(B) = 0$;
3. for all $\ell \neq k$, for all vertices B of Ω_ℓ, $e_\ell^{A,k}(B) = 0$;
4. $e^{A,k}$ is linear on the master edges.

Suppose first that $\Gamma_{k\ell}$ is a slave edge of $\partial\Omega_k$. We can then check, using coordinates such that $\Gamma_{k\ell} = [-H, H]$ with the value H corresponding to A, that

$$e_k^{A,k} = \tilde{e}^{k,\ell,A} = \frac{L_{N_{k\ell}}(\frac{x}{H}) - L_{N_{k\ell}-1}(\frac{x}{H})}{2}, \quad e_\ell^{A,k} = 0, \quad \text{on } \Gamma_{k\ell}, \tag{39}$$

$$e_k^{A,\ell} = \frac{x+H}{2H} - \frac{L_{N_{k\ell}}(\frac{x}{H}) - L_{N_{k\ell}-1}(\frac{x}{H})}{2}, \quad e_\ell^{A,\ell} = \frac{x+H}{2H} \quad \text{on } \Gamma_{k\ell}. \tag{40}$$

Here L_n is the Legendre polynomial of degree n. From (39) and (40), it is straightforward to show that $||e^{A,k}||_{\mathrm{L}^\infty} \leq C$. The next lemma provides estimates of the L^2- and $H_*^{1/2}-$norms of $e^{A,k}$.

LEMMA 10.

$$||\tilde{e}^{k,\ell,A}||_{L^2(\Gamma_{k\ell})} \leq C \left[\frac{H}{N_{k\ell}}\right]^{1/2}$$

$$|\tilde{e}^{k,\ell,A}|_{H^{1/2}(\Gamma_{k\ell})} \leq C\sqrt{\log(N_{k\ell})}$$

Proof. Without loss of generality, we can assume that $\Gamma_{k\ell} =]-1, 1[$, and to simplify the notations, we also set $n = N_{k\ell}$. Recalling that

$$\int_{-1}^{1} L_n^2(x) dx = \frac{2}{2n-1},$$

it is easy to show that

$$||\tilde{e}^{k,\ell,A}||_{L^2(\Gamma_{k\ell})} \leq \frac{C}{n^{\frac{1}{2}}}.$$

In order to evaluate the $H^{1/2}$–norm of $\tilde{e}^{k,\ell,A}$, we compute the $H^{1/2}$–norm of L_n. By the definition of the norm, we have to estimate

$$\|L_n\|^2_{H^{1/2}(-1,1)} = \int_{-1}^{1}\int_{-1}^{1}\left[\frac{L_n(x)-L_n(y)}{x-y}\right]^2 dxdy.$$

Integration by parts, in the case where n is even, leads to

$$\begin{aligned}\int_{-1}^{1}\left[\frac{L_n(x)-L_n(y)}{x-y}\right]^2 dx &= 2\int_{-1}^{1}\frac{(L_n(x)-L_n(y))L_n'(x)}{x-y}dx - \left[\frac{(L_n(x)-L_n(y))^2}{x-y}\right]_{-1}^{1}\\ &= 2\int_{-1}^{1}\frac{(L_n(x)-L_n(y))L_n'(x)}{x-y}dx - \left[\frac{(1-L_n(y))^2}{1-y}-\frac{(1-L_n(y))^2}{-1-y}\right]\\ &= 2\int_{-1}^{1}\frac{(L_n(x)-L_n(y))L_n'(x)}{x-y}dx - 2\frac{(1-L_n(y))^2}{1-y^2},\end{aligned}$$

and similarly, in the case where n is odd,

$$\int_{-1}^{1}\left[\frac{L_n(x)-L_n(y)}{x-y}\right]^2 dx = 2\int_{-1}^{1}\frac{(L_n(x)-L_n(y))L_n'(x)}{x-y}dx - 2\frac{1+L_n^2(y)-2yL_n(y)}{1-y^2}.$$

Hence,

$$\begin{aligned}&\int_{-1}^{1}\int_{-1}^{1}\left[\frac{L_n(x)-L_n(y)}{x-y}\right]^2 dxdy\\ &= 2\int_{-1}^{1}\int_{-1}^{1}\frac{(L_n(x)-L_n(y))L_n'(x)}{x-y}dxdy \quad - \quad 2\begin{cases}\int_{-1}^{1}\frac{(1-L_n(y))^2}{1-y^2}dy & \text{n even;}\\ \int_{-1}^{1}\frac{(1+L_n^2(y))-2yL_n(y)}{1-y^2}dy & \text{n odd.}\end{cases}\end{aligned}$$

In order to estimate the last terms in this formula, we use Gaussian quadrature based on the roots $(\zeta_i)_{1\le i\le n}$ of L_n. It is well known that there exists positive quadrature weights $(\omega_i)_{1\le i\le n}$ such that

$$\forall\varphi\in \mathbb{P}_{2n-1}, \quad \int_{-1}^{1}\varphi(x)dx = \sum_{i=1}^{n}\varphi(\zeta_i)\omega_i.$$

We therefore obtain

$$\int_{-1}^{1}\frac{(1-L_n(y))^2}{1-y^2}dy = \sum_{i=1}^{n}\frac{(1-L_n(\zeta_i))^2}{1-\zeta_i^2}\omega_i = \sum_{i=1}^{n}\frac{1}{1-\zeta_i^2}\omega_i.$$

We now recall, see [25],(thm 6.21.3 & (15.3.14)) that $\zeta_i = \cos\theta_i$ with

$$\frac{(n-i+\frac{1}{2})}{n}\pi \le \theta_i \le \frac{(n-i+1)}{n}\pi,$$

and that there exist two positive constants c and C such that

$$\frac{c}{n}(1-\zeta_i^2)^{1/2} \le \omega_i \le \frac{C}{n}(1-\zeta_i^2)^{1/2}.$$

We find that

$$\int_{-1}^{1} \frac{(1-L_n(y))^2}{1-y^2} dy \leq \frac{C}{n} \sum_{i=1}^{n} \frac{1}{(1-\cos^2\theta_i)^{1/2}}.$$

We recognize this last expression as a Riemann sum of $(\frac{1}{1-\cos^2(\theta)})^{1/2}$. Hence,

$$\int_{-1}^{1} \frac{(1-L_n(y))^2}{1-y^2} dy \leq C\log(n).$$

The same argument leads to

$$\int_{-1}^{1} \frac{1+L_n^2(y)-2yL_n(y)}{1-y^2} dy \leq C\log(n),$$

which in turn leads to

$$(41) \qquad \begin{aligned} &|\int_{-1}^{1}\int_{-1}^{1}\left[\frac{L_n(x)-L_n(y)}{x-y}\right]^2 dxdy - \\ &2\int_{-1}^{1}\int_{-1}^{1}\frac{(L_n(x)-L_n(y))L_n'(x)}{x-y}dxdy| \quad \leq C\log(n). \end{aligned}$$

Finally, we remark that

$$\begin{aligned} &\left[\frac{L_n(x)-L_n(y)}{x-y}\right]^2 - \frac{(L_n(x)-L_n(y))L_n'(x)}{x-y} = \\ &\frac{(L_n(x)-L_n(y))(L_n(x)-L_n(y)-(x-y)L_n'(x))}{(x-y)^2}. \end{aligned}$$

Here $\frac{(L_n(x)-L_n(y)-(x-y)L_n'(x))}{(x-y)^2}$ is a polynomial of degree $< n$ in both x and y. The orthogonality of the Legendre polynomials then leads to

$$\int_{-1}^{1}\int_{-1}^{1}\left[\frac{L_n(x)-L_n(y)}{x-y}\right]^2 - \frac{(L_n(x)-L_n(y))L_n'(x)}{x-y}dxdy = 0.$$

By using (41), we find that

$$\int_{-1}^{1}\int_{-1}^{1}\left[\frac{L_n(x)-L_n(y)}{x-y}\right]^2 dxdy \leq C\log(n).$$

Thus, the square of the $H^{1/2}$–norm of L_n is bounded by $C\log(n)$ and so is the square of the $H^{1/2}$–norm of $\tilde{e}^{k,\ell,A}$. $\square$

The next result is the analogue of Lemma 2:

LEMMA 11. *Let A be a crosspoint and let $k,\ell,m \in \mathcal{K}_A$, with $|\Gamma_{k\ell}| > 0$ and $|\Gamma_{km}| > 0$. Assume that $\partial\Omega_k \cap \Gamma_{k\ell}$ is a slave side of Ω_k and that $\partial\Omega_k \cap \Gamma_{km}$ is a master side of Ω_k. Then,*

$$(42) \qquad |e_k^{A,k}|^2_{H^{1/2}(\partial\Omega_k)} \leq C(1+\log(N_k)).$$

Proof. Since we are interested in the $H^{1/2}(\partial\Omega_k)$ semi-norm, it is possible to rescale the problem and assume that the edge $\Gamma_{k\ell}$ is $(-1,1)$. We again set $n = N_k$. We first use the following inverse inequality,

$$|e_k^{A,k}|^2_{H^{1/2}(\partial\Omega_k)} \le Cn^{4\epsilon}|e_k^{A,k}|^2_{H^{1/2-\epsilon}(\partial\Omega_k)};$$

see [8].

Let us denote by $f^{k,m,A}$ and $f^{k,\ell,A}$ the functions on $\partial\Omega_k$ which are equal to $e_k^{A,k}$ on Γ_{km} and $\Gamma_{k\ell}$, respectively, and vanish on $\partial\Omega_k\backslash\Gamma_{km}$ and $\partial\Omega_k\backslash\Gamma_{k\ell}$, respectively. It is clear that, almost everywhere,

$$e_k^{A,k} = f^{k,m,A} + f^{k,\ell,A}.$$

The semi-norm $|f^{k,m,A}|^2_{H^{1/2-\epsilon}(\partial\Omega_k)}$ can be computed explicitly, because $f^{k,m,A}$ is piecewise linear, and the following estimate is obtained:

$$|f^{k,m,A}|^2_{H^{1/2-\epsilon}(\partial\Omega_k)} \le C\frac{1}{\epsilon}.$$

To evaluate the contribution of $f^{k,\ell,A}$, it is enough to estimate the following quantity

$$\int_{y=-1}^{3}\int_{x=-1}^{3} \frac{[\chi(x)L_n(x) - \chi(y)L_n(y)]^2}{|x-y|^{2-2\epsilon}}.$$

Here χ is the characteristic function of $(-1,1)$. It can be proved, as in Lemma 5.1, that the contribution of $\int_{y=-1}^{1}\int_{x=-1}^{1} \frac{[L_n(x)-L_n(y)]^2}{|x-y|^{2-2\epsilon}}$ is bounded by $C(1+\log(n))$. There remains to estimate

$$\int_{y=1}^{3}\int_{x=-1}^{1} \frac{L_n^2(x)}{(y-x)^{2-2\epsilon}}.$$

This is done as follows:

$$\int_{y=1}^{3}\int_{x=-1}^{1} \frac{L_n^2(x)}{(y-x)^{2-2\epsilon}} \le \frac{1}{1-2\epsilon}\int_{x=-1}^{1} \frac{L_n^2(x)}{(1-x)^{1-2\epsilon}}.$$

Since $L_n^2(x) \le 1,\ |x| \le 1$, it follows that

$$\int_{y=1}^{3}\int_{x=-1}^{1} \frac{L_n(x)^2}{(y-x)^{2-2\epsilon}} \le \frac{C}{\epsilon}.$$

Hence,

$$|e_k^{A,k}|^2_{H^{1/2}(\partial\Omega_k)} \le Cn^{4\epsilon}\left(1 + \frac{1}{\epsilon} + \log(n)\right).$$

Choosing $\epsilon = \frac{1}{\log(n)}$ yields

$$|e_k^{A,k}|^2_{H^{1/2}(\partial\Omega_k)} \le C(1+\log(n)).$$

$\square$

Let $\tilde{S}$ be the matrix of s_N in the new basis described above. The matrix $\tilde{S}$ can be written as

$$\tilde{S} = \begin{pmatrix} \tilde{S}_{NN} & \tilde{S}_{NH} \\ \tilde{S}_{NH}^T & \tilde{S}_{HH} \end{pmatrix}.$$

In order to design a preconditioner for $\tilde{S}$, we replace the block $\tilde{S}_{NH}$ by 0, and the block $\tilde{S}_{NN}$ by its block diagonal with one block for each edge. The resulting preconditioner is

$$\hat{S} = \begin{pmatrix} \hat{S}_{NN} & 0 \\ 0 & \tilde{S}_{HH} \end{pmatrix}.$$

Let $\hat{s}_N$ be the bilinear form corresponding to the matrix $\hat{S}$. The bound,

$$\forall v_N \in \mathcal{Y}_N, \quad s_N(v_N, v_N) \le C\hat{s}_N(v_N, v_N),$$

is obtained exactly as in Lemma 5.

The following lemma will also be needed:

LEMMA 12. *There exists a constant C such that $\forall v_{kN} \in \mathcal{X}_{kN}$,*

$$\|v_{kN}\|^2_{L^\infty(\partial\Omega_k)} \le C(1+\log(N_k))\|v_{kN}\|^2_{H^{1/2}(\partial\Omega_k)}. \tag{43}$$

Similarly, if x_0 is a point of $\partial\Omega_k$, then,

$$\|v_{kN} - v_{kN}(x_0)\|^2_{L^\infty(\partial\Omega_k)} \le C(1+\log(N_k))|v_{kN}|^2_{H^{1/2}(\partial\Omega_k)}. \tag{44}$$

Proof. To prove (43), we first rescale so that the diameter of Ω_k equals 1, and then prove that

$$\|v_{kN}\|^2_{L^\infty(\partial\Omega_k)} \le C\frac{1}{\epsilon}\|v_{kN}\|^2_{H^{1/2+\epsilon}(\partial\Omega_k)}.$$

This formula is true for any function and can be derived by using a Fourier transform argument. From an inverse inequality for polynomials, we also have

$$\|v_{kN}\|^2_{H^{1/2+\epsilon}(\partial\Omega_k)} \le CN_k^{4\epsilon}\|v_{kN}\|^2_{H^{1/2}(\partial\Omega_k)}.$$

Choosing $\epsilon = \frac{1}{1+\log(N_k)}$ yields the desired result. Finally, (44) is obtained by a standard quotient space argument.

Given Lemma 12, analogues of Lemma 6 and Theorem 1 can be obtained as in the finite element case. The only notable difference is when estimating the integral

$$\int_{-H}^{H} \frac{|v_{kN} - v_{kH}|^2}{x+H};$$

cf. (21)-(24) in the proof of Theorem 1. The integral is again split into two, over $(-H, H\xi_1)$ and $(H\xi_1, H)$, respectively. Here the $(\xi_i)_{i\in\{0,N_k\}}$ are the Gauss-Lobatto points in $[-1,1]$. We note that $\xi_1 + 1 = O(\frac{1}{N_k^2})$ and that an inverse inequality for polynomials of degree n is given by

$$\|v_n\|_{W^{1,p}(-1,1)} \le n^2\|v_n\|_{L^p(-1,1)}, \quad p \in (1, +\infty].$$

□

Finally, since all necessary technical tools are at hand, the proof of the following main result can be obtained

THEOREM 5. *There exists a constant C such that*

$$cond(\hat{S}^{-1}\tilde{S}) \leq C \max_{1\leq k\leq K}(1+\log(N_k))^2.$$

5.2. The geometrically nonconforming case. The previous mortar spectral element method can be generalized straightforwardly to the case where the assumption of geometrical conformity is relaxed. The analysis of our preconditioner requires, in Lemma 5.5 below, a bound on the relative size of the intersection of two edges; in order to provide estimates which only depend polylogarithmically on the parameter N, we make the assumption that

$$\forall k,\ell,i,j \text{ such that } \Gamma_k^i \cap \Gamma_\ell^j \neq \emptyset, \quad \frac{|\Gamma_k^i \cap \Gamma_\ell^j|}{|\Gamma_k^i|} \geq c, \tag{45}$$

a condition less standard and more stringent than (28). The mortar method is based on the definition of the master and slave sides as described in Section 4 in the finite element context. The projection that gives slave values in terms of those on the master edges is defined as in Subsection 5.1. The iterative substructuring algorithm now considered is the same as that of Section 4. We now have the new tools of Subsection 5.1 at our disposal, and in order to prove a result equivalent to Theorem 3 for the spectral approximation, we additionally only have to derive an analogue of Lemma 7. We do so after some preliminary work.

The definition of the mortar condition naturally leads to the introduction of a projection operator $\pi_N : L^2(-1,1) \to P_N^0(-1,1)$, given by

$$\forall \varphi \in L^2(-1,1), \int_{-1}^{1} [\varphi - \pi_N(\varphi)]\psi = 0, \quad \forall \psi \in P_{N-2}.$$

This projection operator has already been analyzed in [7] and certain stability and approximation properties are given in that paper. While we do not now know how to prove a uniform bound for the $H_{00}^{1/2}$-norm of this operator, the following results will allow us to derive a strong result on our preconditioner.

LEMMA 13. *Let a be any real number in $]-1,1[$. Let χ_a denote the piecewise linear, discontinuous function on $]-1,1[$ that vanishes on $]-1,a[$ and goes linearly from 1 and 0 over the interval $[a,1]$. There then exists a constant C such that*

$$||\pi_N\chi||^2_{H_{00}^{1/2}} \leq C\frac{\log(N)}{\sqrt{1-a^2}}.$$

Proof. Let φ_N be a polynomial of degree N in one variable that vanishes at ± 1. It is easy to see that it can be written as

$$\varphi_N(x) = \sum_{n=1}^{N-1} a_n(1-x^2)L_n'(x). \tag{46}$$

Recalling that the Legendre polynomials L_n are $L^2(-1,1)$-orthogonal, with norm

$$\int_{-1}^{1} L_n^2(x)dx = \frac{2}{2n+1},$$

and that they satisfy the differential equation

$$\frac{d}{dx}((1-x^2)L_n'(x)) + n(n+1)L_n(x) = 0,$$

we can first show that

$$|\varphi_N|_1^2 = \sum_{n=1}^{N-1} a_n^2 \frac{2n^2(n+1)^2}{2n+1} \leq C \sum_{n=1}^{N-1} n^3 a_n^2.$$

Using now the integral relation

$$\int_{-1}^{x} L_n(s)ds = \frac{L_{n+1}(x) - L_{n-1}(x)}{2n+1}$$

and the differential equation once more, we find that

$$\|\varphi_N\|_0^2 = \| \sum_{n=1}^{N-1} a_n \frac{n(n+1)}{2n+1}(L_{n+1}(x) - L_{n-1}(x))\|_0^2 \leq C \sum_{n=1}^{N-1} n a_n^2.$$

Here we have used the L^2-orthogonality of the L_n and the formula

$$\sum_{n=1}^{N-1} a_n \frac{n(n+1)}{2n+1}(L_{n+1}(x) - L_{n-1}(x)) = \qquad -a_1 \frac{2}{3} L_0(x) +$$

$$\left[\sum_{n=1}^{N-1} (a_{n-1} \frac{(n-1)n}{2n-1} - a_{n+1} \frac{(n+1)(n+3)}{2n+3}) L_n(x) \right] \quad + \quad a_{N-1} \frac{(N-1)N}{2N-1} L_N(x).$$

Let us denote by $[I\!\!P_N(-1,1) \cap L^2(-1,1), I\!\!P_N(-1,1) \cap H_0^1(-1,1)]_\theta$, the interpolation space, with index θ, between the space of polynomials $I\!\!P_N(-1,1)$ of degree N provided with the L^2-norm and the H_0^1-norm, respectively. By a main result of interpolation theory [20], it follows from our two estimates of the norms of φ_N that there exists a constant C such that, for all $\varphi_N \in I\!\!P_N(-1,1)$,

$$\|\varphi_N\|^2_{[I\!\!P_N(-1,1)\cap L^2(-1,1), I\!\!P_N(-1,1)\cap H_0^1(-1,1)]_\theta} \leq C \sum_{n=1}^{N-1} n^{1+2\theta} a_n^2.$$

We recall now that, according to [21], $[I\!\!P_N(-1,1) \cap L^2(-1,1), I\!\!P_N(-1,1) \cap H_0^1(-1,1)]_\theta$ coincides topologically with $I\!\!P_N(-1,1)$ provided with the H_0^θ-norm. In particular, we can conclude that

$$\|\varphi_N\|^2_{H_{00}^{1/2}(-1,1)} \leq C \sum_{n=1}^{N-1} n^2 a_n^2. \tag{47}$$

In order to complete the proof, we now evaluate the coefficients a_n of the decomposition of $\varphi_N = \pi_N \chi$ given by (46). By using the definition of χ_a, we find that

$$\begin{aligned} \forall n,\ 1 \leq n \leq N-1, \quad \int_{-1}^{1} \varphi_N(x) L_n'(x) &= \int_{-1}^{1} \chi_a(x) L_n'(x) \\ &= \int_a^1 \tfrac{1-x}{1-a} L_n'(x) \\ &= \tfrac{1}{1-a} \left[-(1-a)L_n(a) + \int_a^1 L_n(x) \right] \\ &= -L_n(a) - \tfrac{1}{1-a} \tfrac{L_{n+1}(a) - L_{n-1}(a)}{2n+1}. \end{aligned} \tag{48}$$

Combining the integral and the differential formulae for the Legendre polynomials, we deduce

$$\begin{aligned}\forall n,\ 1 \le n \le N-1, \quad \int_{-1}^{1} \varphi_N(x) L_n'(x) &= -L_n(a) + \frac{1}{1-a}\frac{(1-a^2)L_n'(a)}{n(n+1)} \\ &\le |L_n(a)| + |\frac{2L_n'(a)}{n(n+1)}|.\end{aligned}$$

From [1], (22.14.9), we know that

$$|L_n(a)| \le \frac{C}{\sqrt{n}} \frac{1}{\sqrt[4]{1-a^2}}. \tag{49}$$

As another consequence of the integral formula, we find that

$$L_n'(x) = \sum_{k \ge 0; n-2k-1 \ge 0} (2n-4k-1) L_{n-2k-1}(x),$$

which yields

$$|\frac{L_n'(a)}{n(n+1)}| \le \frac{C}{\sqrt{n}} \frac{1}{(1-a^2)^{\frac{1}{4}}}. \tag{50}$$

Finally from (48),(49), and (50), we deduce that

$$\forall n,\ 1 \le n \le N-1, \quad \int_{-1}^{1} \varphi_N(x) L_n'(x) \le \frac{C}{\sqrt{n}} \frac{1}{\sqrt[4]{1-a^2}}.$$

By using (46), we obtain

$$2\frac{n(n+1)}{2n+1} a_n = 2 \int_{-1}^{1} \varphi_N(x) L_n'(x).$$

Thus,

$$a_n = \frac{2n+1}{2n(n+1)}(-L_n(a) - \frac{1}{1-a}\frac{L_{n+1}(a) - L_{n-1}(a)}{2n+1}),$$

and hence

$$|a_n| \le \frac{C}{n\sqrt{n}} \frac{1}{\sqrt[4]{1-a^2}}.$$

By introducing this bound into formula (47), we can conclude that

$$||\pi_N \chi||^2_{H_{00}^{1/2}} \le C \frac{\log(N)}{\sqrt{1-a^2}}.$$

□

It is now an easy matter to state an analogue of the Lemma 7.

Lemma 14. *Assume that (45) holds, let A be a crosspoint, and let Γ_k^j be a master side of Ω_k with A as an end point. Let $\ell \ne k$, $i \in j(\ell)$ satisfy $|\Gamma_\ell^i \cap \Gamma_k^j| > 0$. Then,*

$$|e_\ell^{A,k}|_{\Gamma_\ell^i}|^2_{H^{1/2}(\partial\Omega_\ell)} \le C(1 + \log(N_\ell)). \tag{51}$$

Proof. Let us denote by χ_ℓ the function obtained from χ by a scaling which maps $]-1,1[$ onto Γ_ℓ^i with the further property that $\chi_\ell(A)=1$ and $\chi_\ell(B)=0$. Here A and B are the end points of $\Gamma_\ell^i \cap \Gamma_k^j$. By construction, $e_k^{A,k} - \chi_\ell$ is continuous and it even belongs to $H^{\frac{3}{2}-\varepsilon}$ for any positive ε. We decompose $e_\ell^{A,k}$ into the following sum

$$e_\ell^{A,k} = \pi_N^\ell(\chi_\ell) + \pi_N^\ell(e_k^{A,k} - \chi_\ell).$$

Lemma 13 supplies the desired bound for the first term $|\pi_N^\ell(\chi_\ell)|_{H^{1/2}(\partial\Omega_l)}$. The bound on the second, $|\pi_N^\ell(e_k^{A,k} - \chi_\ell)|_{H^{1/2}(\partial\Omega_l)}$, is obtained as in the geometrically conforming case. □

With Lemma 14, we are ready to prove the following result using the same arguments as in Section 4.

THEOREM 6. *Under the assumption (45), there exists a constant C such that*

$$cond(\hat{S}^{-1}\tilde{S}) \leq C \max_{1\leq k\leq K}(1+\log(N_k))^2.$$

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