# Schwarz Preconditioners for Elliptic Problems with Discontinuous Coefficients Using Conforming and Non-Conforming Elements

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## Abstract

Additive multilevel Schwarz methods are developed for conforming finite element approximations of second order elliptic problems. We focus on problems in three dimensions with possibly large jumps in the coefficients across the interface separating the subregions. We establish condition number estimates for the iterative operators, which are independent of the coefficients, and grow at most as the square of the number of levels $\ell$. For the multiplicative versions, such as the V-cycle multigrid methods using Gauss Seidel and damped Jacobi smoothers, we obtain a rate of convergence bounded from above by $1 - C\,\ell^{-2}$. We also characterize a class of distributions of values of the coefficients, called quasi-monotone, for which the error of the weighted $L^2$-projection is stable and for which we can use the standard piecewise linear functions as a coarse space and obtain condition number estimates independent of the number of levels, subregions, and the coefficients. We also design and analyze multilevel methods with new coarse spaces given by simple explicit formulas. We also consider nonuniform meshes, multilevel iterative substructuring methods, and two-level additive methods with inexact solvers for the local problems.

In a second part, two-level domain decomposition methods are developed for a simple nonconforming approximation of second order elliptic problems. A bound is established for the condition number of these iterative methods that grows only logarithmically with the number of degrees of freedom in each subregion. This bound holds for two and three dimensions and is independent of jumps in the value of the coefficients. For these finite elements methods, a preconditioner is constructed from the restriction of the given elliptic problem to overlapping subregions into which the given region has been decomposed. In addition, in order to enhance the convergence rate, our preconditioners include a nonstandard coarse mesh component of relatively modest dimension. We also consider multilevel Schwarz preconditioners and show that all results for the conforming case also hold for the nonconforming case.

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*Em memoria de Regina, Milton, and Katia*

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# Contents

# Chapter 1

# Introduction

## 1.1 An overview

The development of numerical methods for large algebraic systems is central in the development of efficient codes for computational fluid dynamics, elasticity, and other core problems of continuum mechanics. Many other tasks in such codes parallelize relatively easily. Therefore, the importance of the algebraic system solvers is increasing with the arrival of new parallel computing systems with many fast processors, systems that are at least as capable as traditional supercomputers. The development of these methods makes it possible to carry out simulations in three dimensions, with very high resolution, relatively easily. Much higher resolution than now possible will be required in many applications in order to gain insight into important open problems in science and engineering.

Domain decomposition (DD) methods provide a very natural way of deriving parallel algorithms for the numerical solution of partial differential equations and have recently attracted much theoretical and practical interest; see e.g. [46], [20], [21], [47], [22], [51], [50]. DD methods can often be viewed as preconditioners for iterative methods like the conjugate gradient method or GMRES. Much has recently been learned about how to design these special preconditioned conjugate gradient type methods to obtain very fast convergence.

Such a successful preconditioner method is built mainly from two major components: many local problems and one global problem. The local problems typically correspond to subproblems that correspond to the original (or a similar) problem restricted to subregions into which the given a domain has been divided. In a multiprocessor machine, with distributed memory, one approach is to assign a subregion (of the domain of the PDE) to each of the processors, and then to triangulate these subdomains to obtain a fine mesh and a discretized system. The domain can be decomposed into *overlapping* or *non-overlapping* subregions. In the former case the algorithms are often refered as *Schwarz methods with overlap*, in the latter they are called *iterative substructuring methods*; this distinction is however not always clear because many iterative substructuring methods can be analyzed as Schwarz methods.

When DD algorithms are used, a large number of subproblems can be solved in parallel. The local interaction is through the interexchange of information between neighboring subregions. In addition, in the elliptic case, it is necessary to introduce a global *coarse* part of the preconditioner, with only one or a few degrees of freedom per subregion, to model the global interaction of the subregions and to obtain fast convergence in case of many subregions.

There are several important reasons why domain decomposition methods have became popular in recent years : They

i) are well suited for parallel computers;

ii) are of great intrinsic mathematical interest;

iii) apply to regions with complex geometries;

iv) have a solid theoretical foundation;

v) make it possible to use different numerical schemes for the different subproblems, such as the finite element method, $h$, $p$ and $hp$ versions, fast Poisson solvers, finite differences, spectral and collocation methods;

vi) allow the use of different kinds of equations in different subregions whenever the underlying physics is of a different nature;

vii) can be combined with multiple scale solutions and local refinement;

viii) apply to several important physical problems of nonlinear nature arising from mechanics, elasticity, fluid dynamics, aerodynamics and thermomechanics.

In this thesis, we develop Schwarz methods for conforming and nonconforming finite element approximations to boundary values problems of second order, self-adjoint, linear elliptic PDE's. A special emphasis is placed on problems in three dimensions with possibly large discontinuities in the coefficients of the PDE's. There is a variety of engineering and natural science applications where our methods can be very useful. Among them are problems that arise in the computation of homogenized coefficients. Composite materials, phase transitions, optimal shape design, polycrystaline dielectrics, polyphased fluids, permeability of porous media are examples of such applications. Other important applications arise in contaminant transport, groundwater flow, and oil reservoir simulation.

The thesis is organized as follows. In the remainder of Chapter 1, we review some basic definitions and useful results about Sobolev spaces for scalar and vector functions.

In Chapter 2, we first review some iterative methods for positive definite linear system of equations. We then discuss recent results on Schwarz methods, which are regarded as generalizations of the first domain decomposition method proposed in 1869 by H. A. Schwarz [78]. In particular, a general abstract variational framework developed by Dryja and Widlund is usaed to analyze Schwarz methods in terms of subspaces and bilinear forms.

In Chapter 3, we introduce the elliptic model problem and corresponding discrete system using the standard $h$-version finite element method and the lowest order Raviart-Thomas mixed method; a parallel discussion of well-posedness of these two formulations is given. The latter formulation is useful in applications for which accurate approximation to the flux variable of the elliptic problem is required and the solution (of the elliptic problem) is not sufficiently smooth; this is the case where there are highly discontinuous coefficients. We next review the Arnold-Brezzi theory which takes advantage of an equivalent hybrid formulation of the discrete mixed problem to reduce a symmetric indefinite problem to one which is positive definite. The resulting problem is directly related to the nonconforming $P_1$ finite element problem; many local elementwise problems also need to be solved. We can then apply our Schwarz methods developed in Chapter 5 and recover the flux variable by a simple element-by-element post-processing. This procedure appears to hold the best promise for obtaining an approximation to the flux variable for elliptic problems with highly discontinuous coefficients in three-dimensional space.

The main contributions of our work are discussed primarily in Chapters 4 and 5.

Chapter 5 is an extension of a technical report *Two-Level Schwarz Methods for Nonconforming Finite Elements and Discontinuous Coefficients* which was completed in March 1993 [76]. This work was inspired by earlier multilevel studies, cf. Brenner [13], Oswald [65], as well as by recent work by Dryja, Smith, and Widlund [35], but a number of additional technical difficulties had to be overcome. They are primarily related to our efforts to treat quite general coefficients. One of the main ideas relates to the construction of certain *isomorphisms*, or *local interpolators*, which map between conforming and nonconforming spaces and obtaining several results for the nonconforming case which are known for the conforming case.

These isomorphisms were apparently first used by the author in a short version of [76] that was entered into Copper Mountain student competition in mid-December 1992. We note that recently considerable attention has been focused on related techniques for domain decomposition methods with nonconforming spaces; for second order scalar problems (cf. Brenner [12], Cowsar [28, 29], Cowsar, Mandel, and Wheeler [30], Sarkis [77]), for plate elements (cf. Brenner [10]), and for non-nested meshes (cf. Cai [17, 16], Chan, Smith, and Zou [24]). We also note the *face based* coarse spaces, which were introduced in [76], have also been discussed in the conforming case in the recent work by Dryja, Smith, and Widlund [35]. We also introduce *approximate harmonic extensions*, and *$\beta$-Neumann-Neumann* coarse spaces (generalizations of the Neumann-Neumann coarses space introduced by Dryja and Widlund [41], and Mandel and Brezina [55]). These topics are also discussed in the conforming case in Chapter 4. Finally, at the end of Chapter 5, we use our results of Chapter 4 to analyze some multilevel for hybrid-mixed finite element methods that are insensitive to the jumps of the coefficients across substructure interfaces and also to the number of substructures; see [77].

Chapter 4 is an extension of a technical report *Multilevel Schwarz Methods for Elliptic Problems with Discontinuous Coefficients in Three Dimensions* completed in March 1994 in joint work with Maksymilian Dryja and Olof Widlund [34]. This work follows earlier work on iterative substructuring methods [35] and Neumann-Neumann type methods [41]. It is also focused on making the performance of the algorithms independent of the jumps in the coefficients. We explore the use of *nonstandard*, *exotic* coarse spaces, and also derive a new condition on the coefficients, *quasi-monotonicity*, for which we can establish the same basic results for $L^2$-projection in the constant coefficient case. For such coefficients, we show that our multilevel methods converge at the rate which is independent on the number of refinement levels. A strategy for selecting nonuniform refinement of the mesh near singularities is considered.

## 1.2 Sobolev spaces

Sobolev spaces are of fundamental importance in studying elliptic boundary value problems. The existence for many such problems of generalized solutions is readily established using variational principles. Classical existence is accordingly transformed into the question of regularity of generalized solutions under appropriate boundary conditions. Sobolev spaces are also important for numerical analysts who need to answer questions related to well-posedness of the discrete system and how close the discrete solution is to that of the continuous problem. In this thesis, the main use of Sobolev space theory is to analyze preconditioners for discrete systems.

Many elliptic boundary values problems, which arise in practice, are posed in domains which are simple but not smooth. In finite element studies, for instance, we often face a geometry composed of polyhedra. In domain decomposition methods, we also encounter substructures that are polyhedra. Thus, it is natural to introduce Sobolev spaces for the class of *Lipschitz domains*. For this class of domains, it is possible to obtain several equivalent norms. It is important to have several equivalent norms because that we can then choose the most appropriate one for a certain problem.

Let $\mathcal{G}$ denoted a region, i.e. an open, connected set, in $\Re^d$. In later chapters, $\mathcal{G}$ will be the whole region $\Omega$, a substructure $\Omega_i$, or a finite element $\tau$.

**Definition 1.1 (Lipschitz region)** *Let $\mathcal{G}$ be an open subset of $\Re^d$. $\mathcal{G}$ is a Lipschitz region, if for every $x \in \partial\mathcal{G}$, there exist a neighborhood $\mathcal{O} \subset \Re^d$ of $x$ and a mapping $\psi$ from $\mathcal{O}$ onto a unity cube $\square = \{|\xi_j| < 1, j = 1, \cdots, d\}$ such that*

*i) $\psi$ is injective*

*ii) $\psi$ together with $\psi^{-1}$ (defined on $\square$) is Lipschitz continuous* *iii)* $\mathcal{G} \cap \mathcal{O} = \{y \in \mathcal{G} \,:\, \xi_d = \psi(y)_d < 0\}$ *where* $\psi(y)_d$ *denotes the dth component of* $\psi(y)$.

*A consequence of (iii) is that the boundary* $\partial\mathcal{G}$ *is defined locally by the equation* $\psi(y)_d = 0$.

We remark that if $\mathcal{G}$ is a bounded Lipschitz region, we can, by compactness, select a finite number of pairs $(\mathcal{O}_j, \psi_j)$, $j = 1, \cdots, M$, to cover a neighborhood of $\partial\mathcal{G}$.

**Definition 1.2** ($L^2(\mathcal{G})$) *Let* $u$ *be a Lebesgue measurable function and let* $\mathcal{G}$ *be an open region in* $\Re^d$. *The Hilbert space* $L^2(\mathcal{G})$ *is defined by the norm*

$$\|u\|^2_{L^2(\mathcal{G})} = \int_{\mathcal{G}} u^2 \, dx.$$

**Definition 1.3** ($H^1(\mathcal{G})$) *Let* $\mathcal{G}$ *be an open region in* $\Re^d$. *The Sobolev space* $H^1(\mathcal{G})$ *is defined by the seminorm*

$$|u|^2_{H^1(\mathcal{G})} = \int_{\mathcal{G}} \nabla u \cdot \nabla u \, dx,$$

*and the norm*

$$\|u\|^2_{H^1(\mathcal{G})} = |u|^2_{H^1(\mathcal{G})} + \frac{1}{H^2_{\mathcal{G}}} \|u\|^2_{L^2(\Omega)},$$

*where* $H_{\mathcal{G}}$ *is the diameter of* $\mathcal{G}$. *Here,* $\nabla u$ *has to be understood as distributional derivatives.*

The scale factor $H_{\mathcal{G}}$ is obtained by a change of variable beginning with the standard definition of the norm for a domain with unit diameter.

**Definition 1.4** ($H^s(\mathcal{G}), 0 < s < 1$) *Let* $\mathcal{G}$ *be an open region in* $\Re^d$. *The fractional order Sobolev space* $H^s(\mathcal{G})$, $0 < s < 1$, *is defined as the space of all* $u \in L^2(\mathcal{G})$ *such that*

$$|u|^2_{H^s(\mathcal{G})} = \int_{\mathcal{G}} \int_{\mathcal{G}} \frac{|u(x) - u(y)|^2}{|x - y|^{d+2s}} \, dx \, dy < \infty \tag{1.1}$$

*with norm*

$$\|u\|^2_{H^s(\mathcal{G})} = |u|^2_{H^s(\mathcal{G})} + \frac{1}{H^{2s}_{\mathcal{G}}} \|u\|^2_{L^2(\mathcal{G})}.$$

For a bounded Lipschitz domain $\mathcal{G}$, it can be shown [49, 62] that the space $H^s(\mathcal{G})$ is the completion of the space $C^\infty(\mathcal{G})$ ( or $C^\infty(\bar{\mathcal{G}})$) with respect to $\|\cdot\|_{H^s(\mathcal{G})}$; see also [1]. The space $C^\infty(\mathcal{G})$ consists of the infinitely continuously differentiable functions defined in $\mathcal{G}$. The space $C^\infty(\bar{\mathcal{G}}) \subset C^\infty(\mathcal{G})$ is the restriction of $C^\infty(\Re^d)$ to $\bar{\mathcal{G}}$.

For an open domain $\mathcal{G}$, the space $H_0^s(\mathcal{G}) \subset H^s(\mathcal{G})$ is defined as the completion of $C_0^\infty(\mathcal{G})$ with respect to $\|\cdot\|_{H^s(\mathcal{G})}$. Here, the space $C_0^\infty(\mathcal{G})$ is the subspace of $C^\infty(\mathcal{G})$ of functions with support in $\mathcal{G}$. For a bounded Lipschitz domain, it can be shown [49] that $H^s(\mathcal{G}) = H_0^s(\mathcal{G})$, for $0 \leq s < 1/2$.

We also are interested in studying the behavior of a function $u \in H_0^s(\mathcal{G})$ extended by zero, outside $\mathcal{G}$; let us denote this extension by $\tilde{u}$. It is possible to show [49] that for bounded Lipschitz regions $\mathcal{G}$, and $s \in [0, 1/2) \cup (1/2, 1]$ that

$$\tilde{u} \in H^s(\Re^d) \quad \text{if} \quad u \in H_0^s(\mathcal{G}).$$

For $u \in H_0^{1/2}(\mathcal{G})$, $\tilde{u}$ may not belong to $H^{1/2}(\Re^d)$. However, we can define a subspace $H_{00}^{1/2}(\mathcal{G})$, for which we have a bounded extension.

**Definition 1.5** ($H_{00}^{1/2}(\mathcal{G})$) *Let $\mathcal{G}$ be a bounded Lipschitz region. The Sobolev space $H_{00}^{1/2}(\mathcal{G})$ is defined by*

$$\|u\|^2_{H_{00}^{1/2}(\mathcal{G})} = \|u\|^2_{H^{1/2}(\mathcal{G})} + \int_{\mathcal{G}} \frac{|u(x)|^2}{d(x, \partial\mathcal{G})}\, dx. \tag{1.2}$$

*Here, $d(x, \partial\mathcal{G})$ denotes the distance from $x$ to the boundary $\partial\mathcal{G}$.*

It is possible to show [49] for any bounded Lipschitz domain, that

$$c_1 \|\tilde{u}\|_{H^{1/2}(\Re^d)} \leq \|u\|^2_{H_{00}^{1/2}(\mathcal{G})} \leq c_2 \|\tilde{u}\|_{H^{1/2}(\Re^d)}.$$

The constants $c_i$ only depend on the Lipschitz constants of $\partial\mathcal{G}$.

Next we state a lemma which makes it possible to extend results from a cube or a smooth region to a bounded Lipschitz regions.

**Lemma 1.1** *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be bounded open regions, and let $\psi$ be a bi-Lipschitz mapping from $\bar{\mathcal{G}}_1$ to $\bar{\mathcal{G}}_2$. Then, for $u \in H^s(\mathcal{G}_2)$, $0 \leq s \leq 1$,*

$$c\, |u \circ \psi|_{H^s(\mathcal{G}_1)} \leq |u|_{H^s(\mathcal{G}_2)} \leq C\, |u \circ \psi|_{H^s(\mathcal{G}_1)}.$$

This lemma is proved in Nečas [62] for $s = 0, 1$. For intermediate $s$, we prove it by working straightforwardly with formula (1.1); see also Grisvald [49].

### 1.2.1 Traces spaces

We shall need Sobolev spaces on manifolds such as $\partial\mathcal{G}$, or an open subset $\Gamma \subset \partial\mathcal{G}$. Let us assume that $\mathcal{G}$ is a bounded Lipschitz region in $\Re^d$ with charts $(\mathcal{O}_j, \psi_j)$, $j = 1, \cdots, M$ covering $\partial\mathcal{G}$.

**Definition 1.6** **($H^s(\Gamma)$)** *A distribution $u$ on $\Gamma$ belongs to $H^s(\Gamma)$, $0 \leq s \leq 1$, if $u \circ \psi_j^{-1} \in H^s(\psi_j(\mathcal{O}_j \cap \Gamma))$, for $j = 1, \cdots, M$.*

One possible seminorm for $H^s(\Gamma)$, $0 < s \leq 1$ is given by

$$|u|^2_{H^s(\Gamma)} = \sum_{j=1}^{M} |u \circ \psi_j^{-1}|^2_{H^s(\psi_j(\mathcal{O}_j \cap \Gamma))}, \tag{1.3}$$

the $L^2(\Gamma)$−norm by

$$\|u\|^2_{L^2(\Gamma)} = \sum_{j=1}^{M} |u \circ \psi_j^{-1}|^2_{L^2(\psi_j(\mathcal{O}_j \cap \Gamma))}, \tag{1.4}$$

and the $H^s(\Gamma)$-norm by

$$\|u\|^2_{H^s(\Gamma)} = |u|^2_{H^s(\Gamma)} + \frac{1}{H_{\mathcal{G}}^{2s}} \|u\|^2_{L^2(\Gamma)}. \tag{1.5}$$

Note that the seminorms and norms introduced in (1.3), (1.4), and (1.5), depend on the charts $(\mathcal{O}_j, \psi_j)$ chosen. The definition of the norm $L^2(\Gamma)$ and the seminorm $H^{1/2}(\partial\Omega)$ that we will use in this thesis will be introduced later and they are independent on the charts chosen and are equivalent to (1.4) and (1.3), respectively.

We note that for a bounded Lipschitz region, the outward vector normal to $\partial\mathcal{G}$ is defined almost everywhere with respect to a hypersurface measure $dS$; see Grisvald [49]. This hypersurface measure is uniquely defined in terms of the $d$-Lebesgue measure $dx$ and $\partial\mathcal{G}$. It does not depend on the chosen charts. For a piecewise smooth $\partial\mathcal{G}$, $dS$ coincides with the standard notion of surface area.

The measure associated with the charts $(\mathcal{O}_j, \psi_j)$ can be obtained by

$$dS_\psi = \sum_{j=1}^{M} det|J_{d-1}(\psi_j^{-1})|\, ds.$$

Here, $ds$ is the standard $(d-1)$-Lebesgue measure, and $J_{d-1}(\psi_j^{-1})$ is the $(d-1) \times (d-1)$ matrix obtained by deleting the last row and column from the Jacobian of $\psi_j^{-1}$. Using that $\psi_j$ are bi-Lipschitz, we see that the measures $dS_\psi$ and $dS$ are equivalent. Hence, we modify (1.4) and define

**Definition 1.7** ($\|u\|_{L^2(\Gamma)}$). *Let $\mathcal{G}$ be a bounded Lipschitz region and let $\Gamma$ be an open subset of $\partial\mathcal{G}$. Let $u$ be a measurable function with respect to the hypersurface measure $dS$. The $L^2(\Gamma)$-norm is defined by*

$$\|u\|^2_{L^2(\Gamma)} = \int_\Gamma u^2\, dS. \tag{1.6}$$

Note that $dS_\psi$ becomes less equivalent to $dS$ when $M$ is large or when the Lipschitz constants of the $\psi_j$ and $\psi_j^{-1}$ become larger; i.e. for bad geometries. It is clear that for a cube, tetrahedron, or sphere we obtain good equivalence. Similarly arguments applies to a face of a cube, or a face of a tetrahedron. For instance, we can introduce an equivalent $H^{1/2}$-seminorm for a face $\mathcal{F}$ of a cube (or tetrahedron) by using (1.1) with $d = 2$, $s = 1/2$, and $\mathcal{G} = \mathcal{F}$ obtaining

$$|u|^2_{H^{1/2}(\mathcal{F})} = \int_\mathcal{F}\int_\mathcal{F} \frac{|u(x)-u(y)|^2}{|x-y|^3}\, dS \quad dS. \tag{1.7}$$

Using the same arguments, we also have an equivalent seminorm

$$|u|^2_{H^{1/2}(\partial\mathcal{G})} = \int_{\partial\mathcal{G}}\int_{\partial\mathcal{G}} \frac{|u(x)-u(y)|^2}{|x-y|^3}\, dS \quad dS, \tag{1.8}$$

since the nonlocal contribution of (1.8) is relative unimportant for well shaped regions like cubes, tetrahedra, or spheres.

Finally, we are interested in knowing the behavior of a function $u$ in $H^{1/2}(\Gamma)$ when extended by zero on $\partial\mathcal{G}\backslash\Gamma$; let us denote this extension by $\tilde{u}(\partial\mathcal{G})$. We use the same arguments as above, and see that on a face $\mathcal{F}$ of a cube, we can use the norm

$$\|u\|^2_{H^{1/2}_{00}(\mathcal{F})} = \|u\|^2_{H^{1/2}(\mathcal{F})} + \int_\mathcal{F} \frac{|u(x)|^2}{d(x,\partial\mathcal{F})}\, dS,$$

to obtain a norm equivalent to $\|\tilde{u}\|_{H^{1/2}(\partial\mathcal{G})}$.

We now introduce the concept of trace maps. We have an obvious definition of boundary values, or trace, on $\partial\mathcal{G}$, for functions in $C^\infty(\bar{\mathcal{G}})$. These maps can be generalized to functions in $H^1(\mathcal{G})$ for a bounded Lipschitz region $\mathcal{G}$; see Nečas [62].

**Lemma 1.2** *(***Trace and Extension theorem***) Let $\mathcal{G}$ be a bounded Lipschitz region. The trace map $\gamma : u \rightarrow u|_{\partial\mathcal{G}}$, defined for $C^\infty(\bar{\mathcal{G}})$, has a unique continuous extension from $H^1(\mathcal{G})$ onto $H^{1/2}(\partial\mathcal{G})$. This operator has a right continuous inverse.*

As a consequence, we can easily show that the kernel of $\gamma$ is $H_0^1(\mathcal{G})$, i.e.

$$H_1^0(\mathcal{G}) = \{u \in H^1(\mathcal{G}) \,:\, \gamma u = 0 \text{ on } \partial\mathcal{G}\}.$$

Another important consequence of Lemma 1.2 is that the seminorm $|u|_{H^{1/2}(\partial\mathcal{G})}$ introduced can be replaced by the following equivalent seminorm

**Definition 1.8** ($H^{1/2}(\partial\mathcal{G})$) *Let $\mathcal{G}$ be a bounded Lipschitz region in $\Re^d$. Let $u$ be a square integrable function with respect to the hypersurface measure $dS$. We define the norm and seminorm for the space $H^{1/2}(\partial\mathcal{G})$ by*

$$|u|_{H^{1/2}(\partial\mathcal{G})} = \inf_{v \in H^1(\mathcal{G}), \gamma v = u} |v|_{H^1(\mathcal{G})}, \tag{1.9}$$

*and*

$$\|u\|^2_{H^{1/2}(\partial\mathcal{G})} = |u|^2_{H^{1/2}(\partial\mathcal{G})} + \frac{1}{H_{\mathcal{G}}^{2s}} \|u\|^2_{L^2(\partial\mathcal{G})}, \tag{1.10}$$

*respectively.*

We now introduce spaces that will be used in the mixed formulation of elliptic problems.

**Definition 1.9** ($H^{-1/2}(\partial\mathcal{G})$) *The dual space of $H^{1/2}(\partial\mathcal{G})$, is denoted by $H^{-1/2}(\partial\mathcal{G})$ and is a Hilbert space with the norm*

$$\|u\|_{H^{-1/2}(\partial\mathcal{G})} = \sup_{v \in H^{1/2}(\partial\mathcal{G})} \frac{\int_{\partial\mathcal{G}} u\, v\, dS}{\|v\|_{H^{1/2}(\partial\mathcal{G})}}.$$

**Definition 1.10** ($H(\mathbf{div};\mathcal{G})$) *The space $H(div;\mathcal{G})$ is defined by*

$$H(div;\mathcal{G}) = \{\mathbf{p} = (p_i)_{(1\le i\le d)} \in (L^2(\mathcal{G}))^d : div\,\mathbf{p} \in L^2(\mathcal{G})\}$$

*and is a Hilbert space with the usual graph norm*

$$\|\mathbf{p}\|^2_{H(div;\mathcal{G})} = \sum_{i=1}^{d} \|p_i\|^2_{L^2(\mathcal{G})} + \|div\,\mathbf{p}\|^2_{L^2(\mathcal{G})}.$$

We know that for a Lipschitz region, the unit normal $\mathbf{n}$ to the boundary $\partial\mathcal{G}$ is defined almost everywhere. Thus, for a smooth vector function $\mathbf{p}$ on $\mathcal{G}$, the normal component $\mathbf{p}\cdot\mathbf{n}$ on $\partial\mathcal{G}$ is defined almost everywhere. The following lemma extend the notion of the normal component to $H(\mathrm{div};\mathcal{G})$ functions.

**Lemma 1.3** *(***Trace and Extension theorems for** $H(\mathbf{div};\mathcal{G})$*) Let $\mathcal{G}$ be a bounded Lipschitz region. The trace map $\gamma_n : \mathbf{p} \to \mathbf{p}|_{\partial\mathcal{G}}$, defined for $C^\infty(\bar{\mathcal{G}})$, has a unique continuous extension from $H(div;\mathcal{G})$ onto $H^{-1/2}(\partial\mathcal{G})$. As a consequence, the following characterization of the norm on $H^{-1/2}(\partial\mathcal{G})$*

$$\|\mu\|_{H^{-1/2}(\partial\mathcal{G})} = \inf_{\mathbf{p}\in H(div;\mathcal{G}):\mathbf{p}\cdot\mathbf{n}=\mu} \|\mathbf{p}\|_{H(div;\mathcal{G})}, \tag{1.11}$$

*is valid.*

Moreover, we also have a Green's formula

**Lemma 1.4 (Green's Formula)** *Let $\mathcal{G}$ be a bounded Lipschitz region. Let $\mathbf{p} \in H(div;\mathcal{G})$. Then,*

$$\int_{\mathcal{G}} (\nabla v \cdot \mathbf{p} + v\, div\, \mathbf{p})\, dx = \int_{\partial\mathcal{G}} v\, \mathbf{p}\cdot\mathbf{n}\, dS \quad \forall v \in H^1(\mathcal{G}). \tag{1.12}$$

**Lemma 1.5** *(***Poincaré's inequality***) Let*

$$\bar{u} = \frac{1}{|\mathcal{G}|}\int_{\mathcal{G}} u\, dx,$$

*where we denoted $|\mathcal{G}|$ the volume of $\mathcal{G}$. Then, there exists a constant $C(\mathcal{G})$, which depends only on the Lipschitz constants of $\partial\mathcal{G}$, such that*

$$\|u - \bar{u}\|_{L^2(\mathcal{G})} \le C(\mathcal{G})\, H_{\mathcal{G}} |u|_{H^1(\mathcal{G})}, \qquad \forall u \in H^1(\mathcal{G}).$$

We also need the Poincaré-Friedrichs inequality. The idea of its proof can be found in Ciarlet (Theorem 6.1) [27] and in Nečas (Chapter 2.7.2) [62].

**Lemma 1.6 (Poincaré-Friedrichs' inequality)** *Let $\Gamma$ be a an open subset of $\partial\mathcal{G}$ with positive measure. Then there exists a constant $C_i(\mathcal{G},\Gamma)$ such that $\forall u \in H^1(\mathcal{G})$*

$$\|u\|^2_{L^2(\mathcal{G})} \le C_1(\mathcal{G},\Gamma)\, H^2_{\mathcal{G}} (|u|^2_{H^1(\mathcal{G})} + \frac{1}{H_{\mathcal{G}}}(\int_\Gamma \gamma u\, dx)^2), \tag{1.13}$$

*and*

$$\|u\|^2_{L^2(\mathcal{G})} \le C_2(\mathcal{G},\Gamma)\, H^2_{\mathcal{G}} (|u|^2_{H^1(\mathcal{G})} + \frac{1}{H_{\mathcal{G}}}\int_\Gamma (\gamma u)^2\, dx)). \tag{1.14}$$

*The constants $C_i(\mathcal{G},\Gamma)$ depend only on the Lipschitz constants of $\partial\mathcal{G}$ and on the relative area of $\Gamma$ in comparison to $\partial\mathcal{G}$.*

*Proof.*

We are only going to prove (1.13). The proof of (1.14) is similar. Consider initially a region $\mathcal{G}$ with diameter 1; to get the general result, we use a linear change of variables.

We first prove that the functional $f$, given by

$$f(v) = \int_{\Gamma} v\, dS, \tag{1.15}$$

is continuous on the space $H^1(\mathcal{G})$.

In fact

$$\begin{aligned} |f(v)| \leq \|v\|_{L^1(\Gamma)} &\leq C\,(\Gamma)\,\|v\|_{L^2(\Gamma)} \\ \leq C_1(\partial\mathcal{G},\Gamma)\|v\|_{H^{1/2}(\Gamma)} &\preceq C_2(\partial\mathcal{G},\Gamma)\,\|v\|_{H^1(\mathcal{G})} \end{aligned} \tag{1.16}$$

using the Cauchy-Schwarz inequality and a trace theorem.

We argue by contradiction assuming that (1.13) is false. Then, there exists a sequence $\{v_l\}_{l=1}^{\infty}$ such that

$$\|v_l\|_{H^1(\mathcal{G})} = 1 \quad \forall\ l, \tag{1.17}$$

and

$$\lim_{l\to\infty} (|v_l|^2_{H^1(\mathcal{G})} + (f(v_l))^2) = 0. \tag{1.18}$$

Since the sequence $\{v_l\}$ is bounded in $\|\cdot\|_{H^1(\mathcal{G})}$, we can by Rellich's theorem find a subsequence, again denoted by $\{v_l\}$, and a function $\tilde{v} \in H^1(\mathcal{G})$ such that

$$\lim_{l\to\infty} \|v_l - \tilde{v}\|_{L^2(\mathcal{G})} = 0. \tag{1.19}$$

By using (1.18), we have

$$|\tilde{v}\|^2_{H^1(\mathcal{G})} = 0 \quad \text{and} \quad f(\tilde{v}) = 0.$$

Therefore, $\tilde{v} = 0$ and

$$\lim_{l\to\infty} \|v_l - \tilde{v}\|_{H^1(\mathcal{G})} = 0\,, \tag{1.20}$$

which contradicts (1.17).

□

The following abstract lemma allows us to show well-posedness of certain elliptic problems.

**Lemma 1.7 (Lax-Milgram Lemma)** *Let $B$ be a bilinear form on a Hilbert space $\mathcal{H}$. Assume that $B$ is bounded*

$$|B(w,v)| \leq K\,\|w\|_{\mathcal{H}} \cdot \|v\|_{\mathcal{H}} \qquad \forall w, v \in \mathcal{H}$$

*and coercive, i.e. there exists a $\nu > 0$ such that*

$$B(v,v) \geq \nu\,\|v\|^2_{\mathcal{H}} \qquad \forall v \in \mathcal{H}.$$

*Then, for every bounded functional $f \in \mathcal{H}^*$, there exists a unique element $u_f \in \mathcal{H}$ such that*

$$B(u_f, v) = f(v) \qquad \forall v \in \mathcal{H},$$

*and*

$$\|u_f\|_{\mathcal{H}} \leq \frac{\|f\|_{\mathcal{H}^*}}{\nu}.$$

The counterpart of the Lax–Milgram Lemma for certain saddle point problems is given by (cf. Brezzi and Fortin [15])

**Lemma 1.8 (Babuška-Brezzi Lemma)** *Let $V$ and $Q$ be Hilbert spaces with the norms $\|\cdot\|_V$ and $\|\cdot\|_Q$, respectively. Let $a(\cdot,\cdot)$ be a continuous bilinear form on $V \times V$, let $b(\cdot,\cdot)$ be a continuous bilinear form on $V \times Q$, and Let us suppose that the range of the operator $B : V \to Q'$, defined by $(B\mathbf{p}, v) = b(\mathbf{p}, v)$, is closed in $Q'$, i.e. there exists $k_0 > 0$ such that*

$$\sup_{\mathbf{p}\in V} \frac{b(\mathbf{p},v)}{\|\mathbf{p}\|_V} \geq k_0\,\|v\|_{Q/KerB^T} = k_0\,(\inf_{v_0 \in KerB^T} \|v+v_0\|_Q) \;\; \forall v \in Q. \tag{1.21}$$

*Let us also suppose that the operator given by the bilinear form $a(\cdot,\cdot)$ is elliptic on Ker $B$, i.e. there exists $\alpha_0 > 0$, such that*

$$\left\{ \begin{array}{l} \inf_{\mathbf{q}_0 \in KerB} \sup_{\mathbf{p}_0 \in KerB} \frac{a(\mathbf{q}_0,\mathbf{p}_0)}{\|\mathbf{p}_0\|_V\,\|\mathbf{q}_0\|_V} \geq \alpha_0, \\[2ex] \inf_{\mathbf{p}_0 \in KerB} \sup_{\mathbf{q}_0 \in KerB} \frac{a(\mathbf{p}_0,\mathbf{q}_0)}{\|\mathbf{q}_0\|_V\,\|\mathbf{p}_0\|_V} \geq \alpha_0. \end{array} \right. \tag{1.22}$$

*Then the problem:*

*Find $\mathbf{q} \in V$ and $u \in Q$ such that*

$$\left\{ \begin{array}{lll} a(\mathbf{q},\mathbf{p}) + b(\mathbf{p},u) & = g(\mathbf{p}) & \forall \mathbf{p}, \in V \\ b(\mathbf{q},v) & = f(v) & \forall v \in Q, \end{array} \right.$$

*has a solution* $(\mathbf{q}, u)$ *for any* $g \in V'$ *and for any* $f \in ImB$*. The first component,* $\mathbf{q}$*, is unique and* $u$ *is defined up to an element of Ker* $B^T$*. Furthermore*

$$\|\mathbf{q}\|_V \leq \frac{1}{\alpha_0}\|g\|_{V'} + (1 + \frac{\|a\|}{\alpha_0})\,\frac{1}{k_0}\|f\|_{Q'}, \tag{1.23}$$

*and*

$$\|u\|_{Q/KerB^T} \leq \frac{1}{k_0}(1 + \frac{\|a\|}{\alpha_0})\|g\|_{V'} + \frac{\|a\|}{k_0^2}(1 + \frac{\|a\|}{\alpha_0}\|f\|_{Q'}. \tag{1.24}$$

We note that the conditions (1.21) and (1.22) of this Lemma are not only sufficient but also necessary for the existence of a solution; cf. Brezzi [14].

# Chapter 2

# Domain Decomposition Techniques

## 2.1 Iterative methods

Let

$$Ax = b \tag{2.1}$$

be a linear system of equations arising from a discretization of a symmetric elliptic problem by some finite element method. The stiffness matrix $A$ is real, symmetric, positive definite, and sparse.

*Direct methods* give exact solutions in exact arithmetic. The best known involve a triangular factorization of the matrix $A$ but there are sometimes other options, such as fast Poisson solvers. Methods based on triangular factorizations can be impractical if $n$ is large, because the factors are often much less sparse than $A$, and much more storage is necessary.

In order to fix ideas, let us assume in this paragraph that $A$ arises from a discretization of a Poisson equation with a uniform triangulation of a unit square (cube) using piecewise linear functions. Let us denote by $n$ the total degree of freedom of the system; the degree of freedom are ordered now by row. In two dimensions, the work and the storage required by the band Cholesky method are $O(n^2)$ and $O(n^{3/2})$, respectively, and in three dimensions are $O(n^{7/3})$ and $O(n^{5/3})$, respectively; cf. Golub and Van Loan [48]. For the nested dissection method, a special variant of Cholesky's method (cf. George [44]), the work and the storage required in two dimensions are $O(n^{3/2})$ and $O(n\log(n))$, respectively; see George and Liu [45]. Using the same techniques as in [45], we obtain, for three dimensions, lower bounds for the work and the storage which are $O(n^2)$ and $O(n^{4/3})$, respectively. These bounds reflect the experience that methods which are based on direct factorization of the stiffness matrix can be prohibitively expensive.

Alternatives to the direct methods are given by *iterative methods*. These methods generate a sequence of approximate solutions $\{x_k\}$ and often involve the matrix $A$ only in terms of matrix-vector multiplications and/or inversions of submatrices of $A$. Well known examples are Jacobi, Gauss-Seidel, SOR, Symmetric SOR, Block Jacobi, Block Gauss-Seidel, Chebyshev semi-iterative, and conjugate gradient type methods; cf. Golub and Van Loan [48].

### 2.1.1 Conjugate gradient type methods

The conjugate gradient algorithm (CG) is often an effective iterative algorithm to solve symmetric, positive definite systems because:

i) it does not depend upon parameters that are sometimes hard to choose properly; Chebyshev semi-iterative methods, for example, require parameter input;

ii) it has good properties with respect to complexity and storage since it is based on a three-term recurrence formula;

iii) it is founded solidly in theory;

iv) for a well conditioned matrix $A$, CG converges to a good approximate solution in relative few iterations;

v) the explicit representation of $A$ is not needed. We only need to know how to apply $A$ to a given vector;

When $A$ is not well conditioned, which is generally the case for discretizations of elliptic problems, we can introduce a preconditioner $B$ and solve the preconditioned linear system

$$BAx = Bb. \tag{2.2}$$

To obtain the preconditioned conjugate gradient algorithm (PCG), we first consider the standard CG applied to the transformed system

$$\tilde{A}\tilde{x} = \tilde{b}$$

where $\tilde{A} = B^{1/2}AB^{1/2}\tilde{x}$, $\quad \tilde{b} = B^{1/2}b$, and $\tilde{x} = B^{-1/2}x$.

**Conjugate Gradient Algorithm**

Set $k = 0$; $\tilde{x}_0 = 0$; and $\tilde{r}_0 = \tilde{b}$
**while** $\|\tilde{r}_k\|_{B^{-1}} \geq \epsilon \, \|\tilde{r}_0\|_{B^{-1}}$
  $k = k + 1$
  **if** $k = 1$
    $\tilde{p}_1 = \tilde{r}_0$
  **else**
    $\beta_k = (\tilde{r}_{k-1}, \tilde{r}_{k-1})/(\tilde{r}_{k-2}, \tilde{r}_{k-2})$
    $\tilde{p}_k = \tilde{r}_{k-1} + \beta_k \tilde{p}_{k-1}$
  **end**
  $\alpha_k = (\tilde{r}_{k-1}, \tilde{r}_{k-1})/(\tilde{p}_k, \tilde{A}\tilde{p}_k)$
  $\tilde{x}_k = \tilde{x}_{k-1} + \alpha_k \tilde{p}_k$
  $\tilde{r}_k = \tilde{r}_{k-1} - \alpha_k \tilde{A}\tilde{p}_k$
**end**
Let $m = k$

Next, we eliminate the explicit reference to the matrix $B^{1/2}AB^{1/2}$ by defining $\tilde{p}_k = B^{-1/2}p_k$, $\quad \tilde{x}_k = B^{-1/2}x_k$, $\quad \tilde{r}_k = B^{1/2}r_k$, and $z_k = Br_k$, and obtain

**Preconditioned Conjugate Gradient Algorithm**

Set $k = 0$; $x_0 = 0$; and $r_0 = b$
**while** $\|r_k\|_2 \geq \epsilon \|r_0\|_2$
  Solve $z_k = Br_k$
  $k = k + 1$
  **if** $k = 1$
    $p_1 = z_0$

**else**

$\beta_k = (r_{k-1}^T, z_{k-1})/(r_{k-2}^T, z_{k-2})$

$p_k = z_{k-1} + \beta_k p_{k-1}$

**end**

$\alpha_k = (r_{k-1}^T, z_{k-1})/(p_k, Ap_k)$

$x_k = x_{k-1} + \alpha_k p_k$

$r_k = r_{k-1} - \alpha_k A p_k$

**end**

Let $m = k$

We remark that the $\beta_k$ and $\alpha_k$ are the same for both algorithms. We can now use that $\|x - x_k\|_A = \|\tilde{x} - \tilde{x}_k\|_{\tilde{A}}$, where $\|w\|_A = \sqrt{w^T A w}$, and a well known formula for the reduction in the energy norm of the error after $k$ steps of the standard CG iterations [54], to obtain

$$\|x - x_k\|_A = \|\tilde{x} - \tilde{x}_k\|_{\tilde{A}} \leq 2\,(\frac{\sqrt{\kappa(\tilde{A})} - 1}{\sqrt{\kappa(\tilde{A})} + 1})^k\, \|\tilde{x} - \tilde{x}_0\|_{\tilde{A}}$$

$$= 2\,(\frac{\sqrt{\kappa(\tilde{A})} - 1}{\sqrt{\kappa(\tilde{A})} + 1})^k\, \|x - x_0\|_A.$$

Here, $\kappa = \kappa(\tilde{A})$ is the condition number of $BA$ given by

$$\kappa = \frac{\lambda_{max}(\tilde{A})}{\lambda_{min}(\tilde{A})}.$$

An essential feature of PCG is that an explicit representation of $A$ and $B$ are not needed. We only need to know how to apply them to a given vector. The preconditioner $B$ should be chosen with the following properties:

i) the computation of the $Br_k$ should be easily and efficiently realizable on scalar as well as parallel machines;

ii) the condition number $\kappa(\tilde{A})$ should be small to guarantee that the PCG converges in a small number of iterations.

An effective way of constructing such preconditioner is based on domain decomposition techniques. Other preconditioners that have been used extensively in practice are the diagonal scaling, multigrid, and incomplete Cholesky methods.

### 2.1.2 Extremal eigenvalues by Lanczos algorithm

In this thesis, we will study domain decomposition preconditioners based on an general theoretical framework which has previously been developed by Dryja and Widlund. Using this framework, we will be able to analyze and establish upper bounds for the condition number of our preconditioner systems. To see how sharp these upper bounds are, we may compute $\kappa(\tilde{A})$ approximately by using a generalized Lanczos procedure for eigenvalue problems; see [83], [69]. We note that the Lanczos algorithm is closed related to the CG algorithm. Both algorithms use Krylov subspaces and a three-term recurrence formulas; see [81],[48],[71], [69]. We first define the matrix of normalized residual vectors $\tilde{R}_m \in \Re^{n\times m}$ by

$$\tilde{R}_m = [\frac{\tilde{r}_0}{\|\tilde{r}_0\|_2}, \cdots, \frac{\tilde{r}_{m-1}}{\|\tilde{r}_{m-1}\|_2}],$$

where the vector $\tilde{r}_k$ are the residual vectors obtained in the $CG$ algorithm for solving $\tilde{A}\tilde{x} = \tilde{b}$.

It is possible to show [48] that

$$T_m = \begin{pmatrix} \frac{1}{\alpha_1} & -\frac{\sqrt{\beta_2}}{\alpha_1} & & & \\ -\frac{\sqrt{\beta_2}}{\alpha_1} & \frac{\beta_2}{\alpha_1}+\frac{1}{\alpha_2} & -\frac{\sqrt{\beta_3}}{\alpha_2} & & \\ & -\frac{\sqrt{\beta_3}}{\alpha_2} & \frac{\beta_3}{\alpha_2}+\frac{1}{\alpha_3} & \ddots & \\ & & \ddots & \ddots & -\frac{\sqrt{\beta_m}}{\alpha_{m-1}} \\ & & & -\frac{\sqrt{\beta_m}}{\alpha_{m-1}} & \frac{\beta_k}{\alpha_{k-1}}+\frac{1}{\alpha_k} \end{pmatrix}, \tag{2.3}$$

where $T_m = \tilde{R}_m^T \tilde{A} \tilde{R}_m$ is a $\Re^{m\times m}$ tridiagonal matrix; see [48]. The $\alpha_k$ and $\beta_k$ are the parameters of the CG algorithm, and they are readily available during the PCG algorithm. Thus, we reduce to the problem to finding the condition number of the tridiagonal, and relatively small matrix $T_m$. Questions related to convergence of the extremal eigenvalues of $T_m$ to those of $BA$ are considered in [69].

## 2.2 The Classical alternating Schwarz method

It is believed that the first domain decomposition method was proposed by Hermann Amandus Schwarz in 1869 [78]. It was originally used to show the existence of the solution of an elliptic boundary value problem on domains that consists of the union of simple overlapping subregions. In each subregion it is a priori known how to solve the elliptic problem. For instance, assume that we have two overlapping rectangles $\Omega_1$ and $\Omega_2$; see Fig. 2.1. Since we can use separation of variables techniques to solve in each $\Omega_i'$ separately, we can use Schwarz ideas to solve the elliptic problem in $\Omega_1' \cup \Omega_2'$. (If $\Omega_1'$ were a disk, we could use the Poisson formula for the corresponding subproblem.) For more than two overlapping subregions, we can use recursion.

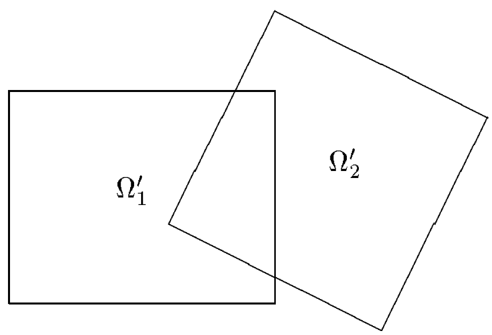


Fig. 2.1

Let $\Omega = \Omega_1' \cup \Omega_2'$ be the domain shown in Fig 2.1. Let $\Gamma_i$, $i = 1, 2$, denote the part of the boundary of $\Omega_i'$ which is in the interior of $\Omega$. Let us assume that we want to solve

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

The classical alternating Schwarz method, also called the *multiplicative Schwarz method* (MSM), consists of constructing a sequence of functions $\{u^k\}$ converging to the solution $u$ of (2.4). The sequence is constructed in the following way:

Step i) Choose a suitable initial guess $u^0$ defined on $\Omega$, and let $k = 0$.

Step ii) Solve

$$\begin{cases} -\Delta u_1^{k+1/2} = f & \text{in } \Omega_1', \\ u_1^{k+1/2} = u^k & \text{on } \Gamma_1 \\ u_1^{k+1/2} = 0 & \text{on } \partial\Omega_1' \backslash \Gamma_1. \end{cases}$$

Step iii) Solve

$$\begin{cases} -\Delta u_2^{k+1} = f & \text{in } \Omega_2', \\ u_2^{k+1} = u_1^{k+1/2} & \text{on } \Gamma_2 \\ u_2^{k+1} = 0 & \text{on } \partial\Omega_2' \backslash \Gamma_2. \end{cases}$$

Step iv) Let $k = k + 1$ and go to Step ii).

The convergence of the sequence $\{u^k\}$ was first established by Schwarz using maximum principle techniques.

## 2.3 A variational formulation

We also can interpret the alternating Schwarz algorithm in a variational framework; see e.g. Dryja and Widlund [36], Matsokin and Nepomnyaschikh [59], Nepomnyaschikh [61], and Lions [53]. Consider the problem:
Find $u \in H_0^1(\Omega)$ such that

$$a(u, v) = f(v) \quad \forall v \in H_0^1(\Omega), \tag{2.5}$$

where

$$a(u, v) = \int_\Omega \rho(x) \, \nabla u \cdot \nabla v \, dx, \qquad \text{and} \quad f(v) = \int_\Omega f \, v \, dx.$$

Here, the coefficient $\rho \in L^\infty$ satisfies $0 < c \leq \rho(x) \leq C < \infty$, almost everwhere on $\Omega$.

In all that follows, we consider $H_0^1(\Omega_1')$ and $H_0^1(\Omega_2')$ as closed subspaces of $H_0^1(\Omega)$ by extending their elements to $\Omega \backslash \Omega_i$ by zero. We can use the alternating Schwarz algorithm for numerical computations by considering finite element subspaces $V^h \subset H_0^1(\Omega)$ and $V_i^h = V^h \cap H_0^1(\Omega_i')$. In order to consider the continuous and the finite element cases simultaneously, we let $V = H_0^1(\Omega)$ and $V_i = H_0^1(\Omega_i')$, or let $V = V^h$ and $V_i = V_i^h$.

The *classical alternating Schwarz* algorithm can be written as

Step i) choose an initial guess $u^0 \in V$. Let $k = 0$.

Step ii) find $u^{k+1/2} - u^k \in V_1$ such that

$$a(u^{k+1/2} - u^k, v) = f(v) - a(u^k, v) \quad \forall v \in V_1.$$

Step iii) find $u^{k+1} - u^{k+1/2} \in V_2$ such that

$$a(u^{k+1} - u^{k+1/2}, v) = f(v) - a(u^{k+1/2}, v) \quad \forall v \in V_2.$$

Step iv) let $k = k + 1$, and go to Step ii).

Let us define the orthogonal projection $P_i : V \to V_i$, by

$$a(P_i u, v) = a(u, v) \quad \forall v \in V_i, \quad i = 1, 2,$$

and let us introduce the error $e^k$, for integer $k \geq 0$, by

$$e^k = u^k - u,$$

$$e^{k+1/2} = u^{k+1/2} - u.$$

Here, $u$ is the solution of (2.5). It is easy to check that

$$e^{k+1/2} = (I - P_1)e^k,$$

$$e^{k+1} = (I - P_2)e^{k+1/2},$$

and that the error propagation operator $E$ for a complete step of the classical alternating Schwarz algorithm is given by

$$E = (I - P_2)(I - P_1).$$

The formalism also allows us to introduce inexact solvers for the subproblems. For each subspace $V_i$, we introduce a symmetric, positive definite bilinear form $b_i(\cdot, \cdot)$ defined on $V_i \times V_i$, and an operator $T_i : V \to V_i$ defined by

$$b_i(T_i u, v) = a(u, v) \quad \forall\ v \in V_i. \tag{2.6}$$

The $b_i(\cdot, \cdot)$ can be regarded as an approximation of $a(\cdot, \cdot)$. When we use exact solvers for the subproblems, i.e. $b_i(\cdot, \cdot) = a(\cdot, \cdot)$, we obtain $T_i = P_i$.

Next, we show how to view $T_i$ in a matrix form. Let us define the linear operator $A : V \to V'$ by

$$(Au, v) = a(u, v) \quad \forall v \in V,$$

and the linear operators $B_i^{-1} : V_i \to V_i'$ by

$$(B_i^{-1} u, v) = b_i(u, v) \quad \forall v \in V_i.$$

Here, the $V_i'$ and $V'$ are the dual spaces of the $V_i$ and $V$, respectively, with respect to the inner product $(\cdot, \cdot)$. It follows from the properties of the bilinear forms $b_i(\cdot, \cdot)$ that the $B_i^{-1}$ are symmetric, positive definite operators with respect to $(\cdot,\cdot)$. It is easy to see that the $T_i$ are given by

$$T_i = B_i A.$$

The product $B_i A$ is well defined since $V' \subset V_i'$. It is also easy to see that the $T_i$ are symmetric with respect to the inner product $a(\cdot,\cdot)$. If we use exact solvers, then $b_i(\cdot,\cdot) = a(\cdot,\cdot)$ and $B_i = A_i^{-1}$. Here, the operator $A_i$ is given by

$$(A_i u, v) = a(u,v) \quad \forall u, v \in V_i.$$

We can regard the classical alternating Schwarz algorithm as a simple iterative method for solving the equation

$$Tu = (I - E)u = (T_1 + T_2 - T_2 T_1)u = g_T,$$

where $g_T = T_1 u + T_2 u - T_2 T_1 u$. We note that $g_T$ can be computed, without the knowledge of the solution of (2.5), since we can find the $g_i = T_i u$ and $g_{21} = T_2 T_1 u$ by solving

$$a(g_i, v) = a(u,v) = f(v), \quad v \in V_i, \quad i = 1, 2,$$

and

$$a(g_{21} u, v) = a(g_1, v), \quad v \in V_2.$$

## 2.4 Multidomain Schwarz methods

We now consider algorithms based on many overlapping open subdomains $\Omega_i'$, $i = 1, \cdots N$, which cover $\Omega$. For each subdomain $\Omega_i'$, $i = 1, \cdots N$, we introduce a space $V_i \subset V$, a symmetric positive definite bilinear form $b_i(\cdot,\cdot)$, and a local operators $T_i$ given as in (2.6). The *multiplicative Schwarz algorithm* is a straightforwardly extension of the classical alternating Schwarz algorithm to $N$ subdomains once has been an order selected for the subproblems. We note, however, that the extension of the theory originally provided some difficulties; see Bramble, Pasciak, Wang, and Xu [7], Cai and Widlund [19], Dryja and Widlund [41], and Xu [91]. If a large number subdomains is used then the convergence rates of this method will typically deteriorate rapidly with the number of subdomains. This happens because such a method does not provide global communication of information in each iteration; information is passed only between neighboring subdomains. The most commonly used mechanism of transmitting global information is to use a *coarse space* $V_0 \subset V$ and solving an appropriate problem on a coarse grid. We also introduce a bilinear form $b_0(\cdot,\cdot)$, and an operator $T_0$ for the space $V_0$.

**Multiplicative Schwarz method**

i) *compute* $g_i = T_i u$, *for* $i = 0, 1, \cdots, N$;

ii) *given* $u^k$, *compute* $u^{k+1}$ *in* $N+1$ *fractional steps*:

$$u^{k+\frac{i+1}{N+1}} = u^{k+\frac{i}{N+1}} + (g_i - T_i u^{k+\frac{1}{N+1}}), \qquad i = 1, \cdots, N.$$

This algorithm can be viewed as a simple iterative method for solving

$$T_{ms} u = (I - (I - T_N) \cdots (I - T_0))u = g_{ms},$$

with an appropriate right-hand side $g_{ms}$.

It is easy to see that the error $e^k = u^k - u$ satisfies

$$e^{k+1} = E_N e^k,$$

where $E_N$ is the error propagation operator

$$E_N = (I - T_N) \cdots (I - T_0).$$

We notice that the operator $T_{ms}$ is generally a nonsymmetric operator. It can be accelerated by the GMRES method or another conjugate gradient type method designed for nonsymmetric operators; cf. Saad and Schultz [75].

**Accelerated multiplicative Schwarz algorithm**

i) *compute* $g = (I - E_N)u$.

ii) *solve the nonsymmetric operator equation*

$$(I - E_N)v = g$$

*by the GMRES algorithm.*

We can solve the problem by PCG using the $a(\cdot,\cdot)$ inner product, by considering the following *symmetrized multiplicative Schwarz* operator:

$$T_{sms} = I - (I - T_0)\cdots(I - T_N)(I - T_N)\cdots(I - T_0).$$

Another possibility (cf. Dryja, Smith, and Widlund [35]) is to replace $T_{sms}$ by

$$I - (I - T_0)\cdots(I - T_N)\cdots(I - T_0),$$

or

$$T_{sm} + T_{sm}^T.$$

We have used that the $T_i$ are symmetric operators with respect to the inner product $a(\cdot,\cdot)$ in order to guarantee that the $T_{sms}$ are symmetric operators with respect to $a(\cdot,\cdot)$.

We note that all preconditioned system that we introduced in this chapter can be written as

$$T = poly(T_0, T_1, \cdots, T_N),$$

a polynomial of the $T_i'$s such that $poly(0, 0, \cdots, 0) = 0$ so that $g = Tu$ can be computed without knowing the solution $u$ itself. It is easy to see that if $T$ is invertible, then the solution $u$ of (2.5) is the only solution of $Tv = g$; see Cai [87].

In the multiplicative Schwarz algorithm, each iteration involves $N + 1$ sequential fractional steps and this is not ideal for parallel computing if $N$ is large; on an abstract level, $T$ is not ideal for parallel computing if the degree of the polynomial $poly$ is large. We can decrease the degree of the polynomial by grouping the subregions using the following coloring strategy. Associate with the decomposition $\{\Omega_i'\}$, an undirected graph in which the nodes represent the subdomains $\{\Omega_i'\}$ and the edges intersections of subdomains. This graph can be colored, using colors $1, \cdots, J$, such that no connected nodes have the same color. We now group the $T_i$ in term of the color that the subdomain has been assigned. We obtain

$$T_{ms} = I - (I - T_J)\cdots(I - T_0).$$

We note that different $T_i$ of the same color correspond to domains that are mutually disconnected; these subproblems can be solved in parallel.

To remove completely the inherent sequential behavior of the $T_{ms}$, Dryja and Widlund [36], and Matsokin and Nepomnyaschikh [59] introduced the *additive Schwarz methods*. The basic idea is to choose the simplest (nontrivial lowest degree) polynomial of operators $T_i$'s, namely

$$T_{as} = T_0 + T_1 + \cdots + T_N.$$

**Accelerated Additive Schwarz method**

i) *compute* $g = T_{as}u$.

ii) *solve the operator equation*

$$Tv = g$$

*by the preconditioned conjugate gradient method using the* $a(\cdot,\cdot)$ *inner product.*

In practice, despite less parallelism, the multiplicative Schwarz methods-(MS) are often substantially faster than the additive Schwarz methods-(AS) since their algebraic convergence rates tend to be higher. As we have noted, the parallelism of a MS results mainly from the fact that for each $j = 1, \cdots, J$, $T_j$ is a sum a number of local, independent subproblems that can be handled in parallel. We note, however, that the coarse problem cannot be solved simultaneously with the local problems. This is in contrast to an AS in which all subproblems, including $T_0$, can be solved in parallel. In a MS, there is, therefore, a potential bottleneck with many processor idly waiting for the solution of the coarse problem. Motivated by this, Cai [18] introduced a hybrid Schwarz preconditioner given by

$$T_{cai} = \gamma T_0 + I - (I - T_J)\cdots(I - T_0)$$

for which combines advantages of the AS and MS. Here, $\gamma > 0$ is a balancing parameter. A symmetrized version is given by

$$\gamma T_0 + I - (I - T_1)\cdots(I - T_J)(I - T_J)\cdots(I - T_1).$$

Another hybrid method was introduced in Mandel and Brezina [55]

$$T_{man} = T_0 + (I - T_0)(T_1 + \cdots + T_N)(I - T_0).$$

This algorithm, however, does not have the same potential for parallelism as Cai's algorithm but has other advantages.

## 2.5 Abstract theorems

### 2.5.1 Additive Schwarz methods

Let $V$ be a finite dimensional space. Consider the following abstract variational problem:

Find $u \in V$ such that

$$a(u,v) = f(v) \quad \forall\ v \in V. \tag{2.7}$$

The bilinear form $a(\cdot,\cdot)$ is symmetric positive definite, and $f$ is a continuous linear functional on $V$.

An additive Schwarz method is defined by decomposing $V$ into a sum of $N+1$ subspaces:

$$V = V_0 + V_1 + \cdots + V_N.$$

We note that this decomposition is not necessarily a direct sum of subspaces; in many applications of interest, the representation of an element of $V$ in terms of components of the $V_i$ is not unique. Often, these subspaces are related to a decomposition of the domain $\Omega$ into overlapping subregions. The space $V_0$ represents the coarse subspace and it is added to the algorithm to provide global communication of information in each iteration. We note that Widlund [87] has shown that, without a global coarse space problem, the condition number of the relevant iteration operator must grow at least like $1/H^2$, where $H$ is the diameter of the subregions.

Let

$$T_{as} = T_0 + T_1 + \cdots + T_N.$$

We now replace (2.7) by the problem:

Find $\tilde{u} \in V$ such that

$$T_{as}\tilde{u} = g_{as}, \quad g_{as} = \sum_{i=0}^{N} g_i, \quad g_i = T_i u. \tag{2.8}$$

By construction, the solution $u$ of (2.7) is also a solution of (2.8). We note that $g_{as}$ can be computed, without knowledge of the solution of (2.7), since we can find the $g_i$ by solving

$$b_i(g_i, v_i) = a(u, v_i) = f(v_i) \quad \forall v_i \in V_i.$$

It is very easy to check that the operator $T_{as}$ is symmetric with respect to the inner product $a(\cdot,\cdot)$. The reason for replacing the problem (2.7) by (2.8) is that, by a suitable choice of subspace $V_i$ and bilinear forms $b_i(\cdot,\cdot)$, we can transform a large ill-conditioned system into a very well conditioned system problem at the expense of solving many small independent linear problems. The equation (2.8) is typically solved by a conjugate gradient method, without further preconditioning, using $a(\cdot,\cdot)$ as the inner product. In order to see that the problem (2.8) has $u$ as the unique solution, and also to estimate the rate of convergence of the preconditioned conjugate gradient method, we need to obtain upper and lower bounds for the spectrum of $T_{as}$. The bounds are obtained by using the following theorem; cf. Dryja and Widlund [37, 38, 41], Zhang [95, 94].

**Theorem 2.1** *Suppose the following three assumptions hold:*

*i) There exists a constant $C_0$ such that for all $v \in V$ there exists a decomposition $v = \sum_{k=0}^{N} u_i,\ \ v_i \in V_i$, such that*

$$\sum_{i=0}^{N} b_i(v_i, v_i) \leq C_0^2 a(v, v).$$

*ii) There exists a constant $\omega > 0$ such that*

$$a(v, v) \leq \omega b_i(v, v) \ \ \forall v \in V_i, \quad i = 1, \cdots, N.$$

*iii) There exist constants $\epsilon_{ij}$, $i, j = 1, \cdots, N$, such that*

$$a(v_i, v_j) \leq \epsilon_{ij}\, a(v_i, v_i)^{1/2} a(v_j, v_j)^{1/2}$$

$$\forall v_i \in V_i \quad \forall v_j \in V_j.$$

*Then,*

$$C_0^{-2} a(v, v) \leq a(T_{as} v, v) \leq (\rho(\mathcal{E}) + 1)\omega a(v, v) \quad \forall v \in V. \tag{2.9}$$

*Here $\rho(\mathcal{E})$ is the spectral radius of the matrix $\mathcal{E} = \{\epsilon_{ij}\}_{i,j=1}^{N}$.*

*Proof.* The left inequality: Using the definition of the $T_i$, the Cauchy-Schwarz inequality, and assumption (i), we obtain

$$a(v, v) = \sum_{i=0}^{N} a(v, v_i) = \sum_{i=0}^{N} b_i(T_i v, v_i)$$

$$\leq (\sum_{i=0}^{N} b_i(T_i v, T_i v))^{1/2} (\sum_{i=0}^{N} b_i(v_i, v_i))^{1/2}$$
$$\leq (\sum_{i=0}^{N} a(v, T_i v))^{1/2}\, C_0 (a(v,v))^{1/2},$$

Therefore,

$$a(v,v) \leq C_0^2 a(T_{as} v, v).$$

The right inequality: We first note that $\|T_i\|_a \leq \omega$. Indeed, using the definition of the $T_i$ and *Assumption* (i), we obtain

$$a(T_i u, T_i u) \leq \omega b_i(T_i u, T_i u) = \omega a(v, T_i v)$$
$$\leq \omega\, a(v,v)^{1/2}\, a(T_i v, T_i v)^{1/2}.$$

Therefore,

$$a(T_i v, T_i v) \leq \omega^2\, a(v,v).$$

To prove the right inequality, we use the *Assumption* (iii) and (ii), and the definition of $T_i$:

$$a(\sum_{i=1}^{N} T_i v, \sum_{i=1}^{N} T_i v) = \sum_{i,j=1}^{N} a(T_i v, T_j v)$$
$$\leq \sum_{i,j=1}^{N} \epsilon_{ij} a(T_i v, T_i v)^{1/2}\, a(T_j v, T_j v)^{1/2} \leq \rho(\mathcal{E}) \sum_{i=1}^{N} a(T_i v, T_i v)$$
$$\leq \rho(\mathcal{E})\omega \sum_{i=1}^{N} b_i(T_i v, T_i v) = \rho(\mathcal{E})\omega \sum_{i=1}^{N} a(v, T_i v)$$
$$= \rho(\mathcal{E})\omega a(v, \sum_{i=1}^{N} T_i v) \leq \rho(\mathcal{E})\omega a(v,v)^{1/2}\, a(\sum_{i=1}^{N} T_i v, \sum_{i=1}^{N} T_i v)^{1/2}.$$

Hence,

$$a(\sum_{i=1}^{N} T_i v, \sum_{i=1}^{N} T_i v) \leq \rho(\mathcal{E})^2 \omega^2 a(v,v),$$

and therefore

$$a(\sum_{i=1}^{N} T_i v, v) \leq \rho(\mathcal{E})\, \omega\, a(v,v).$$

This last inequality, added to

$$a(T_0 v, v) \leq \omega a(v,v),$$

completes the proof.

$\square$

We remark that the abstract theory easily can be extended to cases where there are several coarse spaces. If there are two coarse spaces, we exclude them both when considering the strengthened Cauchy-Schwarz inequalities of the *Assumption* $(iii)$ and the factor $(\rho(\mathcal{E})+1)$ in the theorem is replaced by $(\rho(\mathcal{E})+2)$.

### 2.5.2 Multiplicative Schwarz methods

Since the error $e^k = u^k - u$ satisfies $e^{k+1} = E_N e^k$, the reduction of the error in the MS algorithm can be estimated from above by

$$\|E_N\|_a = sup_{u \in V} \frac{a(E_N u, E_N u)^{1/2}}{a(u,u)^{1/2}}.$$

**Theorem 2.2** *Suppose the three assumptions of Theorem 2.1 hold. Assume further that* $\omega < 2$. *Then,*

$$\|E_N\|_a^2 \leq 1 - \frac{(2-\hat{\omega})}{(1+2\hat{\omega}^2\rho(\mathcal{E})^2)C_0^2} \tag{2.10}$$

*Here,* $\hat{\omega} = \max(1,\omega)$.

We remark that from the definition of $T_i$ that $\|T_i\|_a \leq \omega$; see the proof of Theorem 2.1. The assumption that $\omega < 2$ is natural because $\|T_i\|_a > 2$ implies $\|I - T_i\|_a > 1$, and then we cannot guarantee that $\|E_N\|_a < 1$. If $\omega$ is too large, we can scale the bilinear forms $b_i(\cdot,\cdot)$ properly to obtain $\|T_i\|_a = 1$. If the $\|T_i\|_a$ are very small then $\|E_N\|_a$ is close to 1 and the MS will converge slowly; we also can see this by (2.10) since if we scale the $b_i(\cdot,\cdot)$ to make $\omega$ small, the $C_0$ must become large.

**Lemma 2.1** *Under the same assumptions as in Lemma 2.2, we obtain*

$$\frac{(2-\hat{\omega})}{(1+2\hat{\omega}^2\rho(\mathcal{E})^2)C_0^2}\, a(u,u) \leq a(T_{sms}u,u) \leq a(u,u) \qquad \forall u \in V.$$

*Proof.* Using that $T_{sms} = I - E_N^T E_N$, we obtain

$$a(T_{sms}u,u) = a(u,u) - a(E_N u, E_N u) \geq (1+2\hat{\omega}^2\rho(\mathcal{E})^2)C_0^2\, a(u,u),$$

and

$$a(T_{sms}u,u) = a(u,u) - a(E_N u, E_N u) \leq a(u,u).$$

$\square$

# Chapter 3

# Differential and Finite Element Problems

## 3.1 The elliptic problem

Let us consider the following selfadjoint second order problem:

Find $u \in H_0^1(\Omega)$, such that

$$a(u,v) = f(v) \;\; \forall \; v \in H_0^1(\Omega), \tag{3.1}$$

where

$$a(u,v) = \int_\Omega \mathbf{A}(x)\,\nabla u \cdot \nabla v \, dx \quad \text{and} \quad f(v) = \int_\Omega fv \, dx \;\; \text{for} \;\; f \in L^2(\Omega). \tag{3.2}$$

For simplicity, let $\Omega$ be a bounded polyhedral region in $\Re^3$ with a diameter of order 1.

We assume that $\mathbf{A}$ is a three-by-three symmetric matrix-valued Lebesque measurable function on $\Omega$, such that

$$0 < \lambda_{min}|\xi|^2 \le \xi^T \mathbf{A}(x)\xi \le \lambda_{max}|\xi|^2 \;\; \forall \xi \in \Re^3 \;\; \text{a.e } x \in \Omega.$$

The well-posedness of the problem (3.1) is shown by checking the hypotheses of the Lax-Milgram Lemma with

$$\mathcal{H} = H_0^1(\Omega), \quad \text{and} \quad \|\cdot\|_{\mathcal{H}} = \|\cdot\|_{H^1(\Omega)}.$$

i) The boundness is obtained by

$$a(w,v) \le \lambda_{max}|w|_{H^1(\Omega)} \cdot |v|_{H^1(\Omega)} \le \lambda_{max}\|w\|_{H^1(\Omega)} \cdot \|v\|_{H^1(\Omega)}.$$

ii) The coerciveness is obtained by using Friedrichs' inequality for the space $H_0^1(\Omega)$.

$$a(v,v) \geq \lambda_{min} |v|^2_{H^1(\Omega)} \geq c(\Omega)\, \lambda_{min}\, \|v\|^2_{H^1(\Omega)} \qquad \forall v \in H_0^1(\Omega).$$

Hence, the solution $u$ of (3.1) satisfies

$$\|u\|_{H^1(\Omega)} \leq C(\Omega)\, \frac{\|f\|_{H^{-1}}}{\lambda_{min}} \leq C(\Omega)\, \frac{\|f\|_{L^2(\Omega)}}{\lambda_{min}}. \tag{3.3}$$

## 3.2 The finite element problem

### 3.2.1 The triangulations

A triangulation of $\Omega$ is introduced by dividing the region into nonoverlapping shape regular simplices $\{\Omega_i\}_{i=1}^N$, with diameters of order $H$, which are called substructures or subdomains. This partitioning induces a coarse triangulation $\mathcal{T}^H$ associated with the parameter $H$. We note that our results can be extended straightforwardly without the quasi-uniformity assumption since only local arguments are used; see further Section further 4.8.

#### The two level refinements

In the case of two-level methods, the substructures $\Omega_i$ are further divided into elements $\tau_j^h$ in such way that a conforming triangulation of all of $\Omega$ is obtained. We associate a parameter $h$ to the finest triangulation and denote this triangulation by $\mathcal{T}^h$. Let $V^h(\Omega)$ be the finite element space of continuous, piecewise linear functions, defined on the fine triangulation $\mathcal{T}^h$, and let $V_0^h(\Omega)$ be the subspace of $V^h(\Omega)$ of functions which vanish on $\partial\Omega$, the boundary of $\Omega$. We note that, for *two level methods*, the substructures $\Omega_i$ can be chosen in a more general way; see further Remark 5.2. As in the coarse triangulation, only the shape regularity of the finest triangulation is necessary for our results.

#### The multilevel refinements

In the case of *multilevel methods*, we define a sequence of quasi-uniform nested triangulations $\{\mathcal{T}^k\}_{k=0}^{\ell}$ as follows. We start with a coarse triangulation $\mathcal{T}^0 = \{\Omega_i\}_{i=1}^N$ and set $h_0 = H$. A triangulation $\mathcal{T}^k = \{\tau_j^k\}_{j=1}^{N^k}$ on level $k$ is obtained by subdividing each individual element $\tau_j^{k-1}$ in the set $\mathcal{T}^{k-1}$ into several elements denoted by $\tau_j^k$. We assume that all the triangulations are shape regular and quasi-uniform. Let $h_j^k = \text{diameter}(\tau_j^k)$, $h_k = \max_j h_j^k$, and $h = h_\ell$, where $\ell$ is the number of refinement levels. We denote $\mathcal{T}^h = \mathcal{T}^\ell$, and $\tau_j^h = \tau_j^\ell$.

We also assume that there exist constants $\gamma < 1$, $c > 0$, and $C$, such that if an element $\tau_j^{n+k}$ of level $n+k$ is contained in an element $\tau_j^k$ of level $k$, then

$$c\gamma^n \le \frac{\text{diam}(\tau_j^{n+k})}{\text{diam}(\tau_j^k)} \le C\gamma^n. \tag{3.4}$$

A refinement procedure to obtain shape regularity and (3.4) was introduced by Ong [63]. For $k = 1, \cdots, \ell$, all the tetrahedra $\tau_j^{k-1} \in \mathcal{T}^{k-1}$ can be subdivided into eight tetrahedra (see, Ong [63]); these are elements of level $k$ and belong, by definition, to $\mathcal{T}^k$. A shape regular refinement is obtained by connecting properly the midpoints of the edges of $\tau_j^{k-1}$. We consider in Section 4.8 certain nonuniform refinements where our results can be extended.

For each level of triangulation, we define a finite element space $V^k(\Omega)$ which is the space of continuous piecewise linear functions associated with the triangulation $\mathcal{T}^k$. Let $V_0^k(\Omega)$ be the subspace of $V^k(\Omega)$ of functions which vanish on $\partial\Omega$. We also use the notation $V_0^h(\Omega) = V_0^\ell(\Omega)$.

### 3.2.2 The finite element problem

The discrete problem associated with (3.1) is given by:

Find $u \in V_0^h(\Omega)$, such that

$$a(u,v) = f(v) \quad \forall\ v \in \ V_0^h(\Omega). \tag{3.5}$$

The well-posedness of the problem (3.5) follows directly from Lax-Milgram's Lemma; we also have a stability result as in (3.3), with the constant independent of $h$.

## 3.3 The saddle point problem

### 3.3.1 The motivation

There are many engineering applications in which the main goal is to find a good approximation for $\mathbf{q} = -\mathbf{A}\,\nabla u$. Here, $u$ is the solution of an elliptic problem with the coefficient

$\mathbf{A}$. We can find an approximation for $\mathbf{q}$ by finding an approximation for $u$ and then applying the operator $\mathbf{A}\,\nabla$. This procedure may generate serious errors since when $\mathbf{A}$ becomes more discontinuous, the solution $u$ becomes more singular and the operator $\mathbf{A}\,\nabla$ more numerically unstable. We note that in the interior of $\Omega$ we have, formally, $\mathbf{div}\,\mathbf{q} = f$. Therefore, we expect $\mathbf{q}(x)$ to be less sensitive to variations of $\mathbf{A}(x)$ than $u(x)$, if $f$ is relatively smooth. For instance, if we consider the one-dimensional case with $f = 0$ and inhomogeneous Dirichlet data, we obtain $\mathbf{q}$ = constant. For this reason, mixed methods have been introduced in order to approximate $\mathbf{A}\nabla u$ and $u$, simultaneously. We note that our motivation for considering the nonconforming $P_1$ space in chapter 5 comes primarily from the fact that there is an equivalence between mixed methods and nonconforming methods [2]. In this section, we present all the theoretical steps which establish this equivalence, and note that the analysis is done locally, element-by-element. It follows that the most expensive part in solving the mixed problem numerically is to find the solution of a nonconforming $P_1$ problem; an approximate solution of the mixed problem is then recovered by using only element-by-element computations.

### 3.3.2 The mixed problem

Assume that $f \in L^2(\Omega)$. Then, it is easy to see that if $u$ is the solution of (3.1), then

$$\mathbf{q} = -\mathbf{A}\nabla u \quad \in \quad H(\mathbf{div};\Omega). \tag{3.6}$$

We now use the Green's formula (1.12) and density arguments to see that $(\mathbf{q}, u)$ is a solution of the following mixed formulation of (3.1):
Find $(\mathbf{q}, u) \in H(\mathbf{div};\Omega) \times L^2(\Omega)$ such that

$$\begin{cases} \int_\Omega \mathbf{A}^{-1}\,\mathbf{q}\cdot\mathbf{p}\,dx - \int_\Omega u\,\mathbf{div}\,\mathbf{p}\,dx = 0 & \forall \mathbf{p} \in H(\mathbf{div};\Omega) \\ -\int_\Omega v\,\mathbf{div}\mathbf{q}\,dx \qquad\qquad = -\int_\Omega f\,v\,dx & \forall v \in L^2(\Omega). \end{cases} \tag{3.7}$$

We obtain well-posedness for the problem (3.7) by checking the hypotheses of Lemma 1.8 with

$$Q = L^2(\Omega), \quad V = H(\mathbf{div};\Omega),$$

$$a(\mathbf{q},\mathbf{p}) = \int_\Omega \mathbf{A}^{-1}\,\mathbf{q}\cdot\mathbf{p}\,dx \quad \text{and,} \quad b(\mathbf{q}, v) = -\int_\Omega v\,\mathbf{div}\mathbf{q}\,dx.$$

Using the definition of $B$, we have

$$\mathrm{Ker}B = \{\mathbf{p}^0 \in H(\mathbf{div};\Omega) : \mathbf{div}\,\mathbf{p}^0 = 0\}. \tag{3.8}$$

We also have

$$\mathrm{Im}B = L^2(\Omega). \tag{3.9}$$

To show (3.9), we have to show that for every $f \in L^2(\Omega)$, there exists a $\mathbf{p} \in H(\mathbf{div};\Omega)$ such that $B\,\mathbf{p} = f$. Indeed, in a first step we solve: Find $w \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla\, w \cdot \nabla\, \psi\, dx = \int_\Omega f\, \psi\, dx \qquad \forall \psi \in H_0^1(\Omega),$$

and set $\mathbf{p} = \nabla w$.

We obtain straightforwardly, by using (3.9), that

$$\mathrm{Ker}B^T = \emptyset.$$

We also obtain $\|a\| \leq \frac{1}{\lambda_{min}}$ since

$$a(\mathbf{p},\mathbf{q}) \leq \frac{1}{\lambda_{min}} \|\mathbf{p}\|_{L^2(\Omega)}\, \|\mathbf{q}\|_{L^2(\Omega)}$$

$$\leq \frac{1}{\lambda_{min}} \|\mathbf{p}\|_{H(\mathbf{div};\Omega)}\, \|\mathbf{q}\|_{H(\mathbf{div};\Omega)} \quad \forall \mathbf{p},\mathbf{q} \in H(\mathbf{div};\Omega).$$

We obtain $\alpha_0 \geq \frac{1}{\lambda_{max}}$ since

$$\inf_{\mathbf{q}^0 \in \mathrm{Ker}\,B}\ \sup_{\mathbf{p}^0 \in \mathrm{Ker}\,B} \frac{a(\mathbf{q}^0,\mathbf{p}^0)}{\|\mathbf{p}^0\|_V\, \|\mathbf{q}^0\|_V} \geq \inf_{\mathbf{q}^0 \in \mathrm{Ker}\,B} \frac{a(\mathbf{q}^0,\mathbf{q}^0)}{\|\mathbf{q}^0\|_V\, \|\mathbf{q}^0\|_V}$$

$$= \inf_{\mathbf{q}^0 \in \mathrm{Ker}\,B} \frac{a(\mathbf{q}^0,\mathbf{q}^0)}{\|\mathbf{q}^0\|_{L^2(\Omega)}\, \|\mathbf{q}^0\|_{L^2(\Omega)}} \geq \frac{1}{\lambda_{max}}.$$

To show that $k_0 \geq \frac{1}{C_2(\Omega)} > 0$, we first note that

$$\sup_{\mathbf{p} \in V} \frac{b(\mathbf{p},v)}{\|\mathbf{p}\|_V} \geq \frac{b(\mathbf{r},v)}{\|\mathbf{r}\|_V},$$

where $\mathbf{r} = \nabla w$, and $w$ is the solution of

$$\int_\Omega \nabla\, w \cdot \nabla\, \psi\, dx = \int_\Omega -v\, \psi\, dx \qquad \forall \psi \in H_0^1(\Omega).$$

Using the estimate

$$\|\nabla w\|_{L^2(\Omega)} \leq C_1(\Omega)\,\|v\|_{L^2(\Omega)},$$

we obtain

$$\frac{b(\mathbf{r},v)}{\|\mathbf{r}\|_V} = \frac{-\int_\Omega v\,\mathbf{div}\mathbf{r}\,dx}{\|\mathbf{r}\|_V} = \frac{\int_\Omega v^2\,dx}{\|\nabla w\|_{H(\mathbf{div};\Omega)}} =$$

$$\frac{\int_\Omega v^2\,dx}{(\|\Delta w\|^2_{L^2(\Omega)} + \|\nabla w\|^2_{L^2(\Omega)})^{1/2}} \geq \frac{1}{C_2(\Omega)}\,\frac{\int_\Omega v^2\,dx}{\|v\|_{L^2(\Omega)}} = \frac{1}{C_2(\Omega)}\,\|v\|_{L^2(\Omega)}.$$

Using the bounds for $\|a\|$, $\alpha_0$, and $k_0$ in (1.23) and (1.24), we obtain

$$\|\mathbf{q}\|_{H(\mathbf{div};\Omega)} \leq C_3(\Omega)\,\frac{\rho_{max}}{\rho_{min}}\,\|f\|_{L^2(\Omega)}, \tag{3.10}$$

and

$$\|u\|_{L^2(\Omega)} \leq C_4(\Omega)\,\frac{\rho_{max}}{\rho^2_{min}}\,\|f\|_{L^2(\Omega)}. \tag{3.11}$$

We note that the estimate (3.10) can be recovered from (3.3); we only use that the solution $(\mathbf{q},u)$ of (3.7) is unique and satisfies the relation (3.6). We note, however, that we must use the present technique above to derive (3.10) and (3.11) to obtain stability results, independent of $h$, for the discrete mixed problem.

We note that the problem (3.7) also can be viewed as the Euler-Lagrange equation of the following saddle point problem:

$$\inf_{\mathbf{p}\in H(\mathbf{div};\Omega)}\ \sup_{v\in L^2(\Omega)}\ \frac{1}{2}\int_\Omega \mathbf{A}^{-1}\,\mathbf{p}\cdot\mathbf{p}\,dx + \int_\Omega f\,v\,dx + \int_\Omega v\,\mathbf{div}\,\mathbf{p}\,dx. \tag{3.12}$$

### 3.3.3 The discrete mixed problem

Let $\hat{\tau}$ be the unit reference tetrahedron with vertices

$$\hat{a}_0 = (0,0,0),\ \ \hat{a}_1 = (1,0,0),\ \ \hat{a}_2 = (0,1,0),\ \ \text{and}\ \ \hat{a}_3 = (0,0,1).$$

The lowest order Raviart-Thomas velocity space on $\hat{\tau}$ is defined by

$$RT^0_{-1}(\hat{\tau}) = \{\mathbf{p}:\ \ \mathbf{p} = \begin{bmatrix} a_{\hat{\tau}} \\ b_{\hat{\tau}} \\ c_{\hat{\tau}} \end{bmatrix} + d_{\hat{\tau}} \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix}\}.$$

Let $\mathcal{T}^h$ be a triangulation, as before, of the three dimensional region. For a tetrahedron $\tau \in \mathcal{T}^h$ with vertices $a_0, a_1, a_2$, and $a_3$, we define an invertible, affine linear map

$F_\tau : \tau \to \hat{\tau}$, such that $F_\tau(a_i) = \hat{a}_i, i = 1, \cdots 4$. Here, $F_\tau(\hat{x}) = B_\tau \, \hat{x} + b_\tau$, where $B_\tau$ is a $3 \times 3$ invertible matrix and $b_\tau$ a 3-vector. For any scalar function $\hat{v}$ defined on $\hat{\tau}$ (resp. on $\partial\hat{\tau}$), we associate the function $v$ defined on $\tau$ (resp. on $\partial\tau$) by

$$v = \hat{v} \circ F_\tau^{-1} \quad (\hat{v} = v \circ F_\tau),$$

and for any vector-valued function $\hat{\mathbf{p}}$ defined on $\hat{\tau}$, we associate the function $\mathbf{p}$ on $\tau$ by

$$\mathbf{p} = \frac{1}{det(B_\tau)} B_\tau \, \hat{\mathbf{p}} \circ F_\tau^{-1} \qquad (\hat{\mathbf{p}} = det(B_\tau) B_\tau^{-1} \mathbf{p} \circ F_\tau). \tag{3.13}$$

The choice of the transformation (3.13) is based on the following results:

$$\int_{\hat{\tau}} \hat{v} \, \mathbf{div} \, \hat{\mathbf{p}} \, d\hat{x} = \int_\tau v \, \mathbf{div} \, \mathbf{p} \, dx \quad \forall \hat{v} \in L^2(\hat{\tau}) \;\; \forall \hat{\mathbf{p}} \in (H^1(\hat{\tau}))^3, \tag{3.14}$$

and

$$\int_{\partial\hat{\tau}} \hat{v} \, \hat{\mathbf{p}} \cdot \hat{\mathbf{n}} \, d\hat{S} = \int_{\partial\tau} v \, \mathbf{p} \cdot \mathbf{n} \, dS \quad \forall \hat{v} \in L^2(\hat{\tau}) \;\; \forall \hat{\mathbf{p}} \in (H^1(\hat{\tau}))^3.$$

The space $RT_{-1}^0(\tau)$ is defined by

$$RT_{-1}^0(\tau) = \frac{1}{det(B_\tau)} B_\tau \, RT_{-1}^0(\hat{\tau}) \circ F_\tau^{-1}. \tag{3.15}$$

It is easy to show that $RT_{-1}^0(\tau)$ consists of linear vector functions which have a constant normal component on the faces of $\tau$.

We introduce the spaces

$$RT_{-1}^0(\mathcal{T}^h) = \{\mathbf{p} : \mathbf{p} \in (L^2(\Omega))^3, \mathbf{p}|_\tau \in RT_{-1}^0(\tau) \;\; \forall \tau \in \mathcal{T}^h\},$$

$$\begin{aligned} RT_0^0(\mathcal{T}^h) = \{\mathbf{p} : \mathbf{p} \in RT_{-1}^0(\mathcal{T}^h), \; & \text{the normal component of } \mathbf{p} \\ & \text{is continuous across the interelements boundaries}\}, \end{aligned}$$

and

$$M_{-1}^0(\mathcal{T}^h) = \{v : v \in L^2(\Omega), v|_\tau = c_\tau \;\; \forall \tau \in \mathcal{T}^h\}.$$

Here, $c_\tau$ is a constant that only depends on the element $\tau$. It is easy to check that $RT_0^0(\mathcal{T}^h) = RT_{-1}^0(\mathcal{T}^h) \cap H(\mathbf{div}; \Omega)$.

The lowest order Raviart-Thomas mixed element method is given by:
Find $(\mathbf{q}_h, u_h) \in RT_0^0(\mathcal{T}^h) \times M_{-1}^0(\mathcal{T}^h)$ such that

$$\left\{ \begin{array}{ll} \int_\Omega \mathbf{A}^{-1} \, \mathbf{q}_h \cdot \mathbf{p}_h \, dx - \int_\Omega u_h \, \mathbf{div} \, \mathbf{p}_h \, dx = 0 & \forall \, \mathbf{p}_h \in RT_0^0(\mathcal{T}^h) \\ -\int_\Omega v_h \, \mathbf{div} \mathbf{q}_h \, dx \qquad\qquad = -\int_\Omega f \, v_h \, dx & \forall v_h \in M_{-1}^0(\mathcal{T}^h). \end{array} \right. \tag{3.16}$$

Let $\mathbf{q}_h = \sum_{i=1}^{n} q_i \mathbf{\Phi}_i$, and $u_h = \sum_{i=1}^{m} u_i \chi_i$, where the $\mathbf{\Phi}_i$ and the $\chi_i$ are basis elements of $RT^0_{-1}(\mathcal{T}^h)$ and $M^0_{-1}(\mathcal{T}^h)$, respectively. In this basis, the mixed problem (3.16) is of the form

$$\begin{bmatrix} A_h & B_h^T \\ B_h & 0 \end{bmatrix} \begin{bmatrix} \mathbf{q}_h \\ u_h \end{bmatrix} = \begin{bmatrix} 0 \\ f_h \end{bmatrix}. \tag{3.17}$$

Here, $A_h$ is a symmetric, positive definite matrix with $A_{hij} = \int_\Omega \mathbf{A}^{-1} \mathbf{\Phi}_i \cdot \mathbf{\Phi}_j dx$ and $B_h$ is an approximation of the divergence map which is given by $B_{hij} = -\int_\Omega \chi_j \mathbf{div} \mathbf{\Phi}_i dx$. The system (3.17) is a saddle point problem, a discrete version of (3.12). Hence, (3.17) is symmetric but indefinite and cannot be solved safely by the standard conjugate gradient method.

We again use Lemma 1.8 to show well-posedness for the discrete problem (3.16). We show that the stability results (1.23) and (1.24) are uniform in $h$. The spaces $Q$ and $V$ are given by

$$Q = M^0_{-1}(\mathcal{T}^h), \qquad V = RT^0_0(\mathcal{T}^h),$$

and the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ by (3.8).

We first note that the *discrete divergence free* space $\mathrm{Ker} B_h$ is *divergence free* in the $L^2(\Omega)$ sense, i.e.

$$\mathrm{Ker} B_h \subset \mathrm{Ker} B = \{\mathbf{p}^0 \in H(\mathbf{div};\Omega) : \nabla \cdot \mathbf{p}^0 = 0\}. \tag{3.18}$$

To show (3.18), we first reduce the arguments to one on the reference element. Hence,

$$\int_\Omega v_h (\nabla \cdot \mathbf{p}_h)\, dx = 0 \quad \forall v_h \in M^0_{-1}(\mathcal{T}^h)$$

$$\Longleftrightarrow$$

$$\int_\tau v_h (\nabla \cdot \mathbf{p}_h)\, dx = 0 \quad \forall v_h \in M^0_{-1}(\tau) \;\; \forall \tau \in \mathcal{T}^h.$$

Using (3.14), we have for any $\tau \in \mathcal{T}^h$

$$\int_\tau v_h (\nabla \cdot \mathbf{p}_h)\, dx = 0 \Longleftrightarrow \int_{\hat\tau} \hat v_h (\nabla \cdot \hat{\mathbf{p}}_h)\, d\hat x = 0.$$

We now set $\hat v_h = \nabla \cdot \hat{\mathbf{p}}_h$ to obtain

$$\nabla \cdot \hat{\mathbf{p}}_h = 0 \;\text{ in }\; \hat\tau,$$

which implies that

$$\nabla \cdot \mathbf{p}_h = 0 \;\text{ in }\; \Omega.$$

Using the same ideas as in the continuous case, we have $\|a\| \leq \frac{1}{\rho_{min}}$.

Using (3.18) and the same ideas as in the continuous case, we obtain $\alpha_0 \geq \frac{1}{\rho_{max}}$.

To show that $k_0 \geq \tilde{c}(\Omega) > 0$, with $\tilde{c}(\Omega)$ independent of $h$, we use the same ideas as the continuous case and the following lemma:

**Lemma 3.1** *For any function $v_h \in M_{-1}^0(\mathcal{T}^h)$, there exists a function $\mathbf{p}_h \in RT_0^0(\mathcal{T}^h)$ such that*

$$\mathbf{div}\,\mathbf{p}_h = v_h \quad in \quad \Omega,$$

*and*

$$\|\mathbf{p}_h\|_{H(\mathbf{div};\Omega)} \leq \tilde{C}(\Omega)\,\|v_h\|_{L^2(\Omega)}.$$

The proof of this lemma is given in Raviart and Thomas [73, 72]. The arguments in [73] are for the two-dimensional case and they can be extended, straightforwardly, to the three-dimensional case.

## 3.4 The nonconforming formulation

### 3.4.1 The motivation

There are several approaches to solving the system (3.16). We now discuss some good iterative methods which do not depend upon iteration parameters that be sometimes hard to select properly.

i) We can eliminate the variable $\mathbf{q}_h$ obtaining

$$S_h u_h = -B_h A_h^{-1} B_h^T u_h = f_h. \tag{3.19}$$

The matrix $S_h$ is symmetric, negative definite and hence a conjugate gradient method can be used. Each matrix-vector product with $S_h$ can be computed essentially at a cost of solving exactly a linear system with the matrix $A_h$. We note however that $S_h$ is not well conditioned. Drawback of this algorithm are that the action of $A_h^{-1}$ on a vector may be very expensive and must be computed for a substantial numbers of vectors. One alternative would be the use of a preconditioned conjugate gradient iteration to evaluate $A_h^{-1}$. The problem with this approach is that to guarantee convergence of the outer iteration, it becomes necessary to run the inner iteration to great accuracy, thus making the overall solution process expensive. We note that for problems with small variations in $\mathbf{A}$, it can be shown that $S_h$ is spectrally equivalent to a discrete Laplacian of finite difference type (cf. Wheeler and Gonzalez [86]); therefore, there is no difficulty finding an effective preconditioner for $S_h$ and fewer iterations are needed in the outer iteration. To find and analyze an effective preconditioner for $S_h$ becomes much harder when we work with large discontinuities in the coefficients because when we compute explicitly $B_h A_h^{-1} B_h^T$, we have a mixing of the coefficients near their discontinuities.

ii) We can also solve the system (3.16) by an iterative method for symmetric, nonsingular indefinite linear system such as the minimum residual method for indefinite system studied by Paige and Saunders [70]. Its convergence rate can be estimated in terms of the spectral properties of the matrix in (3.16). We note, however, that its convergence rate will typically deteriorate rapidly when the discretization is refined. In order to speed up the convergence a preconditioned version of the method should be considered. For problems with small variations in $\mathbf{A}$, Rusten and Winther [74] analyze, successfully, block diagonal preconditioners for which the convergence rate of the preconditioned minimum residual is bounded independently of the discretization parameter $h$.

iii) Another approach was adopted by Mathew [56, 57, 58, 23]. In a first step, by solving exactly a coarse space problem and many local problems in parallel, Mathew reduces the problem (3.16) to a case for which $f_h = B_h \mathbf{q}_h = 0$, i.e.

$$\begin{bmatrix} A_h & B_h^T \\ B_h & 0 \end{bmatrix} \begin{bmatrix} \mathbf{q}_h \\ u_h \end{bmatrix} = \begin{bmatrix} g_h \\ 0 \end{bmatrix}, \tag{3.20}$$

for some appropriate right-hand side $g_h$. Since $B_h \mathbf{q}_h = 0$, the problem (3.20) can be solved in the subspace of divergence free velocities which satisfy $B_h \mathbf{p}_h = 0$. This new problem becomes positive definite since

$$\begin{bmatrix} \mathbf{p}_h \\ v_h \end{bmatrix}^T \begin{bmatrix} A_h & B_h^T \\ B_h & 0 \end{bmatrix} \begin{bmatrix} \mathbf{p}_h \\ v_h \end{bmatrix} = \tag{3.21}$$

$$\mathbf{p}_h^T A_h \mathbf{p}_h + 2\, v_h^T B_h \mathbf{p}_h = \mathbf{p}_h^T A_h \mathbf{p}_h > 0.$$

In a second step, Mathew constructs a Schwarz preconditioners which preserves the divergence free velocities in each iteration of the PCG. The divergence free invariance is obtained at the expense of solving a coarse problem and the local problems (in parallel) exactly in each iteration. In the two-dimensional case, Ewing and Wang [43] introduced a stream function to reduce (3.16) to a standard finite element elliptic problem. Therefore, if we have discontinuous coefficients across subregion interfaces, we can solve the problem by several well-known domain decomposition methods, or by the two-dimensional version of the algorithms developed in Chapter 4 of this thesis. In the three-dimensional case, the theory is still incomplete; the difficulty is how to find a decomposition as in *Assumption* i) of Theorem 2.1 with $v$ and the $v_i$ belonging to the divergence free space.

iv) The method that we adopt in this thesis is based on the Arnold-Brezzi [2] theory and on our preconditioners for nonconforming $P_1$ finite elements problems [76, 77]; see also Chapter 5 of this thesis. We can then handle two- and three-dimensional problems (3.16) with large variations of the coefficients across the subregion interfaces.

### 3.4.2 The hybrid-mixed formulation

Let $\mathcal{F}_h$ be the set of faces in $\mathcal{T}^h$, and let $\mathcal{F}_h^\partial = \{f \in \mathcal{F}_h : f \subseteq \partial\Omega\}$, and $\mathcal{F}_h^0 = \mathcal{F}_h \backslash \mathcal{F}_h^\partial$. We introduce the space of Langrange multipliers $M_{-1}^0(\mathcal{F}_h^0)$ as the set of all functions on $\cup\mathcal{F}_h$ that are constant on each face $f \in \mathcal{F}_h^0$, and that vanish on $\mathcal{F}_h^\partial$. The *hybrid-mixed discrete* formulation is given by:

Find $(\mathbf{q}_h^*, p_h^*, \lambda_h) \in RT_{-1}^0(\mathcal{T}^h) \times M_{-1}^0(\mathcal{T}^h) \times M_{-1}^0(\mathcal{F}_h^0)$ such that

$$\begin{cases} \int_\Omega \mathbf{C}\, \mathbf{q}_h^* \cdot \mathbf{p}_h \, dx - \sum_{\tau \in \mathcal{T}^h} \left( \int_\tau u_h^* \, \mathbf{div}\, \mathbf{p}_h \, dx - \int_{\partial\tau} \lambda_h \, \mathbf{p}_h \cdot \mathbf{n}_\tau \, ds \right) = 0 \\ - \sum_{\tau \in \mathcal{T}^h} \int_\tau v_h \, \mathbf{div}\, \mathbf{q}_h^* \, dx = - \int_\Omega f\, v_h \, dx \\ \sum_{\tau \in \mathcal{T}^h} \int_{\partial\tau} \mu_h \, \mathbf{q}_h^* \cdot \mathbf{n}_\tau \, ds = 0 \end{cases} \tag{3.22}$$

$$\forall (\mathbf{p}_h, v_h, \mu_h) \in RT_{-1}^0(\mathcal{T}^h) \times M_{-1}^0(\mathcal{T}^h) \times M_{-1}^0(\mathcal{F}_h^0).$$

Note that if $\mathbf{p}_h \in RT^0_{-1}(\mathcal{T}^h)$, then

$$\mathbf{p}_h \in RT^0_0(\mathcal{T}^h) \quad \text{iff} \quad \Big( \sum_{\tau \in \mathcal{T}^h} \int_{\partial\tau} \mu_h \, \mathbf{p}_h \cdot \mathbf{n}_\tau \, ds = 0 \ \ \forall \mu_h \in M^0_{-1}(\mathcal{F}^0_h) \Big).$$

Therefore, using element-by-element arguments, it is easy to check, that the system (3.22) has a unique solution with $\mathbf{q}^*_h = \mathbf{q}_h$ and $u^*_h = u_h$, where $(\mathbf{q}_h, u_h)$ is the solution of (3.16). $\lambda_h$ is then uniquely determined from the first equation of (3.22). Hence, the systems (3.16) and (3.22) are equivalent, and we can therefore drop the superscript, *, in (3.22).

In matrix notation, the system (3.22) is of the form

$$\begin{bmatrix} \bar{A}_h & \bar{B}^T_h & \bar{C}^T_h \\ \bar{B}_h & 0 & 0 \\ \bar{C}_h & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{q}_h \\ u_h \\ \lambda_h \end{bmatrix} = \begin{bmatrix} 0 \\ f_h \\ 0 \end{bmatrix}. \tag{3.23}$$

**Remark 3.1** *An advantage of the hybrid-mixed formulation is that the matrix $\bar{A}_h$ is block diagonal, with each block corresponding to a single element. Hence, $\bar{A}_h$ can be inverted easily and in parallel. After eliminating the velocity in (3.23), we obtain a symmetric positive definite system*

$$\begin{bmatrix} \bar{B}_h \bar{A}_h^{-1} \bar{B}^T_h & \bar{B}_h \bar{A}_h^{-1} \bar{C}^T_h \\ \bar{C}_h \bar{A}_h^{-1} \bar{B}^T_h & \bar{C}_h \bar{A}_h^{-1} \bar{C}^T_h \end{bmatrix} \begin{bmatrix} u_h \\ \lambda_h \end{bmatrix} = \begin{bmatrix} -f_h \\ 0 \end{bmatrix}. \tag{3.24}$$

### 3.4.3 The Arnold-Brezzi theory

As in the previous subsection, all the arguments given in this subsection are local, element-by-element. We now note that the weak formulation for$\mathbf{q} = \mathbf{A}\,\nabla u$, on a single element $\tau$, is given by:

$$\int_\tau \mathbf{A}^{-1}\, \mathbf{q} \cdot \mathbf{p} \, dx - \int_\tau u \, \mathbf{div}\, \mathbf{p} \, dx + \int_{\partial\tau} u \, \mathbf{p} \cdot \mathbf{n}_\tau \, ds = 0 \quad \forall \mathbf{p} \in H(\mathbf{div}; \tau). \tag{3.25}$$

Hence, by comparing (3.25) with the first equation of (3.22), we may interpret the Lagrange multiplier $\lambda_h$ as an approximation of the trace of $p$ on the boundaries of the elements. This observation motivated Arnold and Brezzi [2] to obtain, from $u_h$ and $\lambda_h$, an asymptoticaly more accurate approximation of the original field $u$. Furthermore, the variables $u_h$ and $\lambda_h$ can be treated as one variable. Let us therefore introduce the spaces

$$V^h_N(\mathcal{T}^h) = V^h_B(\mathcal{T}^h) \oplus V^h_{CR}(\mathcal{T}^h).$$

Here,

$$V_B^h(\mathcal{T}^h) = \{\psi \,:\, \psi|_\tau \text{ is a cubic polynomial vanishing on } \partial\tau \quad \forall \tau \in \mathcal{T}^h\},$$

and

$$\begin{aligned} V_{CR}^h(\mathcal{T}^h) = \{\psi \,:\, & \psi|_\tau \text{ is linear in every tetrahedron } \tau \in \mathcal{T}^h, \\ & \psi \text{ continuous at the barycenter of the faces in } \mathcal{F}_h^0, \text{ and} \\ & \psi = 0 \text{ at the barycenter of faces in } \mathcal{F}_h^\partial\}. \end{aligned}$$

We note that $V_B^h(\mathcal{T}^h)$ consists of bubble functions, i.e. functions that vanishes on the boundaries of the elements, and the space $V_{CR}^h(\mathcal{T}^h)$ is the classical nonconforming $P_1$ space introduced by Crouziex and Raviart [31].

Let $b_f$ be the barycenter of a face $f \in \mathcal{F}_h$. Let $\mathcal{Q}_h$ be the local $L_2$ projection for an individual element $\tau$ and $\Pi_h$ the local $L_2$ projection for an individual face of $\mathcal{F}_h^0$. We introduce the mapping $\mathcal{S}_h : V_N^h(\mathcal{T}^h) \to M_{-1}^0(\mathcal{T}^h) \times M_{-1}^0(\mathcal{F}_h^0)$, defined by

$$\mathcal{S}_h(\psi_h) = (v_h, \mu_h) = (\mathcal{Q}_h \psi_h, \Pi_h \psi_h)$$

where

$$v_h|_\tau = \frac{1}{|\tau|} \int_\tau \psi_h \, dx \quad (\text{i.e. } v_h = \mathcal{Q}_h \psi_h),$$

and

$$\mu_h|_f = \psi(b_f) \quad (\text{i.e. } \mu_h = \Pi_h \psi_h).$$

It is easy to see that $\mathcal{S}_h$ is an isomorphism. Therefore, we can define $\psi_h$ by $\psi_h = \mathcal{S}_h^{-1}(u_h, \lambda_h)$, where $(\mathbf{q}_h, u_h, \lambda_h)$ is the unique solution of (3.22). Using the Green's Formula (1.12) with $\mathcal{G} = \tau$, and obvious properties of projections, we show that $(\mathbf{q}_h, \psi_h)$ is the unique solution of the following problem:

Find $(\mathbf{q}_h, \psi_h) \in RT_{-1}^0(\mathcal{T}^h) \times V_N^h(\mathcal{T}^h)$ such that:

$$\left\{ \begin{array}{ll} \sum_{\tau \in \mathcal{T}^h} \int_\tau \mathbf{A}^{-1} \mathbf{q}_h \cdot \mathbf{p}_h \, dx + \sum_{\tau \in \mathcal{T}^h} \int_\tau \nabla \psi_h \cdot \mathbf{p}_h = 0 & \forall \mathbf{p}_h \in RT_{-1}^0(\mathcal{T}^h) \\ \sum_{\tau \in \mathcal{T}^h} \int_\tau \mathbf{q}_h \cdot \nabla \chi_h \, dx \qquad = - \int_\Omega (\mathcal{Q}_h^T f) \, \chi_h \, dx & \forall \chi_h \in V_N^h(\mathcal{T}^h). \end{array} \right. \tag{3.26}$$

In matrix form, the system (3.26) has the form

$$\begin{bmatrix} \bar{A}_h & \bar{B}_h^T \mathcal{Q}_h + \bar{C}_h^T \Pi_h \\ \mathcal{Q}_h^T \bar{B}_h + \Pi_h^T \bar{C}_h & 0 \end{bmatrix} \begin{bmatrix} \mathbf{q}_h \\ \psi_h \end{bmatrix} = \begin{bmatrix} 0 \\ \mathcal{Q}_h^T f_h \end{bmatrix}. \tag{3.27}$$

We next show that the system (3.26) can be reduced to a positive definite system by eliminating the velocity $\mathbf{q}_h$.

Let $\mathcal{Q}^{\tau}_{RT,A^{-1}}$ be defined as the orthogonal projection from $(L_2(\tau))^3$ onto $RT^0_{-1}(\tau)$ with respect to the inner product

$$(\mathbf{p}_h, \mathbf{r}_h)_{A^{-1}(\tau)} = \int_\tau \mathbf{A}^{-1}\, \mathbf{p}_h \cdot \mathbf{r}_h \, dx, \quad \mathbf{p}_h, \mathbf{r}_h \in RT^0_{-1}(\tau).$$

We note that the projection $\mathcal{Q}^{\tau}_{RT,A^{-1}}$ is local in each element and can be computed easily and in parallel by inverting a block diagonal matrix with $4 \times 4$ blocks corresponding to individual tetrahedra of $\mathcal{T}^h$. We now use the first equation of (3.26) and obtain

$$\mathbf{q}_h = -\mathcal{Q}^{\tau}_{RT,A^{-1}}(\mathbf{A}\nabla\psi_h), \qquad \text{on} \ \ \tau. \tag{3.28}$$

We substitute (3.28) into the second equation of (3.26) and use obvious properties of projections to obtain the following equivalent problem:
Find $\psi_h \in V^h_N(\mathcal{T}^h)$ such that:

$$c^h(\psi_h, \chi_h) = \int_\Omega (\mathcal{Q}^T_h f)\, \chi_h \, dx \quad \forall \chi_h \in V^h_N(\mathcal{T}^h), \tag{3.29}$$

where

$$c^h(\psi_h, \chi_h) = \sum_{\tau \in \mathcal{T}^h} (\mathcal{Q}^{\tau}_{RT,A^{-1}}(\mathbf{A}\, \nabla\psi_h), \mathcal{Q}^{\tau}_{RT,A^{-1}}(\mathbf{A}\, \nabla\chi_h))_{A^{-1}(\tau)}.$$

The next lemma show that the inner product $c^h(\cdot,\cdot)$ is equivalent to a more easy computable inner product $a^h(\cdot,\cdot)$. The technique used in the proof is due to Brenner [11].

**Lemma 3.2** *Let us assume that* $\mathbf{A}$ *is a three-by-three symmetric matrix-valued sufficiently function inside each element* $\tau \in \mathcal{T}^h$ *and satisfies*

$$0 < \lambda_{min}(\tau)|\xi|^2 \le \xi^T \mathbf{A}(x)\xi \le \lambda_{max}(\tau)|\xi|^2 \quad \forall \xi \in \Re^3 \quad a.e \ x \in \tau \quad .$$

*Then, there exist a positive constant* $C$*, independent of the coefficient* $\mathbf{A}(x)$*, such that*

$$c^h(\psi, \psi) \le a^h(\psi, \psi) \qquad \forall \psi \in H^1_0(\Omega) + V^h_N(\mathcal{T}^h), \tag{3.30}$$

*and*

$$c^h(\psi, \psi) \ge C\, a^h(\psi, \psi) \qquad \forall \psi \in V^h_N(\mathcal{T}^h), \tag{3.31}$$

*where*

$$a^h(\psi, \psi) = \sum_{\tau \in \mathcal{T}^h} \int_\tau \mathbf{A}\, \nabla\psi \cdot \nabla\psi \, dx.$$

*Proof.* The first inequality: Using that $\mathcal{Q}^{\tau}_{RT,A^{-1}}$ is a projection, we have

$$c^h(\psi,\psi) = \sum_{\tau\in\mathcal{T}^h} (\mathcal{Q}^{\tau}_{RT,A^{-1}}(\mathbf{A}\,\nabla\psi), \mathcal{Q}^{\tau}_{RT,A^{-1}}(\mathbf{A}\,\nabla\psi))_{A^{-1}(\tau)}$$

$$\leq \sum_{\tau\in\mathcal{T}^h} (\mathbf{A}\,\nabla\psi, \mathbf{A}\,\nabla\psi)_{A^{-1}(\tau)} = \sum_{\tau\in\mathcal{T}^h} \int_{\tau} \mathbf{A}\,\nabla\psi\cdot\nabla\psi\,dx = a^h(\psi,\psi).$$

The second inequality: Using element-by-element local arguments, we have

$$c^h(\psi,\psi) = \sum_{\tau\in\mathcal{T}^h} (\mathcal{Q}^{\tau}_{RT,A^{-1}}(\mathbf{A}\,\nabla\psi), \mathcal{Q}^{\tau}_{RT,A^{-1}}(\mathbf{A}\,\nabla\psi))_{A^{-1}(\tau)}$$

$$= \sum_{\tau} \sup_{\mathbf{p}_h\in RT^0_{-1}(\tau)} \frac{(\mathcal{Q}^{\tau}_{RT,A^{-1}}(\mathbf{A}\,\nabla\psi), \mathbf{p}_h)^2_{A^{-1}(\tau)}}{(\mathbf{p}_h,\mathbf{p}_h)_{A^{-1}(\tau)}} = \sum_{\tau} \sup_{\mathbf{p}_h\in RT^0_{-1}(\tau)} \frac{(\mathbf{A}\,\nabla\psi, \mathbf{p}_h)^2_{A^{-1}(\tau)}}{(\mathbf{p}_h,\mathbf{p}_h)_{A^{-1}(\tau)}}$$

$$= \sum_{\tau} \sup_{\mathbf{p}_h\in RT^0_{-1}(\tau)} \frac{(\nabla\psi, \mathbf{p}_h)^2}{(\mathbf{p}_h,\mathbf{p}_h)_{A^{-1}(\tau)}} \geq \sum_{\tau} \lambda_{min}(\mathbf{A}(\tau)) \sup_{\mathbf{p}_h\in RT^0_{-1}(\tau)} \frac{(\nabla\psi_h, \mathbf{p}_h)^2}{(\mathbf{p}_h,\mathbf{p}_h)}$$

$$= \sum_{\tau} \lambda_{min}(\mathbf{A}(\tau)) \|\mathcal{Q}^{\tau}_{RT,\mathbf{I}}(\nabla\psi)\|^2_{L^2(\tau)}.$$

There remains to show that

$$\|\mathcal{Q}^{\tau}_{RT,\mathbf{I}}(\nabla\psi)\|^2_{L^2(\tau)} \geq \tilde{C}\,\|(\nabla\psi\|^2_{L^2(\tau)} \quad \forall\tau\in\mathcal{T}^h, \tag{3.32}$$

and then use that the coefficient $\mathbf{A}$ is sufficiently smooth inside each element $\tau$ to complete the proof. We note that the dimension of $\nabla\,V^h_N(\tau) = 4$ and that of $RT^0_{-1}(\tau) = 4$. Hence, to prove (3.32), we only need to show that

$$\mathcal{Q}^{\tau}_{RT,\mathbf{I}}(\nabla\psi) = 0 \Longrightarrow \nabla\psi = 0 \quad \forall\psi\in V^h_N(\tau).$$

Using that $\mathcal{Q}^{\tau}_{RT,\mathbf{I}}$ is the $L^2$-orthogonal projection onto $RT^0_{-1}(\tau)$, we have

$$\mathcal{Q}^{\tau}_{RT,\mathbf{I}}(\nabla\psi) = 0 \Longleftrightarrow \int_{\tau} \nabla\psi\cdot\mathbf{p}_h\,dx = 0 \quad \forall\mathbf{p}_h\in RT^0_{-1}(\tau),$$

and using (3.14)

$$\Longleftrightarrow \int_{\hat{\tau}} \nabla\hat{\psi}\cdot\hat{\mathbf{p}}_h\,dx = 0 \;\; \forall\hat{\mathbf{p}}_h\in RT^0_{-1}(\hat{\tau}).$$

Using simple calculations, we find that $\hat{\psi}$ is a constant. Therefore, $\psi$ is a constant and $\nabla\psi = 0$. For details, see Brenner [11], or Meddahi [60])

□

We conclude this subsection by discussing three cases:

i) $\mathbf{A} = \rho\,\mathbf{I}$, where $\rho$ is a scalar constant on each element $\tau$. Two interesting things happen: We can show that $C = 1$, where $C$ is the constant of (3.31), and we can also show that the problem (3.29) can be reduced to a non-conforming $P_1$ finite element problem and many local elementwise problems. Using the first equation of (3.26), we obtain

$$\rho^{-1}\,\mathbf{q}_h = -\mathcal{Q}^{\tau}_{RT,I}(\nabla\psi_h) \quad \forall\tau \in \tau T^h,$$

and the problem (3.26) is equivalent to:
Find $\psi_h \in V^h_N(\mathcal{T}^h)$ such that

$$\sum_{\tau\in\mathcal{T}^h}\int_\tau \rho\,\mathcal{Q}^{\tau}_{RT,I}(\nabla\psi_h)\cdot\nabla\chi_h\,dx = \int_\Omega(\mathcal{Q}^T_h f)\,\chi_h\,dx \quad \forall\chi_h \in V^h_N(\mathcal{T}^h). \tag{3.33}$$

We now use that $V^h_N(\mathcal{T}^h) = V^h_B(\mathcal{T}^h) \oplus V^h_{CR}(\mathcal{T}^h)$ decomposes uniquely, i.e.,

$$\psi_h = v_h + \varphi_h, \;\text{ with }\; v_h \in V^h_{CR}(\mathcal{T}^h) \;\text{ and }\; \varphi_h \in V^h_B(\mathcal{T}^h), \tag{3.34}$$

and

$$\chi_h = z_h + \phi_h, \;\text{ with }\; z_h \in V^h_{CR}(\mathcal{T}^h) \;\text{ and }\; \phi_h \in V^h_B(\mathcal{T}^h).$$

For $z_h \in V^h_{CR}(\mathcal{T}^h)$, we find that $\mathcal{Q}^{\tau}_{RT,I}(\nabla z_h) = \nabla z_h$ is piecewise constant. The same property holds for $v_h$. For $\phi_h \in V^h_B(\mathcal{T}^h)$, we find that $\nabla\phi_h$ has zero mean value on each element $\tau$. The same property holds for $\varphi_h$. Hence, the problem (3.33) is equivalent to:
Find $\psi_h = v_h + \varphi_h$, where $(v_h,\varphi_h) \in V^h_{CR}(\mathcal{T}^h) \times V^h_B(\mathcal{T}^h)$ is the unique solution of

$$\begin{cases} \sum_{\tau\in\mathcal{T}^h}\int_\tau \rho\nabla v_h\cdot\nabla z_h\,dx = \int_\Omega(\mathcal{Q}^T_h f)\,z_h\,dx \quad \forall z_h \in V^h_{CR}(\mathcal{T}^h) \\ \sum_{\tau\in\mathcal{T}^h}\int_\tau \rho\mathcal{Q}^{\tau}_{RT,I}(\nabla\varphi_h)\cdot\nabla\phi_h\,dx = \int_\Omega(\mathcal{Q}^T_h f)\,\phi_h\,dx \quad \forall\phi_h \in V^h_B(\mathcal{T}^h). \end{cases} \tag{3.35}$$

Note that $v_h$ and $\varphi_h$ can be computed independently; in fact $\varphi_h$ can be computed locally and in parallel, for each element $\tau$. For the computation of $v_h$, we can, if $\rho$ is nearly constant in each substructure, use one of our

Schwarz methods for nonconforming $P_1$ finite element methods developed in Chapter 5 of this thesis.

ii) $\mathbf{A}$ is a constant matrix on each element $\tau$. Using Lemma 3.2 and the decomposition (3.34), we find that there exist constants $C_i > 0$ such that

$$C_1\, a^h(\psi_h, \psi_h) \leq a^h(v_h, v_h) + a^h(\varphi_h, \varphi_h) \leq C_2\, a^h(\psi_h, \psi_h) \quad \forall \psi_h \in V_N^h(\mathcal{T}^h), \tag{3.36}$$

and then it is straightforward to apply Schwarz techniques to solve (3.29). Again, the hardest part is how to solve a problem like
Find $v_h \in V_{CR}^h(\mathcal{T}^h)$ such as

$$a^h(v_h, z_h) = g(z_h) \quad \forall z_h \in V_{CR}^h(\mathcal{T}^h),$$

for some appropriate right hand side $g$. If in addition, the matrix $\mathbf{A}$ is nearly constant in each substructure, we can use one of our Schwarz methods for nonconforming $P_1$ finite element methods. We note, however, that our analysis does not extend trivially to the case in which $\mathbf{A}$ is highly anisotropic.

iii) $\mathbf{A}$ is a smooth and is small pertubation of a constant matrix on each element $\tau$. In this case, the results of ii) hold.

# Chapter 4

# Multilevel Schwarz Methods for Elliptic Problems with Discontinuous Coefficients in Three Dimensions-Conforming Version

## 4.1 Introduction

In this chapter, we develop multilevel Schwarz methods for a conforming finite element approximation of second order elliptic partial differential equations. A special emphasis is placed on problems in three dimensions with possibly large jumps in the coefficients across the interface separating the subregions. To simplify the presentation only piecewise linear finite elements are considered. Our goal is to design and analyze methods with a rate of convergence which is independent of the jumps of the coefficients, the number of substructures, and the number of levels.

We consider two classes of the methods, additive and multiplicative. The multiplicative methods are variants of the multigrid V-cycle method. In our design and analysis, we use a general Schwarz method framework developed in Chapter 2. Among the particular cases, discussed here, are the BPX algorithm, cf. Bramble, Pasciak and Xu [8], and Xu [91], and the multilevel Schwarz method with one-dimensional subspaces considered by Zhang [94, 95]; see also Dryja and Widlund [39, 40]. It is well known that these methods are optimal when the coefficients are regular.

The problems become quite challenging for problems with highly discontinuous coefficients. Pioneering work was carried out in Dryja and Widlund [40], where the BPX method was modified and applied to a Schur complement system obtained after that the unknowns of the interior nodal points of the substructures had been eliminated. In that case, the condition number of the preconditioned system was shown to be bounded from above by $C\,(1+\ell)^2$, where $\ell$ is the number of level of the refinement; see further Section 4.9.

The main question for problems with discontinuous coefficients is the choice of a coarse space. We introduce a coarse triangulation given by the substructures and assume that the coefficients can have large variations only across the interfaces of these substructures. We then design methods with several coarse spaces, sometimes known as *exotic coarse spaces*; cf. Widlund [88]. Some are new and others have previously been discussed; see Dryja, Smith, and Widlund [35], Dryja and Widlund [41], and Sarkis [76]. One of our main results is that the condition number of the resulting systems can be estimated from above by $C\,(1+\ell)^2$ with $C$ independent of the jumps of coefficients, of the number of substructures, and also of $\ell$; see Section 4.4. For multiplicative variants such as the V-cycle multigrid, the rate of convergence is bounded from above by $1 - C\,(1+\ell)^{-2}, C > 0$; see Section 4.6.

In Section 4.5, we study in detail the weighted $L^2$ projection with weights given by the discontinuous coefficients of the elliptic problem. Bramble and Xu [9], and Xu [90] have considered this problem and established that the weighted $L^2$ projection is not always stable in the presence of interior cross points. Here, we introduce a new concept called *quasi-monotone distribution of coefficients* which characterizes the cases for which certain optimal estimates for the weighted $L^2$ projection are possible. For problems with quasi-monotone coefficients, the standard piecewise linear functions can be used as the coarse space and optimal multilevel algorithms are obtained.

In Section 4.7, we introduce *approximate discrete harmonic extensions* and define new coarses spaces by modifying previously known exotic coarse spaces; see Sarkis [76] and Chapter 5 for a case of nonconforming elements. Using these extensions, we can avoid solving a local Dirichlet problem for each substructure when using exotic coarse spaces [88]. We show that the converge rate estimate of our new iterative methods, with approximate discrete harmonic extensions, are comparable to those using exact discrete harmonic extensions. The use of approximate discrete harmonic extensions results in algorithms where the work per iteration is linear in the number of degrees of freedom with the possible exception of the cost of solving the coarse problem.

Elliptic problems with discontinuous coefficients have solutions with singular behavior. Therefore, in Section 4.8, we consider nonuniform refinements. We begin with a coarse triangulation that is shape regular and possibly nonuniform and then refine it using a local refinement scheme analyzed by Bornemann and Yserentant [3]. We establish a condition number estimate for the iteration operator which is bounded from above by $C(1+\ell)^2$. For quasi-monotone coefficients, we obtain an optimal multilevel preconditioner. We conclude this chapter by an analysis of multilevel iterative substructuring methods in Section 4.9, and by analyzing two-level Schwarz methods with inexact local solvers in Section 4.10.

Our results have been obtained jointly with Maxsymilian Dryja and Olof Widlund. This work has already submitted for publication; see Dryja, Sarkis and Widlund [34]. See also Dryja [33], Sarkis [77], and Widlund [88].

## 4.2 Assumptions and notation

We assume that $\mathbf{A}(x) = \rho(x) > 0$, and $\rho(x)$ is constant, in each substructure, with possibly large jumps occurring only across substructure boundaries. Therefore, $\rho(x) = \rho_i$ in each substructure $\Omega_i$. The analysis of our methods can easily be extended to the case when $\rho(x)$ varies moderately in each subregion $\Omega_i$.

The bilinear form $a(u,v)$ defined in (3.2) is directly related to a weighted Sobolev space $H^1_\rho(\Omega)$ defined by the seminorm

$$|u|^2_{H^1_\rho(\Omega)} = a(u,u).$$

We also define a weighted $L^2$ norm by:

$$||u||^2_{L^2_\rho(\Omega)} = \int_\Omega \rho(x)\,|u(x)|^2\,dx \quad \text{for } u \in L^2(\Omega). \tag{4.1}$$

Let $\mathcal{T}^k, k = 0,\cdots,\ell$ be a triangulation defined in Subsection 3.2.1, and let $\Sigma$ be a region contained in $\Omega$ such that $\partial\Sigma$ does not cut through any element $\tau^k_j \in \mathcal{T}^k$. We denote by $V^k(\Sigma)$ the restriction of $V^k(\Omega)$ to $\bar{\Sigma}$, and by $V_0^k(\Sigma)$ the subspace of $V^k(\Sigma)$ of functions which vanish on $\partial\Sigma$. We also define $H^1_\rho(\Sigma)$ and $L^2_\rho(\Sigma)$ by restricting the domain of integration of the weighted norms to $\Sigma$. To avoid unnecessary notations, we drop the parameter $\rho$ when $\rho = 1$, and $\Sigma$ when the domain of integration is $\Omega$.

In the case of a region $\Sigma$ of diameter of order $h_k$, such as an element $\tau_j^k$ or the union of few elements, we use a weighted norm,

$$\|u\|^2_{H^1_\rho(\Sigma)} = |u|^2_{H^1_\rho(\Sigma)} + \frac{1}{h_k^2}\|u\|^2_{L^2_\rho(\Sigma)}. \tag{4.2}$$

We introduce the following notations: $u \preceq v$ , $w \succeq x$, and $y \asymp z$ meaning that there are positive constants $C$ and $c$ such that

$$u \leq C\, v, \quad w \geq c\, x \quad \text{and} \quad c\, z \leq y \leq C\, z, \quad \text{respectively.}$$

Here $C$ and $c$ are independent of the variables appearing in the inequalities and the parameters related to meshes, spaces and, especially, the weight $\rho$. Sometimes, we will use $\leq$ to stress that $C = 1$.

## 4.3 Multilevel additive Schwarz method

Any Schwarz method can be defined by a splitting of the space $V_0^h$ into a sum of subspaces, and by bilinear forms associated with each of these subspaces; see Subsection 2.5.1. We first consider certain multilevel methods based on the MDS-multilevel diagonal scaling introduced by Zhang [95], enriched with a coarse space as in Dryja and Widlund [41], Dryja, Smith, and Widlund [35], or Sarkis [76, 77].

Let $\mathcal{N}^k$ and $\mathcal{N}_0^k$ be the set of nodes associated with the space $V^k$ and $V_0^k$, respectively. Let $\phi_j^k$ be a standard nodal basis function of $V_0^k$, and let $V_j^k = \mathrm{span}\{\phi_j^k\}$. We decompose $V_0^h$ as

$$V_0^h = V_{-1}^X + \sum_{k=0}^{\ell} V_0^k = V_{-1}^X + \sum_{k=0}^{\ell} \sum_{j \in \mathcal{N}_0^k} V_j^k.$$

We note that this decomposition is not a direct sum and that $\dim(V_j^k) = 1$. Four different types of coarse spaces $V_{-1}^X$, and associated bilinear forms $b_{-1}^X(u,u) : V_{-1}^X \times V_{-1}^X \to \Re$, with $X =\ F, E, NN$, and $W$, will be considered; see Section 4.4. We also consider variants of these four coarses spaces using spaces of approximate discrete harmonic extensions given by simple explicit formulas; see Section 4.7. The case when the coarse space is $V_0^0 = V_0^H$ is considered in Section 4.5.

We introduce operators $P_j^k : V_0^h \rightarrow V_j^k$, by

$$a(P_j^k u, v) = a(u, v) \quad \forall\ v \in\ V_j^k,$$

and an operator $T_{-1}^X : V_0^h \rightarrow V_{-1}^X$, by an inner product $b_{-1}^X(\cdot,\cdot)$ and the formula

$$b_{-1}^X(T_{-1}^X u, v) = a(u, v) \quad \forall\ v \in\ V_{-1}^X. \tag{4.3}$$

The analysis can easily be extended to the case when we use approximate solvers for the spaces $V_j^k$. Thus, we do not need to save, in memory, or recompute, all the values of $a(\phi_j^k, \phi_j^k)$, for $k = 1, \cdots, \ell$, and $\forall j \in \mathcal{N}_0^k$.

Let

$$T^X = T_{-1}^X + \sum_{k=0}^{\ell} \sum_{j \in \mathcal{N}_0^k} P_j^k. \tag{4.4}$$

We now replace (3.5) by

$$T^X u = g, \quad g = T_{-1}^X u + \sum_{k=0}^{\ell} \sum_{j \in \mathcal{N}_0^k} P_j^k u. \tag{4.5}$$

Equation (4.5) is typically solved by a conjugate gradient method. In order to estimate its rate of convergence, we need to obtain upper and lower bounds for the spectrum of $T^X$. The bounds are obtained by using a variant of Theorem 2.1 for which is more suitable in the present context.

**Theorem 4.1** *Suppose the following three assumptions hold:*

*i) There exists a constant $C_0$ such that for all $u \in V_0^h$ there exists a decomposition $u = u_{-1} + \sum_{k=0}^{\ell} \sum_{j \in \mathcal{N}_0^k} u_j^k$, with $u_{-1} \in V_{-1}^X$, $u_j^k \in V_j^k$, such that*

$$b_{-1}^X(u_{-1}^X, u_{-1}^X) + \sum_{k=0}^{\ell} \sum_{j \in \mathcal{N}_0^k} a(u_j^k, u_j^k) \leq C_0^2 a(u, u).$$

*ii) There exists a constant $\omega$ such that*

$$a(u, u) \leq \omega\, b_{-1}^X(u, u) \quad \forall u \in V_{-1}^X.$$

*iii) There exist constants* $\epsilon_{ij}^{mn}$, $m, n = 0, \cdots, \ell$ *and*
$\forall i \in \mathcal{N}_0^m$, $\forall j \in \mathcal{N}_0^n$ *such that*

$$a(u_i^m, u_j^n) \leq \epsilon_{ij}^{mn}\, a(u_i^m, u_i^m)^{1/2} a(u_j^n, u_j^n)^{1/2}$$

$$\forall u_i^m \in V_i^m \quad \forall u_j^n \in V_j^n.$$

*Then,* $T$ *is invertible,* $a(Tu, v) = a(u, Tv)$*, and*

$$C_0^{-2} a(u,u) \leq a(Tu, u) \leq (\rho(\mathcal{E}) + 1)\omega a(u,u) \quad \forall u \in V_0^h. \tag{4.6}$$

*Here* $\rho(\mathcal{E})$ *is the spectral radius of the tensor* $\mathcal{E} = \{\epsilon_{ij}^{mn}\}_{i,j,m,n=0}^{\ell}$.

## 4.4 Exotic coarse spaces and condition numbers

We now introduce certain geometrical objects in preparation for the description of our exotic coarse spaces $V_{-1}^X$. Let $\mathcal{F}_{ij}$ represent the open *face* which is shared by two substructures $\Omega_i$ and $\Omega_j$. Let $\mathcal{E}_l$ represent an open *edge*, and $\mathcal{V}_m$ a *vertex* of the substructure $\Omega_i$. Let $\mathcal{W}_i$ denote the *wire basket* of the subdomain $\Omega_i$, i.e. the union of the closures of the edges of $\partial\Omega_i$. We denote the *interface* between the subdomains by $\Gamma = \cup \partial\Omega_i \backslash \partial\Omega$, and the *wire basket* by $\mathcal{W} = \cup \mathcal{W}_i \backslash \partial\Omega$. The sets of nodes belonging to $\partial\Omega$, $\bar{\Omega}_i$, $\partial\Omega_i$, $\mathcal{F}_{ij}$, $\mathcal{E}_l$, $\mathcal{W}_i$, and $\Gamma$ are denoted by $\partial\Omega_h$, $\partial\Omega_{i,h}$, $\bar{\Omega}_{i,h}$, $\mathcal{F}_{ij,h}$, $\mathcal{E}_{l,h}$, $\mathcal{W}_{i,h}$, and $\Gamma_h$, respectively.

We now proceed to discuss several alternative coarse spaces and to establish bounds for the condition numbers of the corresponding additive multilevel methods.

### 4.4.1 Neumann-Neumann coarse spaces

We first consider the Neumann-Neumann coarse spaces which have been analyzed in Dryja and Widlund [41], Mandel and Brezina [55], and Sarkis [76]. An interesting feature of these coarse spaces is that they are of minimal dimension with only one degree of freedom per substructure, even in the case when the substructures are not simplices. For any $\beta \geq 1/2$, we introduce the weighted counting functions $\mu_{i,\beta}$, for all $i = 1, \cdots, N$, defined by

$$\mu_{i,\beta}(x) = \sum_j \rho_j^\beta, \ x \in \partial\Omega_{i,h} \backslash \partial\Omega_h, \quad \mu_{i,\beta}(x) = 0, \ x \in (\Gamma_h \backslash \partial\Omega_{i,h}) \cup \partial\Omega_h.$$

For each $x \in \partial\Omega_{i,h} \backslash \partial\Omega_h$, the sum is taken over the values of $j$ for which $x \in \partial\Omega_{j,h}$.

The pseudo inverse $\mu_{i,\beta}^{+}$ of $\mu_{i,\beta}$ is defined by

$$\mu_{i,\beta}^{+}(x) = (\mu_{i,\beta}(x))^{-1}, \; x \in \partial\Omega_{i,h} \backslash \partial\Omega_h,$$

and

$$\mu_{i,\beta}^{+}(x) = 0, \; x \in (\Gamma_h \backslash \partial\Omega_{i,h}) \cup \partial\Omega_h.$$

We extend $\mu_{i,\beta}^{+}$ elsewhere in $\Omega$ as a minimal energy, *discrete harmonic* function using the values on $\Gamma_h \cup \partial\Omega_h$ as boundary values. The resulting functions belong to $V_0^h(\Omega)$ and are also denoted by $\mu_{i,\beta}^{+}$.

We can now define the coarse space $V_{-1}^{NN} \subset V_0^h$ by

$$V_{-1}^{NN} = \text{span}\{\rho_i^{\beta} \, \mu_{i,\beta}^{+}\}, \tag{4.7}$$

i.e. we use one basis functions for each substructure $\Omega_i$. We remark that we can even define a Neumann-Neumann coarse space for $\beta = \infty$ by considering, the limit of the space $V_{-1}^{NN}$ when $\beta$ approaches $\infty$, i.e.

$$\rho_i^{\infty} \, \mu_{i,\infty}^{+} = \lim_{\beta \to \infty} \rho_i^{\beta} \, \mu_{i,\beta}^{+}.$$

For instance, for $x \in \mathcal{F}_{ij,h}$, we obtain

$$\rho_i^{\infty} \, \mu_{i,\infty}^{+}(x) = 1, \quad \text{if} \quad \rho_i > \rho_j,$$

$$\rho_i^{\infty} \, \mu_{i,\infty}^{+}(x) = 0, \quad \text{if} \quad \rho_i < \rho_j,$$

and

$$\rho_i^{\infty} \, \mu_{i,\infty}^{+}(x) = 1/2, \quad \text{if} \quad \rho_i = \rho_j.$$

We note that $V_{-1}^{NN}$ is also the range of an *interpolator* $I_h^{NN} : V_0^h \to V_{-1}^{NN}$, given by

$$u_{-1} = I_h^{NN} u(x) = \sum_i u_{-1}^{(i)} = \sum_i \bar{u}_i^h \, \rho_i^{\beta} \mu_{i,\beta}^{+}. \tag{4.8}$$

Here, $\bar{u}_i^h$ is the discrete average value of $u$ over $\partial\Omega_{i,h}$.

We note the coarse spaces defined with $\beta = 1/2$, $\beta = 1$, and $\beta \geq 1/2$ have been used by Dryja and Widlund [41], Mandel and Brezina [55], and Sarkis [76], respectively.

Recently, Wang and Xie [85] introduced another coarse space which is similar to ours with $\beta = \infty$. However, their basis functions only take the value 0 or 1 on $\Gamma_h$.

We introduce the bilinear form $b_{-1}^{NN}(u,v) : V_{-1}^{NN} \times V_{-1}^{NN} \to \Re$, defined by

$$b_{-1}^{NN}(u,v) = a(u,v). \tag{4.9}$$

**Theorem 4.2** *Let $T^{NN}$ be defined by (4.4), and let $1/2 \leq \beta \leq \infty$. Then for any $u \in V_0^h(\Omega)$, we have*

$$(1+\ell)^{-2} a(u,u) \preceq a(T^{NN}u,u) \preceq a(u,u).$$

*The bounds are also independent of $\beta$.*

*Proof.* We use Theorem 4.1

*Assumption i).* For notational convenience, we introduce, for $k = 0, \cdots, \ell$, the bilinear forms $b_k(u_k, u_k)$: $V_0^k \times V_0^k \to \Re$, defined by

$$b_k(u_k,u_k) = \sum_{j \in \mathcal{N}_0^k} u_k^2(x_j) a(\phi_j^k, \phi_j^k) = \sum_{j \in \mathcal{N}_0^k} a(u_j^k, u_j^k), \tag{4.10}$$

We use a level $k$ decomposition given by $u_k = \sum_j u_j^k$, with $u_j^k = u_k(x_j)\phi_j^k$. Here, $x_j$ is the position of the node $j \in \mathcal{N}_0^k$.

We decompose $u \in V_0^h(\Omega)$ as

$$u = \mathcal{H}u + \mathcal{P}u \ \text{ in } \Omega,$$

where

$$u = \mathcal{H}^{(i)}u + \mathcal{P}^{(i)}u \ \text{ in } \Omega_i. \tag{4.11}$$

Here, $\mathcal{H}^{(i)}u$ is the discrete harmonic part of $u$, i.e.

$$(\nabla \mathcal{H}^{(i)}u, \nabla v)_{L^2(\Omega_i)} = 0 \ \ \forall v \in V_0^h(\Omega_i),$$

$$\mathcal{H}^{(i)}u = u \text{ on } \partial\Omega_i,$$

and $\mathcal{P}^{(i)}u \in V_0^h(\Omega_i)$ is the $H^1-$projection associated with the space $V_0^h(\Omega_i)$, i.e.

$$(\nabla(\mathcal{P}^{(i)}u), \nabla\phi)_{L^2(\Omega_i)} = (\nabla u, \nabla\phi)_{L^2(\Omega_i)} \ \ \forall \phi \in V_0^h(\Omega_i).$$

We decompose $\mathcal{P}^{(i)}u$ and $\mathcal{H}^{(i)}u$ separately. We start by decomposing $v^{(i)} = \mathcal{P}^{(i)}u$ in $\Omega_i$ as

$$v^{(i)} = \mathcal{P}_0^{(i)} v^{(i)} + (\mathcal{P}_1^{(i)} - \mathcal{P}_0^{(i)})\, v^{(i)} + \cdots + (\mathcal{P}_\ell^{(i)} - \mathcal{P}_{\ell-1}^{(i)})\, v^{(i)},$$

where $\mathcal{P}_k^{(i)} : V_0^h(\Omega_i) \to V_0^k(\Omega_i)$, is the $H^1-$projection defined by

$$(\nabla(\mathcal{P}_k^{(i)} v^{(i)}), \nabla\phi)_{L^2(\Omega_i)} = (\nabla v^{(i)}, \nabla\phi)_{L^2(\Omega_i)} \quad \forall \phi \in V_0^k(\Omega_i).$$

We extend $\mathcal{P}_k^{(i)} v^{(i)}$ by zero to $\Omega \backslash \bar{\Omega}_i$, and also denote this extension by $\mathcal{P}_k^{(i)}\, v^{(i)}$. Thus, $\mathcal{P}_k^{(i)} v^{(i)} \in V_0^k(\Omega)$. Let

$$v_0^{(i)} = \mathcal{P}_0^{(i)} v^{(i)}, \quad v_k^{(i)} = (\mathcal{P}_k^{(i)} - \mathcal{P}_{k-1}^{(i)})\, v^{(i)},\ k = 1, \cdots, \ell.$$

Hence

$$v^{(i)} = v_0^{(i)} + v_1^{(i)} + \cdots + v_k^{(i)} + \cdots + v_{\ell-1}^{(i)} + v_\ell^{(i)}. \tag{4.12}$$

We use the decomposition (4.12) for all $i = 1, \cdots, N$. The global decomposition of $v$ is equal to $v^{(i)}$ in $\Omega_i$, and is defined by

$$v = \sum_{k=0}^{\ell} v_k, \quad v_k = \sum_{i=1}^{N} v_k^{(i)}. \tag{4.13}$$

We now decompose $\mathcal{H}u$. Let

$$w = \mathcal{H}u - u_{-1},$$

where $u_{-1}$ is defined in (4.8).

We decompose $w$ as

$$w = \sum_{i=1}^{N} w^{(i)},$$

where

$$w^{(i)} = I_h(u\, \rho_i^\beta \mu_{i,\beta}^+) - \bar{u}_i^h\, \rho_i^\beta \mu_{i,\beta}^+ = I_h(\rho_i^\beta \mu_{i,\beta}^+ (u - \bar{u}_i^h)) \quad \text{on} \quad \Gamma \cup \partial\Omega$$

and extended as a discrete harmonic function in each $\Omega_j$, $j = 1, \cdots, N$.
Here, $I_h$ is the standard linear interpolant based on the h-triangulation of $\Gamma$. It is easy to show that $w(x) = \sum_i w^{(i)}(x) \ \ \forall x \in \bar{\Omega}$. Note that the support of $w^{(i)}$ is the union of the $\bar{\Omega}_j$ which have a vertex, edge, or face in common with $\Omega_i$.

We decompose $w^{(i)}$ further as

$$w^{(i)} = \sum_{\mathcal{F}_{ij}\subset\partial\Omega_i} w^{(i)}_{\mathcal{F}_{ij}} + \sum_{\mathcal{E}_l\subset\partial\Omega_i} w^{(i)}_{\mathcal{E}_l} + \sum_{\mathcal{V}_m\subset\partial\Omega_i} w^{(i)}_{\mathcal{V}_m}, \tag{4.14}$$

where $\mathcal{F}_{ij}, \mathcal{E}_l$, and $\mathcal{V}_m$ are the faces, edges, and vertices of $\partial\Omega_i$. Here, $w^{(i)}_{\mathcal{F}_{ij}}, w^{(i)}_{\mathcal{E}_l}$, and $w^{(i)}_{\mathcal{V}_m}$ are the discrete harmonic extensions into $\Omega$ with possibly nonzero interface values only on $\mathcal{F}_{ij,h}$, $\mathcal{E}_{l,h}$, and $\mathcal{V}_m$, respectively. The support of each function on the right hand side of (4.14) is the union of a few $\bar{\Omega}_j$ and its interior is denoted by $\Omega_{\mathcal{F}_{ij}}$, $\Omega_{\mathcal{E}_l}$, or $\Omega_{\mathcal{V}_m}$, respectively. We can assume that the regions $\Omega_{\mathcal{F}_{ij}}$, $\Omega_{\mathcal{E}_l}$, and $\Omega_{\mathcal{V}_m}$ are convex. If not, we can extend them to convex regions and use a trick developed in Lemma 3.6. of Zhang [95].

We now decompose $w^{(i)}_{\mathcal{F}_{ij}}$, $w^{(i)}_{\mathcal{E}_\ell}$, and $w^{(i)}_{\mathcal{V}_m}$ in the same way as the $v^{(i)}$. Let us first consider $w^{(i)}_{\mathcal{F}_{ij}}$. We obtain

$$w^{(i)}_{\mathcal{F}_{ij}} = w^{(i)}_{0,\mathcal{F}_{ij}} + w^{(i)}_{1,\mathcal{F}_{ij}} + \cdots + w^{(i)}_{k,\mathcal{F}_{ij}} + \cdots + w^{(i)}_{\ell-1,\mathcal{F}_{ij}} + w^{(i)}_{\ell,\mathcal{F}_{ij}}. \tag{4.15}$$

Here

$$w^{(i)}_{0,\mathcal{F}_{ij}} = \mathcal{P}^{(i)}_{0,\mathcal{F}_{ij}} w^{(i)}_{\mathcal{F}_{ij}}, \quad w^{(i)}_{k,\mathcal{F}_{ij}} = (\mathcal{P}^{(i)}_{k,\mathcal{F}_{ij}} - \mathcal{P}^{(i)}_{k-1,\mathcal{F}_{ij}})\, w^{(i)}_{\mathcal{F}_{ij}},\ k = 1, \cdots, \ell.$$

$\mathcal{P}^{(i)}_{k,\mathcal{F}_{ij}} : V_0^h(\Omega_{\mathcal{F}_{ij}}) \to V_0^k(\Omega_{\mathcal{F}_{ij}})$, is the $H^1$-projection. As before, we extend $\mathcal{P}^{(i)}_{k,\mathcal{F}_{ij}} w^{(i)}_{\mathcal{F}_{ij}}$ by zero outside $\Omega_{\mathcal{F}_{ij}}$.

We decompose $w^{(i)}_{\mathcal{E}_l}$ and $w^{(i)}_{\mathcal{V}_m}$ in the same way, and obtain

$$w^{(i)}_{\mathcal{E}_l} = w^{(i)}_{0,\mathcal{E}_l} + w^{(i)}_{1,\mathcal{E}_l} + \cdots + w^{(i)}_{k,\mathcal{E}_l} + \cdots + w^{(i)}_{\ell-1,\mathcal{E}_l} + w^{(i)}_{\ell,\mathcal{E}_l}, \tag{4.16}$$

where

$$w^{(i)}_{0,\mathcal{E}_l} = \mathcal{P}^{(i)}_{0,\mathcal{E}_l} w^{(i)}_{\mathcal{E}_l}, \quad w^{(i)}_{k,\mathcal{E}_l} = (\mathcal{P}^{(i)}_{k,\mathcal{E}_l} - \mathcal{P}^{(i)}_{k-1,\mathcal{E}_l})\, w^{(i)}_{\mathcal{E}_l},\ k = 1, \cdots, \ell,$$

and

$$w^{(i)}_{\mathcal{V}_m} = w^{(i)}_{0,\mathcal{V}_m} + w^{(i)}_{1,\mathcal{V}_m} + \cdots + w^{(i)}_{k,\mathcal{V}_m} + \cdots + w^{(i)}_{\ell-1,\mathcal{V}_m} + w^{(i)}_{\ell,\mathcal{V}_m}. \tag{4.17}$$

Here,

$$w^{(i)}_{0,\mathcal{V}_m} = \mathcal{P}^{(i)}_{0,\mathcal{V}_m} w^{(i)}_{\mathcal{V}_m}, \quad w^{(i)}_{k,\mathcal{V}_m} = (\mathcal{P}^{(i)}_{k,\mathcal{V}_m} - \mathcal{P}^{(i)}_{k-1,\mathcal{V}_m})\, w^{(i)}_{\mathcal{V}_m},\ k = 1, \cdots, \ell.$$

We can now define a global decomposition of the function $w$ as

$$w = w_0 + w_1 + \cdots + w_k + \cdots + w_{\ell-1} + w_\ell, \tag{4.18}$$

where

$$w_k = \sum_i \Big( \sum_{\mathcal{F}_{ij} \subset \partial\Omega_i} w^{(i)}_{k,\mathcal{F}_{ij}} + \sum_{\mathcal{E}_l \subset \partial\Omega_i} w^{(i)}_{k,\mathcal{E}_l} + \sum_{\mathcal{V}_m \subset \partial\Omega_i} w^{(i)}_{k,\mathcal{V}_m} \Big).$$

We check straightforwardly that (4.18) is valid $\forall x \in \Omega$. Using (4.18) and (4.13), we define a decomposition for $u \in V_0^h$ as

$$u = u_{-1} + u_0 + \cdots + u_k + \cdots + u_{\ell-1} + u_\ell, \tag{4.19}$$

where

$$u_k = v_k + w_k, \quad k = 0, \cdots, \ell.$$

We obtain the desired decomposition in Assumption i) by decomposing $u_k$ as in (4.10). We will now prove that

$$\sum_{k=-1}^{\ell} b_k(u_k, u_k) \preceq (1 + \log(H/h))^2 a(u, u). \tag{4.20}$$

Therefore, in view of (4.10), we obtain $C_0^2 = C\,(1+\ell)^2$ in Assumption i) of Theorem 4.1.

We start with the decomposition of $v$.

**Lemma 4.1** *For the decomposition of $v$ given by (4.13), we have*

$$\sum_{k=0}^{\ell} b_k(v_k, v_k) \preceq a(u, u). \tag{4.21}$$

*Proof.* We first show that

$$\sum_{k=0}^{\ell} b_k(v_k^{(i)}, v_k^{(i)}) \preceq \rho_i \, |u|^2_{H^1(\Omega_i)}. \tag{4.22}$$

Note that for $k \geq 1$, $v_k^{(i)} = (\mathcal{P}_k^{(i)} - \mathcal{P}_{k-1}^{(i)})\, v^{(i)} = (I - \mathcal{P}_{k-1}^{(i)})\, v_k^{(i)}$.
Hence,

$$b_k(v_k^{(i)}, v_k^{(i)}) \preceq \frac{1}{h_k^2}\, \rho_i \|v_k^{(i)}\|^2_{L^2(\Omega_i)} \preceq \rho_i \, |v_k^{(i)}|^2_{H^1(\Omega_i)}.$$

The last inequality follows from the well known error estimate for $H^1-$projections on convex domains; see Ciarlet [27]. For $k = 0$,

$$b_0(v_0^{(i)}, v_0^{(i)}) \preceq \frac{1}{h_0^2} \rho_i \, \|v_0^{(i)}\|^2_{L^2(\Omega_i)} \preceq \rho_i \, |v_0^{(i)}|^2_{H^1(\Omega_i)},$$

using Friedrichs' inequality.

Adding the above inequalities, we obtain

$$\sum_{k=0}^{\ell} b_k(v_k^{(i)}, v_k^{(i)}) \preceq$$

$$\rho_i \{ (\nabla \mathcal{P}_0^{(i)} v^{(i)}, \nabla v^{(i)})_{L^2(\Omega_i)} + \sum_{k=1}^{\ell} (\nabla (\mathcal{P}_k^{(i)} - \mathcal{P}_{k-1}^{(i)})\, v^{(i)}, \nabla v^{(i)})_{L^2(\Omega_i)} \}$$

$$= \rho_i \, \|\nabla v^{(i)}\|_{L^2(\Omega_i)}^2 = \rho_i \, \|\nabla \mathcal{P}^{(i)} u\|_{L^2(\Omega_i)}^2 \leq \rho_i \, \|\nabla u\|_{L^2(\Omega_i)}^2 .$$

Thus

$$\sum_{i=1}^{N} \sum_{k=0}^{\ell} b_k(v_k^{(i)}, v_k^{(i)}) \preceq a(u, u).$$

□

**Lemma 4.2** *For the decomposition of* $w_{\mathcal{F}_{ij}}^{(i)}$*, given by (4.15), we have*

$$\sum_{k=0}^{\ell} b_k(w_{k,\mathcal{F}_{ij}}^{(i)}, w_{k,\mathcal{F}_{ij}}^{(i)}) \preceq (1+\ell)^2 \rho_i \, |u|_{H^1(\Omega_i)}^2 . \tag{4.23}$$

*Proof.* Note that

$$b_k(w_{k,\mathcal{F}_{ij}}^{(i)}, w_{k,\mathcal{F}_{ij}}^{(i)}) \preceq \frac{1}{h_k^2} (\rho_i + \rho_j) \, \|w_{\mathcal{F}_{ij,k}}^{(i)}\|_{L^2(\Omega_{\mathcal{F}_{ij}})} .$$

Here, $\Omega_{\mathcal{F}_{ij}} = \Omega_i \cup \Omega_j \cup \mathcal{F}_{ij}$. Note that $w_{\mathcal{F}_{k,ij}}^{(i)} = 0$ in $\Omega \backslash \Omega_{\mathcal{F}_{ij}}$. Using the same arguments as in the proof of Lemma 4.1, we obtain

$$\sum_{k=0}^{\ell} b_k(w_{k,\mathcal{F}_{ij}}^{(i)}, w_{k,\mathcal{F}_{ij}}^{(i)}) \preceq (\rho_i + \rho_j) |w_{\mathcal{F}_{ij}}^{(i)}|_{H^1(\Omega_{\mathcal{F}_{ij}})}^2 . \tag{4.24}$$

Note that

$$(\rho_i + \rho_j) |w_{\mathcal{F}_{ij}}^{(i)}|_{H^1(\Omega_{\mathcal{F}_{ij}})}^2 \preceq (\rho_i + \rho_j) \|w_{\mathcal{F}_{ij}}^{(i)}\|_{H_{00}^{1/2}(\mathcal{F}_{ij})}^2 =$$

$$(\rho_i + \rho_j) \|(I_h(\rho_i^{\beta} \mu_{i,\beta}^{+} (u - \bar{u}_i^h)))_{\mathcal{F}_{ij,h}}\|_{H_{00}^{1/2}(\mathcal{F}_{ij})}^2 =$$

$$\rho_i \frac{(1 + \frac{\rho_j}{\rho_i})}{(1 + (\frac{\rho_j}{\rho_i})^{\beta})^2} \|(u - \bar{u}_i^h)_{\mathcal{F}_{ij,h}}\|_{H_{00}^{1/2}(\mathcal{F}_{ij})}^2 \preceq \rho_i \|(u - \bar{u}_i^h)_{\mathcal{F}_{ij,h}}\|_{H_{00}^{1/2}(\mathcal{F}_{ij})}^2 .$$

The first inequality above follows from an extension theorem for finite element functions given in Bramble, Pasciak, and Schatz [6], and from properties of $H_{00}^{1/2}$-norm given in Chapter 2; see also Dryja [32]. A simple proof for this extension theorem is given in Lemma 4.14. Here, $(v)_{\mathcal{F}_{ij,h}}$ is the piecewise linear function on $\mathcal{F}_{ij} \cap \mathcal{T}^h$ such that $(v)_{\mathcal{F}_{ij,h}} = v$ at the nodal points $\mathcal{F}_{ij,h}$ and $(v)_{\mathcal{F}_{ij}} = 0$ on $\partial\mathcal{F}_{ij,h}$. For the last inequality, we have also used the fact that $\beta \geq 1/2$. For $\beta = \infty$, we use a limiting process.

Let $\theta_{\mathcal{F}_{ij}}$ be the discrete harmonic function, defined in $\Omega_i$, which equals 1 on $\mathcal{F}_{ij,h}$ and zero on $\partial\Omega_{i,h} \backslash \mathcal{F}_{ij,h}$. Using properties of the $H_{00}^{1/2}$-norm and a trace theorem given in Chapter 1, we obtain

$$\rho_i \|(u - \bar{u}_i^h)_{\mathcal{F}_{ij,h}}\|^2_{H_{00}^{1/2}(\mathcal{F}_{ij})} \preceq \rho_i \, \|\gamma(I_h(\theta_{\mathcal{F}_{ij}}(u - \bar{u}_i^h)))\|_{H^{1/2}(\partial\Omega_i)}$$

$$\preceq \rho_i \, \|(I_h(\theta_{\mathcal{F}_{ij}}(u - \bar{u}_i^h)))\|_{H^1(\Omega_i)}$$

and using Lemma 4.5 of Dryja, Smith, and Widlund [35] and a Poincaré Friedrichs' inequality, we obtain

$$\preceq \rho_i (1 + \log H/h)^2 \|u - \bar{u}_i^h\|^2_{H^1(\Omega_i)} \preceq \rho_i (1 + \ell)^2 |u|^2_{H^1(\Omega_i)}.$$

Combining this inequality with (4.24), we obtain (4.23).

□

**Remark 4.1** *The Poincaré Friedrichs inequality that we have just used does not fit, straightforwardly, into the hypotheses of Lemma 1.6 since in (1.13) we possibly may have*

$$f(\gamma(u - \bar{u}_i^h)) = \int_{\partial\Omega_i} \gamma(u - \bar{u}_i^h) \, dx) \neq 0.$$

*We note, however, that for quasi-uniform triangulations, we can define*

$$f_h(v) = h^2 \sum_{p \in \partial\Omega_{i,h}} v(x_p) \qquad \forall v \in V^h(\Omega_i),$$

*and use the same arguments as in the proof of Lemma 1.6, and show that*

$$\|v\|^2_{L^2(\Omega_i)} \leq C \, H^2 \, (|v|^2_{H^1(\Omega_i)} + \frac{1}{H} \, (f_h(\gamma v))^2 \qquad \forall v \in V^h(\Omega_i).$$

*Hence, we obtain $f_h(\gamma(u - \bar{u}_i^h)) = 0$.*

The next two lemmas are proved in the same way as Lemma 4.2; see a similar argument in the proof of Theorem 4.4

**Lemma 4.3** *For the decomposition of $w^{(i)}_{\mathcal{E}_l}$, given in (4.16), we have*

$$\sum_{k=0}^{\ell} b_k(w^{(i)}_{k,\mathcal{E}_l}, w^{(i)}_{k,\mathcal{E}_l}) \preceq (1+\ell)\,\rho_i\,|u|^2_{H^1(\Omega_i)}. \tag{4.25}$$

**Lemma 4.4** *For the decomposition of $w^{(i)}_{\mathcal{V}_m}$, given in (4.17), we have*

$$\sum_{k=0}^{\ell} b_k(w^{(i)}_{k,\mathcal{V}_m}, w^{(i)}_{k,\mathcal{V}_m}) \preceq \rho_i\,|u|^2_{H^1(\Omega_i)}. \tag{4.26}$$

**Corollary 4.1** *For the decomposition of $w$, given in (4.18), we have*

$$\sum_{k=0}^{\ell} b_k(w_k, w_k) \preceq (1+\ell)^2 a(u,u). \tag{4.27}$$

The proof of Corollary 4.1 follows from Lemmas 4.2, 4.3, and 4.4.

We now estimate $a(u_{-1}, u_{-1})$.

**Lemma 4.5** *For $u_{-1} = \sum_i u^{(i)}_{-1}$, $u^{(i)}_{-1} = \bar{u}^h_i\,\rho^\beta_i\,\mu^+_{i,\beta}$,*

$$a(u_{-1}, u_{-1}) \preceq (1+\log H/h)^2 a(u,u) \asymp (1+\ell)^2 a(u,u) \tag{4.28}$$

The proof of this result, for $\beta = 1/2$, can be found in the proofs of Theorem 6 and 7 of Dryja and Widlund [41]. For different values of $\beta$, we use an argument similar to that of the proof of Lemma 4.2.

Returning to the proof of Theorem 4.2, we find that (4.20) follows from Corollary 4.1, and Lemmas 4.1 and 4.5. The bound for $C_0$ has then been established.

*Assumption ii).* Trivially, we have $\omega = 1$.

*Assumption iii)* We need to show that $\rho(\mathcal{E}) \leq$ const. This has been established in Remark 3.3 in Zhang [95].

$\square$

**Remark 4.2** *Recently, Maxsymilian Dryja has proved that the estimate (4.28) can be sharpened to*

$$a(u_{-1}, u_{-1}) \preceq (1+\log H/h) a(u,u). \tag{4.29}$$

*We note that this result is in agreement with a result obtained by Sarkis [76] for nonconforming spaces; see further Chapter 5. The proof of 4.29 will be published elsewhere.*

### 4.4.2 A face based coarse space

The next exotic coarse space is denoted by $V_{-1}^F \subset V_0^h$, and is based on values on the wire basket $\mathcal{W}_h$ and averages over the faces $\mathcal{F}_{ij}$. This coarse space can conveniently be defined as the range of an interpolation operator $I_h^F : V_0^h \to V_{-1}^F$, defined by

$$I_h^F u(x)_{|\bar{\Omega}_i} = \sum_{p \in \mathcal{W}_{i,h}} u(x_p)\varphi_p(x) + \sum_{\mathcal{F}_{ij} \subset \partial\Omega_i} \bar{u}_{\mathcal{F}_{ij}}^h \theta_{\mathcal{F}_{ij}}(x).$$

Here, $\varphi_p(x)$ is the discrete harmonic extension, into $\Omega_i$, of the standard nodal basis function associated with a node $p$. $\bar{u}_{\mathcal{F}_{ij}}^h$ is the discrete average value of $u$ over $\mathcal{F}_{ij,h}$.

We define the bilinear form by

$$b_{-1}^F(u,u) = \sum_i \rho_i \{ \sum_{p \in \mathcal{W}_{i,h}} h\,(u(x_p) - \bar{u}_i^h)^2$$

$$+ \;\; H(1+\ell) \sum_{\mathcal{F}_{ij} \subset \partial\Omega_i} (\bar{u}_{\mathcal{F}_{ij}}^h - \bar{u}_i^h)^2 \},$$

where $\bar{u}_i^h$ is the discrete average value of $u$ over $\partial\Omega_{i,h}$.

**Theorem 4.3** *Let $T^F$ be defined by (4.4). Then for any $u \in V_0^h(\Omega)$, we have*

$$(1+\ell)^{-2} a(u,u) \preceq a(T^F u, u) \preceq a(u,u).$$

*Proof. Assumption i).* The decomposition here is the same as before, except that we now choose $w^{(i)} = I_h(\rho_i^{1/2} \mu_{i,1/2}^+ (u - I_h^F u))$ on $\Gamma \cup \partial\Omega$, and extend these boundary values as a discrete harmonic function elsewhere. We note that this decomposition is simpler since $w^{(i)}$ vanishes on the wire basket. Therefore, $w_{\mathcal{E}_l}^{(i)} = 0$ and $w_{\mathcal{V}_m}^{(i)} = 0$. A counterpart of Lemma 4.2 holds since

$$(\rho_i + \rho_j) |w_{\mathcal{F}_{ij}}^{(i)}|_{H^1(\Omega_{\mathcal{F}_{ij}})}^2 \preceq (\rho_i + \rho_j) \| (I_h(\rho_i^{1/2} \mu_{i,1/2}^+ (u - I_F^h u)))_{\mathcal{F}_{ij,h}} \|_{H_{00}^{1/2}(\mathcal{F}_{ij})}^2$$

$$\preceq \rho_i \, \| (u - \bar{u}_{\mathcal{F}_{ij}}^h)_{\mathcal{F}_{ij,h}} \|_{H_{00}^{1/2}(\mathcal{F}_{ij})}^2 \preceq \rho_i \, (1 + \log H/h)^2 \| u - \bar{u}_{\mathcal{F}_{ij}}^h \|_{H^1(\Omega_i)}^2$$

$$\preceq \rho_i \, (1 + \log H/h)^2 |u|_{H^1(\Omega_i)}^2 \asymp \rho_i \, (1+\ell)^2 |u|_{H^1(\Omega_i)}^2.$$

Here we have used the same arguments as in the proof of Lemma 4.2. Finally, Lemma 4.5 is replaced by

**Lemma 4.6** *For $u \in V_0^h(\Omega)$*

$$b_{-1}^F(I_h^F u, I_h^F u) \preceq (1 + \log H/h) a(u,u) \asymp (1+\ell) a(u,u).$$

The proof of this result can be found in the proof of Theorem 6.7 of Dryja, Smith, and Widlund [35].

*Assumption ii).* We have $\omega \preceq 1$; see Dryja, Smith, and Widlund [35].

*Assumption iii).* As in Subsection 4.4.1.

$\square$

**Remark 4.3** *Another possible decomposition for $w$ is given by*

$$w = \sum_{\mathcal{F}_{ij} \subset \Gamma} w_{\mathcal{F}_{ij}}.$$

*Here, $w_{\mathcal{F}_{ij}}$ is the discrete harmonic function on $\Omega$ with possibly nonzero interface values only on $\mathcal{F}_{ij,h}$. We note that the support of $w_{\mathcal{F}_{ij}}$ is $\bar{\Omega}_{\mathcal{F}_{ij}}$. We can decompose $w_{\mathcal{F}_{ij}}$ as in (4.15), and obtain*

$$b_k(w_{k,\mathcal{F}_{ij}}, w_{k,\mathcal{F}_{ij}}) \preceq (\rho_i + \rho_j) |w_{\mathcal{F}_{ij}}|^2_{H_{00}^{1/2}(\mathcal{F}_{ij})} =$$

$$(\rho_i + \rho_j) \, \|(u - \bar{u}^h_{\mathcal{F}_{ij}})_{\mathcal{F}_{ij,h}}\|^2_{H_{00}^{1/2}(\mathcal{F}_{ij})} \preceq (1 + \log H/h)^2 \|u - \bar{u}^h_{\mathcal{F}_{ij}}\|^2_{H^1_\rho(\Omega_{\mathcal{F}_{ij}})}$$

$$\preceq (1 + \log H/h)^2 |u|^2_{H^1_\rho(\Omega)} \asymp (1+\ell)^2 |u|^2_{H^1_\rho(\Omega)}.$$

### 4.4.3 An edge based coarse space

We can decrease the dimension of $V_{-1}^F$ and define another coarse space. Rather than using the values at all the nodes on the edges as degrees of freedom, only one degree of freedom per edge, an average value, is used. The resulting space, denoted by $V_{-1}^E \subset V_0^h$, is the range of the interpolation operator $I_h^E : V_0^h \to V_{-1}^E$, defined by

$$I_h^E u(x)_{|\bar{\Omega}_i} = \sum_{\mathcal{V}_m \in \partial\Omega_i} u(\mathcal{V}_m) \varphi_m(x) +$$

$$\sum_{\mathcal{E}_l \subset \mathcal{W}_i} \bar{u}^h_{\mathcal{E}_l} \theta_{\mathcal{E}_l}(x) + \sum_{\mathcal{F}_{ij} \subset \partial\Omega_i} \bar{u}^h_{\mathcal{F}_{ij}} \theta_{\mathcal{F}_{ij}}(x).$$

Here, $\bar{u}^h_{\mathcal{E}_l}$ is the discrete average value of $u$ over $\mathcal{E}_{l,h}$, and $\theta_{\mathcal{E}_l}$ the discrete harmonic function which equals 1 on $\mathcal{E}_{l,h}$ and is zero on $\partial\Omega_{i,h} \setminus \mathcal{E}_{l,h}$. $\varphi_m(x)$ is the discrete harmonic extension into $\Omega_i$ of the boundary values of standard nodal basis function associated with the vertex $\mathcal{V}_m$.

We define a bilinear form by

$$b_{-1}^E(u,u) = \sum_i \rho_i \{ h \sum_{\mathcal{V}_m \in \partial\Omega_i} (u(\mathcal{V}_m) - \bar{u}_i^h)^2 +$$

$$H \sum_{\mathcal{E}_m \subset \partial\Omega_i} (\bar{u}_{\mathcal{E}_m}^h - \bar{u}_i^h)^2 + H(1+\ell) \sum_{\mathcal{F}_{ij} \subset \partial\Omega_i} (\bar{u}_{\mathcal{F}_{ij}}^h - \bar{u}_i^h)^2 \}.$$

**Theorem 4.4** *Let $T^E$ be defined by (4.4). Then, for any $u \in V_0^h(\Omega)$, we have*

$$(1+\ell)^{-2} a(u,u) \preceq a(T^E u, u) \preceq a(u,u).$$

*Proof. Assumption i).* Here, we choose $w^{(i)} = I_h(\rho_i^{1/2} \mu_{i,1/2}^{+}(u - I_h^E u))$ on $\Gamma \cup \partial\Omega$ and extend these boundary values as a discrete harmonic function elsewhere. Note also that, we have $w_{\mathcal{V}_m}^{(i)} = 0$. The proof of a counterpart of Lemma 4.2 is similar to that given in the proof of Theorem 4.3.

The proof of a variant of Lemma 4.3 for this case proceeds as follows. Let $w_{\mathcal{E}_l}^{(i)}$ be the edge component of $w^{(i)}$, given similarly as in (4.14), and let $w_{k,\mathcal{E}_l}^{(i)}$ be the decomposition of $w_{\mathcal{E}_l}^{(i)}$ given as in (4.16). Using similar arguments as in Lemma 4.2, we have

$$\sum_{k=0}^{\ell} b_k(w_{k,\mathcal{E}_l}^{(i)}, w_{k,\mathcal{E}_l}^{(i)}) \preceq \sum_m \rho_m |w_{\mathcal{E}_l}^{(i)}|_{H^1(\Omega_{\mathcal{E}_l})}^2,$$

where the sum $\sum_m$ is taken over all substructures, which share the open edge $\mathcal{E}_l$. We now use the fact that $\mu_{i,1/2}^{+} = (\sum_m \rho_m^{1/2})^{-1}$ on $\mathcal{E}_{l,h}$, and an inverse inequality, and obtain

$$\sum_m \rho_m |w_{\mathcal{E}_l}^{(i)}|_{H^1(\Omega_{\mathcal{E}_l})}^2 \preceq \rho_i \sum_{p \in \mathcal{E}_{l,h}} h\,(\bar{u}_{\mathcal{E}_l}^h - u(x_p))^2.$$

We next use a Sobolev type inequality to obtain

$$\rho_i \sum_{p \in \mathcal{E}_{l,h}} h\,(\bar{u}_{\mathcal{E}_l}^h - u(x_p))^2 \preceq \rho_i\,(1 + \log H/h)|u|_{H^1(\Omega_i)}^2. \tag{4.30}$$

Results very similar to (4.30) can be found in Bramble, Pasciak, and Schatz [6], Bramble and Xu [9], Dryja [32], Dryja, Smith, and Widlund [35], and Dryja and Widlund [41]. Therefore,

$$\sum_m \rho_m |w_{\mathcal{E}_l}^{(i)}|_{H^1(\Omega_{\mathcal{E}_l})}^2 \preceq \rho_i\,(1+\ell)|u|_{H^1(\Omega_i)}^2. \tag{4.31}$$

Finally, we use

**Lemma 4.7** *For* $u \in V_0^h$

$$b_{-1}^E(I_h^E u, I_h^E u) \preceq (1+\ell)\, a(u,u).$$

The proof of this result can be found in the proof of Theorem 6.10 of Dryja, Smith, and Widlund [35].

*Assumption ii).* We have $\omega \preceq 1$; see Dryja, Smith, and Widlund [35].

*Assumption iii).* As in Subsection 4.4.1.

□

**Remark 4.4** *We can also simplify the proof by decomposing* $w$ *as*

$$w = \sum_{\mathcal{F}_{ij} \subset \Gamma} w_{\mathcal{F}_{ij}} + \sum_{\mathcal{E}_l \subset \Gamma} w_{\mathcal{E}_l}.$$

*Here,* $w_{\mathcal{F}_{ij}}$ *is chosen as in Remark 4.3 and* $w_{\mathcal{E}_l}$ *is the piecewise discrete harmonic function with possibly nonzero interface values only on* $\mathcal{E}_{l,h}$*. We note that the support of* $w_{\mathcal{E}_l}$ *is in* $\bar{\Omega}_{\mathcal{E}_l}$*. We decompose* $w_{\mathcal{E}_l}$ *as in (4.16) and obtain*

$$\sum_m \rho_m |w_{\mathcal{E}_l}|^2_{H^1(\Omega_{\mathcal{E}_l})} \preceq \sum_m \rho_m \sum_{p \in \mathcal{E}_{l,h}} h\,(\bar{u}^h_{\mathcal{E}_l} - u(x_p))^2$$

$$\preceq (1+\ell)\, |u|^2_{H^1_\rho(\Omega_{\mathcal{E}_l})}.$$

### 4.4.4 A wire basket based coarse space

Finally, we consider a coarse space $V_{-1}^W \subset V_0^h$, due to Smith [79]. It is based only on the values on the wire basket $\mathcal{W}_h$. The interpolation operator $I_h^W : V_0^h \to V_{-1}^W$, and is defined by

$$I_W^h u(x)_{|\bar{\Omega}_i} = \sum_{p \in \mathcal{W}_{i,h}} u(x_p)\varphi_p(x) + \sum_{\mathcal{F}_{ij} \subset \partial\Omega_i} \bar{u}^h_{\partial\mathcal{F}_{ij}} \theta_{\mathcal{F}_{ij}}(x).$$

Here, $\bar{u}^h_{\partial\mathcal{F}_{ij}}$ is the discrete average value of $u$ on $\partial\mathcal{F}_{ij,h}$. Let $\bar{u}^h_{\mathcal{W}_i}$ be the discrete average value of $u$ on $\mathcal{W}_{i,h}$. We define the bilinear form by

$$b_{-1}^W(u,u) = (1+\ell)\sum_i \rho_i \sum_{p \in \mathcal{W}_{i,h}} h(u(p) - \bar{u}^h_{\mathcal{W}_i})^2.$$

**Theorem 4.5** *Let $T^W$ be defined by (4.4). Then, for any $u \in V_0^h(\Omega)$, we have*

$$(1+\ell)^{-2} a(u,u) \preceq a(T^W u, u) \preceq a(u,u).$$

*Proof. Assumption i).* Let $w^{(i)} = I_h(\rho_i^{1/2} \mu_{i,1/2}^{+} (u - I_h^W u))$. Therefore, we have $w_{\mathcal{E}_l}^{(i)} = 0$ and $w_{\mathcal{V}_m}^{(i)} = 0$. The proof of the counterpart of Lemma 4.2 is as follows:

$$\begin{aligned}
&(\rho_i + \rho_j) |w_{\mathcal{F}_{ij}}^{(i)}|^2_{H^1(\Omega_{\mathcal{F}_{ij}})} \preceq \rho_i \, \|((u - \bar{u}^h_{\partial \mathcal{F}_{ij}})\, \theta_{\mathcal{F}_{ij}})_{\mathcal{F}_{ij,h}}\|^2_{H_{00}^{1/2}(\mathcal{F}_{ij})} \\
&\preceq \rho_i \{ \|(u\, \theta_{\mathcal{F}_{ij}})_{\mathcal{F}_{ij,h}}\|^2_{H_{00}^{1/2}(\mathcal{F}_{ij})} + (\bar{u}^h_{\partial \mathcal{F}_{ij}})^2 \|(\theta_{\mathcal{F}_{ij}})_{\mathcal{F}_{ij,h}}\|^2_{H_{00}^{1/2}(\mathcal{F}_{ij})} \} \\
&\preceq \rho_i \, \{ (1 + \log H/h)^2 \|u\|^2_{H^1(\Omega_i)} + (1 + \log H/h) \|u\|^2_{L^2(\partial \mathcal{F}_{ij})} \} \\
&\quad \preceq \rho_i (1 + \log H/h)^2 \|u\|^2_{H^1(\Omega_i)} \asymp \rho_i (1+\ell)^2 \|u\|^2_{H^1(\Omega_i)}
\end{aligned} \tag{4.32}$$

Here we have used ideas of the proof of Lemma 4.2, and the following results:

$$|\theta_{\mathcal{F}_{ij}}|^2_{H^1(\Omega_i)} \preceq H\,(1 + \log H/h), \tag{4.33}$$

$$(\bar{u}^h_{\partial \mathcal{F}_{ij}})^2 \preceq \frac{1}{H} \, \|u\|^2_{L^2(\partial \mathcal{F}_{ij})}, \tag{4.34}$$

and

$$\|u\|^2_{L^2(\partial \mathcal{F}_{ij})} \preceq (1 + \log H/h) \, \|u\|^2_{H^1(\Omega_i)}. \tag{4.35}$$

For the proof of (4.33), see Lemma 4.4 of [35]. The proof of (4.34) is a direct consequence of Schwarz inequality. The proof of (4.35) is related to the proof of (4.30).

To get the seminorm bound in (4.32), we use the same arguments as for (4.31). Finally, we also use

**Lemma 4.8** *For $u \in V_0^h(\Omega)$*

$$b_{-1}^W(I_h^W u, I_h^W u) \preceq (1 + \log H/h)^2 a(u,u) \asymp (1+\ell)^2 a(u,u).$$

The proof of this result can be found in the proof of Theorem 6.4 of Dryja, Smith, and Widlund [35].

*Assumption ii).* We have $\omega \preceq 1$; see Dryja, Smith, and Widlund [35].

*Assumption iii).* As in Subsection 4.4.1.

□

Remark 4.3 also applies in this case.

## 4.5 Special coefficients and an optimal algorithm

In this section, we show that if the coefficients $\rho_i$ satisfy certain assumptions, the $L^2_\rho-$projection is $H^1_\rho$-stable and we can use the space of piecewise linear functions $V^H(\Omega)$ as a coarse space and obtain an *optimal* multilevel preconditioner. It should be pointed out that the $L^2_\rho-$projection is not $H^1_\rho$-stable in general; see the counterexample given in Xu [90].

### 4.5.1 Quasi-monotone coefficients

Let $\mathcal{V}_m, m = 1, \cdots, L$, be the set of substructure vertices. We also include the vertices on $\partial\Omega$. Let $\Omega_{m_i},\ i = 1, \cdots, s(m)$, denote the substructures that have the vertex $\mathcal{V}_m$ in common, and let $\rho_{m_i}$ denote their coefficients. Let $\Omega_{\mathcal{V}_m}$ be the interior of the closure of the union of the substructures $\Omega_{m_i}$, i.e. the interior of $\cup_{i=1}^{s(m)}\bar{\Omega}_{m_i}$. By using the fact that all substructures are simplices, we see that each $\Omega_{m_i}$ has a whole face in common with $\partial\Omega_{\mathcal{V}_m}$. Two-dimensional illustrations of $\bar{\Omega}_{\mathcal{V}_m} = \cup_i\bar{\Omega}_{m_i}$ are given by Fig. 4.1 and Fig. 4.2.

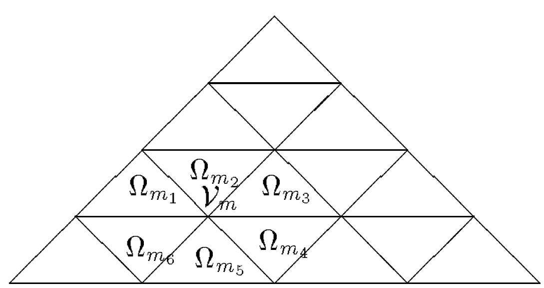


Fig. 4.1

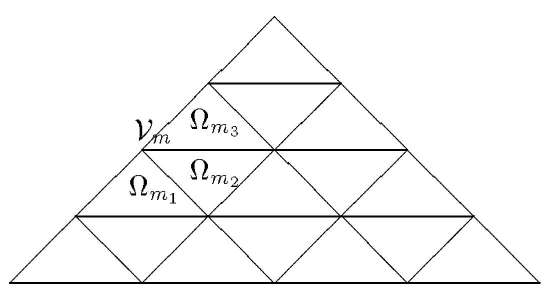


Fig. 4.2

**Definition 4.1** *For each $\Omega_{\mathcal{V}_m}$, order its substructures such that $\rho_{m_1} = \max_{i:1,\cdots,s(m)} \rho_{m_i}$. We say that a distribution of the $\rho_{m_i}$ is quasi-monotone in $\Omega_{\mathcal{V}_m}$ if for every $i = 1, \cdots, s(m)$, there exists a sequence $i_j,\ j = 1, \cdots, R$, with*

$$\rho_{m_i} = \rho_{m_{i_R}} \leq \cdots \leq \rho_{m_{i_{j+1}}} \leq \rho_{m_{i_j}} \leq \cdots \leq \rho_{m_{i_1}} = \rho_{m_1}, \tag{4.36}$$

*where the substructures $\Omega_{m_{i_j}}$ and $\Omega_{m_{i_{j+1}}}$ have a face in common. If the vertex $\mathcal{V}_m \in \partial\Omega$, then we additionally assume that $\partial\Omega_{m_1} \cap \partial\Omega$ contains a face for which $\mathcal{V}_m$ is a vertex.*

*A distribution $\rho_i$ on $\Omega$ is quasi-monotone with respect to the coarse triangulation $\mathcal{T}^0$ if it is quasi-monotone for each $\Omega_{\mathcal{V}_m}$.*

*We also define quasi-monotonicity with respect to a triangulation $\mathcal{T}^k$, as before, by replacing the $\Omega_i$ and the substructure vertices $\mathcal{V}_m$ by elements $\tau_j^k$ and nodes in $\mathcal{N}^k$, respectively.*

**Remark 4.5** *The analysis and results can easily be extended to the case in which*

$$\rho_{m_i} = \rho_{m_{i_R}} \preceq \cdots \preceq \rho_{m_{i_{j+1}}} \preceq \rho_{m_{i_j}} \preceq \cdots \preceq \rho_{m_{i_1}} = \rho_{m_1}.$$

In two dimensions, quasi-monotonicity with respect to $\mathcal{T}^0$ implies, for the same distribution of the $\rho_i$, quasi-monotonicity with respect to $\mathcal{T}^k$. We can show this as follows. The nodes of $\mathcal{N}^k$ divide into three sets: i) those which coincide with vertices of the substructures (nodes of the coarse triangulation), ii) those which belong to edges of the substructures, and iii) those which belong to the interior of the substructures. By examing the three cases, it is now easy to see that a distribution of the coefficients is quasi-monotone with respect to $\mathcal{T}^k$ if it is quasi-monotone with respect to $\mathcal{T}^0$.

In three dimensions there are cases in which a distribution of $\rho_i$ is quasi-monotone with respect to $\mathcal{T}^0$ but not quasi-monotone with respect to $\mathcal{T}^k$. In this case, the nodes of $\mathcal{N}^k$ are divided in four sets: those at vertices, edges, faces, and interiors of the substructures. There are no problems for those of the vertices, faces and interior sets but there can be complications with the edge set of nodes. Quasi-monotonicity of the coefficients for nodes belonging to the edge set does not follow from the quasi-monotonicity with respect to the coarse triangulation. To see that, let $\mathcal{E}_l$ be an edge of a substructure $\Omega_i$, and let $\mathcal{V}_m$ be a vertex of $\Omega_i$ and an end point of $\mathcal{E}_l$. There are more substructures sharing $\mathcal{V}_m$ than substructures sharing the whole edge $\mathcal{E}_l$. From this observation, it easy to distribute coefficients in such a way that they are quasi-monotone with respect to the coarse triangulation but not quasi-monotone with respect to a finer triangulation.

We note that in Theorem 4.6 only the quasi-monotonicity of the coefficients $\rho_i$ with respect to $\mathcal{T}^0$ is needed. We have introduced quasi-monotonicity with respect to $\mathcal{T}^k$ only to obtain a complete theory for the stability of the $L^2_\rho$-projection on finer levels; see Lemma 4.9 and Corollary 4.2

### 4.5.2 A new interpolator

We define an interpolation operator $I_k^M : V^h(\Omega) \rightarrow V^k(\Omega)$, as follows. This operator is central in our study of the properties of the weighted $L^2$-projection.

**Definition 4.2** *Given $u \in V^h(\Omega)$, define $u_k = I_k^M u \in V^k(\Omega)$ by the values of $u_k$ at two sets of nodes of $\mathcal{N}^k$:*

*i) For a nodal point $P \in \mathcal{N}_0^k$, let $u_k(P)$ be the average of $u$ over an element $\tau_{j_P}^k \in \mathcal{T}^k$.*

*ii) For a nodal point $P \in \mathcal{N}_{\partial\Omega}^k$, let $u_k(P)$ be the average of $u$ over $\bar{\tau}_{j_P}^k \cap \partial\Omega$.*

*Here, $\tau_{j_P}^k$ is the element, or one of the elements, with the vertex $P$ with the largest coefficient $\rho_i$. $\mathcal{N}_{\partial\Omega}^k$ is the set of nodes of $\mathcal{N}^k$ which belong to $\partial\Omega$, and $\mathcal{N}_0^k = \mathcal{N}^k \backslash \mathcal{N}_{\partial\Omega}^k$.*

It is easy to see that, for any constant $c$, $I_k^M(u - c) = I_k^M(u) - c \quad \forall c$, and also that $u_k$ vanishes on $\partial\Omega$ whenever $u$ vanishes on $\partial\Omega$.

We note that $\bar{\tau}_{j_P}^k \cap \partial\Omega$ is a face of $\tau_{j_P}^k$ for a quasi-monotone distribution of coefficients $\rho_i$ with respect to the triangulation $\mathcal{T}^0$, but that $\bar{\tau}_{j_P}^k \cap \partial\Omega$ might be just an edge or a vertex for coefficients that are not quasi-monotone with respect to $\mathcal{T}^0$.

**Lemma 4.9** *For a quasi-monotone distribution of coefficients $\rho_i$ with respect to the triangulation $\mathcal{T}^k$, we have $\forall u \in V^h(\Omega)$*

$$\|(I - I_k^M)u\|_{L^2_\rho(\tau_j^k)} \preceq h_k\, |u|_{H^1_\rho(\bar{\tau}_j^{k,M})} \quad \forall \tau_j^k \in \mathcal{T}^k, \tag{4.37}$$

*and*

$$|I_k^M u|_{H^1_\rho(\tau_j^k)} \preceq |u|_{H^1_\rho(\bar{\tau}_j^{k,M})} \quad \forall \tau_j^k \in \mathcal{T}^k. \tag{4.38}$$

*Here,* $\bar{\tau}_j^{k,M} \subset \bar{\tau}_j^{k,ext}$ *is a connected Lipschitz region given explicitly in the proof of this lemma.* $\bar{\tau}_j^{k,ext}$ *is the union of the* $\bar{\tau}_i^k$ *which have a vertex, edge, or face in common with* $\tau_j^k$*.*

*Furthermore,*

$$I_k^M u \in V_0^k(\Omega) \quad \textit{if} \quad u \in V_0^h(\Omega).$$

*Proof.* We have

$$\|u - I_k^M u\|^2_{L^2_\rho(\tau_j^k)} = \rho(\tau_j^k)\,\|u - I_k^M u\|^2_{L^2(\tau_j^k)}$$

$$\preceq \rho(\tau_j^k)\,(\|I_k^M u - c\|^2_{L^2(\tau_j^k)} + \|u - c\|^2_{L^2(\tau_j^k)}).$$

Using the definition and properties of $I_k^M$, we obtain

$$\|I_k^M u - c\|^2_{L^2(\tau_j^k)} = \|I_k^M (u-c)\|^2_{L^2(\tau_j^k)} \asymp \sum_{P \in \tau_j^k} h_k^3\,|(I_k^M(u-c))(P)|^2.$$

Here, each $P$ is a vertex of the element $\tau_j^k$.

For a case in which $P \in \mathcal{N}_0^k$, $I_k^M u(P)$, the average value of $u$ over an element $\tau_{j_P}^k$, can be bounded from above in terms of the $L^2$ norm of $u$ in $\tau_{j_P}^k$, i.e.

$$h_k^3\,|(I_k^M(u-c))(P)|^2 \preceq \|u-c\|^2_{L^2(\tau_{j_P}^k)}.$$

Here, $\tau_{j_P}^k$ is the element given in Definition 4.2.

For a case in which $P \in \mathcal{N}_{\partial\Omega}^k$, $I_k^M u(P)$, the average value of $u$ over a triangle $\bar{\tau}_{j_P}^k \cap \partial\Omega$, can be bounded from above in terms of the energy norm (4.2) of $u$ in $\tau_{j_P}^k$. Indeed,

$$h_k^3\,|(I_k^M(u-c))(P)|^2 \preceq h_k\,\|u-c\|^2_{L^2(\bar{\tau}_{j_P}^k \cap \partial\Omega)} \preceq h_k^2\,\|u-c\|^2_{H^1(\tau_{j_P}^k)}.$$

From the definition of quasi-monotonicity with respect to the triangulation $\mathcal{T}^k$, there exists for each $P$ a sequence of elements $\tau_{j_i}^k$, $i = 1, \cdots, n$, with

$$\rho(\tau_j^k) = \rho(\tau_{j_n}^k) \le \cdots \le \rho(\tau_{j_2}^k) \le \rho(\tau_{j_1}^k) = \rho(\tau_{j_P}^k). \tag{4.39}$$

Let $\bar{\tau}_{j_P}^{k,M} = \cup_{i=1}^n \bar{\tau}_{j_n}^k$ and $\bar{\tau}_j^{k,M} = \cup_{P \in \tau_j^k} \bar{\tau}_{j_P}^{k,M}$. Then,

$$\|u - I_k^M u\|^2_{L^2_\rho(\tau_j^k)} \preceq \rho(\tau_j^k)\,h_k^2\,\|u-c\|^2_{H^1(\bar{\tau}_j^{k,M})}.$$

Note that $\bar{\tau}_j^{k,M}$ is a connected Lipschtz region with a diameter of order $h_k$. Thus, we can use Poincaré's inequality to obtain

$$\inf_c \rho(\tau_j^k)\, h_k^2\, \|u - c\|^2_{H^1(\bar{\tau}_j^{k,M})} \preceq \rho(\tau_j^k)\, h_k^2\, |u|^2_{H^1(\bar{\tau}_j^{k,M})},$$

and use (4.39) to obtain

$$\rho(\tau_j^k)\, h_k^2\, |u|^2_{H^1(\bar{\tau}_j^{k,M})} \leq h_k^2\, |u|^2_{H^1_\rho(\bar{\tau}_j^{k,M})}.$$

To obtain (4.38), we use

$$|I_k^M u|^2_{H^1_\rho(\tau_j^k)} \preceq \sum_{i=1}^{4} \rho(\tau_j^k)\, h_k\, |(I_k^M(u-c))(P_i)|^2.$$

Here, the $P_i$ are the vertices of the element $\tau_j^k$. For the rest of the proof, we use the same arguments as before.

□

**Corollary 4.2** *For a quasi-monotone distribution of coefficients $\rho_i$ with respect to $\mathcal{T}^k$, we have*

$$\|(I - Q_\rho^k)u\|_{L^2_\rho(\Omega)} \preceq h_k\, |u|_{H^1_\rho(\Omega)} \quad \forall u \in V_0^h(\Omega), \tag{4.40}$$

*and*

$$|Q_\rho^k u|_{H^1_\rho(\Omega)} \preceq |u|_{H^1_\rho(\Omega)} \quad \forall u \in V_0^h(\Omega). \tag{4.41}$$

*Here, $Q_\rho^k$ is the weighted $L^2$-projection from $V_0^h(\Omega)$ to $V_0^k(\Omega)$.*

*Proof.* We obtain (4.40) from (4.37), since $Q_\rho^k$ gives the best approximation with respect to $L^2_\rho(\Omega)$. Finally, we note that (4.41) follows from (4.40); see Theorem 3.4 in Bramble and Xu [9].

□

**Remark 4.6** *The Lemma 4.9 and Corollary 4.2 can easily be extended to functions which do not vanish on the whole boundary $\partial\Omega$. Using Lemma 4.9, we can also establish optimal multilevel algorithms for problems with Neumann or mixed boundary conditions, and quasi-monotone coefficients with respect to $\mathcal{T}^0$.*

### 4.5.3 An optimal algorithm

We prove that the MDS algorithm, using the space of piecewise linear functions, $V_0^0$, as a coarse space, is optimal if the coefficient is quasi-monotone with respect to the coarse mesh $\mathcal{T}^0$. It is important to note that to prove our next theorem, we do not need to have quasi-monotonicity with respect to the fine meshes $\mathcal{T}^k$.

**Theorem 4.6** *Let $T^0$ be defined by (4.4) with $V_{-1} = V_0^0 = V_0^H$, $b_{-1}(\cdot,\cdot) = a(\cdot,\cdot)$. For a quasi-monotone distribution of the coefficients $\rho_i$ with respect to $\mathcal{T}^0$, we have*

$$a(T^0 u, u) \asymp a(u,u) \quad \forall u \in V_0^h(\Omega).$$

*Proof.* We only need to consider *Assumption i)*; *Assumptions ii)* and *iii)* have been checked in the proofs of the previous theorems.

Let the $\{\theta_m\}$ be a partition of unity over $\Omega$ with $\theta_m \in C_0^\infty(\Omega_{\mathcal{V}_m})$. Because of the size of the overlap of the subregions $\Omega_{\mathcal{V}_m}$, these functions can be chosen such that $|\nabla\theta_m|$ is bounded by $C/H$. We decompose $w = u - I_0^M u$ as

$$w = \sum_{m=1}^{L} w_m, \quad \text{where} \quad w_m = I_h(\theta_m w). \tag{4.42}$$

Here, $I_h$ is the standard linear interpolant with respect to the triangulation $\mathcal{T}^\ell$.

We note that $w_m = 0$, on and outside of $\partial\Omega_{\mathcal{V}_m}$, $m = 1,\cdots,L$. Using standard arguments, cf. Dryja and Widlund [37], we can show that

$$|w_m|^2_{H^1_\rho(\Omega_{\mathcal{V}_m})} \preceq |w|^2_{H^1_\rho(\Omega_{\mathcal{V}_m})} + \frac{1}{H^2}\|w\|^2_{L^2_\rho(\Omega_{\mathcal{V}_m})},$$

and by using Lemma 4.9, we obtain

$$|w_m|^2_{H^1_\rho(\Omega_{\mathcal{V}_m})} \preceq |u|^2_{H^1_\rho(\bar\Omega^{ext}_{\mathcal{V}_m})}.$$

Here, $\bar\Omega^{ext}_{\mathcal{V}_m}$ is the closure of the union of $\Omega_{\mathcal{V}_m}$ and the $\Omega_i$ which have a vertex, edge, or face in common with $\partial\Omega_{\mathcal{V}_m}$.

By assumption, we have quasi-monotone coefficients with respect to $\mathcal{T}^0$. We now remove the substructure $\Omega_{m_1}$ from $\Omega_{\mathcal{V}_m}$ obtaining $\Omega^c_{m_1} = \Omega_{\mathcal{V}_m}\backslash\bar\Omega_{m_1}$.

We decompose $w_m$ as

$$w_m = \mathcal{H}^{(m)} w_m + (w_m - \mathcal{H}^{(m)} w_m).$$

Here, $\mathcal{H}^{(m)} w_m$ is the piecewise discrete harmonic function on $\Omega_{m_1}$ and $\Omega_{m_1}^c$ that equals $w_m$ on $\partial\Omega_{m_1} \cup \partial\Omega_{\mathcal{V}_m}$. We stress that we use the weight $\rho = 1$ in obtaining this piecewise discrete harmonic function.

We decompose $\mathcal{H}^{(m)} w_m$ as in (4.17), and obtain

$$\mathcal{H}^{(m)} w_m = (\mathcal{H}^{(m)} w_m)_\ell + \cdots + (\mathcal{H}^{(m)} w_m)_1 + (\mathcal{H}^{(m)} w_m)_0. \tag{4.43}$$

Therefore, by using that $\rho_{m_1}$ is maximal and the arguments in the proof of Lemma 4.2, we obtain

$$\begin{aligned}\sum_{k=0}^{\ell} b_k((\mathcal{H}^{(m)} w_m)_k, (\mathcal{H}^{(m)} w_m)_k) &\preceq \rho_{m_1} |\mathcal{H}^{(m)} w_m|^2_{H^1(\Omega_{\mathcal{V}_m})} \\ \preceq \rho_{m_1} \|\mathcal{H}^{(m)} w_m\|^2_{H_{00}^{1/2}(\partial\Omega_{m_1} \cap \partial\Omega_{m_1}^c)} &\preceq \rho_{m_1} |\mathcal{H}^{(m)} w_m|^2_{H^1(\Omega_{m_1})} \\ \leq \rho_{m_1} |w_m|^2_{H^1(\Omega_{m_1})} &\leq |w_m|^2_{H_\rho^1(\Omega_{\mathcal{V}_m})}.\end{aligned} \tag{4.44}$$

Let $\tilde{w}_m = w_m - \mathcal{H}^{(m)} w_m$. Using the triangular inequality, we obtain

$$|\tilde{w}_m|_{H_\rho^1(\Omega_{\mathcal{V}_m})} \preceq |w_m|_{H_\rho^1(\Omega_{\mathcal{V}_m})}.$$

Note that $\tilde{w}_m$ vanishes on $\partial\Omega_{m_1} \cup \partial\Omega_{\mathcal{V}_m}$. Therefore, we can decompose $\tilde{w}_m$ in $\Omega_{m_1}$ and $\Omega_{m_1}^c$, independently. For the decomposition in $\Omega_{m_1}$, we have no difficulties, since we have constant coefficients. For the decomposition in $\Omega_{m_1}^c$, we can try to remove the substructure with largest coefficient $\rho_{m_i}$ in $\Omega_{m_1}^c$, and repeat the analysis just described. It is easy to show that we can remove all substructures, recursively, if we have a quasi-monotone distribution with respect to $\mathcal{T}^0$. We note that the argument in (4.44) is invalid if we do not have quasi-monotone coefficients.

$\square$

**Remark 4.7** *In the proof of Theorem 4.6, we just need to carry out the analysis locally for each $\Omega_{\mathcal{V}_m}$. In a case of quasi-monotone coefficients with respect to $\mathcal{T}^0$ and with the coarse spaces $V_{-1}^F$ or $V_{-1}^E$, we can derive a bound on the condition number of the multilevel additive Schwarz algorithm that is linear with respect to the number of levels $\ell$. The analysis also works if we use different coarse spaces in different parts of the domain $\Omega$. We can also use the coarse space $V_0^0$ and an exotic space $V_{-1}^X$ simultaneously. The resulting multilevel algorithm is optimal if the coefficient is quasi-monotone, and is* *almost optimal with a condition number bounded in terms of* $(1+\ell)^2$ *otherwise. The same arguments can also be used to prove that we also have an optimal multilevel algorithm with Neumann or mixed boundary condition and quasi-monotone coefficients.*

## 4.6 Multiplicative versions

In this section, we discuss some multiplicative versions of the multilevel additive Schwarz methods; they correspond to certain multigrid methods. Let $X = NN, F, E$, or $W$. Following Zhang [95], we consider two algorithms defined by their error propagation operators

$$E_G = (\prod_{k=0}^{\ell} \prod_{j\in\mathcal{N}_0^k} (I - P_j^k))(I - \eta T_{-1}^X), \tag{4.45}$$

and

$$E_J = \prod_{k=-1}^{\ell} (I - \tilde{T}^k) = (\prod_{k=0}^{\ell}(I - \eta \sum_{j\in\mathcal{N}_0^k} P_j^k))(I - \eta T_{-1}^X), \tag{4.46}$$

where $\eta$ is a damping factor chosen such that $\|\tilde{T}^k\|_{H_\rho^1} \leq w < 2$.

The products in the above expressions can be arranged in any order; different orders result in different schemes; see Zhang [95]. When the product is arranged in an appropriate order, the operators $E_G$ and $E_J$ correspond to the error propagation operators of V-cycle multigrid methods using Gauss-Seidel and damped Jacobi method as smoothers for the refined spaces, respectively.

By applying techniques developed in Zhang [95], and Dryja and Widlund [41], we can show that the norm of the error propagation operators $\|E_G\|_{H_\rho^1}$ and $\|E_F\|_{H_\rho^1}$ can be estimated from above by $1 - C\,(1+\ell)^{-2}$. In a case in which we have quasi-monotone coefficients and use the standard coarse space $V^H$, we can establish that the V-cycle multigrid methods, given by (4.45) and (4.46), are *optimal.*

## 4.7 Approximate discrete harmonic extensions

A disadvantage of using the coarse spaces $V_{-1}^X$, with $X = F, E, NN$, and $W$, is that we have to solve a local Dirichlet problem exactly for each substructure to obtain the discrete harmonic extensions. However, we can define new exotic coarse spaces, denoted by $V_{-1}^{\tilde{X}}$, with $\tilde{X} = \tilde{F}, \tilde{E}, \widetilde{NN}$, and $\tilde{W}$ by introducing *approximate discrete harmonic* *extensions.* They are given by simple explicit formulas [35, 76] and have the same $H^1_\rho$−stability estimates as the discrete harmonic extensions. Here we use strongly the fact that our exotic spaces $V_{-1}^{\tilde{X}}$ have constant values at the nodal points of the faces of the substructures. We prove that the MDS, with these new coarse spaces, have condition number estimate proportional to $(1+\ell)^2$.

Let $C_k$, $k = 1, \cdots, 4$, be the barycenters of the faces $\mathcal{F}_{ik}$ of $\partial\Omega_i$, and let $V_k$ be the vertex of $\Omega_i$ that is opposite to $C_k$. Let $C$ be the centroid of $\Omega_i$, i.e. the intersection of the line segments connecting the $V_k$ to the $C_k$. Let $E_{kl}$, $l = 1, 2, 3$ , be the open edges of $\partial\mathcal{F}_{ik}$; see Fig. 4.3.

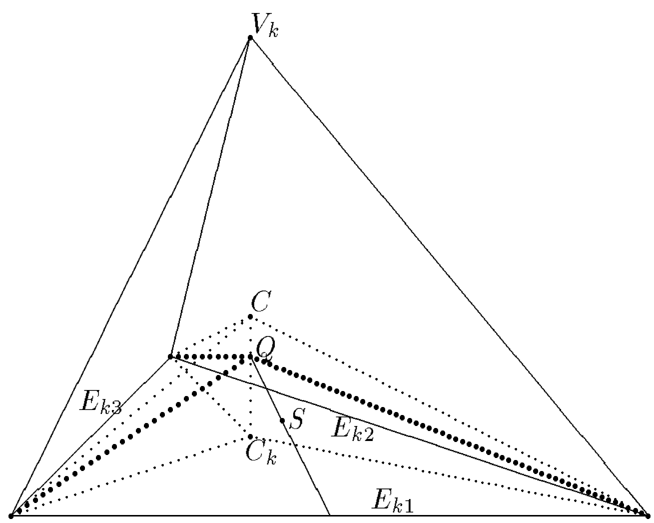


*Fig.* 4.3

To approximate the discrete harmonic function $\theta_{\mathcal{F}_{ij}}$ in $\Omega_i$, we use the finite element function $\vartheta_{\mathcal{F}_{ij}}$ introduced in the proof of Lemma 4.4 of Dryja, Smith, and Widlund [35] (see also Extension 3 in Sarkis [76]), given by (see Fig. 4.3).

**Definition 4.3** *The finite element function $\vartheta_{\mathcal{F}_{ij}} \in V^h(\Omega_i)$ is given by the following steps:*

*i) Let*

$$u_{ij}(C) = \frac{1}{4}.$$

*ii) For a point $Q$ that belongs to a line segment connecting $C$ to $C_k, k = 1, \cdots, 4$, define $u_{ij}(Q)$ by linear interpolation between the values $u_{ij}(C) = 1/4$ and $u_{ij}(C_k) = \delta_{jk}$, i.e. by*

$$u_{ij}(Q) = \lambda(Q)\frac{1}{4} + (1 - \lambda(Q))\delta_{jk}.$$

*Here $\lambda(Q)$ =distance$(Q, C_k)$/distance$(C, C_k)$ and $\delta_{jk} = 1$ if $j = k$ and $\delta_{jk} = 0$, otherwise.*

*iii) For a point $S$ that belongs to a triangle defined by the previous $Q$ as a vertex, and the edge $E_{kl}$ as a side opposite to $Q$, $l = 1, \cdots, 3$, let*

$$u_{ij}(S) = u_{ij}(Q).$$

*iv) Let $\vartheta_{\mathcal{F}_{ij}} = I_h^0 u_{ij}$, where $I_h^0$ is the interpolation operator into the space $V^h(\Omega_i)$ that preserves the values of a function $u_{ij}$ at the nodal points of $\bar{\Omega}_{i,h} \backslash \mathcal{W}_{i,h}$ and set them to zero on $\mathcal{W}_{i,h}$.*

*v) In $\Omega_j$, which has a common face $\mathcal{F}_{ij}$ with $\Omega_i$, $\vartheta_{\mathcal{F}_{ij}}$ is defined as in $\Omega_i$. Finally, $\vartheta_{\mathcal{F}_{ij}}$ is extended by zero outside $\Omega_{\mathcal{F}_{ij}}$.*

Note that $\vartheta_{\mathcal{F}_{ij}} \in V_0^h$, and

$$\sum_{\mathcal{F}_{ij} \subset \partial\Omega_i} \vartheta_{\mathcal{F}_{ij}} = 1 \quad \text{on} \quad \bar{\Omega}_{i,h} \backslash \mathcal{W}_{i,h}.$$

We remark that other extensions are also possible; see, e.g., Extension 2 of [76].

Using ideas in the proof of Lemma 4.4 in [35], we obtain

**Lemma 4.10**

$$|\theta_{\mathcal{F}_{ij}}|^2_{H^1(\Omega_i)} \leq |\vartheta_{\mathcal{F}_{ij}}|^2_{H^1(\Omega_i)} \preceq H\,(1 + \log H/h).$$

For the wire basket contributions, we replace the piecewise discrete harmonic function $\sum_{p \in \mathcal{W}_{i,h}} u(x_p)\varphi_p$, by $\sum_{p \in \mathcal{W}_{i,h}} u(x_p)\phi_p^\ell$, where $\phi_p^\ell$ is the standard nodal basis function associated with a node $p$. Using the definition of $\phi_p^\ell$ and a Sobolev type inequality (see Lemma 4.3 of [35]), we obtain

**Lemma 4.11**

$$| \sum_{p \in \mathcal{W}_{i,h}} (u(x_p) - \bar{u}_i^h)\, \phi_p^\ell|^2_{H^1(\Omega_i)} \preceq \|u - \bar{u}_i^h\|^2_{L^2(\mathcal{W}_i)}$$

$$\preceq (1+\ell)\, |u|^2_{H^1(\Omega_i)}.$$

New exotic coarse spaces $V_{-1}^{\tilde{X}}$, with $\tilde{X} = \tilde{F}, \tilde{E}, \widetilde{NN}$, and $\tilde{W}$ are introduced by combining the approximate discrete harmonic functions $\vartheta_{\mathcal{F}_{ij}}$ and $\sum_{p \in \mathcal{W}_{i,h}} u(x_p)\phi_p^\ell$. We define $V_{-1}^{\tilde{X}}$ as the range of the following interpolators $I_h^{\tilde{X}} : V_0^h \to V_{-1}^{\tilde{X}}$:

- **Modified Neumann-Neumann coarse spaces**

$$I_h^{\widetilde{NN}} u = \tilde{u}_{-1} = \sum_i \tilde{u}_{-1}^i = \sum_i \bar{u}_i^h\, \rho_i^\beta \tilde{\mu}_{i,\beta}^+.$$

Here, $\tilde{\mu}_{i,\mu}^+ = \mu_{i,\beta}^+$ on $\Gamma_h \cup \partial\Omega$ and is extended elsewhere in $\Omega$ as an approximate discrete harmonic function given by:

$$\tilde{\mu}_{i,\beta}^+(x) = \sum_{p \in \mathcal{W}_{i,h}} \mu_{i,\beta}^+(x_p)\phi_p^\ell(x) + \sum_{\mathcal{F}_{ij} \subset \partial\Omega_i} \mu_{i,\beta}^+(\mathcal{F}_{ij})\, \vartheta_{\mathcal{F}_{ij}}(x) \quad \forall x.$$

- **A modified face based coarse space**

$$I_h^{\tilde{F}} u(x)_{|\bar{\Omega}_i} = \sum_{p \in \mathcal{W}_{i,h}} u(x_p)\phi_p^\ell(x) + \sum_{\mathcal{F}_{ij} \subset \partial\Omega_i} \bar{u}_{\mathcal{F}_{ij}}^h \vartheta_{\mathcal{F}_{ij}}(x).$$

- **A modified edge based coarse space**

$$I_h^{\tilde{E}} u(x)_{|\bar{\Omega}_i} = \sum_{\mathcal{V}_m \in \partial\Omega_i} u(\mathcal{V}_m)\phi_{\mathcal{V}_m}^\ell(x) +$$

$$\sum_{\mathcal{E}_l \subset \mathcal{W}_i} \bar{u}_{\mathcal{E}_l} \sum_{p \in \mathcal{E}_{l,h}} \phi_p^\ell(x) + \sum_{\mathcal{F}_{ij} \subset \partial\Omega_i} \bar{u}_{\mathcal{F}_{ij}} \vartheta_{\mathcal{F}_{ij}}(x).$$

- **A modified wire basket based coarse space**

$$I_h^{\tilde{W}} u(x)_{|\Omega_i} = \sum_{p \in \mathcal{W}_{i,h}} u(x_p)\phi_p^\ell(x) + \sum_{\mathcal{F}_{ij} \subset \partial\Omega_i} \bar{u}_{\partial\mathcal{F}_{ij}} \vartheta_{\mathcal{F}_{ij}}(x).$$

It is important to note that our approximate discrete harmonic extensions recover constant functions, because

$$\sum_{p \in \mathcal{W}_{i,h}} \phi_p^\ell(x) + \sum_{\mathcal{F}_{ij} \subset \partial\Omega_i} \vartheta_{\mathcal{F}_{ij}}(x) = 1 \quad \forall x \in \bar{\Omega}_i.$$

We define the bilinear forms exactly as before, i.e.

$$b_{-1}^{\tilde{X}} = b_{-1}^{X},$$

and operators $T_{-1}^{\tilde{X}} : V^h \rightarrow V_{-1}^{\tilde{X}}$, by

$$b_{-1}^{\tilde{X}}(T_{-1}^{\tilde{X}} u, v) = a(u, v) \quad \forall \ v \in \ V_{-1}^{\tilde{X}}.$$

Let

$$T^{\tilde{X}} = T_{-1}^{\tilde{X}} + \sum_{k=0}^{\ell} \sum_{j \in \mathcal{N}_0^k} P_j^k.$$

**Theorem 4.7** *For any $u \in V_0^h(\Omega)$, we have*

$$(1+\ell)^{-2} a(u, u) \preceq a(T^{\tilde{X}} u, u) \preceq a(u, u).$$

*Proof.* Let us first consider a case with $V_{-1}^{\tilde{F}}$ as the coarse space.

*Assumption ii)* Using the triangle inequality, the explicit formulas for the approximate discrete harmonic functions, and Lemma 4.10, we obtain

$$|u|^2_{H^1_\rho(\Omega_i)} \preceq \sum_i \rho_i \{ \sum_{p \in \mathcal{W}_{i,h}} h \, (u(x_p))^2$$

$$+ \quad H(1+\ell) \sum_{\mathcal{F}_{ij} \subset \partial\Omega_i} (\bar{u}^h_{\mathcal{F}_{ij}})^2 \} \quad \forall u \in V_{-1}^{\tilde{F}}.$$

We now use that the approximate discrete harmonic extension recovers constant functions to obtain

$$a(u, u) \preceq b_{-1}^{\tilde{F}}(u, u). \tag{4.47}$$

*Assumption i)* Note that the bilinear form $b_{-1}^{\tilde{F}}(u, u)$ depends only on the values of $u$ on $\Gamma_h$, and let $\tilde{u}_{-1} = I_h^{\tilde{F}} u$ and $u_{-1} = I_h^F u$. We can therefore use Lemma 4.6 to obtain

$$b_{-1}^{\tilde{F}}(\tilde{u}_{-1}, \tilde{u}_{-1}) = b_{-1}^{F}(u_{-1}, u_{-1}) \preceq (1+\ell) \, a(u, u) \quad \forall u \in V_0^h. \tag{4.48}$$

We now modify the decomposition in the proof of Theorem 4.3

$$u = u_{-1} + \sum_k \sum_j u_j^k$$

and construct a decomposition for the current theorem by

$$u = \tilde{u}_{-1} + (u_{-1} - \tilde{u}_{-1}) + \sum_k \sum_j u_j^k =$$

$$\tilde{u}_{-1} + \sum_k \sum_j \tilde{u}_j^k + \sum_k \sum_j u_j^k = \tilde{u}_{-1} + \sum_k \sum_j v_j^k.$$

Here we use that $(u_{-1} - \tilde{u}_{-1})$ vanishes on $\Gamma_h \cup \partial\Omega_h$, and then decompose $(u_{-1} - \tilde{u}_{-1})$ as in Lemma 4.1 to obtain

$$\sum_k \sum_j a(\tilde{u}_j^k, \tilde{u}_j^k) \preceq a(u_{-1} - \tilde{u}_{-1}, u_{-1} - \tilde{u}_{-1}) \preceq a(\tilde{u}_{-1}, \tilde{u}_{-1}).$$

We now use (4.47) and (4.48) and obtain

$$a(\tilde{u}_{-1}, \tilde{u}_{-1}) \preceq b_{-1}^{\tilde{F}}(\tilde{u}_{-1}, \tilde{u}_{-1}) \preceq (1+\ell) a(u,u).$$

Finally, we use the proof of Theorem 4.3 to obtain

$$\sum_k \sum_j a(u_j^k, u_j^k) \preceq (1+\ell)^2 \, a(u,u).$$

The proof of this theorem for $V_{-1}^{\tilde{E}}$ or $V_{-1}^{\tilde{W}}$ as the coarse space is quite similar.

Let us finally consider the case with $V_{-1}^{\widetilde{NN}}$ as the coarse space. For *Assumption ii)*, we trivially have $\omega = 1$. *Assumption i)* is handled exactly as before. The only nontrivial part is to show that

$$a(I_h^{\widetilde{NN}} u, I_h^{\widetilde{NN}} u) \preceq (1+\ell)^2 a(u,u) \quad \forall u \in V_0^h(\Omega). \tag{4.49}$$

The idea of the proof of (4.49) is the same as in Dryja and Widlund [41]. We reduce the estimates to bounds related to the vertices, edges, and faces and use Lemma 4.5 in [35], 4.10 and 4.11.

□

## 4.8 Nonuniform refinements

We now consider finite element approximation with locally nested refinement. Such refinements can be used to improve the accuracy of the solutions of problems with singular behavior which arise in elliptic problems with discontinuous coefficients, nonconvex domains, or singular data. We note, in particular, that solutions of elliptic problems with highly discontinuous coefficients are very likely to become increasingly singular when we approach the wire basket.

Nested local refinements have previously been analyzed by Bornemann and Yserentant [3], Bramble and Pasciak [5], Cheng [25, 26], Oswald [66, 67], and Yserentant [92, 93]. By nested local refinement we mean that an element, which is not refined at level $j$, cannot be a candidate for further refinement. Under certain assumptions on the local refinement, optimal multilevel preconditioners previously have been obtained for problems with nearly constant coefficients in two and three dimensions. For problems in two dimensions with highly discontinuous coefficients, the standard piecewise linear function can be used as a coarse space to design multilevel preconditioners. A bound on the condition number can be derived, which is independent of the coefficients, and which grows at most as the square of the number of levels; see, e.g., Yserentant [93]. Here, we extend the analysis to the case where the coefficients are quasi-monotone with respect to the coarse triangulation or are highly discontinuous in two or in three dimensions.

Let us begin by a shape regular but possibly nonuniform coarse triangulation $\mathcal{T}^0 = \mathcal{T}^{0*}$, which defines substructures $\Omega_i$ with diameters $H_i$. It follows from shape regularity that neighboring substructures are of comparable size.

We introduce the following refinement procedure: For $k = 1, \cdots, \ell$, subdivide all the tetrahedra $\tau_j^{k-1} \in \mathcal{T}^{k-1}$ into eight tetrahedra (see, e.g., Ong [63]); these are elements of level $k$ and belong, by definition, to $\mathcal{T}^k$. A shape regular refinement is obtained by connecting properly the midpoints of the edges $\tau_j^{k-1}$. We note that this refinement, restricted to each $\bar{\Omega}_i$, is quasi-uniform.

Let $V_0^k$ be the space of piecewise linear functions associated with $\mathcal{T}^k$, which vanish on $\partial\Omega$, and let $\mathcal{N}_0^k$ be the set of nodal points associated with the space $V_0^k$. Let $\phi_j^k$, $j \in \mathcal{N}_0^k$, be a standard nodal basis function of $V_0^k$, and let $V_j^k = \text{span}\{\phi_j^k\}$.

We define a locally nested refinement in terms of a sequence of open subregions $\mathcal{O}_k \subset \Omega$ such that

$$\mathcal{O}_\ell \subset \mathcal{O}_{\ell-1} \subset \cdots \subset \mathcal{O}_k \subset \cdots \subset \mathcal{O}_1 \subset \mathcal{O}_0 = \Omega,$$

and assume that the $\partial\mathcal{O}_k$, the boundary of $\mathcal{O}_k$, align with element boundaries of $\mathcal{T}^{k-1}$, for $k \geq 1$.

We define a nested, nonconforming triangulations $\mathcal{T}^{k^*}$, $k = 0, \cdots, \ell$, as follows:

$$\mathcal{T}^{k^*} = \mathcal{T}^k \quad \text{on} \quad \bar{\mathcal{O}}_k,$$

and

$$\mathcal{T}^{k^*} = \mathcal{T}^j \quad \text{on} \quad \bar{\mathcal{O}}_j \backslash \mathcal{O}_{j+1} \quad \forall j < k.$$

**Assumption 4.1** *The levels of two elements of a triangulation $\mathcal{T}^{k^*}$, which have at least one common point, differ by at most one.*

We note that Assumption 4.1 guarantees that all elements in $\mathcal{T}^{\ell^*}(\Omega_i)$ with a common vertex are of comparable size. This type of refinement is exactly the same as that analyzed by Bornemann and Yserentant [3].

Let $V_0^{k^*}$ be the space of piecewise linear functions associated with the triangulation $\mathcal{T}^{k^*}$, which are continuous on $\Omega$ and vanish on $\partial\Omega$. By construction, $V_0^{k^*} \subset V_0^{k+1^*} \subset V_0^{k+1}$. The vertices of the elements of $\mathcal{T}^{k^*}$ are called nodes. From the requirement of continuity, it follows that we can distinguish between the set of free nodes $\mathcal{N}_0^{k^*}$ and the remaining set of slave nodes. A function $u \in V_0^{\ell^*}$ is determined uniquely by its values at the free nodes $\mathcal{N}_0^{k^*}$; the values of $u$ at the slave nodes are determined, by interpolation, from the values at $\mathcal{N}_0^{k^*}$. Therefore, we have the following representation:

$$u = \sum_{j \in \mathcal{N}_0^{\ell^*}} u(x_j)\, \phi_j^{\ell^*} \quad \forall u \in V_0^{\ell^*} \tag{4.50}$$

Here, $\phi_j^{\ell^*} \in V_0^{\ell^*}$ is a nodal basis function, with respect to $\mathcal{T}^{\ell^*}$, which equals 1 at one free node and vanishes at all other free nodes of $\mathcal{N}_0^{\ell^*}$.

The discrete problem is given by:

Find $u \in V_0^{\ell^*}$, such that

$$a(u, v) = f(v) \quad \forall\ v \in\ V_0^{\ell^*}. \tag{4.51}$$

In order to obtain a preconditioner, we consider the following splitting:

$$V_0^{\ell^*} = V_{-1}{}^* + \sum_{k=0}^{\ell} \sum_{j \in \mathcal{V}^{k^*}} V_j^k. \tag{4.52}$$

Here, $\mathcal{V}^{k^*}$ is the set of nodes of $\mathcal{N}_0^k$ which belong to the interior of $\mathcal{O}_k$.

We consider in detail only the exotic coarse space $V_{-1}{}^* = V_{-1}^{\tilde{F}}{}^*$, i.e. the counterpart of the modified face based coarse space introduced in Section 4.7. We note that the same ideas can be extended straightforwardly to define and analyze algorithms using the other exotic coarse spaces introduced in Sections 4.4 and 4.7.

The coarse space $V_{-1}^{\tilde{F}}{}^*$ can be defined as the range of an interpolation operator $I_\ell^{\tilde{F}^*} : V_0^{\ell *} \rightarrow V_{-1}^{\tilde{F}}{}^*$, defined by

$$I_\ell^{\tilde{F}^*} u(x)_{|\bar{\Omega}_i} = \sum_{p \in (\mathcal{W}_i \cap \mathcal{N}_0^{\ell *})} u(x_p) \phi_j^{\ell *}(x) + \sum_{\mathcal{F}_{ij} \subset \partial\Omega_i} \bar{u}^*_{\mathcal{F}_{ij}} \vartheta^*_{\mathcal{F}_{ij}}(x),$$

where

$$\bar{u}^*_{\mathcal{F}_{ij}} = \frac{\sum_{p \in (\mathcal{F}_{ij} \cap \mathcal{N}_0^{\ell *})} u(x_p) \int_{\text{supp } (\phi_p^{\ell *}(\mathcal{F}_{ij}))} \phi_p^{\ell *} \, dS}{\sum_{p \in (\mathcal{F}_{ij} \cap \mathcal{N}_0^{\ell *})} \int_{\text{supp } (\phi_p^{\ell *}(\mathcal{F}_{ij}))} \phi_p^{\ell *} \, dS}.$$

$\vartheta^*_{\mathcal{F}_{ij}}$ is defined in a way similar to $\vartheta_{\mathcal{F}_{ij}}$ except that in Step iv), of Definition 4.3, we interpolate at the free nodes $\mathcal{N}_0^{\ell *}$ which belong to $\bar{\Omega}_i \setminus \mathcal{W}_i$ and set $\vartheta^*_{\mathcal{F}_{ij}}$ to zero on $\mathcal{W}_i$.

We consider the following bilinear form:

$$b_{-1}^{\tilde{F}}{}^*(u,u) = \sum_i \rho_i \{ \|u - \bar{u}_i^*\|^2_{L^2(\mathcal{W}_i)} + \quad H_i(1+\ell) \sum_{\mathcal{F}_{ij} \subset \partial\Omega_i} (\bar{u}^*_{\mathcal{F}_{ij}} - \bar{u}_i^*)^2 \},$$

where,

$$\bar{u}_i^* = \frac{\sum_{p \in (\partial\Omega_i \cap \mathcal{N}_0^{\ell *})} u(x_p) \int_{\text{supp } (\phi_p^{\ell *}(\partial\Omega_i))} \phi_p^{\ell *} \, dS}{\sum_{p \in (\partial\Omega_i \cap \mathcal{N}_0^{\ell *})} \int_{\text{supp } (\phi_p^{\ell *}(\partial\Omega_i))} \phi_p^{\ell *} \, dS},$$

and introduce an operator $T_{-1}^{\tilde{F}}{}^* : V_0^{\ell *} \rightarrow V_{-1}^{\tilde{F}}{}^*$, by

$$b_{-1}^{\tilde{F}}{}^*(T_{-1}^{\tilde{F}}{}^* u, v) = a(u,v) \quad \forall \; v \in \; V_{-1}^{\tilde{F}}{}^*.$$

Let

$$T^{\tilde{F}^*} = T_{-1}^{\tilde{F}}{}^* + \sum_{k=0}^{\ell} \sum_{j \in \mathcal{V}^{k^*}} P_j^k. \tag{4.53}$$

**Theorem 4.8** *For any $u \in V_0^{\ell *}$, we have*

$$(1+\ell)^{-2} a(u,u) \preceq a(T^{\tilde{F}^*} u, u) \preceq a(u,u).$$

*Proof.* *Assumptions i) and ii).* We first introduce pseudo inverses $\mu_{i,1/2}^{+*}$ and apply Sobolev type inequalities and extension theorems to obtain the required results on the coarse space approximation (see Lemma 4.16) and to reduce our problem to local problems with constant coefficients. For the local problems with constant coefficients, we use the decomposition given in Bornemann and Yserentant [3]. Note that our refinement is a particular case of the local refinement NLR2 considered by Oswald [67]; we can then apply Theorem 6 of Oswald [67] to obtain a good decomposition.

The Sobolev type inequalities and extension theorems required for our particular nonuniform refinement are given below in several lemmas.

*Assumption iii).* Note that, on each $\bar{\Omega}_i$, the strengthened Cauchy-Schwarz tensor $\mathcal{E}^*$ associated with the splitting (4.52) can be obtained by symmetrically deleting columns and rows from the tensor $\mathcal{E}$ associated with the case of quasi-uniform refinement. Therefore, we obtain $\rho(\mathcal{E}^*) \leq C$ by a standard Rayleigh quotient argument.

□

We now slightly modify some lemmas that are well known for quasi-uniform refinement to show that they hold in our nonuniform refinement case.

**Lemma 4.12** *For $u \in V^{\ell^*}(\Omega_i)$,*

$$|\sum_{p \in (\mathcal{W}_i \cap \mathcal{N}^{\ell^*})} (u(x_p) - \bar{u}_i^*)\, \phi_p^{\ell^*}|_{H^1(\Omega_i)}^2 \preceq \|u - \bar{u}_i^*\|_{L^2(\mathcal{W}_i)}^2$$

$$\preceq (1+\ell)\, |u|_{H^1(\Omega_i)}^2.$$

*Proof.* For the proof of the first inequality, we use Assumption 4.1 and the inverse inequality. For the second inequality, we use that $V^{\ell^*}(\Omega_i) \subset V^{\ell}(\Omega_i)$ and then apply a standard Sobolev type inequality (see Lemma 4.3 of [35]). To obtain the estimate with the seminorm, we use the fact that for any constant $c$, $\bar{u}_i^* = c$, if $u = c$ on $\mathcal{W}_i \cap \mathcal{N}^{\ell^*}$.

□

**Lemma 4.13**

$$|\theta_{\mathcal{F}_{ij}}^*|_{H^1(\Omega_i)}^2 \leq |\vartheta_{\mathcal{F}_{ij}}^*|_{H^1(\Omega_i)}^2 \preceq H_i\,(1+\ell),$$

*where $\theta_{\mathcal{F}_{ij}}^* \in V^{\ell^*}(\Omega_i)$ is the discrete harmonic function, in the sense of $V^{\ell^*}(\Omega_i)$, which equals 1 on $(\mathcal{F}_{ij} \cap \mathcal{N}^{\ell^*})$ and is 0 on $(\partial\Omega_i \backslash \mathcal{F}_{ij})$.*

*Proof.* The first inequality is trivial since the discrete harmonic function has minimal energy. For the second inequality, we use the same ideas as in Lemma 4.4 of Dryja, Smith, and Widlund [35]. Assumption 4.1 is crucial in this proof.

$\square$

We denote by $V^{\ell^*}(\partial\Omega_i)$ the restriction of $V^{\ell^*}(\Omega_i)$ to $\partial\Omega_i$. Let $\mathcal{H}^{(i)^*} : V^{\ell^*}(\partial\Omega_i) \to V^{\ell^*}(\Omega_i)$, be the discrete harmonic extension operator in the sense of $V^{\ell^*}(\Omega_i)$.

**Lemma 4.14** *Let $u \in V^{\ell^*}(\partial\Omega_i)$. Then*

$$|\mathcal{H}^{(i)^*} u|_{H^1(\Omega_i)} \preceq |u|_{H^{1/2}(\partial\Omega_i)}. \tag{4.54}$$

*Proof.* Let $\tilde{u} \in H^1(\Omega_i)$ be the harmonic extension of $u$ defined by

$$(\nabla\tilde{u}, \nabla v)_{L^2(\Omega_i)} = 0 \quad \forall v \in H_0^1(\Omega_i),$$

$$\tilde{u} = u \text{ on } \partial\Omega_i.$$

Therefore, by the definition of the $H^{1/2}-$seminorm,

$$|\tilde{u}|_{H^1(\Omega_i)} = |u|_{H^{1/2}(\partial\Omega_i)}. \tag{4.55}$$

We now slightly modify the interpolator $I_k^M$ introduced in Definition 4.2 and define another interpolation operator $I_\ell^{M^*} : H^1(\Omega_i) \to V^{\ell^*}(\Omega_i)$, as follows

**Definition 4.4** *Given $\tilde{u} \in H^1(\Omega_i)$, such that $\tilde{u}|_{\partial\Omega_i} \in V^{\ell^*}(\partial\Omega_i)$, define $u^* = I_k^M\tilde{u} \in V^{\ell^*}(\Omega_i)$ by the values of $u^*$ at two sets of free nodal points $\mathcal{N}^{\ell^*}(\bar{\Omega}_i)$:*

*i) For a free nodal point $P \in \mathcal{N}^{\ell^*}(\bar{\Omega}_i)\backslash\mathcal{N}^{\ell^*}(\partial\Omega_i)$, let $u^*(P)$ be the average of $\tilde{u}$ over an element $\tau_{j_P}^{\ell} \in \mathcal{T}^{\ell^*}(\Omega_i)$.*

*ii) For a free nodal point $P \in \mathcal{N}^{\ell^*}(\partial\Omega_i)$, let $u^*(P) = \tilde{u}(P)$.*

*Here, $\tau_{j_P}^k$ is any element $\mathcal{T}^{\ell^*}(\Omega_i)$ with vertex $P$.*

Using the same arguments as in Lemma 4.9, we obtain

$$|u^*|_{H^1(\Omega_i)} \preceq |\tilde{u}|_{H^1(\Omega_i)}. \tag{4.56}$$

Finally, we use (4.55), (4.56), and the fact that $\mathcal{H}^{(i)^*}u$ has minimal energy, to obtain (4.54).

$\square$

Let $I^{\ell^*}$ be the interpolation operator into the space $V^{\ell^*}$ that preserves the values of a function at the free nodal points $\mathcal{N}^{\ell^*}$. Using the same ideas as in Lemma 4.4 of Dryja, Smith, and Widlund [35], we obtain

**Lemma 4.15** *Let $u \in V^{\ell^*}(\Omega_i)$. Then*

$$|I^{\ell^*}(\vartheta^*_{\mathcal{F}_{ij}}\, u)|^2_{H^1(\Omega_i)} \preceq (1+\ell)^2 \|u\|^2_{H^1(\Omega_i)}.$$

Using Lemmas 4.12, we obtain the necessary results on the coarse space approximation:

**Lemma 4.16** *Let $u \in V_0^{\ell^*}$. Then*

$$b^{\tilde{F}}_{-1}{}^{*}(I_\ell^{\tilde{F}^*} u, I_\ell^{\tilde{F}^*} u) \preceq (1+\ell)\, a(u,u).$$

We now consider the case of quasi-monotone coefficients with respect to the coarse triangulation $\mathcal{T}^{0^*}$. Let

$$T^{0^*} = P^{0^*} + \sum_{k=0}^{\ell} \sum_{j \in \mathcal{V}^{k^*}} P_j^k.$$

Here, $P^{0^*} : V_0^{\ell^*} \to V_0^{0^*}$, is the $H^1_\rho(\Omega)$-projection.

**Theorem 4.9** *For a quasi-monotone distribution of the coefficients $\rho_i$ with respect to $\mathcal{T}^{0^*}$, we have*

$$a(\mathcal{T}^{0^*} u, u) \asymp a(u,u) \quad \forall u \in V_0^{\ell^*}.$$

*Proof.* The proof is very similar to that of Theorem 4.6 using now Lemmas 4.12-4.16. We note that Lemma 4.9, for $k=0$, holds for a locally quasi-uniform triangulation $\mathcal{T}^{0^*}$. We now replace the decomposition (4.43) by a decomposition analyzed by Bornemann and Yserentant [3], or Oswald [67].

We note that Assumption 4.1 is needed to prove the first inequality in (4.44).

$\square$

## 4.9 Multilevel methods on the interface

In this section, we extend our results to multilevel iterative substructuring algorithms for problems with discontinuous coefficients. We recall that iterative substructuring methods provide preconditioners for the reduced system of equations that remains after that all the interior variables of the substructures have been eliminated. We focus only on variants of the algorithms developed in Section 4.4. Other algorithms, based on other coarse spaces of Sections 4.4 and with nonuniform refinements, can be designed and analyzed in the same way.

Let $V_0^h(\Gamma)$ be the restriction of $V_0^h(\Omega)$ to $\Gamma$. The iterative substructuring method associated with (3.5) is of the form: Find $u \in V_0^h(\Gamma)$ such that

$$s(u,v) = \tilde{f}(v) \quad \forall v \in V_0^h(\Gamma), \tag{4.57}$$

where

$$s(u,v) = a(\mathcal{H}u, \mathcal{H}v) = \sum_i \rho_i \int_{\Omega_i} \nabla \mathcal{H}^{(i)} u \cdot \nabla \mathcal{H}^{(i)} v \, dx,$$

and

$$\tilde{f}(v) = \sum_i \int_{\Omega_i} f \, \mathcal{H}^{(i)} v.$$

Let $V_0^k(\Gamma)$, $k = 0, \cdots, \ell$, be the restriction of $V_0^k(\Omega)$ to $\Gamma$ and let $\mathcal{N}_0^k(\Gamma)$ be the set of nodal points associated with the space $V_0^k(\Gamma)$. Let $V_j^k(\Gamma), j \in \mathcal{N}_0^k(\Gamma)$, be the restriction of $V_j^k(\Omega)$ to $\Gamma$.

We introduce the bilinear forms $b_k^j(u,v)$: $V_j^k(\Gamma) \times V_j^k(\Gamma) \to \Re$, for $k = 0, \cdots, \ell$, and $j \in \mathcal{N}_0^k(\Gamma)$ by

$$b_k^j(u,v) = u(x_j) v(x_j) \, a(\phi_j^k, \phi_j^k). \tag{4.58}$$

Here, $\phi_j^k$ is the nodal basis functions that span $V_j^k(\Omega)$. We can easily extend the analysis to the case in which we use a good approximation of $a(\phi_j^k, \phi_j^k)$.

Let $V_{-1}^X(\Gamma)$, with $X = F, E, NN$, and $W$, be the restriction of $V_{-1}^X(\Omega)$, to $\Gamma$, and let the associated bilinear form be given by $b_{-1}^X(\cdot,\cdot)$. Note that $b_{-1}^X$ is well defined for $u \in V_0^h(\Gamma)$, since the computation of $b_{-1}^X(u,u)$ depends only on the values of $u$ on $\Gamma_h$.

We introduce the operators $T_j^k : V_0^h(\Gamma) \to V_j^k(\Gamma)$, $k = 0, \cdots, \ell$, and $j \in \mathcal{N}_0^k(\Gamma)$, by

$$b_k^j(T_j^k \, u, v) = s(u,v) \quad \forall \; v \in V_j^k(\Gamma),$$

and the operator $T_{-1}^{X,\Gamma} : V_0^h(\Gamma) \to V_{-1}^X(\Gamma)$, by

$$b_{-1}^X(T_{-1}^{X,\Gamma} u, v) = s(u,v) \ \ \forall \ v \in V_{-1}^X(\Gamma).$$

Let

$$T^{X,\Gamma} = T_{-1}^{X,\Gamma} + \sum_{k=0}^{\ell} \sum_{j \in \mathcal{N}_0^k(\Gamma)} T_j^k$$

**Theorem 4.10** *For $u \in V_0^h(\Gamma)$*

$$(1+\ell)^2 s(u,u) \preceq s(T^{X,\Gamma} u, u) \preceq s(u,u).$$

*Proof.* The proof for $X = W$, with a condition number estimate of $C(1+\ell)^3$ is given in Dryja and Widlund [40] (Theorem 6.2). To obtain an improved, quadratic estimate, we can use a result of Zhang (see Remark 3.3 in [95]). Using similar arguments as in [40] and in previous sections, we can prove our current theorem for the other exotic coarse spaces as well.

Another technique for estimating condition numbers for preconditioned Schur complement systems was introduced by Smith, and Widlund [80]. They showed that the condition number of the preconditioned Schur complement is bounded from above by the condition number of the full linear system preconditioned by a related preconditioner. Using the same technique, Tong, Chan, and Kuo [82] gave an upper bound for the condition number for a Schur complement system preconditioned by a BPX preconditioner. They only considered elliptic problems with nearly constant coefficients. Here, we can also use the same technique to prove our theorem.

□

## 4.10 Two level methods

We study several two-level overlapping domain decomposition methods with a multilevel algorithm solver for the local problems. This approach is quite attractive because we can introduce more parallelism in our algorithms. The possible variants of the algorithms depend on:

i) Which coarse space we use: $V_{-1}^{NN}$, $V_{-1}^{F}$, $V_{-1}^{E}$, $V_{-1}^{W}$, $V_0^{H}$, $V_{-1}^{\widetilde{NN}}$, $V_{-1}^{\tilde{F}}$, $V_{-1}^{\tilde{E}}$, or $V_{-1}^{\tilde{W}}$.

ii) How we cover $\Omega$ with overlapping subregions: $\Omega_{\mathcal{V}_m}$, $\Omega_{\mathcal{F}_{ij}}$, $\Omega_i'$, or respective subregions with small overlap. Here, $\Omega_i'$ is the union of $\Omega_i$ and its next neighboring substructures. We note that to use the covering $\Omega_{\mathcal{F}_{ij}}$ we must use $V_{-1}^F$, $V_{-1}^W$, $V_{-1}^{\tilde{F}}$, or $V_{-1}^{\tilde{W}}$ as a coarse space.

iii) Which method we use for the local problems: exact solvers, multilevel diagonal scaling, one symmetrized multigrid V-cycle, $n$ symmetrized multigrid V-cycle, etc.

**Algorithm 4.1** *Let us first consider a case in which the coarse space is given by $V_{-1}^{NN}$, the local problems are given by a multilevel diagonal scaling-(MDS) on $\Omega_{\mathcal{V}_m}$, $m = 1, \cdots, M$. Therefore, we introduce a subspace splitting by*

$$V_0^h = V_{-1}^{NN} + \sum_{k=1}^{\ell} \sum_{j \in \mathcal{N}_1^k} V_j^k + \cdots + \sum_{k=1}^{\ell} \sum_{j \in \mathcal{N}_m^k} V_j^k + \cdots \sum_{k=1}^{\ell} \sum_{j \in \mathcal{N}_M^k} V_j^k,$$

*and a preconditioned operator by*

$$T_{NNadd} = T_{-1}^{NN} + \sum_{k=1}^{\ell} \sum_{j \in \mathcal{N}_1^k} P_j^k + \cdots + \sum_{k=1}^{\ell} \sum_{j \in \mathcal{N}_m^k} P_j^k + \cdots \sum_{k=1}^{\ell} \sum_{j \in \mathcal{N}_M^k} P_j^k. \tag{4.59}$$

*Here, $\mathcal{N}_m^k$ is the set of $k$-nodal points associated to the space $V_0^k(\Omega_{\mathcal{V}_m})$.*

**Theorem 4.11** *Let $T$ be defined by (4.59). Then for any $u \in V_0^h(\Omega)$, we have*

$$(1+\ell)^{-2} a(u,u) \preceq a(T_{NNadd}\, u, u) \preceq a(u,u).$$

*Proof. Assumption (i)* In a first step, we decompose $u$ as in the proof of Theorem 4.2, and obtain

$$u - u_{-1} = \sum_{i=1}^{N} (v^{(i)} + \sum_{\mathcal{F}_{ij} \subset \partial\Omega_i} w_{\mathcal{F}_{ij}}^{(i)} + \sum_{\mathcal{E}_l \subset \partial\Omega_i} w_{\mathcal{E}_l}^{(i)} + \sum_{\mathcal{V}_n \subset \partial\Omega_i} w_{\mathcal{V}_n}^{(i)}). \tag{4.60}$$

Let $\theta_m$, $m = 1, \cdots, M$ be the partition of unity introduced in the proof of Theorem 4.6. We decompose each right hand side term of (4.60) as

$$v^{(i)} = \sum_m I_h(\theta_m v^{(i)}), \quad w_{\mathcal{F}_{ij}}^{(i)} = \sum_m I_h(\theta_m w_{\mathcal{F}_{ij}}^{(i)}),$$

$$w_{\mathcal{E}_l}^{(i)} = \sum_m I_h(\theta_m w_{\mathcal{E}_l}^{(i)}), \quad \text{and} \quad w_{\mathcal{V}_n}^{(i)} = \sum_m I_h(\theta_m w_{\mathcal{V}_n}^{(i)}).$$

We only focus on the analysis for the face part $I_h(\theta_m w_{\mathcal{F}_{ij}}^{(i)})$, since the arguments are similar for the other parts. The idea is to modify slightly the arguments given in the proof of Theorem 4.2. Note that $I_h(\theta_m w_{\mathcal{F}_{ij}}^{(i)})$ is not piecewise discrete harmonic on $\Omega_{\mathcal{F}_{ij}}$. Therefore, we decompose $I_h(\theta_m w_{\mathcal{F}_{ij}}^{(i)})$ as

$$I_h(\theta_m w_{\mathcal{F}_{ij}}^{(i)}) = \mathcal{H}\, I_h(\theta_m w_{\mathcal{F}_{ij}}^{(i)}) + \mathcal{P}\, I_h(\theta_m w_{\mathcal{F}_{ij}}^{(i)}).$$

We decompose the piecewise discrete harmonic part $\mathcal{H}\, I_h(\theta_m w_{\mathcal{F}_{ij}}^{(i)})$ as in (4.15), and we decompose the interior part $\mathcal{P}\, I_h(\theta_m w_{\mathcal{F}_{ij}}^{(i)})$ as in (4.12). By using the same arguments as in the proof of Lemma 4.2, we obtain

$$\sum_{k=0}^{\ell} b_k(w_{\mathcal{F}_{ij,k}}^{(i)}, w_{\mathcal{F}_{ij,k}}^{(i)}) \preceq (\rho_i + \rho_j) |\, \mathcal{H}\, I_h(\theta_m w_{\mathcal{F}_{ij}}^{(i)})|_{H^1(\Omega_{\mathcal{F}_{ij}})}^2 \leq$$

$$\begin{aligned}(\rho_i + \rho_j)\, |I_h(\theta_m w_{\mathcal{F}_{ij}}^{(i)})|_{H^1(\Omega_{\mathcal{F}_{ij}})}^2 &\preceq (\rho_i + \rho_j)(|w_{\mathcal{F}_{ij}}^{(i)}|_{H^1(\Omega_{\mathcal{F}_{ij}})}^2 + \frac{1}{H^2}\|w_{\mathcal{F}_{ij}}^{(i)}\|_{L^2(\Omega_{\mathcal{F}_{ij}})}^2) \\ &\preceq (\rho_i + \rho_j)|w_{\mathcal{F}_{ij}}^{(i)}|_{H^1(\Omega_{\mathcal{F}_{ij}})}^2.\end{aligned}$$

In the last step, we use Friedrich's inequality since $w_{\mathcal{F}_{ij}}^{(i)}$ vanishes on $\partial\Omega_{\mathcal{F}_{ij}}$. The rest of the proof is straightforward.

$\square$

**Algorithm 4.2** *The next algorithm is a combination of an additive and a multiplicative Schwarz method. Let us consider a case in which the coarse space is $V_{-1}^{NN}$, and the local problems given by one symmetrized multigrid V-cycle on $\Omega_{\mathcal{V}_m}$ with Gauss Siedel or damped Jacobi method as smoothers. The preconditioned operator is given by*

$$T_{NNmult} = T_{-1}^{NN} + T_1 + \cdots + T_m + \cdots + T_M, \tag{4.61}$$

*where*

$$T_m = I - E_{G,\Omega_{\mathcal{V}_m}}^T E_{G,\Omega_{\mathcal{V}_m}}, \ \ \textit{or} \ \ T_m = I - E_{J,\Omega_{\mathcal{V}_m}}^T E_{J,\Omega_{\mathcal{V}_m}}.$$

*Here,*

$$E_{G,\Omega_{\mathcal{V}_m}} = (\prod_{k=0}^{\ell} \prod_{j \in \mathcal{N}_m^k} (I - P_j^k)),$$

*and*

$$E_{J,\Omega_{\mathcal{V}_m}} = \prod_{k=0}^{\ell}(I - \tilde{T}_k^m) = (\prod_{k=0}^{\ell}(I - \eta \sum_{j \in \mathcal{N}_m^k} P_k^j)),$$

*where $\eta$ is a damping factor chosen such that $\|\tilde{T}_m^k\|_{H_\rho^1} \le w < 2$.*

**Theorem 4.12** *Let $T_{NNmult}$ be defined by (4.61). Then for any $u \in V_0^h(\Omega)$, we have*

$$(1+\ell)^{-2} a(u,u) \preceq a(T_{NNmult}\, u, u) \preceq a(u,u).$$

*Proof.* By using the theory developed by Bramble, Pasciak, Wang, and Xu [7], Dryja, and Widlund [41], and Zhang [95], we have

$$\frac{(2-\hat{w})}{1+2\hat{w}^2\rho(\mathcal{E})^2} \sum_{k=1}^{\ell} \sum_{j \in \mathcal{N}_m^k} P_k^j \le T_m. \tag{4.62}$$

Here, $\hat{w} = \max(1,w) = 1$ for the Gauss Seidel case, and $\hat{w} = \max(1,w) < 2$ for the damped Jacobi case; see Widlund [41]. We also have $\rho(\mathcal{E}) \le C$ for both cases; see Zhang [95]. Therefore, by using Theorem 4.11 and (4.62), we obtain

$$(1+\ell)^{-2} a(u,u) \preceq a(T_{NNadd}\, u, u) \preceq a(T_{NNmult}\, u, u).$$

To obtain the upper bound of this Theorem is straightforward since $\Omega_{\mathcal{V}_m}$ form a finite overlapping covering and $\|E_{J,\Omega_{\mathcal{V}_m}}\|_{H_\rho^1}, \|E_{G,\Omega_{\mathcal{V}_m}}\|_{H_\rho^1} \le 1$.

□

Consider the case of the coarse space is $V_{-1}^{NN}$. The local problems are solved by a linear, iterative method; hence the preconditioned conjugate gradient method is not a candidate. Use $n$ multigrid V-cycles on $\Omega_{\mathcal{V}_m}$. $n$ is chosen a priori; therefore our local solvers are linear. Note that the norm of the error propagation operator of one multigrid V-cycle on $\Omega_{\mathcal{V}_m}$ can be estimate from above by $1 - C(1+\ell)^{-2}$. Therefore, we choose $n$ on the order of $(1+\ell)^2$ to obtain $\tilde{T}_m$, the operator associated with $n$ multigrid V-cycles on $\Omega_{\mathcal{V}_m}$, which is spectrally equivalent to $P_m$, the $H_\rho^1$-projection from $V^h(\Omega_{\mathcal{V}_m})$ to $V_0^h(\Omega_{\mathcal{V}_m})$. For a quasi-monotone distribution of the coefficients $\rho_i$, we can choose $n = 1$.

**Algorithm 4.3** *Let operators $T_{-1}^{NN}$ and $T_{-1}^{H}$ be given by (4.3) with $V_{-1}^{X} = V_{-1}^{NN}$ and $V_{-1}^{X} = V_0^H$, respectively. We introduce the preconditioned operators $T_{NN_n}$ and $T_{H1}$ by*

$$T_{NN_n} = T_{-1}^{NN} + \sum_{m=1}^{M} \tilde{T}_m, \tag{4.63}$$

*and*

$$T_{H_1} = T_{-1}^{H} + \sum_{m=1}^{M} \tilde{T}_m. \tag{4.64}$$

**Theorem 4.13** *Let $T_{NN_n}$ be defined by (4.63), and $T_{H_1}$ defined by (4.64). Then for any $u \in V_0^h(\Omega)$, we have*

$$(1+\ell)^{-1} a(u,u) \preceq a(T_{NN_n} u, u) \preceq a(u,u). \tag{4.65}$$

*For a quasi-monotone distribution of the coefficients $\rho_i$, we have*

$$a(T_{H_1} u, u) \asymp a(u,u). \tag{4.66}$$

*Proof.* By construction, we have $P_m \asymp \tilde{T}_m$. Therefore,

$$T_{-1}^{NN} + \sum_{m=1}^{M} \tilde{T}_m \asymp T_{-1}^{NN} + \sum_{m=1}^{M} P_m,$$

and

$$T_{-1}^{H} + \sum_{m=1}^{M} \tilde{T}_m \asymp T_{-1}^{H} + \sum_{m=1}^{M} P_m.$$

Hence, (4.65) and (4.66) follow directly from Remark 4.2, Corollary 4.2, and results on the related methods with exact local solvers; cf. Dryja and Widlund [37], or Section 5.1.9.

□

**Remark 4.8** *Let us consider algorithms in which we replace the coarse space $V_{-1}^{NN}$ by $V_{-1}^{F}$, $V_{-1}^{E}$, $V_{-1}^{\widetilde{NN}}$, $V_{-1}^{\tilde{F}}$, or $V_{-1}^{\tilde{E}}$ in the Algorithms 4.1, 4.2, and 4.3. Then, it is easy to prove that we obtain the same condition number estimates as in Theorems 4.11, 4.12, and 4.13. If we replace to $V_{-1}^{W}$, or $V_{-1}^{\tilde{W}}$, we obtain condition number estimates of the form $(1+\ell)^2$ in Theorems 4.11, 4.12, and 4.13.*

**Remark 4.9** *Consider algorithms in which we replace the covering* $\Omega_{\mathcal{V}_m}$ *by* $\Omega_i'$ *in the Algorithms 4.1, 4.2, and 4.3, and in the variants introduced in Remark 4.8. We can then also show that we obtain the same condition number estimates as in Theorems 4.11, 4.12, 4.13, and as in the Remark 4.8. We can also consider algorithms in which we replace the covering* $\Omega_{\mathcal{V}_m}$ *by* $\Omega_{\mathcal{F}_{ij}}$, *and replace the coarse space* $V_{-1}^{NN}$ *by* $V_{-1}^{F}$, $V_{-1}^{W}$, $V_{-1}^{\tilde{F}}$, *or* $V_{-1}^{\tilde{W}}$ *in the Algorithms 4.1, 4.2, and 4.3. Again, we obtain the same condition number estimates as in Theorems 4.11, and 4.12.*

# Chapter 5

# Schwarz Methods for Nonconforming Finite Elements

## 5.1 Two-level Schwarz methods for nonconforming $P_1$ finite elements with discontinuous coefficients

### 5.1.1 Introduction

The purpose of this chapter is to develop a domain decomposition methods for second order elliptic partial differential equations approximated by a simple nonconforming finite element method, the nonconforming $P_1$ elements. We first consider a variant of a two-level additive Schwarz method introduced in 1987 by Dryja and Widlund [36] for a conforming case. In this method, a preconditioner is constructed from the restriction of the given elliptic problem to overlapping subregions into which the given region has been decomposed. In addition, in order to enhance the convergence rate, the preconditioner includes a coarse problem with lower dimension. The construction of this component is the most interesting part of our work; we introduced nonstandard coarses spaces for problems with discontinuous coefficients. Here we have been able to draw on earlier multilevel studies, cf. Brenner [13], Oswald [68], as well as on recent work by Dryja, Smith, and Widlund [35], and Dryja and Widlund [41]. We show that the condition number of the corresponding iterative methods is bounded by $C\,(1 + log(H/h))$, where $H$ and $h$ are the mesh sizes of the global and local problems, respectively. We also note that this bound is independent of the variations of the coefficients across the subregion interfaces.

The *face based* and the *Neumann-Neumann* coarse spaces, that we are introducing, have the following characteristics. The values at the nonconforming nodes are constant on each edge (face) of the subregions and the values at the other nonconforming nodes are given by a piecewise discrete nonconforming $P_1$ harmonic extension, or by a simple but nonstandard interpolation formula. In the latter case and for triangular (tetrahedral) substructures, the value at any nonconforming node in the interior of a subregion is a convex combination of the three (four) values given on the boundary. We note that an important difference between the nonconforming and conformings case is that there are no nodes at the vertices (wire basket) of the subregions.

Another interesting and original part of our work is the technique that we introduce to analyze algorithms with nonconforming spaces; see Section 1.1. We introduce nonstandard local operators (see the *isomorphisms* in [76]) in order to map between conforming and nonconforming spaces and then obtain several results for the nonconforming case which are known for the conforming case. For instance, we can obtain extension theorems, Poincaré' inequalities, trace theorems, and partitions of unity for nonconforming spaces, and then analyze nonconforming versions of domain decomposition methods which have already analyzed for conforming cases.

### 5.1.2 Differential and finite element model problems

To simplify the presentation, we assume that $\Omega$ is an open, bounded, polygonal region of diameter 1 in the plane, with boundary $\partial\Omega$. In a separate subsection, we extend all our results to the three dimensional case.

We introduce a partition of $\Omega$ as follows. In a first step, we divide the region $\Omega$ into nonoverlapping triangular substructures $\Omega_i, i = 1, \cdots, N$. Adopting common assumptions in finite element theory, cf. Ciarlet [27], all substructures are assumed shape regular, quasi-uniform and to have no dead points, i.e. each interior edge is the intersection of the boundaries of two triangular regions. We can show that the theory also holds if we choose nontriangular substructures, where the boundary of each substructure is a composition of several polygonal edges, and each of them is the intersection of two substructures. Naturally, we need assumptions related to the nondegeneracy of this partition. Initially, we restrict our exposition to the case of triangular substructures since the main ideas can be seen in this case. This partition induces a coarse mesh $\mathcal{T}^H$ and we also introduce a mesh parameter $H = \max\{H_1, \cdots, H_N\}$ where $H_i$ is the diameter of $\Omega_i$. Later, we will extend the results to nontriangular substructures. We can show that the quasi-uniformity assumption is not needed since all arguments are local.

We study the following selfadjoint second order elliptic problem:
Find $u \in H_0^1(\Omega)$, such that

$$a(u,v) = f(v) \ \ \forall\ v \in \ H_0^1(\Omega)\ , \tag{5.1}$$

where

$$a(u,v) = \sum_{i=1}^{N} \int_{\Omega_i} \rho_i \, \nabla u \cdot \nabla v \, dx \quad \text{and} \quad f(v) = \int_{\Omega} f v \, dx \quad \text{for } f \in \ L^2 \, .$$

Here, the $\rho_i$ are constants values such that $\rho_i \geq \lambda_{min} > 0 \ \ \forall i$. This includes cases where there is a great variation in the value of the $\rho_i$ across substructures interfaces. We remark that there is no difficulty in extending the analysis and the results to the case where each $\rho_i$ varies mildly inside each substructure $\Omega_i$.

In a second step, we obtain the elements $\tau_j^h$ in the finest triangulation by subdividing the substructures into triangles in such a way that they are shape regular, and quasi-uniform. The mesh parameter $h$ is the diameter of the smallest element and denote this triangulation by $\mathcal{T}^h$. Similarly, we assume the triangulation $\mathcal{T}^h$ has no dead points.

Let $\mathcal{E}_{ij}$ represents an open *edge* of a substructure $\Omega_i$; sometimes we use $\mathcal{E}_{ij}$ to represent an open edge shared by two substructures $\Omega_i$ and $\Omega_j$. We denote the interface between the subdomains by $\Gamma = \cup \partial\Omega_i \backslash \partial\Omega$. We denote by *CR nodal points* the nonconforming nodal points, i.e. the midpoints of the edges of elements in $\mathcal{T}^h$. The sets of CR nodal points belonging to $\bar{\Omega}$, $\partial\Omega$, $\mathcal{E}_{ij}$, $\partial\Omega_i$, and $\Gamma$ are denoted by $\Omega_h^{CR}$, $\partial\Omega_h^{CR}$, $\mathcal{E}_{ij,h}^{CR}$, $\partial\Omega_{i,h}^{CR}$, and $\Gamma_h^{CR}$, respectively. In order to preserve the notations of the previous chapters and to distinguish between conforming and nonconforming spaces, we use the superscript, , for a nonconforming space.

**Definition 5.1** *The nonconforming $P_1$ element spaces (cf. Crouzeix and Raviart [31]) on the $h$-mesh and $H$-mesh are given by*

$\tilde{V}^h(\Omega) = \{v \,|\, v$ *linear in each triangle* $\tau_j^h \in \mathcal{T}^h$,
$\quad\quad v$ *continuous at the CR nodes of* $\Omega_h^{CR}\}$,
$\tilde{V}_0^h(\Omega) = \{v \,|\, v \in \tilde{V}^h(\Omega)$ *and* $v = 0$ *at the CR nodes of* $\partial\Omega_h^{CR}\}$,
$\tilde{V}^H(\Omega) = \{v \,|\, v$ *linear in each triangle in* $\mathcal{T}^H$,

*$v$ continuous at the midpoints of the edges of $\mathcal{T}^H\}$, and*

$$\tilde{V}_0^H(\Omega) = \{v \mid v \in \tilde{V}^H(\Omega) \text{ and } v = 0 \text{ at the midpoints of edges of } \mathcal{T}^H \text{ that belong to } \partial\Omega\}.$$

These spaces are nonconforming since $\tilde{V}_0^H(\Omega) \not\subset \tilde{V}_0^h(\Omega) \not\subset H_0^1(\Omega)$.

Let $\Sigma$ be a region contained in $\Omega$ such that $\partial\Sigma$ does not cut through any element. Denote by $\tilde{V}^h(\Sigma)$ and $\mathcal{T}^h(\bar{\Sigma})$ the space $\tilde{V}^h$ and the triangulation $\mathcal{T}^h$ restricted to $\bar{\Sigma}$, respectively.

Given $u \in \tilde{V}^h(\Sigma)$, we define a discrete weighted energy seminorm by:

$$|u|^2_{H^1_{\rho,h}(\Sigma)} = a^h_\Sigma(u,u)\,, \tag{5.2}$$

where

$$a^h_\Sigma(u,v) = \sum_{T \in \mathcal{T}^h|_\Sigma} \int_T \rho(x)\; \nabla u \cdot \nabla u \; dx. \tag{5.3}$$

In a similar fashion, we define the inner product $a^H_\Omega(u,v)$ and the seminorm $|u|_{H^1_{a,H}(\Sigma)}$ for $u, v \in \tilde{V}^H(\Sigma)$. In order not to use unnecessary notation, we drop the subscript $\Sigma$ when the integration is over $\Omega$ and the subscript $\rho$ when $\rho = 1$.

The discrete problem associated with (5.1) is given by:
Find $u \in \tilde{V}_0^h$, such that

$$a^h(u,v) = f(v) \;\; \forall \; v \in \tilde{V}_0^h(\Omega). \tag{5.4}$$

We note that $|\cdot|_{H^1_{\rho,h}(\Omega)}$ is a norm, because if $|u|_{H^1_{\rho,h}(\Omega)} = 0$, then $u$ is constant in each element. By the continuity at the CR nodes and the zero boundary condition, we obtain $u = 0$. Therefore, we obtain uniqueness and existence of a solution for (5.4). To show well-posedness independent of $h$, we use a Friedrichs' inequality for nonconforming $P_1$ functions and repeat the same arguments as given in Section 3.1. This Friedrich's inequality is established by using a variant of Lemma 5.9 with $\Omega_i = \Omega$, and using a standard Friedrich's inequality for $H^1(\Omega)$ functions. We obtain similar stability results as for the conforming case.

We also define the weighted $L^2$ norm by:

$$\|u\|^2_{L^2_\rho(\Sigma)} = \int_\Sigma \rho(x)\; |u(x)|^2 \, dx \;\; \text{for } u \in (\tilde{V}^h(\Sigma) + \tilde{V}^H(\Sigma) + L^2(\Sigma)), \tag{5.5}$$

and a weighted energy norm by:

$$\|u\|^2_{H^1_{\rho,h}(\Sigma)} = |u|^2_{H^1_{\rho,h}(\Sigma)} + \frac{1}{|\Sigma|^2}\|u\|^2_{L^2_\rho(\Sigma)}. \tag{5.6}$$

.

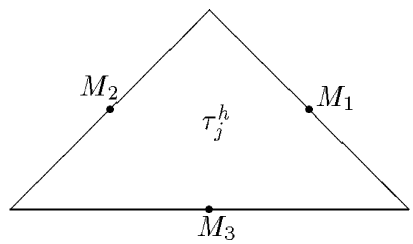


Fig. 5.1

Sometimes is more convenient to evaluate a norm of a finite element function in terms of the values of this function at the CR nodal points. By first working on a reference element and then using the assumption that the elements are shape regular, we obtain the following lemma:

**Lemma 5.1** *For* $u \in \tilde{V}^h(\Sigma)$,

$$\|u\|_{L^2_{\rho,h}(\Sigma)} \asymp h^2 \sum_{\tau_j^h \in \mathcal{T}^h(\Sigma)} \rho(\tau_j^h)\,(u^2(M_1) + u^2(M_2) + u^2(M_3)) \tag{5.7}$$

*and*

$$|u|^2_{H^1_{\rho,h}(\Sigma)} \asymp \sum_{\tau_j^h \in \mathcal{T}^h(\Sigma)} \rho(\tau_j^h)\,\{(u(M_1) - u(M_2))^2 + (u(M_2) - u(M_3))^2 + (u(M_3) - u(M_1))^2\}. \tag{5.8}$$

*where* $M_1, M_2, M_3$ *are the CR nodes of the triangle* $\tau_j^h$ *as in Fig. 5.1.*

We stress the the formulas (5.7) and (5.8) are for the two-dimensional case. There are three-dimensional version of these formulas; the main difference is that we have to multiply the right hand sides by an additional factor $h$.

An inverse inequality can be obtained by using only local properties. It is easy to see that for $u \in \tilde{V}^h(\Sigma)$,

$$|u|_{H^1_{\rho,h}(\Sigma)} \preceq h^{-1}\|u\|_{L^2_\rho(\Sigma)}. \tag{5.9}$$

We note that (5.9) also holds for the three-dimensional case.

### 5.1.3 Additive Schwarz schemes

We now describe the special additive Schwarz method introduced by Dryja and Widlund; see e.g. [37, 42]. In this method, we cover $\Omega$ by overlapping subregions obtained by extending each substructure $\Omega_i$ to a larger region $\Omega_i'$. We assume that the overlap is $\delta_i$, where $\delta_i$ is the distance between the boundaries $\partial\Omega_i$ and $\partial\Omega_i'$, and we denote by $\delta$ the minimum of the $\delta_i$. We also assume that $\partial\Omega_i'$ does not cut through any element. We make the same construction for the substructures that meet the boundary except that we cut off the part of $\Omega_i'$ that is outside of $\Omega$.

For each $\Omega_i'$, a nonconforming $P_1$ finite element triangulation is inherited from $\mathcal{T}^h(\Omega)$. The corresponding finite element space is defined by

$$\tilde{V}_i = \{v \,|\, v \in \ \tilde{V}_0^h, \text{ support of } v \subset \Omega_i'\}, \quad i = 1, \cdots, N. \tag{5.10}$$

A coarse space $\tilde{V}_{-1} \subset \tilde{V}_0^h(\Omega)$ is given as the range of a certain interpolation (or prolongation) operator. The most fundamental and original coarse space for nonconforming $P_1$ spaces is the *face based* coarse space with an *approximate discrete harmonic extension.* This space, which we denote by $\tilde{V}_{-1}^{\tilde{F}}$, can also be defined as the range of $\tilde{I}_H^h$. The *prolongation operator* $\tilde{I}_H^h$ will be defined later.

Our finite element space is represented as a sum of $N+1$ subspaces

$$\tilde{V}_0^h = \tilde{V}_{-1} + \tilde{V}_1 + \cdots + \tilde{V}_N. \tag{5.11}$$

We introduce projection operators $\tilde{P}_i : \tilde{V}_0^h \to \tilde{V}_i$, $i = -1$ and $1, \cdots, N$, by

$$a^h(\tilde{P}_i w, v) = a^h(w, v) \quad \forall\ v \in \tilde{V}_i, \tag{5.12}$$

and the operator $\tilde{P} : \tilde{V}_0^h \to \tilde{V}_0^h$, by

$$\tilde{P} = \tilde{P}_{-1} + \tilde{P}_1 + \cdots + \tilde{P}_N. \tag{5.13}$$

In a case in which $\tilde{V}_{-1} = \tilde{V}_{-1}^{\tilde{F}}$, in matrix form, $\tilde{P}_{-1}$ is given by

$$\tilde{P}_{-1} = \tilde{I}_H^h(\, \tilde{I}_H^{h\ T} K\, \tilde{I}_H^h\,)^{-1} \tilde{I}_H^{h\ T}\, K \tag{5.14}$$

where $K$ is the global stiffness matrix associated with $a_h(\cdot,\cdot)$.

We replace the problem (5.4) by

$$\tilde{P}u = g, \quad g = \sum_{i=0}^{N} g_i \quad \text{where} \quad g_i = \tilde{P}_i u. \tag{5.15}$$

An upper and lower bounds for the spectrum of $\tilde{P}$ is obtained by using Theorem 2.5.1.

### 5.1.4 Properties of the nonconforming $P_1$ finite element space

We first define two *local equivalence maps* (*isomorphisms*) in order to obtain some inequalities and local properties of our nonconforming space. With the help of these mappings, we can extend some results that are known for the piecewise linear conforming elements to our nonconforming case. We point out that all the isomorphisms introduced in this chapter are carefully defined so that our analysis also holds for any triangulation that is shape regular and nonuniform.

Let $V^{\frac{h}{2}}(\Omega_i)$ be the conforming space of piecewise linear functions on the triangulation $\mathcal{T}^{\frac{h}{2}}(\Omega_i)$, where the $h/2$-mesh is obtained by joining midpoints of the edges of elements of $\mathcal{T}^h(\bar{\Omega}_i)$.

We define the *local equivalence map* $\mathcal{M}_i : \tilde{V}^h(\Omega_i) \to V^{\frac{h}{2}}(\Omega_i)$, as follows:

**Isomorphism 5.1** *Given $u \in \tilde{V}^h(\Omega_i)$, define $\mathcal{M}_i u \in V^{\frac{h}{2}}(\Omega_i)$ by the values of $\mathcal{M}_i u$ at the three sets of points (cf. Fig. 5.2):*

*i) If $P \in \bar{\Omega}^{CR}_{i,h}$, then*

$$\mathcal{M}_i u(P) = u(P).$$

*ii) If $P \in \bar{\Omega}_{i,h} \backslash \partial\Omega_{i,h}$, and the $\tau_j^h$ are the elements that have $P$ as a vertex, then*

$$\mathcal{M}_i u(P) = \text{mean of} \quad u|_{\tau_j^h}(P).$$

*Here $u|_{\tau_j^h}(P)$, is the limit value of $u(x)$ when $x \in \tau_j^h$ approaches $P$.*

*iii) If $Q \in \partial\Omega_{i,h}$, and $Q_l$ and $Q_r$ the two CR nodes of $\bar{\Omega}^{CR}_{i,h}$ that are the next neighbors of $Q$, then*

$$\mathcal{M}_i u(Q) = \frac{|Q_l Q|}{|Q_l Q_r|} u(Q_l) + \frac{|Q_r Q|}{|Q_l Q_r|} u(Q_r).$$

*Here $|Q_r Q|$ is the length of the segment $Q_r Q$.*

In a case where the triangulation $\mathcal{T}^h(\bar{\Omega}_i)$ is uniform, e.g. as in Fig. 5.2, Case ii) becomes

$$\mathcal{M}_i u(P) = \frac{1}{6} \sum_{i=j}^{6} u|_{\tau_j^h}(P).$$

Case iii) is required in order to have property (5.18), which will be very important in our analysis.

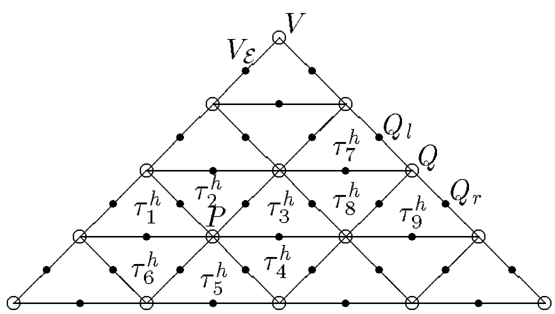


Fig. 5.2

**Lemma 5.2** *Let $u \in \tilde{V}^h(\Omega_i)$. Then*

$$|\mathcal{M}_i u|_{H^1_\rho(\Omega_i)} \asymp |u|_{H^1_{\rho,h}(\Omega_i)} , \tag{5.16}$$

$$\|\mathcal{M}_i u\|_{L^2_\rho(\Omega_i)} \asymp \|u\|_{L^2_\rho(\Omega_i)} , \tag{5.17}$$

*and*

$$\int_{\partial\Omega_i} \mathcal{M}_i u(s)\, ds = \int_{\partial\Omega_i} u(s)\, ds. \tag{5.18}$$

*Here $|\cdot|_{H^1_\rho(\Omega_i)}$ is the standard weighted energy seminorm for conforming functions.*

*Proof.* We first note that we have results similar to (5.7) and (5.8) for the conforming space $V^{\frac{h}{2}}(\Omega_i)$, where now $M_1$, $M_2$, and $M_3$ are the vertices of a triangle in $\mathcal{T}^{\frac{h}{2}}$. In order to prove (5.16), we compare (5.8) with the analogous formula for the piecewise linear conforming space. We find (see Fig. 5.2),

$$|\mathcal{M}_i u(Q) - \mathcal{M}_i u(Q_r)|^2 = \frac{|Q_l Q|}{|Q_l Q_r|} |u(Q_l) - u(Q_r)|^2.$$

The right hand side can be bounded by the energy seminorm of $u$ restricted to the union of the triangles $\tau_7^h$, $\tau_8^h$ and $\tau_9^h$.

We also prove that if we take two next neighboring vertices of $\mathcal{T}^{\frac{h}{2}}$ in the interior of $\Omega_i$, the energy seminorm can be bounded locally. If $\rho(x)$ does not vary a great deal, we can work with weighted seminorms. Using the fact that our arguments are local, it is easy to obtain the upper bound of (5.16).

The lower bound is easy to obtain since the degrees of freedom of $\tilde{V}^h(\Omega_i)$ are contained in those of $V^{\frac{h}{2}}(\Omega_i)$.

Similar arguments can also be used to obtain (5.17).

Finally, it is easy to see that (5.18) follows directly from iii) even if the refinement is not uniform.

$\square$

**Remark 5.1** *The Isomorphism 5.1 is a local construction. Therefore, Lemma 5.2 holds locally in each $\Omega_i$ but not simultaneously for every $i$. In fact, we can prove that it is impossible to construct a global Isomorphism $\mathcal{M} : \tilde{V}^h(\Omega) \rightarrow V^{\frac{h}{2}}(\Omega)$ such that (5.19) holds for every $i$ simultaneously. The problem is how to define the values at the vertices of the substructures that are cross points. We note, however, that for problems with constant or quasi-monotone coefficients, a global Isomorphism $\mathcal{M}$ can be defined by replacing $\Omega_i$ by $\Omega$ in Isomorphism 5.1. Using the same ideas as in the proof of Lemma 5.17, we obtain*

$$(\mathcal{M}\tilde{V}_0^h(\Omega)) \subset V_0^{\frac{h}{2}}(\Omega),$$

$$\|\mathcal{M}u\|_{L^2_\rho(\Omega)} \asymp \|u\|_{L^2_\rho(\Omega)},$$

$$\|u - \mathcal{M}u\|_{L^2_\rho(\Omega)} \preceq h\,|u|_{H^1_\rho(\Omega)},$$

*and*

$$|\mathcal{M}u|_{H^1_\rho(\Omega)} \asymp |u|_{H^1_{\rho,h}(\Omega)}.$$

We define another local equivalence map $\mathcal{M}_i^{\mathcal{E}} : \tilde{V}^h(\Omega_i) \rightarrow V^{\frac{h}{2}}(\Omega_i)$, by:

**Isomorphism 5.2** *Given $u \in \tilde{V}^h(\Omega_i)$ and an edge $\mathcal{E}$ of the substructure $\Omega_i$, define $\mathcal{M}_i^{\mathcal{E}}u \in V^{\frac{h}{2}}(\Omega_i)$ by the values of $\mathcal{M}_i^{\mathcal{E}}u$ at the three sets of points (cf. Fig. 5.2):*

*i) Same as step i) of Isomorphism 5.1.*

*ii) Same as step ii) of Isomorphism 5.1.*

*iii) If $V$ is a vertex of the substructure $\Omega_i$ and an end point of $\mathcal{E}$, and $V_{\mathcal{E}}$ the CR node on $\mathcal{E}_h^{CR}$ that is the next neighbor of $V$, then*

$$\mathcal{M}_i^{\mathcal{E}} u(V) = u(V_{\mathcal{E}}).$$

*iv) If $Q \in \partial\Omega_{i,h}$ and is not in the case iii), then*

$$\mathcal{M}_i^{\mathcal{E}} u(Q) = \frac{|Q_l Q|}{|Q_l Q_r|} u(Q_l) + \frac{|Q_r Q|}{|Q_l Q_r|} u(Q_r).$$

Using the same ideas as in Lemma 5.2, we can prove:

**Lemma 5.3** *Given $u \in \tilde{V}^h(\Omega_i)$. Then,*

$$|\mathcal{M}_i^{\mathcal{E}} u|_{H^1_\rho(\Omega_i)} \asymp |u|_{H^1_{\rho,h}(\Omega_i)}, \tag{5.19}$$

$$\|\mathcal{M}_i^{\mathcal{E}} u\|_{L^2_\rho(\Omega_i)} \asymp \|u\|_{L^2_\rho(\Omega_i)}, \tag{5.20}$$

*and*

$$\int_{\mathcal{E}} \mathcal{M}_i^{\mathcal{E}} u(s)\, ds = \int_{\mathcal{E}} u(s)\, ds. \tag{5.21}$$

### 5.1.5 Interpolation operators

**Definition 5.2** *Let $v \in \tilde{V}^h(\Omega)$. The Interpolation operator $I_h^H : \tilde{V}^h(\Omega) \to \tilde{V}^H(\Omega)$, is given by:*

*i) If $P_{ij}$ is the midpoint of the edge $\mathcal{E}_{ij}$ common to $\bar{\Omega}_i$ and $\bar{\Omega}_j$, then*

$$(I_h^H v)(P_{ij}) = \frac{1}{|\mathcal{E}_{ij}|} \int_{\mathcal{E}_{ij}} v|_{\bar{\Omega}_i}(x)\; dx = \frac{1}{|\mathcal{E}_{ij}|} \int_{\mathcal{E}_{ij}} v|_{\bar{\Omega}_j}(x)\; dx. \tag{5.22}$$

*ii) If $P$ is the midpoint of an edge $\mathcal{E}$ common to $\bar{\Omega}_i$ and $\partial\Omega$, then*

$$(I_h^H v)(P) = \frac{1}{|\mathcal{E}|} \int_{\mathcal{E}} v|_{\bar{\Omega}_i}(x)\; dx. \tag{5.23}$$

Here, $v|_{\bar{\Omega}_k}(x)$ is the limit value of $v(y)$ when $y \in \Omega_i$ approaches $x$. This limit is defined a.e. We note that the second equality in (5.22) follows from the fact that the mean of $v$ on each edge of an element $\tau_j^h \in \mathcal{T}^h(\Omega)$ is equal to $v(M_1)$, where $M_1$ is the CR node of this edge. It is important to note that the value of $(I_h^H v)(P_{ij})$ depends only on the values of $v$ on the interface $\mathcal{E}_{ij}$. This allows us to obtain stability properties that are independent of the differences of $\rho(x)$ across the substructure interfaces. It is easy to check from (5.23) that

$$(I_h^H \tilde{V}_0^h(\Omega)) \subset \tilde{V}_0^H(\Omega).$$

The next lemma is a Poincaré-Friedrichs inequality for nonconforming $P_1$ elements. It is obtained by using Lemmas 5.2, 5.3 and 1.6.

**Lemma 5.4** *Let $u \in \tilde{V}^h(\Omega_i)$, where $\Omega_i$ is a triangular substructure of diameter O(H). Let $\Gamma$ be $\partial\Omega_i$ (or an edge of $\partial\Omega_i$). Then,*

$$\|u\|^2_{L^2(\Omega_i)} \preceq H^2 |u|^2_{H^1_h(\Omega_i)} + (\int_\Gamma u(s)\, ds)^2. \tag{5.24}$$

*As a consequence, if $\int_\Gamma u(s)\, ds = 0$, we have the Poincaré inequality*

$$\|u\|_{L^2_{\rho,h}(\Omega_i)} \preceq H\, |u|_{H^1_{\rho,h}(\Omega_i)}. \tag{5.25}$$

The next lemma gives an example of an operator that is locally $L^2_\rho-$ and $H^1_\rho-$stable.

**Lemma 5.5** *Let $\bar{u} \in H^1(\Omega_i)$, where $\Omega_i$ is a triangular substructures of diameter of O(H). Define a linear function $\bar{u}_H$ in $\Omega_i$ by*

$$\bar{u}_H(P_{ij}) = \frac{1}{|\mathcal{E}_{ij}|} \int_{\mathcal{E}_{ij}} \bar{u}(s)\; ds \qquad j = 1, 2, 3\,, \tag{5.26}$$

*where the $\mathcal{E}_{ij}$ are the edges of $\Omega_i$, and $P_{ij}$ is the midpoint of $\mathcal{E}_{ij}$. Then*

$$|\bar{u}_H(P_{ij})|^2 \preceq |\bar{u}|^2_{H^1(\Omega_i)} + \frac{1}{H^2}\|\bar{u}\|^2_{L^2(\Omega_i)}. \tag{5.27}$$

$$|\bar{u}_H|_{H^1_\rho(\Omega_i)} \preceq |\bar{u}|_{H^1_\rho(\Omega_i)}, \tag{5.28}$$

*and*

$$\|\bar{u}_H - \bar{u}\|_{L^2_\rho(\Omega_i)} \preceq H|\bar{u}|_{H^1_\rho(\Omega_i)}. \tag{5.29}$$

*Proof.* Consider initially a subregion $\Omega_i$ with diameter of 1. Using that $|\mathcal{E}_{ij}| = \mathrm{O}(1)$, the Cauchy-Schwarz inequality and a trace theorem, we have

$$|\bar{u}_H(P_{ij})|^2 \asymp |\int_{\mathcal{E}_{ij}} \bar{u}(x)\; dx|^2 \preceq \|\bar{u}\|^2_{L^2(\mathcal{E}_{ij})}$$

$$\preceq \|\bar{u}\|^2_{H^{\frac{1}{2}}(\mathcal{E}_{ij})} \preceq \|\bar{u}\|^2_{H^1(\mathcal{E}_{ij})} \preceq \|\bar{u}\|^2_{L^2(\mathcal{E}_{ij})} + |\bar{u}|^2_{H^1(\mathcal{E}_{ij})}.$$

We obtain (5.27) by returning to a region of diameter $H$.

Note that for any constant $c$

$$|\bar{u}_H|^2_{H^1(\Omega_i)} \asymp \tag{5.30}$$

$$|\bar{u}_H(P_{i1}) - \bar{u}_H(P_{i2})|^2 + |\bar{u}_H(P_{i2}) - \bar{u}_H(P_{i3})|^2 + |\bar{u}_H(P_{i3}) - \bar{u}_H(P_{i1})|^2$$

$$\preceq \|\bar{u} - c\|^2_{H^1(\Omega_i)}.$$

By choosing $c = \bar{u}(P_{i1})$ and $\Gamma = \mathcal{E}_{i1}$, we can apply Lemma 1.6 and obtain the $H^1_\rho(\Omega)$-stability (5.28).

We now prove the error $L^2_\rho(\Omega)$-stability. Since $\bar{u} - \bar{u}_H$ has mean zero on $\partial\Omega_i$, we can apply the Poincaré inequality (5.25) and obtain

$$\|\bar{u} - \bar{u}_H\|_{L^2(\Omega_i)} \preceq H\, |\bar{u} - \bar{u}_H|_{H^1(\Omega_i)}.$$

Using the first part of this lemma, we obtain the error $L^2_\rho(\Omega)$-stability (5.29).

□

The next lemma shows that the interpolation operator $I_h^H$, defined by (5.22) and (5.23), is locally $L^2_\rho-$ and $H^1_\rho-$stable.

**Lemma 5.6** *Let $u \in \tilde{V}^h(\Omega)$. Then $u_H = I_h^H u \in \tilde{V}^H(\Omega)$ satisfies the following properties*

$$(I_h^H \tilde{V}_0^h(\Omega)) \subset \tilde{V}_0^H(\Omega), \tag{5.32}$$

$$|u_H|_{H^1_{\rho,H}(\Omega_i)} \preceq |u|_{H^1_{\rho,h}(\Omega_i)}\;, \tag{5.33}$$

*and*

$$\|u_H - u\|_{L^2_\rho(\Omega_i)} \preceq H\, |u|_{H^1_{\rho,h}(\Omega_i)}\,, \quad i = 1, \cdots, N. \tag{5.34}$$

*Proof.* Let $u_H = I_h^H u \in \tilde{V}^H(\Omega)$ and $u_{i1} = \mathcal{M}_i^{\mathcal{E}_{i1}} u \in V^{\frac{h}{2}}(\Omega_i)$, and let $\bar{u}_H(P_{i1}) \in H^1(\Omega_i)$ be given by (5.26). Using the properties (5.21) and (5.22), we have

$$u_H(P_{i1}) = \bar{u}_H(P_{i1}). \tag{5.35}$$

Therefore, from (5.35), (5.27) and Lemma 5.3, we have

$$\begin{aligned} |u_H(P_{i1})|^2 = |\bar{u}_H(P_{i1})|^2 &\preceq |\bar{u}|^2_{H^1(\Omega_i)} + \frac{1}{H^2}\|\bar{u}\|^2_{L^2(\Omega_i)} \qquad (5.36) \\ &\preceq |u|^2_{H^1(\Omega_i)} + \frac{1}{H^2}\|u\|^2_{L^2(\Omega_i)}. \end{aligned}$$

We also obtain the same estimate for $|u_H(P_{i2})|$ and $|u_H(P_{i3})|$.

The rest of the proof is similar to that of Lemma 5.5. We now use the Poincaré inequality for nonconforming elements.

□

### 5.1.6 The Face based basis

In this subsection, we introduce several prolongation operators and establish that they are stable. The range of each of these operators will serve as a coarse space in our algorithms.

**Definition 5.3** *The Prolongation Operator, $I_H^h : \tilde{V}^H \to \tilde{V}^h$, is given by:*

*i) Let $\mathcal{E}_{ij}, j = 1,2,3$ be the edges of a substructures $\Omega_i$. For all CR nodes $P \in \mathcal{E}_{ij,h}^{CR}$, $j = 1,2,3$ and $i = 1,\cdots,N$, let $(I_H^h u_H)(P) = u_H(P_{ij})$, where $P_{ij}$ is the midpoint of the edge $\mathcal{E}_{ij}$.*

*ii) Given $I_H^h u_H$ at the CR nodal points of $\Gamma = \cup_i \partial\Omega_i$ from i), let $I_H^h u_H(\Omega)$ be the nonconforming $P_1$ harmonic extension inside each $\Omega_i$.*

It is easy to check that $u_h = I_H^h u_H \in \tilde{V}^h(\Omega)$. A disadvantage of step *ii)* is that we have to solve exactly a local Dirichlet problem for each substructure in order to obtain the harmonic extension. Other extensions can be used, which we call *approximate harmonic extensions.* They are given by simple explicit formulas with the same $L^2_\rho$ and $H^1_{\rho,h}$ stability properties as the harmonic one.

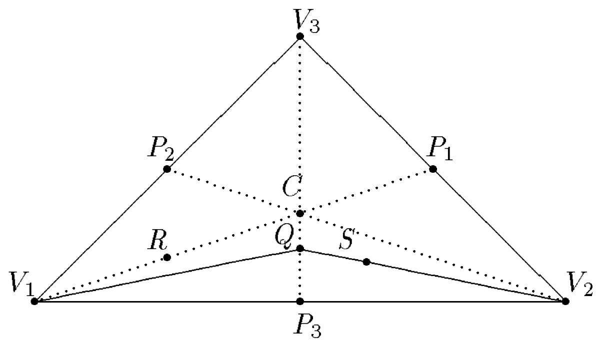

Fig. 5.3

Our first construction is a natural generalization of the partition of unity introduced by Dryja and Widlund in [37]; this partition of unity will provide the basis of our approximate extensions. Let $P_j$, $j = 1, 2, 3$, be the midpoints of the edges of $\Omega_i$, and let $V_j$ be the vertex of $\Omega_i$ that is opposite to $P_j$. Let $C$ be the barycenter of the triangle $\Omega_i$, i.e. the intersection of the line segments connecting $V_j$ to $P_j$.

**Extension 5.1** *The construction of an approximate harmonic extension in $\Omega_i$ is defined by the following steps (see Fig. 5.3):*

*i) Let*

$$\bar{u}(C) = \frac{1}{3}\left\{u_H(P_1) + u_H(P_2) + u_H(P_3)\right\}.$$

*ii) For a point $R$ that belongs to a line segment that connects $C$ to a vertex $V_j$, let*

$$\bar{u}(R) = \bar{u}(C).$$

*iii) For a point $Q$ that belongs to a line segments connecting $C$ to $P_k$, define $\bar{u}(Q)$ by linear interpolation between the values $\bar{u}(C)$ and $u_H(P_k)$, i.e by*

$$\bar{u}(Q) = \lambda(Q)\bar{u}(C) + (1 - \lambda(Q))u_H(P_k).$$

*Here $\lambda(Q) =$distance$(Q, P_k)/$distance$(C, P_k)$.*

*iv) For a point $S$ that belongs to the line segment connecting the previous point $Q$ to a vertex $V_\ell$, with $\ell \neq k$, let*

$$\bar{u}(S) = \bar{u}(Q).$$

*v) Finally, let $\tilde{I}_H^h u_H(\Omega_i) = I_{i,h}^{CR}\bar{u}$, where $I_{i,h}^{CR}$ is the interpolation operator into the space $\tilde{V}^h(\Omega_i)$ that preserves the values of a function at the CR nodes $\bar{\Omega}_{i,h}^{CR}$.*

Note that the function $\bar{u}$ just constructed is continuous except at the vertices $V_j$ of $\Omega_i$. The step $i$) can be viewed as emulating the mean value theorem for harmonic functions. However, near the vertices, $\bar{u}$ is a bad approximation of the harmonic extension. We know that the local behavior of the harmonic extension near a vertex $V_j$ depends primarily on the boundary values in the vicinity of $V_j$. For instance, if $u_H(P_1) = 0, u_H(P_3) = 0$, and $u_H(P_2) = 1$, we should obtain $u_h \simeq 0$ near $V_2$; in addition, by using symmetry arguments, we should have $u_h \simeq 1/2$ for points near $V_1$ that lie on the bisector that passes through $V_1$ . With this in mind, we now construct an alternative approximate harmonic extension.

We change notation in order to be able to use Fig. 5.3. Let now $C$ be the point where the three bisectors intersect.

**Extension 5.2** *The construction of the approximate harmonic extension in $\Omega_i$ is defined by (see Fig. 5.3):*

*i) Same as Step i) of Extension 5.1.*

*ii) Define $\bar{u}(V_j) = \frac{1}{2}\sum_{l\neq j}\bar{u}(P_l)$. For a point $R$ that belongs to a line segment connecting $C$ to $V_j$, define $\bar{u}(R)$ by linear interpolation between the values $\bar{u}(C)$ and $\bar{u}(V_j)$.*

*iii) Same as Step iii) of Extension 5.1.*

*iv) For a point $S$ that belongs to a line segment connecting the previous point $Q$ to $V_\ell$, $\ell \neq k$, $\bar{u}(S)$ is defined by linear interpolation between the values $\bar{u}(Q)$ at $Q$ and $f(Q,k,\ell)$ at $V_\ell$. Here,*

$$f(Q,k,\ell) = \lambda(Q)\,\bar{u}(V_\ell) + (1-\lambda(Q))\,\bar{u}(P_k).$$

*v) Same as Step v) of Extension 5.1.*

A disadvantage of this extension is that we cannot just work in a reference triangle, since the angles are not preserved under a linear transformation. This is similar to the fact that under a linear transformation a harmonic function does not necessarily remain harmonic. We can construct other approximate harmonic extensions which combine the properties of the two extensions, given so far. For instance, we can construct Extension 5.3 by working with the barycenter $C$ as in Extension 5.2 and replacing the weight 1/2 in Step ii).

The next lemma shows that the extensions given above have quasi-optimal energy stability. Using ideas of Dryja and Widlund [37], we prove the following lemma.

**Lemma 5.7** *Let* $u_H \in \tilde{V}^H(\Omega)$*. Then*

$$|\tilde{I}_H^h u_H|_{H^1_{\rho,h}(\Omega_i)} \preceq (1 + log(H/h))^{\frac{1}{2}} \, |u_H|_{H^1_{\rho,H}(\Omega_i)} \tag{5.37}$$

*and*

$$\|\tilde{I}_H^h u_H - u_H\|_{L^2_\rho(\Omega_i)} \preceq H \, |u_H|_{H^1_{\rho,H}(\Omega_i)} \qquad i = 1, \cdots, N. \tag{5.38}$$

*Proof.* Let $\vartheta^{CR}_{\mathcal{E}_{ij}} \in \tilde{V}^h(\Omega_i), j = 1, 2, 3$, be the approximate harmonic extensions, e.g. by Extension 5.1, constructed from the boundary values $\vartheta^{CR}_{\mathcal{E}_{ij}} = 1$ at the CR nodes $\mathcal{E}^{CR}_{ij,h}$, and $\vartheta^{CR}_{\mathcal{E}_{ij}} = 0$ at the other CR nodes of $\partial\Omega^{CR}_{i,h}$. It is easy to see that the $\vartheta^{CR}_{\mathcal{E}_{ij}}$ form a basis of all approximate harmonic extensions that take constant values on the edges of the substructure. It is easy to show that if a point $x$ belongs to the interior of an element of $\Omega_i$, then $|\nabla \, \vartheta^{CR}_{\mathcal{E}_{ij}}(x)|$ is bounded by $C/r$, where $r$ is the minimum distance from $x$ to any vertex of $\Omega_i$. Note that any element that touches a vertex of $\Omega_i$ provides an order one contribution to the energy seminorm. To estimate the contribution to the energy seminorm from the rest of the substructure, we introduce polar coordinate systems centered at the vertices of $\Omega_i$. Then,

$$|\vartheta^{CR}_{\mathcal{E}_{ij}}|^2_{H^1_h(\Omega_i)} \preceq 1 + \int \int_h^H \, r^{-2} \, r \, dr \, d\varphi \preceq 1 + log(H/h). \tag{5.39}$$

Since the partition of unity $\vartheta^{CR}_{\mathcal{E}_{ij}}$ forms a basis, it is easy to see that

$$|\tilde{I}_H^h u_H|^2_{H^1_h(\Omega_i)} \preceq \tag{5.40}$$

$$(1 + log(H/h)) \left\{ |u_H(P_1)|^2 + |u_H(P_2)|^2 + |u_H(P_3)|^2 \right\}$$

and using ideas similar to that of Lemma 5.5, we have

$$|\tilde{I}_H^h u_H|^2_{H^1_h(\Omega_i)} \preceq (1 + log(H/h)) \left\{ |u_H(P_1) - u_H(P_2)|^2 + \right.$$

$$\left. |u_H(P_2) - u_H(P_3)|^2 + |u_H(P_3) - u_H(P_1)|^2 \right\}$$

$$\asymp (1 + log(H/h)) |u_H|^2_{H^1_h(\Omega_i)}.$$

By construction, it is easy to see that

$$|(\tilde{I}_H^h u_H)(x)| \leq \max_{i=1,2,3} |u_H(P_i)|.$$

Therefore

$$\|\tilde{I}_H^h u_H - u_H\|^2_{L^2(\Omega_i)} \preceq \sum_i H^2 \, |u_H(P_i)|^2 \, ,$$

and by using (5.36) and (5.25), we obtain (5.38).

Since, by assumption, $\rho(x) = \rho_i$ in each $\Omega_i$, these arguments are also valid for the weighted norms and we obtain (5.37).

$\square$

Let us denote $\tilde{I}_h^{\tilde{F}} = \tilde{I}_H^h I_h^H$. Using Lemmas 5.4 and 5.7 and the triangular inequality, we have:

**Theorem 5.1** *Let $u \in \tilde{V}^h(\Omega)$. Then*

$$(\tilde{I}_h^{\tilde{F}} \tilde{V}_0^h(\Omega)) \subset \tilde{V}_0^h(\Omega),$$

$$\|\tilde{I}_h^{\tilde{F}} u - u\|_{L^2_\rho(\Omega_i)} \preceq H \, |u|_{H^1_{\rho,h}(\Omega_i)}, \tag{5.41}$$

*and*

$$|\tilde{I}_h^{\tilde{F}} u|_{H^1_{\rho,h}(\Omega_i)} \preceq (1 + log(H/h))^{\frac{1}{2}} \, |u|_{H^1_{\rho,h}(\Omega_i)} \qquad i = 1, \cdots, N. \tag{5.42}$$

Let $\theta_{\mathcal{E}_{ij}}^{CR} \in \tilde{V}^h(\Omega_i)$, $\quad j = 1, 2, 3$, be the nonconforming $P_1$ harmonic functions in $\Omega_i$ constructed with the values $\theta_{\mathcal{E}_{ij}}^{CR} = 1$ at the CR nodes $\mathcal{E}_{ij,h}^{CR}$, and $\theta_{\mathcal{E}_{ij}}^{CR} = 0$ at the other CR nodes of $\partial\Omega_{i,h}^{CR}$. It is easy to see that the $\theta_{\mathcal{E}_{ij}}^{CR}$ form a basis of all nonconforming $P_1$ harmonic functions in $\Omega_i$ that take constant values on the edges of the substructure. Hence, the Interpolation Operator $I_H^h$ given in Defintion 5.3 can be given in terms of these $\theta_{\mathcal{E}_{ij}}^{CR}$ functions. Let us denote $\tilde{I}_h^F = I_H^h I_h^H$. As in Theorem 5.1, we obtain

**Theorem 5.2** *Let $u \in \tilde{V}^h(\Omega)$. Then*

$$\tilde{I}_h^F(\tilde{V}_0^h(\Omega)) \subset \tilde{V}_0^h(\Omega),$$

$$\|\tilde{I}_h^F u - u\|_{L^2_\rho(\Omega_i)} \preceq H\, |u|_{H^1_{\rho,h}(\Omega_i)}, \tag{5.43}$$

*and*

$$|\tilde{I}_h^F u|_{H^1_{\rho,h}(\Omega_i)} \preceq (1 + log(H/h))^{\frac{1}{2}}\, |u|_{H^1_{\rho,h}(\Omega_i)} \qquad i = 1, \cdots, N. \tag{5.44}$$

*Proof.* The inequality (5.44) follows trivially from (5.42) since the nonconforming $P_1$ harmonic function has minimal energy seminorm.

Using trivial arguments, we can show a weaker result than (5.43), given by

$$\|I_H^h I_h^H u - u\|_{L^2_\rho(\Omega_i)} \preceq H\,(1 + log(H/h))^{\frac{1}{2}} |u|_{H^1_{\rho,h}(\Omega_i)}. \tag{5.45}$$

We note that (5.45) will be enough for our purposes. To prove (5.43), we use the same ideas as in the proof of Lemma 4.3 of Dryja, Smith, and Widlund [35]; we note $\Omega_i$ convex is used.

□

**Remark 5.2** *It is easy to see that we do not need to use the fact that $u_H \in V_H(\Omega)$; we only need to calculate the values $\bar{u}_{\mathcal{E}_{ij}} = u_H(P_{ij})$ by formula (5.22). The next step is to provide the constant value $\bar{u}_{\mathcal{E}_{ij}}$ to all CR nodes on $\mathcal{E}_{ij,h}^{CR}$ and perform a discrete nonconforming $P_1$ harmonic extension, or an approximate harmonic extension. An important observation is that these extensions can be constructed for nontriangular substructures. In a case of approximate harmonic extension, in a first step, we construct a partition of unity in $\Omega_i$. This can be done by using ideas similar to those of the triangular case, if*

*$\Omega_i$ is not too degenerate. By using the same technique as in the proof of Lemma 5.7, we can show that*

$$|\tilde{I}_h^{\tilde{F}} u|^2_{H^1_{\rho,h}(\Omega_i)} \preceq \rho_i \left(1 + log(H/h)\right) \sum_{\mathcal{E}_{ij} \subset \partial\Omega_i} (\bar{u}_{\mathcal{E}_{ij}} - \bar{u}_{\partial\Omega_i})^2. \tag{5.46}$$

*Here, $\bar{u}_{\partial\Omega_i}$ is the average of $u$ over $\partial\Omega_i$. We obtain (5.42) by showing that each term inside the sum can be bounded by $C\,|u|^2_{H^1_{\rho,h}(\Omega_i)}$.*

### 5.1.7 The Neumann-Neumann basis

In this subsection, we consider Neumann-Neumann coarse spaces. This is the nonconforming $P_1$ version of a coarse space studied in Dryja and Widlund [41], and Mandel and Brezina [55]. However, here we use an approximate harmonic extension inside the substructures. We note that the coarse spaces considered by these authors differ only in how certain weights are chosen. Mandel and Brezina use weights that are convex combinations of the coefficient $\rho(x)$, while Dryja and Widlund use $\rho^{\frac{1}{2}}(x)$. Here we show that any convex combination of $\rho^{\beta}(x)$, for $\beta \geq 1/2$, leads to stability. We remark that we can even define a Neumann-Neumann coarse space for $\beta = \infty$ by considering the limit of $\beta \to \infty$; see Subsection 4.4.1. We point out that the choice $\beta = 1/2$ can be viewed as a $L^2$-average, while $\beta = 1$ is an average in the $L^1$ sense.

The coarse spaces of the previous subsection are *face based.* There are some differences between Neumann-Neumann and face based coarse spaces. A Neumann-Neumann coarse spaces has one degree of freedom per substructure, while a face based uses one degree of freedom per edge. A Neumann-Neumann basis function associated with the substructure $\Omega_i$, has support in $\Omega_i$ and its neighboring substructures, while a face based function basis, associated with an edge of a substructure, has support in just two substructures. The face based coarse spaces appear to be more stable since all the estimates, related to the jumps of the coefficients, are tight. In the lemmas that we have proved for the face based methods, all the stability results were derived in individual substructures, while in the Neumann-Neumann cases, we need to work with the neighboring substructures as well.

**Definition 5.4** *The Neumann-Neumann interpolation operators, $\widetilde{I_h^{NN}} : \tilde{V}_0^h \to \tilde{V}_0^h$, are given as follows:*

*i) For each substructure $\Omega_i$, calculate the mean value on $\partial\Omega_i$, i.e.*

$$\bar{u}_i = \frac{1}{|\partial\Omega_i|} \int_{\partial\Omega_i} u(s)\, ds.$$

*Here $|\partial\Omega_i|$ is the length size of $\partial\Omega_i$.*

*ii) Let $P_{ij}$ be the midpoint of the edge $\mathcal{E}_{ij}$ common to $\bar{\Omega}_i$ and $\bar{\Omega}_j$. Then for all CR node $P \in \mathcal{E}_{ij,h}^{CR}$, let $\widetilde{\tilde{I}_h^{NN}} u(P) = (\hat{I}_h^H u)(P_{ij})$, where*

$$(\hat{I}_h^H u)(P_{ij}) = \frac{\rho_i^\beta}{\rho_i^\beta + \rho_j^\beta}\, \bar{u}_i + \frac{\rho_j^\beta}{\rho_i^\beta + \rho_j^\beta}\, \bar{u}_j.$$

*iii) Let $P_{ij}$ be the midpoint of the edge $\mathcal{E}_{ij}$ common to $\bar{\Omega}_i$ and $\partial\Omega$. Then for all CR node $P \in \mathcal{E}_{ij,h}^{CR}$, let $\widetilde{\tilde{I}_h^{NN}} u(P) = (\hat{I}_h^H u)(P_{ij})$, where*

$$(\hat{I}_h^H u)(P_{ij}) = I_h^H u)(P_{ij}).$$

*iv) Perform an approximate harmonic extension to define $\widetilde{\tilde{I}_h^{NN}} u$ in the interior of the substructures.*

Note that we can also calculate $\bar{u}_i$ by:

$$\bar{u}_i = \sum_{j=1}^{3} \frac{|\mathcal{E}_{ij}|}{|\partial\Omega_i|} (I_h^H u)(P_{ij}). \tag{5.47}$$

Therefore, there exists a linear transformation $I_H^H : \tilde{V}_H(\Omega) \to \tilde{V}_H(\Omega)$, such that $\hat{I}_h^H u = I_H^H I_h^H u$. The next lemma establishes stability properties for $I_H^H$.

**Lemma 5.8** *Let $u_H \in \tilde{V}^H(\Omega)$ and $1/2 \le \beta \le \infty$. Then*

$$|I_H^H u_H|_{H^1_{\rho,H}(\Omega_i)} \preceq |u_H|_{H^1_{\rho,H}(\Omega_i^{ext})}, \tag{5.48}$$

*and*

$$\|I_H^H u_H - u_H\|_{L^2_\rho(\Omega_i)} \preceq H\, |u_H|_{H^1_{\rho,H}(\Omega_i^{ext})}. \tag{5.49}$$

*Here the extended domain $\Omega_i^{ext}$ is the union of $\Omega_i$ and the substructures that share an edge with $\Omega_i$.*

*Proof.* Let us first prove the $L^2_\rho$ stability. We focus on the case where $P_{ij}$ does not belongs to $\partial\Omega$. The other case is trivial. Note that (see Fig. 5.4)

$$|u_H(P_{ij}) - (I_H^H u_H)(P_{ij})|^2 = |u_H(P_{ij}) - \frac{{\rho_i}^\beta\,\bar{u}_i + \rho_j^\beta\,\bar{u}_j}{\rho_i^\beta + \rho_i^\beta}|^2.$$

By using (5.47) and simple calculations, this quantity is equal to

$$\frac{1}{|\rho_i^\beta + \rho_j^\beta|^2} *$$

$$|\rho_i^\beta\,\{\frac{|\mathcal{E}_{ik}|}{|\partial\Omega_i|}(u_H(P_{ij}) - u_H(P_{ik})) + \frac{|\mathcal{E}_{il}|}{|\partial\Omega_i|}(u_H(P_{ij}) - u_H(P_{il}))\}+$$

$$\rho_j^\beta\,\{\frac{|\mathcal{E}_{js}|}{|\partial\Omega_j|}(u_H(P_{ij}) - u_H(P_{js})) + \frac{|\mathcal{E}_{jr}|}{|\partial\Omega_j|}(u_H(P_{ij}) - u_H(P_{jr}))\}|^2.$$

Using the shape regularity of the subdomains, it is easy to see that

$$\rho_i\,|u_H(P_{ij}) - (I_H^H u_H)(P_{ij})|^2 \preceq \tag{5.50}$$

$$\frac{\rho_i^{2\beta}}{|\rho_i^\beta + \rho_j^\beta|^2}\,|u_H|^2_{H^1_{\rho,H}(\Omega_i)} + \frac{\rho_i\,\rho_j^{2\beta-1}}{|\rho_i^\beta + \rho_j^\beta|^2}\,|u_H|^2_{H^1_{\rho,H}(\Omega_j)}$$

and using the fact that $\beta \geq 1/2$, we can bound this quantity by

$$\preceq |u_H|^2_{H^1_{\rho,H}(\Omega_i \cup \Omega_j)}.$$

We note that the constant related to the last inequality $\preceq$, is also independent of $\beta$. Therefore, our results also holds for $\beta = \infty$. We obtain (5.49) by adding all the contributions (5.50) to the $L^2_\rho(\Omega_i)$ norm. We prove (5.48) by using the triangular inequality, an inverse inequality, and (5.49).

□

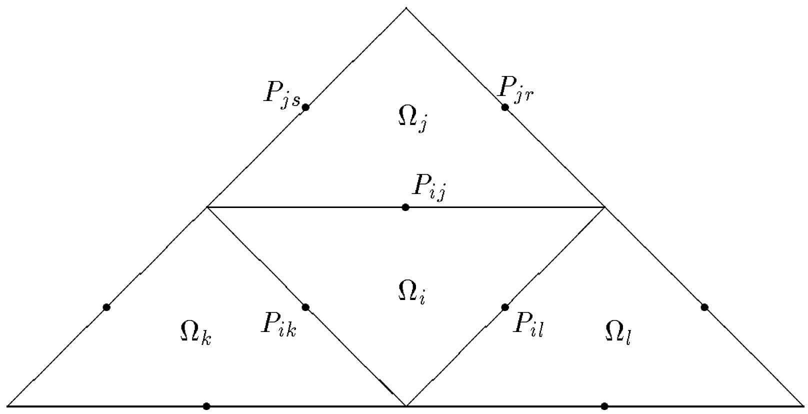


Fig. 5.4

**Theorem 5.3** *Let $u \in \tilde{V}^h(\Omega)$ and $1/2 \le \beta \le \infty$. Then*

$$\|\widetilde{\tilde{I}_h^{NN}} u - u\|_{L^2_\rho(\Omega_i)} \preceq H\, |u|_{H^1_{\rho,h}(\Omega_i^{ext})}\,, \tag{5.51}$$

*and*

$$|\widetilde{\tilde{I}_h^{NN}} u|_{H^1_{\rho,h}(\Omega_i)} \preceq (1 + log(H/h))^{\frac{1}{2}}\, |u|_{H^1_{\rho,h}(\Omega_i^{ext})}. \tag{5.52}$$

*Proof.* Using Lemmas 5.7, 5.8, and 5.6, we have

$$|\widetilde{\tilde{I}_h^{NN}} u|_{H^1_{\rho,h}(\Omega_i)} \preceq (1 + log(H/h))^{\frac{1}{2}}\, |I_H^H I_h^H u|_{H^1_{\rho,H}(\Omega_i)} \le$$

$$\preceq (1 + log(H/h))^{\frac{1}{2}}\, |I_h^H u|_{H^1_{a,H}(\Omega_i^{ext})} \le$$

$$\preceq (1 + log(H/h))^{\frac{1}{2}}\, |u|_{H^1_{\rho,h}(\Omega_i^{ext})}.$$

The $L^2_\rho$-stability is obtained by

$$\|\widetilde{\tilde{I}_h^{NN}} u - u\|_{L^2_\rho(\Omega_i)} \le \|\widetilde{\tilde{I}_h^{NN}} u - I_H^H I_h^H u\|_{L^2_\rho(\Omega_i)} +$$

$$\|I_H^H I_h^H u - I_h^H u\|_{L^2_\rho(\Omega_i)} + \|I_h^H u - u\|_{L^2_\rho(\Omega_i)}\,,$$

and by using Lemmas 5.7, 5.8, and 5.6.

$\square$

### 5.1.8 The Three-dimensional case

We show in this subsection that the methods developed above can be extended to three dimensions.

For simplicity, we assume that $\Omega$ is a polyhedral region of diameter 1 in three dimensional space. Let $\mathcal{T}^H$ and $\mathcal{T}^h$ be the coarse and the fine triangulations, respectively, given in Subsection 3.2.1. Define the nonconforming $P_1$ finite element spaces $\tilde{V}^h(\Omega)$, $\tilde{V}_0^h(\Omega)$, $\tilde{V}^H(\Omega)$, $\tilde{V}_0^H(\Omega)$ as the three-dimensional counterpart of Definition 5.1. Here, the continuity is enforced at the barycenter of the faces of the triangulations; we note that the space $\tilde{V}_0^h(\Omega)$ has already been defined in Subsubsection 3.4.3 as $V_{CR}^h(\mathcal{T}^h)$.

The local equivalence maps are given by the following procedure. In each tetrahedral element of $\mathcal{T}^h$ (cf. Fig. 4.3), we connect the centroid to the four vertices and to the barycenters of the four faces. We also connect each face barycenter to the three vertices of the face. Thus, we subdivide each tetrahedral element into twelve subtetrahedra. We denote this new triangulation by $\mathcal{T}^{\hat{h}}$. The vertices of $\mathcal{T}^{\hat{h}}$ are the vertices, barycenters, and centroids of the elements of $\mathcal{T}^h$. Denote by $V^{\hat{h}}(\Sigma)$ the conforming space of piecewise linear functions on the triangulation $\mathcal{T}^{\hat{h}}$ restricted to a region $\Sigma$. We use notation similar to those of Section 4.4 to denote, e.g., by $\bar{\Omega}_{i,\hat{h}}$ the set of nodes of $\mathcal{T}^{\hat{h}}$ belonging to $\bar{\Omega}_i$. We denote by CR nodal points, the barycenters of the faces of elements in $\mathcal{T}^h$. We preserve the same notation as used for two dimensions. $\mathcal{F}_{ij,h}^{CR}$ represents the set of CR nodal points belonging to a face $\mathcal{F}_{ij}$ of $\Omega_i$.

The counterpart of the Isomorphism 5.1 is given as follows:

**Isomorphism 5.3** *Given $u \in \tilde{V}^h(\Omega_i)$, define $\mathcal{M}_i u \in V^{\hat{h}}(\Omega_i)$ by the value of $\mathcal{M}_i u$ at the following sets of points:*

*i) If $P \in \bar{\Omega}_{i,\hat{h}} \backslash \partial\Omega_{i,\hat{h}}$ and the $\tau_j^{\hat{h}}$ are the elements in $\mathcal{T}^{\hat{h}}(\Omega_i)$ that have $P$ as a vertex, then*

$$\mathcal{M}_i u(P) = \text{mean of} \quad u|_{\tau_j^{\hat{h}}}(P).$$

*Here $u|_{\tau_j^{\hat{h}}}(P)$ is the limit of $u(x)$ when $x \in \tau_j^{\hat{h}}$ approaches $P$.*

*ii) If $P \in \partial\Omega_{i,h}^{CR}$, then*

$$\mathcal{M}_i u(P) = u(P).$$

*iii) If $P \in \partial\Omega_{i,h}$ and and $T_j$, $j = 1, \cdots, N_P$, are the triangles of $\mathcal{T}^h \cap \partial\Omega_i$ that have $P$ as a vertex, then*

$$\mathcal{M}_i u(P) = \sum_{k=1}^{N_P} \frac{|T_k|}{| \cup_{j=1}^{N_P} T_j|} u(C_i).$$

*Here $C_i$ and $|T_i|$ are the barycenter and the area of the triangle $T_i$, respectively.*

**Remark 5.3** *It is easy to check that Lemma 5.2 holds, if we replace $V^{h/2}(\Omega_i)$ by $V^{\hat{h}}(\Omega_i)$.*

The counterpart of Isomorphims 5.2 is given by the following local equivalence map $\mathcal{M}_i^{\mathcal{F}} : \tilde{V}^h(\Omega_i) \to V^{\hat{h}}(\Omega_i)$, by:

**Isomorphism 5.4** *Given $u \in \tilde{V}^h(\Omega_i)$ and a face $\mathcal{F}$ of $\partial\Omega_i$, define $\mathcal{M}_i^{\mathcal{F}} u \in V^{\hat{h}}(\Omega_i)$ by the values $\mathcal{M}_i^{\mathcal{F}} u$ at the following sets of points:*

*i) Same as step i) of Isomorphism 5.3.*

*ii) Same as step ii) of Isomorphism 5.3.*

*iii) Let $P \in \partial\Omega_{i,h} \cap \partial\mathcal{F}$, and let $T_j$, $j = 1, \cdots, N_P^F$, be the triangles of $\mathcal{T}^h \cap \mathcal{F}$ that have $P$ as a vertex. Then*

$$\mathcal{M}_i^{\mathcal{F}} u(P) = \sum_{k=1}^{N_P^F} \frac{|T_k|}{| \cup_{j=1}^{N_P^F} T_j|} u(C_i).$$

*iv) Let $P \in \partial\Omega_{i,h} \backslash \partial\mathcal{F}$, and let $T_j$, $j = 1, \cdots, N_P$, be the triangles of $\mathcal{T}^h \cap \mathcal{F}$ that have $P$ as a vertex. Then*

$$\mathcal{M}_i^{\mathcal{F}} u(P) = \sum_{k=1}^{N_P} \frac{|T_k|}{| \cup_{j=1}^{N_P} T_j|} u(C_i).$$

**Remark 5.4** *It is easy to check that Lemma 5.3 holds, if we replace $V^{h/2}(\Omega_i)$ by $V^{\hat{h}}(\Omega_i)$, and let the faces play the role previously played by the edges.*

**Definition 5.5** *Let $v \in \tilde{V}^h(\Omega)$. The Interpolation operator $I_h^H : \tilde{V}^h(\Omega) \to \tilde{V}^H(\Omega)$, is given by:*

*i) If $C_{ij}$ is the barycenter of the face $\mathcal{F}_{ij}$ common to $\bar{\Omega}_i$ and $\bar{\Omega}_j$, then*

$$(I_h^H v)(C_{ij}) = \frac{1}{|\mathcal{F}_{ij}|} \int_{\mathcal{F}_{ij}} v|_{\bar{\Omega}_i}(x)\, dx = \frac{1}{|\mathcal{F}_{ij}|} \int_{\mathcal{F}_{ij}} v|_{\bar{\Omega}_j}(x)\, dx. \qquad (5.53)$$

*ii) If $P$ is the barycenter of a face $\mathcal{F}$ common to $\bar{\Omega}_i$ and $\partial\Omega$, then*

$$(I_h^H v)(P) = \frac{1}{|\mathcal{F}|} \int_{\mathcal{F}} v|_{\bar{\Omega}_i}(x)\, dx. \qquad (5.54)$$

Using the same ideas as in two dimensions, we can prove lemmas analogous to Lemmas 5.4-5.6.

**Definition 5.6** *Let the prolongation operators $I_H^h, \tilde{I}_H^h : \tilde{V}^H(\Omega) \to \tilde{V}^h(\Omega)$, be defined as in the two dimensional case. In a first step, let define $(I_H^h u_H)(P) = (\tilde{I}_H^h u_H)(P) = u_H(C_{ij})$ for all CR nodes $P \in \mathcal{F}_{ij,h}^{CR}$. Finally, perform a nonconforming $P_1$ harmonic or approximate harmonic extension.*

We describe the three dimensional version of Extension 5.1. This is a generalization of the partition of unity introduced by Dryja, Smith, and Widlund [35]. Let $C_j$, $j = 1, \cdots, 4$, be the barycenters of the faces $F_j$ of $\partial\Omega_i$, and let $V_j$ be the vertex of $\Omega_i$ that is opposite to $C_j$. Let $C$ the centroid of $\Omega_i$, i.e. the intersection of the line segments connecting the $V_j$ to the $C_j$. Let $E_{jk}$, $k = 1, 2, 3$, be the edges of $\partial F_j$.

**Extension 5.3** *The construction of an approximate harmonic extension in $\Omega_i$ is defined by the following steps (see Fig. 4.3):*

*i) Let*

$$\bar{u}(C) = \frac{1}{4} \sum_{j=1}^{4} u_H(C_j).$$

*ii) For a point $Q$ that belongs to a line segment connecting $C$ to $C_j$, define $\bar{u}(Q)$ by linear interpolation between the values $\bar{u}(C)$ and $u_H(C_j)$, i.e. by*

$$\bar{u}(Q) = \lambda(Q)\bar{u}(C) + (1 - \lambda(Q))u_H(C_j).$$

*Here $\lambda(Q) =$distance$(Q, C_j)/$distance$(C, C_j)$.*

*iii) For a point $S$ that belongs to any of the three triangles defined by the previous $Q$, and the edges $E_{jk}$, $k = 1, \cdots, 3$, let*

$$\bar{u}(S) = \bar{u}(Q).$$

*iv) Finally, let $I_H^h u_H(\Omega_i) = I_{i,h}^{CR} \bar{u}$, where $I_{i,h}^{CR}$ is the interpolation operator into the space $\tilde{V}^h(|\Omega_i)$ that preserves the values of a function at the CR nodes of $\bar{\Omega}_{i,h}^{CR}$.*

We can also construct an approximate harmonic extension similar to that of Extension 5.2 or 5.3. This gives a better approximate harmonic extension near the edges.

The prolongation operator $\tilde{I}_H^h$ in three dimensions has the same stability properties as in the two dimensional case, i.e. Lemma 5.7 still holds. The idea of the proof is the following. Consider the case where $u_H(\Omega_i)$ is given by $u_H(C_{i1}) = 1$ and $u_H(C_{i2}) = u_H(C_{i3}) = u_H(C_{i4}) = 0$. This gives an element of a partition of the unity introduced by Dryja, Smith, and Widlund [35] and the energy seminorm of $u_H$ is of order $H$. Let $\vartheta_{\mathcal{F}_{i1}}^{CR}(\Omega_i) = I_H^h u_H(\Omega_i)$. We note that $|\nabla \vartheta_{\mathcal{F}_{i1}}^{CR}(x)|$ is bounded by $C/r$, where $r$ is the distance to the nearest edge of $\Omega_i$. The contribution to the energy seminorm from the union of the elements with at least one vertex on the edge of the substructure can be bounded by $CH$, using that the extension is given by a convex combination of the boundary values. To estimate the contribution to the energy from the rest of the substructure, we introduce cylindrical coordinates using the appropriate substructure edge as the $z$-axis. Integrating $|\nabla \vartheta_{\mathcal{F}_{i1}}^{CR}(x)|^2$ over this region, we find that is bounded by $C\,(1 + log(H/h))\,H$. See Dryja, Smith, and Widlund [35] for more details. The arguments can be extended easily for the nonconforming $P_1$ case.

To prove Lemma 5.7 for a general $u_H \in \tilde{V}^H(\Omega_i)$, we use the same ideas as for two dimensions.

Similarly to the two-dimensional case, we can extend the results to nontriangular substructures and to the Neumann-Neumann case.

### 5.1.9 Main result for Two-level method

In this subsection, we consider the Schwarz method introduced in the previous subsections and prove the following result.

**Theorem 5.4** *The operator $\tilde{P}$ of the additive Schwarz algorithm, defined by the spaces $\tilde{V}_0^h(\Omega)$ and $\tilde{V}_i$, satisfies:*

$$\kappa(\tilde{P}) \preceq (1 + log(\frac{H}{h}))\,(1 + \frac{H}{\delta}). \tag{5.55}$$

*Here $\kappa(\tilde{P})$ is the condition number of $\tilde{P}$. Therefore, if we use a generous overlapping, i.e. $H/\delta$ is uniformly bounded, then*

$$\kappa(\tilde{P}) \preceq 1 + log(\frac{H}{h}).$$

The proof of this theorem is essentially the same as in the case of a conforming space; see Dryja and Widlund [42].

*Proof.* The lower bound is obtained by using *Assumption* i) of Theorem 2.1. We partition the finite element function $u \in \tilde{V}_0^h(\Omega)$, as follows. We first choose $u_0 = \tilde{I}_h^X u$, with $X = F$, $\tilde{F}$, $NN$, or $\widetilde{NN}$, and set $w = u - u_0$. The other terms in the representation of $u$ are defined by $u_i = I_h^{CR}(\theta_i w), i = 1, \cdots, N$. Here $I_h^{CR}$ is the linear interpolation operator into the space $\tilde{V}_0^h(\Omega)$ that preserves the values at the CR nodes of $\Omega_h^{CR}$, and $\{\theta_i\}$ is a partition of unity with $\theta_i \in C_0^\infty(\Omega_i')$ and $\sum \theta_i(x) = 1$.

For a relatively generous overlap of the subdomains, these functions can be chosen so that $\nabla\theta_i$ is bounded by $C/H$. By using the linearity of $I_h^{CR}$, we can show that we have a correct partition of $u$. In order to estimate the seminorm of $u_i$, we work on one element $\tau_j^h$ at a time. We obtain

$$|u_i|^2_{H^1_{\rho,h}(\tau_j^h)} \leq 2\,|\bar{\theta}_i w|^2_{H^1_{\rho,h}(\tau_j^h)} + 2\,|I_h^{CR}((\theta_i - \bar{\theta}_i)w)|^2_{H^1_{\rho,h}(K)}$$

.

Here $\bar{\theta}_i$ is the average value of $\theta_i$ over $\tau_j^h$. It is easy to see, by using the inverse inequality (5.9), that

$$|I_h^{CR}((\theta_i - \bar{\theta}_i)w)|^2_{H^1_{\rho,h}(\tau_j^h)} \preceq h^{-2}\,\|I_h^{CR}((\theta_i - \bar{\theta}_i)w)\|^2_{L^2_\rho(\tau_j^h)}.$$

We can now use the fact that on $\tau_j^h$, $\theta_i$ differs from its average by at most $C\,h/H$. After summing over all elements of $\Omega_i'$, we arrive at the inequality

$$|u_i|^2_{H^1_{\rho,h}(\Omega_i')} \preceq |w|^2_{H^1_{\rho,h}(\Omega_i')} + H^{-2}\,\|w\|^2_{L^2_\rho(\Omega_i')}.$$

We sum over all $i$ and use that each point in $\Omega$ is covered only a fixed number of times and we obtain a uniform bound on $C_0^2$. We conclude the proof, by estimating the two terms of

$$|w|^2_{H^1_{\rho,h}(\Omega)} + H^{-2} \|w\|^2_{L^2_\rho(\Omega)}$$

by $|u|^2_{H^1_{\rho,h}(\Omega)}$. The bounds follow by using the stability results of Theorem 5.1 or 5.3.

For the case of small overlap, the proof is similar to that of the case of piecewise linear conforming space considered in Dryja and Widlund [42].

An upper bound on the spectrum is obtained by bounding

$$a^h(\tilde{P}v, v) = a^h(\tilde{P}_{-1}v, v) + a^h(\tilde{P}_1 v, v) + \cdots + a^h(\tilde{P}_N v, v), \tag{5.56}$$

from above in terms of $a^h(v,v)$. Using Schwarz's inequality, the fact that the $\tilde{P}_i$ are projections, and that the maximum number of regions that intersect at any point is uniformly bounded, it is easy to show that the spectrum of $\tilde{P}$ is bounded above by

$$\max_{p \in \Omega} \{\#(i : p \in \Omega_i') + 1\}.$$

□

**Remark 5.5** *The proof of Theorem 5.4 holds also for triangulations $\mathcal{T}^H$ and $\mathcal{T}^h$ that are shape regular and nonuniform. We can show that*

$$\kappa(\tilde{P}) \preceq \max_{\{1,\cdots,N\}} (1 + log(\frac{H_i}{h_i}))(1 + \frac{H_i}{\delta_i}).$$

*Here, $H_i$ is the diameter of $\Omega_i$, and $h_i$ is the diameter of the smallest element in $\mathcal{T}^h(\Omega_i)$.*

## 5.2 Nonconforming coarse spaces

In this section, we introduce some modifications to our two level methods in order to have more efficient algorithms.

The first modification is related to the average value $\bar{u}_{\mathcal{F}_{ij}^{CR}}$ defined earlier. To calculate $\bar{u}_{\mathcal{F}_{ij}^{CR}}$, it is necessary to know the exact area of each face of $\mathcal{T}^h \cap \mathcal{F}_{ij}$. Therefore, we define another quantity, $\bar{u}^h_{\mathcal{F}_{ij}^{CR}}$, as the *discrete* average value of $u$ over the CR nodes on $\mathcal{F}_{ij,h}^{CR}$, by

$$\bar{u}^h_{\mathcal{F}_{ij}^{CR}} = \frac{\sum_{p \in \mathcal{F}_{ij,h}^{CR}} u(x_p)}{\#(p \in \mathcal{F}_{ij,h}^{CR})}.$$

Similar considerations also apply to $\bar{u}_{\partial\Omega_i^{CR}}$ and $\bar{u}^h_{\partial\Omega_i^{CR}}$. We note that, for quasi-uniform triangulations $\mathcal{T}^h(\Omega_i)$, the bounds in the estimates that we obtain here are the same as before.

Motivated by Remark 5.2, we make another simplification of our algorithm introducing an inexact solver for the face based coarse problem.

We also describe the Neumann-Neumann coarse spaces from a different point of view.

### 5.2.1 A face based coarse space

The space $\tilde{V}^F_{-1}$ can conveniently be defined as the range of an interpolation operator $\tilde{I}^F_h : \tilde{V}^h_0(\Omega) \rightarrow \tilde{V}^F_{-1}$, defined by

$$\tilde{I}^F_h u(x)_{|\bar{\Omega}_i} = \sum_{\mathcal{F}_{ij} \subset \partial\Omega_i} \bar{u}^h_{\mathcal{F}^{CR}_{ij}} \; \theta^{CR}_{\mathcal{F}_{ij}}(x).$$

We define a bilinear form by

$$b^{CR}_{-1,F}(u,u) = \sum_i \rho_i \{ H(1 + \log H/h) \sum_{\mathcal{F}_{ij} \subset \partial\Omega_i} (\bar{u}^h_{\mathcal{F}^{CR}_{ij}} - \bar{u}^h_{\partial\Omega_i^{CR}})^2 \},$$

and the operator $\tilde{T}^F_{-1} : \tilde{V}^h_0 \rightarrow \tilde{V}^F_{-1}$, by

$$b^{CR}_{-1,F}(\tilde{T}^F_{-1}\, u, v) = a^h(u,v) \;\; \forall v \in \tilde{V}^F_{-1}.$$

### 5.2.2 Neumann-Neumann coarse spaces

We consider a family of coarse spaces with only one degree of freedom per substructure; cf. Chapter 4.

For each $\beta \geq 1/2$, we define the pseudo inverses $\mu^+_{i,\beta}$, $i = 1, \cdots, N$, by

$$\mu^+_{i,\beta}(x) = \frac{1}{(\rho_i)^\beta + (\rho_j)^\beta}, \;\; x \in \mathcal{F}^{CR}_{ij,h} \;\; \forall \mathcal{F}_{ij} \subset (\partial\Omega_i \backslash \partial\Omega)$$

and

$$\mu^+_{i,\beta}(x) = 0, \;\; x \in (\Gamma^{CR}_h \backslash \partial\Omega^{CR}_{i,h}) \cup \partial\Omega^{CR}_h.$$

We extend $\mu^+_{i,\beta}$ elsewhere in $\Omega$ as a nonconforming discrete harmonic function using the data on $\Gamma^{CR}_h \cup \partial\Omega^{CR}_h$. The resulting functions belong to $\tilde{V}^h_0(\Omega)$ and are also denoted by $\mu^+_{i,\beta}$.

We can now define the coarse space $\tilde{V}_{-1}^{NN} \subset \tilde{V}_0^h(\Omega)$ by

$$\tilde{V}_{-1}^{NN} = \text{span}\{\mu_{i,\beta}^{+}\},$$

where the span is taken over all the substructures $\Omega_i$. The associated bilinear form is defined by $b_{-1,NN}^{CR}(\cdot,\cdot) = a^h(\cdot,\cdot)$.

We note that $\tilde{V}_{-1}^{NN}$ is also the range of the interpolation operator $\tilde{I}_h^{NN}$ given by

$$u_{-1} = \tilde{I}_h^{NN} u(x) = \sum_i u_{-1}^{(i)} = \sum_i \bar{u}_{\partial\Omega_i^{CR}}^h (\rho_i)^\beta \mu_{i,\beta}^{+}. \tag{5.57}$$

Similarly, we can define our coarse spaces and bilinear forms with approximate harmonic extensions. We only need to replace $\theta_{\mathcal{F}_{ij}}^{CR}$ to $\vartheta_{\mathcal{F}_{ij}}^{CR}$, and define the $\mu_{i,\beta}^{+}$ in $\Omega_i$ as the approximate harmonic extension using the data in $\Gamma_h^{CR} \cup \partial\Omega_h^{CR}$. The operators associated to the coarse problems are denoted by $\tilde{T}_{-1}^{\tilde{F}}$ and $\tilde{T}_{-1}^{\widetilde{NN}}$.

For $X = F, \tilde{F}, NN$, or $\widetilde{NN}$, let

$$\tilde{T}^X = \tilde{T}_{-1}^X + \sum_{i=1}^{N} \tilde{P}_i. \tag{5.58}$$

We can show:

**Theorem 5.5** *Assume that the triangulations $\mathcal{T}^H(\Omega)$ and that $\mathcal{T}^h(\Omega)$ are shape regular, and $\mathcal{T}^h(\Omega_i)$ is quasi-uniform for $i = 1, \cdots, N$. Then, for any $u \in \tilde{V}_0^h(\Omega)$*

$$\max_{i=1,\cdots,N}(1+\log(H_i/h_i))^{-1}(1+\frac{H_i}{\delta_i})a^h(u,u) \preceq a^h(\tilde{T}_{-1}^X u, u) \preceq a^h(u,u).$$

## 5.3 Multilevel additive Schwarz method

### 5.3.1 Overview

The first multigrid methods for nonconforming finite elements were introduced by Braess and Verfüth [4], and Brenner [13]. The existing convergence results are based on assuming $H^2-$regularity for the continuous problem. Later, Oswald [64], and Vassilevski and Wang [84] proposed optimal multilevel BPX-preconditioners for nonconforming $P_1$ elements in the three-dimensional case by using a sequence of nested conforming subspaces. No regularity beyond $H^1$ is used. We note, however, that we cannot guarantee that the rate of convergence of these methods are insensitive to large variations in the coefficients of the differential equation; see also [68], [89], [29], and [52]. In this section, we modify Oswald's preconditioner by introducing our nonstandard coarse spaces in Section 5.2, and establish that its condition number grows at most in proportion to $(1+\ell)^2$, and does not depend on the number of substructures and the jumps of the coefficients. To analyze our methods, we introduce additional isomorphisms besides those of previous section.

### 5.3.2 Additive version

Let $\mathcal{T}^k, k = 0, \cdots, \ell$ be a three-dimensional triangulation defined in Subsection 3.2.1. We use the same notations as in Chapter 4 and in previous section.

As we have noted repeatedly, any Schwarz method can be defined by the underlying splitting of the discretization space $\tilde{V}_0^h(\Omega)$ into a sum of subspaces, and by bilinear forms associated with these subspaces. Let $X = F, NN, \tilde{F}$, or $\widetilde{NN}$. The splitting of $\tilde{V}_0^h(\Omega)$ that we consider is given by

$$\tilde{V}_0^h = \tilde{V}_{-1}^X + \sum_{k=0}^{\ell} \sum_{j \in \mathcal{N}^k} V_j^k + \sum_{j \in \mathcal{N}_h^{CR}} \tilde{V}_j^h.$$

Here, $\mathcal{N}_h^{CR} = \bar{\Omega}_h^{CR} \backslash \partial \Omega_h^{CR}$, i.e. the set of barycentrical nodal points associated with the space $\tilde{V}_0^h(\Omega)$, and the $\tilde{\phi}_j^h$ are the standard nonconforming $P_1$ basis functions associated with these nodes. The space $\tilde{V}_j^h \subset \tilde{V}_0^h(\Omega)$ is the one-dimensional space spanned by $\tilde{\phi}_j^h$. See Chapter 4 for the definitions of the $V_j^k$ and the $\mathcal{N}^k$.

We introduce the following operators:

i) $\tilde{T}_{-1}^X : \tilde{V}_0^h \to \tilde{V}_{-1}^X$, is given by

$$a^h(\tilde{T}_{-1}^X u, v) = a^h(u, v) \ \ \forall v \in \tilde{V}_{-1}^X.$$

ii) $\tilde{P}_j^k : \tilde{V}_0^h \to V_j^k$, $k = 0, \cdots, \ell$, $j \in \mathcal{N}^k$ is given by

$$a^h(\tilde{P}_j^k u, v) = a^h(u, v) \ \ \forall v \in V_j^k.$$

iii) $\tilde{P}_j^h : \tilde{V}_0^h \to \tilde{V}_j^h$, $j \in \mathcal{N}_h^{CR}$, is given by

$$a^h(\tilde{P}_j^h u, v) = a^h(u, v) \ \ \forall v \in \tilde{V}_j^h.$$

Let

$$\tilde{T}^X = \tilde{T}^X_{-1} + \sum_{k=0}^{\ell} \sum_{j\in\mathcal{N}^k} \tilde{P}^k_j + \sum_{j\in\mathcal{N}^{CR}_h} \tilde{P}^h_j. \tag{5.59}$$

**Theorem 5.6** *For any* $u \in \tilde{V}^h_0(\Omega)$

$$(1+\log H/h)^{-2} a^h(u,u) \preceq a^h(\tilde{T}^X u, u) \preceq a^h(u,u).$$

The proof of this lemma is postponed to the end of next subsection.

### 5.3.3 Technical tools

In order to analyze our algorithms for the nonconforming case, we introduce interpolators with which we can map results from the conforming to the nonconforming case. We define a local interpolator $\mathcal{M}^h_i : \tilde{V}^h(\Omega_i) \to V^h(\Omega_i)$, as follows:

**Definition 5.7** *Given* $u \in \tilde{V}^h(\Omega_i)$*, we define a conforming piecewise linear function* $\bar{u} = \mathcal{M}^h_i u$ *in terms of values of* $\bar{u}$*;*

*i) If* $P \in \bar{\Omega}_{i,h} \partial\Omega_{i,h}$*, then let*

$$\bar{u}(P) = \textit{mean of} \quad u|_{\tau^h_j}(P).$$

*Here, the* $\tau^h_j$ *are the elements in* $\mathcal{T}^h(\bar{\Omega}_i)$ *that have* $P$ *as a vertex, and* $u|_{\tau^h_j}(P)$ *is defined as the limit value of* $u(x)$ *when* $x \in \tau^h_j$ *approaches* $P$*.*

*ii) If* $P \in \partial\Omega_{i,h}$*, and* $T_j$*,* $j = 1, \cdots, N_P$*, are the triangles of* $\mathcal{T}^h \cap \partial\Omega_i$ *that have* $P$ *as a vertex, then*

$$\bar{u}(P) = \sum_{k=1}^{N_P} \frac{|T_k|}{|\cup_{j=1}^{N_P} T_j|}\, u(C_i).$$

*Here* $C_i$ *and* $|T_i|$ *are the barycenter and the area of the triangle* $T_i$*, respectively.*

Using the same ideas as in Lemma 5.2 we can show:

**Lemma 5.9** *Given* $u \in \tilde{V}^h(\Omega_i)$*, let* $\bar{u} \in V^h(\Omega_i)$ *be given by* $\bar{u} = \mathcal{M}_i^h u$*. Then*

$$|\bar{u}|_{H^1(\Omega_i)} \asymp |u|_{H_h^1(\Omega_i)}, \tag{5.60}$$

$$\|\bar{u}\|_{L^2(\Omega_i)} \asymp \|u\|_{L^2(\Omega_i)}, \tag{5.61}$$

$$\|u - \bar{u}\|_{L^2(\Omega_i)} \preceq h\, |u|_{H_h^1(\Omega_i)}, \tag{5.62}$$

$$\|u\|_{L^2(\tau_j^h)} \preceq \|\bar{u}\|_{L^2(\tau_j^{h^{ext}} \cap \Omega_i)}, \tag{5.63}$$

*and*

$$\|\bar{u}\|_{L^2(\tau_j^h)} \preceq \|u\|_{L^2(\tau_j^{h^{ext}} \cap \Omega_i)}. \tag{5.64}$$

*Here,* $\tau_j^{h^{ext}}$ *is the union of a finite element* $\tau_j^h$ *and its next-neighbor finite elements in* $\bar{\Omega}_i$*.*

*In addition, if* $u$ *vanishes on the* $\partial\Omega_{i,h}^{CR}$*, then* $\bar{u}$ *vanishes on* $\partial\Omega_i$*.*

We have the following trace lemma for nonconforming $P_1$ functions. It is similar to a trace lemma for the conforming case; see e.g. (4.35).

**Lemma 5.10** *For* $u \in \tilde{V}^h(\Omega_i)$

$$\sum_{\tau_j^h \cap \mathcal{E}_l \neq \emptyset} \|u\|^2_{L^2(K)} \preceq h^2(1 + \log H/h)\, \|u\|^2_{H_h^1(\Omega_i)}.$$

*Note that the summation* $\sum$ *is taken over the finite elements in which intersect the edge* $\mathcal{E}_l$*.*

*Proof.* In fact,

$$\sum_{\tau_j^h \cap \mathcal{E}_l \neq \emptyset} \|u\|^2_{L^2(\tau_j^h)} \preceq \sum_{\tau_j^h \cap \mathcal{E}_l \neq \emptyset} \|\mathcal{M}_i^h u\|^2_{L^2(\tau_j^{h^{ext}})}$$

$$\preceq h^2\,(1 + \log H/h)\, \|\mathcal{M}_i^h u\|^2_{H^1(\Omega_i)} \asymp h^2\,(1 + \log H/h)\, \|u\|^2_{H_h^1(\Omega_i)}$$

Here we have used Lemma 5.9 and a bound on the trace of conforming piecewise linear functions.

□

The following lemma is the nonconforming version of Lemma 4.5 in Dryja, Smith, Widlund [35].

**Lemma 5.11** *For $u \in \tilde{V}^h(\Omega)$, we have*

$$|I_{CR}^h(\vartheta_{\mathcal{F}_{ij}}^{CR} u)|_{H_h^1(\Omega_i)}^2 \preceq (1 + \log H/h)^2 \|u\|_{H_h^1(\Omega_i)}^2, \qquad i = 1, \cdots, N.$$

*Here, $I_{CR}^h$ is the standard interpolation operator into the nonconforming space $\tilde{V}^h(\Omega)$ which preserves the value of a function at the CR nodes $\bar{\Omega}_h^{CR}$.*

The idea of the proof is the same as for the conforming case; see Dryja, Smith, Widlund [35]. All the steps in the proof are the same except that we use Lemma 5.10 for nonconforming $P_1$ functions.

Let $\tilde{V}_0^h(\Omega_{\mathcal{F}_{ij}})$ be the subspace of $\tilde{V}^h(\Omega_{\mathcal{F}_{ij}})$ of functions which vanish at the barycenter nodes on $\partial\Omega_{\mathcal{F}_{ij}}^{CR}$.

Let $\mathcal{T}^{\hat{h}}$ be the triangulation obtained by subdividing each tetrahedral element of $\mathcal{T}^h$ into twelve subtetrahedra. Let $V_0^{\hat{h}}(\Omega_{\mathcal{F}_{ij}})$ be the conforming space of piecewise linear functions ( with respect to the triangulation $\mathcal{T}^{\hat{h}}$) which vanish on $\partial\Omega_{\mathcal{F}_{ij}}$.

We introduce an local interpolator $\mathcal{M}_{ij}^{\hat{h}} : \tilde{V}_0^h(\Omega_{\mathcal{F}_{ij}}) \to V_0^{\hat{h}}(\Omega_{\mathcal{F}_{ij}})$, as follows:

**Definition 5.8** *Given $u \in \tilde{V}_0^h(\Omega_{\mathcal{F}_{ij}})$, define a conforming function $u_c = \mathcal{M}_{ij}^{\hat{h}} u$ by the values of $u_c$ at the following sets of points:*

*i) If $P \in \bar{\Omega}_{i,\hat{h}} \backslash \partial\Omega_{i,\hat{h}}$ or $P \in \bar{\Omega}_{j,\hat{h}} \backslash \partial\Omega_{j,\hat{h}}$, then*

$$u_c(P) = \text{mean of} \quad u|_{\tau_k^{\hat{h}}}(P).$$

*Here, ${\tau_k}^{\hat{h}}$ are the elements in $\mathcal{T}^{\hat{h}}(\bar{\Omega}_i)$ (or $\mathcal{T}^{\hat{h}}(\bar{\Omega}_j)$) that have $P$ as a vertex, and $u|_{\tau_k^{\hat{h}}}(P)$ is the limit value of $u(x)$ when $x \in \tau_k^{\hat{h}}$ approaches $P$.*

*ii) If $P \in \partial\Omega_{\mathcal{F}_{ij},\hat{h}}$, then*

$$u_c(P) = 0.$$

*iii) If $P \in \mathcal{F}_{ij,h}^{CR}$, then*

$$u_c(P) = u(P).$$

*iv) If $P \in \mathcal{F}_{ij,h} \backslash \partial\mathcal{F}_{ij,h}$, then*

$$u_c(P) = \sum_{k=1}^{N_P} \frac{|T_k|}{|\cup_{j=1}^{N_P} T_j|} u(C_i).$$

*Here, the $T_j$, $j = 1, \cdots, N_P$, are the triangles of $\mathcal{T}^h \cap \mathcal{F}_{ij}$ that have $P$ as a vertex.*

The interesting feature about this interpolator $\mathcal{M}_{ij}^{\hat{h}}$ is that we have approximation properties simultaneously in the two domains $\Omega_i$ and $\Omega_j$. Using the same idea as in Lemma 5.2, we can show:

**Lemma 5.12** *Given $u \in \tilde{V}_0^h(\Omega_{\mathcal{F}_{ij}})$, let $u_c \in V_0^{\hat{h}}(\Omega_{\mathcal{F}_{ij}})$ be given by $u_c = \mathcal{M}_{ij}^{\hat{h}} u$. Then, for $k = i, j$, simultaneously, we have*

$$|u_c|_{H^1(\Omega_k)} \asymp |u|_{H_h^1(\Omega_k)} \tag{5.65}$$

$$\|u_c^{(ij)}\|_{L^2(\Omega_k)} \asymp \|u\|_{L^2(\Omega_k)}. \tag{5.66}$$

We define an pseudo inverse map ${\mathcal{M}_{ij}^{\hat{h}}}^{\dagger} : V_0^{\hat{h}}(\Omega_{\mathcal{F}_{ij}}) \to \tilde{V}_0^h(\Omega_{\mathcal{F}_{ij}})$, by

$$({\mathcal{M}_{ij}^{\hat{h}}}^{\dagger})v(P) = v(P).$$

Here, $P$ is any CR node of $\mathcal{T}^h(\bar{\Omega}_{\mathcal{F}_{ij}})$.

By using the fact that the nodal points associated with the space $\tilde{V}_0^h(\Omega_{\mathcal{F}_{ij}})$ are also nodal points associated with the space $V_0^{\hat{h}}(\Omega_{\mathcal{F}_{ij}})$, we obtain, for $k = i, j$,

$$|{\mathcal{M}_{ij}^{\hat{h}}}^{\dagger} v|_{H_h^1(\Omega_k)} \preceq |v|_{H^1(\Omega_k)}, \tag{5.67}$$

and

$$\|{\mathcal{M}_{ij}^{\hat{h}}}^{\dagger} v\|_{L^2(\Omega_k)} \preceq \|v\|_{L^2(\Omega_k)}. \tag{5.68}$$

The next lemma is of fundamental importance in the analysis of our algorithms.

**Lemma 5.13** *Let $u \in \tilde{V}_0^h(\Omega_{\mathcal{F}_{ij}})$ be a nonconforming piecewise discrete harmonic function in $\Omega_{ij}$. Then*

$$|u|_{H_h^1(\Omega_i)} \asymp |u|_{H_h^1(\Omega_j)}.$$

It is easy to prove this lemma when the substructure are simplices. We reflect $u$ from $\Omega_i$ into $\Omega_j$, interpolate into $\tilde{V}_0^h(\Omega_j)$, and perform comparisons of norms.

For general substructures, we proceed as follows. Note that

$$|u|_{H_h^1(\Omega_j)} \leq |{\mathcal{M}_{ij}^{\hat{h}}}^{\dagger} \mathcal{H}_{\hat{h}}^{(j)}(\mathcal{M}_{ij}^{\hat{h}} u)_{|\partial\Omega_j}|_{H_h^1(\Omega_j)} \preceq |\mathcal{H}_{\hat{h}}^{(j)}(\mathcal{M}_{ij}^{\hat{h}} u)_{|\partial\Omega_j}|_{H^1(\Omega_j)}$$

$$\preceq \|(\mathcal{M}_{ij}^{\hat{h}} u)_{\mathcal{F}_{ij}}\|_{H_{00}^{1/2}(\mathcal{F}_{ij})} \preceq \|\mathcal{H}_{\hat{h}}^{(i)}(\mathcal{M}_{ij}^{\hat{h}} u)_{|\partial\Omega_i}\|_{H^1(\Omega_i)}.$$

Here, $\mathcal{H}_{\hat{h}}^{(i)}$ is the discrete harmonic extension operator on the mesh $\hat{h}$ in $\Omega_i$. Note that $u$ and ${\mathcal{M}_{ij}^{\hat{h}}}^{\dagger}\mathcal{H}_{\hat{h}}^{(j)}(\mathcal{M}_{ij}^{\hat{h}}u)_{|\partial\Omega_j}$ have, by construction, the same boundary values on $\partial\Omega_{j,h}^{CR}$. Thus, the first inequality follows from the fact that the nonconforming discrete harmonic function has the minimal energy. The second inequality follows from (5.67). In the third and the fourth inequality, we use properties of the $H_{00}^{1/2}$-norm and the fact that $(\mathcal{M}_{ij}^{\hat{h}}u)$ vanishes on $\partial\Omega_{\mathcal{F}_{ij}}$. For the fourth inequality, we use an extension theorem for piecewise linear, discrete harmonic function.

Hence,

$$\|\mathcal{H}_{\hat{h}}^{(i)}(\mathcal{M}_{ij}^{\hat{h}}u)_{|\partial\Omega_i}\|_{H^1(\Omega_i)} \preceq |\mathcal{H}_{\hat{h}}^{(i)}(\mathcal{M}_{ij}^{\hat{h}}u)_{|\partial\Omega_i}|_{H^1(\Omega_i)}$$

$$\leq |\mathcal{M}_{ij}^{\hat{h}}u|_{H^1(\Omega_i)} \preceq |u|_{H_h^1(\Omega_i)}$$

by using Poincaré's inequality, the minimal energy for conforming harmonic function, and finally, Lemma 5.12.

The proof is completed, by using the previous arguments, to show the reverse inequality

$$|u|_{H_h^1(\Omega_i)} \preceq |u|_{H_h^1(\Omega_j)}.$$

$\square$

*Proof of Theorem* 5.6.
We use Theorem 2.1; see also Theorem 4.1. The proof is given for $\tilde{T}^X = \tilde{T}^{NN}$. Similar arguments can also be used to prove the other cases.

*Assumption (i).* Let

$$u = \mathcal{H}_{CR}u + v \quad \text{in} \quad \Omega,$$

where

$$\mathcal{H}_{CR}u = \mathcal{H}_{CR}^{(i)}u \quad \text{and} \quad v = \mathcal{P}_{CR}^{(i)}u \quad \text{on } \bar{\Omega}_i.$$

Here, $\mathcal{H}_{CR}^{(i)}u \in \tilde{V}^h(\bar{\Omega}_i)$ is the nonconforming discrete harmonic part of $u$ in $\Omega_i$. The remaining part $\mathcal{P}_{CR}^{(i)}u$ is $H_h^1$-projection into $\tilde{V}_0^h(\Omega_i)$.

We decompose $v^{(i)} = \mathcal{P}_{CR}^{(i)}u$ on $\Omega_i$ as

$$v^{(i)} = \bar{v}^{(i)} + (v^{(i)} - \bar{v}^{(i)}) = \bar{v}^{(i)} + \tilde{v}^{(i)}.$$

Here, $\bar{v}^{(i)} = \mathcal{M}_i^h v^{(i)}$ belongs to the space of conforming functions $V_0^h(\Omega_i)$. Therefore, we can decompose $\bar{v}^{(i)}$ further, as in (4.13), and obtain

$$\bar{v}^{(i)} = \bar{v}_0^{(i)} + \bar{v}_1^{(i)} + \cdots + \bar{v}_k^{(i)} + \cdots + \bar{v}_{\ell-1}^{(i)} + \bar{v}_\ell^{(i)}.$$

As in the conforming case, we work with the bilinear forms $b_k(\cdot,\cdot)$ defined by (4.10). We use the same arguments as in the proof of Lemma 4.1, and (5.60), to obtain

$$\sum_{k=0}^{\ell} b_k(\bar{v}_k^{(i)}, \bar{v}_k^{(i)}) \preceq a_{\Omega_i}(\bar{v}^{(i)}, \bar{v}^{(i)}) \preceq a_{\Omega_i}^h(v^{(i)}, v^{(i)}) \leq a_{\Omega_i}^h(u, u).$$

We use that $\mathcal{P}_{CR}^{(i)}$ is a projection to obtain the last inequality.

For the $\tilde{v}^{(i)}$, we use (5.62) to obtain

$$\sum_{j \in \bar{\Omega}_{i,h}^{CR}} a^h(\tilde{v}^{(i)}(x_j)\tilde{\phi}_j^h, \tilde{v}^{(i)}(x_j)\tilde{\phi}_j^h) \asymp$$

$$\frac{\rho_i}{h^2} \|v^{(i)} - \hat{v}^{(i)}\|_{L^2(\Omega_i)}^2 \preceq a_{\Omega_i}^h(v^{(i)}, v^{(i)}) \preceq a_{\Omega_i}^h(u, u).$$

We now decompose $\mathcal{H}_{CR} u$. Let

$$w = \mathcal{H}_{CR} u - u_{-1},$$

where $u_{-1}$ is defined in (5.57). We decompose $w$ as

$$w = \sum_{i=1}^{N} w^{(i)}.$$

Here,

$$w^{(i)} = u\, \rho_i^\beta \mu_{i,\beta}^+ - \bar{u}_{\partial\Omega_i^{CR}}^h\, \rho_i^\beta \mu_{i,\beta}^+ = \rho_i^\beta\,(u - \bar{u}_{\partial\Omega_i^{CR}}^h)\mu_{i,\beta}^+ \text{ on } \Gamma_h^{CR} \cup \partial\Omega_h^{CR}$$

and nonconforming $P_1$ discrete harmonic function into the interior of the $\{\Omega_j\}_{j=1}^N$. It is easy to see that the support of $w^{(i)}$ is a union of $\bar{\Omega}_i$ and the $\bar{\Omega}_j$ which have a face in common with $\bar{\Omega}_i$. We further decompose $w^{(i)}$ as

$$w^{(i)} = \sum_{\mathcal{F}_{ij} \subset \partial\Omega_i} w_{\mathcal{F}_{ij}}^{(i)}. \tag{5.69}$$

Here, $w^{(i)}_{\mathcal{F}_{ij}}$ is the nonconforming $P_1$ discrete harmonic function on $\Omega$, with possible nonzero value, on $\Gamma_h^{CR} \cup \partial\Omega_h^{CR}$, equal to $w^{(i)}$ only on $\mathcal{F}^{CR}_{ij,h}$. Note that the support of each $w^{(i)}_{\mathcal{F}_{ij}}$ is $\bar{\Omega}_{\mathcal{F}_{ij}}$. Therefore, we can decompose $w^{(i)}_{\mathcal{F}_{ij}}$ as

$$w^{(i)}_{\mathcal{F}_{ij}} = \bar{w}^{(i)}_{\mathcal{F}_{ij}} + (w^{(i)}_{\mathcal{F}_{ij}} - \bar{w}^{(i)}_{\mathcal{F}_{ij}}) = \bar{w}^{(i)}_{\mathcal{F}_{ij}} + \tilde{w}^{(i)}_{\mathcal{F}_{ij}}.$$

Here, $\bar{w}^{(i)}_{\mathcal{F}_{ij}} = \mathcal{M}^h_{ij} w^{(i)}_{\mathcal{F}_{ij}}$, where $\mathcal{M}^h_{ij} : \tilde{V}^h_0(\Omega_{\mathcal{F}_{ij}}) \to V^h_0(\Omega_{\mathcal{F}_{ij}})$, is the local interpolator defined as in Definition 5.7 by changing $\Omega_i$ and $\partial\Omega_i$, to $\Omega_{\mathcal{F}_{ij}}$ and $\partial\Omega_{\mathcal{F}_{ij}}$, respectively. The stability properties of $\mathcal{M}^h_{ij}$ that we use are

$$|\bar{w}^{(i)}_{\mathcal{F}_{ij}}|^2_{H^1(\Omega_{\mathcal{F}_{ij}})} \preceq |w^{(i)}_{\mathcal{F}_{ij}}|^2_{H^1_h(\Omega_{\mathcal{F}_{ij}})} \tag{5.70}$$

and

$$\|w^{(i)}_{\mathcal{F}_{ij}} - \bar{w}^{(i)}_{\mathcal{F}_{ij}})\|^2_{L^2(\Omega_{\mathcal{F}_{ij}})} \preceq h^2\, |w^{(i)}_{\mathcal{F}_{ij}}|^2_{H^1_h(\Omega_{\mathcal{F}_{ij}})}. \tag{5.71}$$

We note that we cannot work with the operator $\mathcal{M}^{\hat{h}}_{ij}$ because we need to interpolate into $V^h(\Omega_{\mathcal{F}_{ij}})$, not $V^{\hat{h}}(\Omega_{\mathcal{F}_{ij}})$.

We decompose $\bar{w}^{(i)}_{\mathcal{F}_{ij}}$ as in (4.15) and obtain

$$\bar{w}^{(i)}_{\mathcal{F}_{ij}} = \bar{w}^{(i)}_{0,\mathcal{F}_{ij}} + \cdots + \bar{w}^{(i)}_{k,\mathcal{F}_{ij}} + \cdots + \bar{w}^{(i)}_{\ell,\mathcal{F}_{ij}}.$$

By using the same arguments as in the proof of Lemma 4.2, we obtain

$$\sum_{k=0}^{\ell} b_k(\bar{w}^{(i)}_{k,\mathcal{F}_{ij}}, \bar{w}^{(i)}_{k,\mathcal{F}_{ij}}) \preceq (\rho_i + \rho_j)|\bar{w}^{(i)}_{\mathcal{F}_{ij}}|^2_{H^1(\Omega_{\mathcal{F}_{ij}})}.$$

By using (5.70), Lemma 5.13, the fact that $\beta \geq 1/2$, the fact that the nonconforming discrete harmonic function has the minimal energy seminorm $H^1_h(\Omega_i)$, Lemma 5.11, and a Poincaré inequality for nonconforming $P_1$ functions (see Lemma 5.4)we obtain

$$(\rho_i + \rho_j)|\bar{w}^{(i)}_{\mathcal{F}_{ij}}|^2_{H^1(\Omega_{\mathcal{F}_{ij}})} \preceq (\rho_i + \rho_j)|w^{(i)}_{\mathcal{F}_{ij}}|^2_{H^1_h(\Omega_{\mathcal{F}_{ij}})} \preceq$$

$$(\rho_i + \rho_j)|w^{(i)}_{\mathcal{F}_{ij}}|^2_{H^1_h(\Omega_i)} = (\rho_i + \rho_j)|\mathcal{H}^{(i)}_{CR}(\rho_i^{\beta}(u - \bar{u}^h_{\partial\Omega_i^{CR}})\mu^{+}_{i,\beta})_{\mathcal{F}^{CR}_{ij,h}}|^2_{H^1_h(\Omega_i)}$$

$$\preceq \rho_i|\mathcal{H}^{(i)}_{CR}(u - \bar{u}^h_{\partial\Omega_i^{CR}})_{\mathcal{F}^{CR}_{ij,h}}|^2_{H^1_h(\Omega_i)} \preceq \rho_i|I^h_{CR}(\vartheta^{CR}_{\mathcal{F}_{ij}}(u - \bar{u}^h_{\partial\Omega_i^{CR}}))|^2_{H^1_h(\Omega_i)}$$

$$\preceq \rho_i(1 + \log H/h)^2\|u - \bar{u}^h_{\partial\Omega_i^{CR}}\|^2_{H^1_h(\Omega_i)} \preceq \rho_i(1 + \log H/h)^2|u|^2_{H^1_h(\Omega_i)}.$$

Here, $(v)_{\mathcal{F}_{ij,h}^{CR}}$ is defined by $(v)_{\mathcal{F}_{ij,h}^{CR}} = v$ at the CR nodes $\mathcal{F}_{ij,h}^{CR}$ and $(v)_{\mathcal{F}_{ij,h}^{CR}} = 0$ on $\partial\Omega_{i,h}^{CR} \backslash \mathcal{F}_{ij,h}^{CR}$. For the nonconforming part $\tilde{w}_{\mathcal{F}_{ij}}^{(i)}$, we use (5.71) to obtain

$$\sum_{j \in \mathcal{N}_h^{CR}(\Omega_{ij})} a^h(\tilde{w}_{\mathcal{F}_{ij}}^{(i)}(x_j)\tilde{\phi}_j^h, \tilde{w}_{\mathcal{F}_{ij}}^{(i)}(x_j)\tilde{\phi}_j^h) \asymp$$

$$(\rho_i + \rho_j)\frac{1}{h^2}\|w_{\mathcal{F}_{ij}}^{(i)} - \bar{w}_{\mathcal{F}_{ij}}^{(i)})\|_{L^2(\Omega_{\mathcal{F}_{ij}})}^2 \preceq$$

$$(\rho_i + \rho_j)\,|w_{\mathcal{F}_{ij}}^{(i)}|_{H_h^1(\Omega_{\mathcal{F}_{ij}})} \preceq \rho_i(1 + \log H/h)^2 |u|_{H_h^1(\Omega_i)}^2.$$

*Assumption (ii).* We have trivially $\omega = 1$.

*Assumption (iii).* We have $\rho(\mathcal{E}) \preceq 1$ by applying Remark 3.3 in Zhang [95] for the multilevel conforming part, and using that we have a finite covering for the nonconforming part.

□

### 5.3.4 Multiplicative versions

Let $X = F, \tilde{F}, NN$, or $\widetilde{NN}$. We consider two versions:

$$E_G = (\prod_{j \in \mathcal{N}_h^{CR}} (I - \tilde{P}_j^h))(\prod_{k=0}^{\ell} \prod_{j \in \mathcal{N}^k} (I - \tilde{P}_j^k))(I - \eta \tilde{T}_{-1}^X), \tag{5.72}$$

and

$$E_J = (I - \eta \sum_{j \in \mathcal{N}_h^{CR}} \tilde{P}_j^h)(\prod_{k=0}^{\ell}(I - \eta \sum_{j \in \mathcal{N}^k} \tilde{P}_j^k))(I - \eta \tilde{T}_{-1}^X), \tag{5.73}$$

where $\eta$ is a damping factor such that

$$\|\eta \tilde{T}_{-1}^X\|_{H_{\rho,h}^1}, \ \|\eta \sum_{j \in \mathcal{N}_h^{CR}} \tilde{P}_j^h\|_{H_{\rho,h}^1}, \|\eta \sum_{j \in \mathcal{N}^k} \tilde{P}_j^k\|_{H_{\rho,h}^1} \leq w < 2.$$

When the product is arranged in an appropriated order, the operators $E_G$ and $E_J$ correspond to the error propagation operator of V-cycle multigrid methods using Gauss Seidel and damped Jacobi smoothers, respectively. Using the same techniques as in Zhang [95], we can show that the norm of the error propagation operators $\|E_G\|_{H_{\rho,h}^1}$ and $\|E_J\|_{H_{\rho,h}^1}$ can be estimated from above by $1 - C_2(1 + \ell)^{-2}$.

**Remark 5.6** *In the case of quasi-monotone coefficients (cf. Chapter 4), we can replace our nonstandard coarse spaces by the piecewise linear continuous function space* $V_0^H(\Omega)$*, and show that the algorithms in this chapter are optimal.*

**Remark 5.7** *We also consider algorithms that are straightforward variants of the algorithms developed in Section 4.8, 4.9, and 4.10; we replace the exotic spaces based on conforming* $P_1$ *functions by our nonstandard nonconforming coarse spaces, and add the one-dimensional spaces spanned by the* $\tilde{\phi}_j^h$*. We can then establish the same condition number estimates as obtained in Chapter 4. The techniques needed for these proofs have already been described in this thesis.*

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