MULTILEVEL SCHWARZ METHODS WITH PARTIAL REFINEMENT

HSUANJEN CHENG *

Abstract. We consider multilevel additive Schwarz methods with partial refinement. These algorithms are generalizations of the multilevel additive Schwarz methods developed by Dryja and Widlund and many others. We will give two different proofs by using quasi-interpolants under two different assumptions on selected refinement subregions to show that this class of methods has an optimal condition number. The first proof is based purely on the localization property of quasi-interpolants. However, the second proof use some results on iterative refinement methods. As a by-product, the multiplicative versions which corresponds to the FAC algorithms with inexact solvers consisting of one Gauss-Seidel or damped Jacobi iteration have optimal rates of convergence. Finally, some numerical results are presented for these methods.

Key words. Schwarz Methods, Preconditioned Conjugate Gradient Method, Elliptic Regularity, Iterative Refinement Methods, Finite Elements.

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1. Introduction. In this paper, we consider some solution methods of the large linear systems of algebraic equations which arise when working with elliptic finite element approximations on composite meshes. We consider the following linear, self-adjoint, elliptic problems discretized by finite element methods on a bounded Lipschitz polyhedral region Ω in R^n.

$$\begin{cases} -\sum_i \sum_j \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial u}{\partial x_j} = f & \text{in } \Omega, \\ u = u_0 \quad \text{on } \Gamma_D \subset \partial\Omega, \\ \sum_j \sum_i a_{ij}(x) \frac{\partial u}{\partial x_j} n_i = g & \text{on } \Gamma_N = \partial\Omega \backslash \Gamma_D. \end{cases} \tag{1}$$

Here the matrix $\{a_{ij}(x)\}$ is positive definite with a positive uniform lower bound c for almost all x in Ω. Each $a_{ij}(x)$ is a bounded measurable function in Ω and $\vec{n}$ is the unit outward normal to $\partial\Omega$. We assume that the measure of Γ_D is strictly greater than zero. This insures a unique solution to problem (1).

We will assume, without loss of generality, that $u_0 = 0$. If not, we can always substract an arbitrary function w that equals u_0 on Γ_D from u. Let

$$V = H_D^1(\Omega) = \{u \in H^1(\Omega) | \gamma u = 0 \quad \text{on } \Gamma_D\}.$$

Here γ is the trace operator. The standard continuous and discrete weak formulations for the above elliptic problem (1) then consist of

$$a(u,v) = f(v), \qquad \forall v \in V, \tag{2}$$

* Courant Institute of Mathematical Science, New York University, 251 Mercer Street, New York, N.Y. 10012. The work of author, while a Courant Institute doctoral student, was supported in part by the National Science Foundation under grant NSF-CCR-9204255 and in part by the U.S. Department of Energy under contract DE-FG02-92ER25127.

and

$$(3) \qquad a(u_h, v_h) = f(v_h), \qquad \forall v_h \in V^h,$$

respectively. Here

$$a(u,v) = \int_\Omega \sum_{i,j} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx \qquad \text{and} \qquad f(v) = \int_\Omega f v dx + \int_{\Gamma_N} g v ds.$$

The space V^h will be defined in the next few paragraphs. It is easy to see that the norm $(a(u,u))^{1/2}$ is equivalent to the seminorm $|u|_{H^1(\Omega)}$ in $H^1(\Omega)$.

To simplify the presentation, we use continuous, Lagrange finite element of type 1 and only consider homogeneous Dirichlet boundary value problems. Then we will remark how to proceed our analysis to more general mixed type boundary condition and more general Lagrange elements.

The space V^h is defined on a composite triangulation, which is possibly the result of a large number of successive refinements. The triangulation of Ω is given in the following way.

We first introduce a relatively coarse triangulation of Ω, also denoted by Ω_1, and denote the corresponding space of finite element functions by V^{h_1}. We can think of this space as having a relatively uniform mesh size h_1. Let Ω_2 be a subregion where we wish to increase the resolution. We do so by subdividing the elements and introducing an additional finite element space V^{h_2}. We assure that the resulting composite space $V^{h_1} + V^{h_2}$ is conforming by having the functions of V^{h_2} vanish on $\partial\Omega_2$. We repeat this process by selecting a subregion Ω_3 of Ω_2 and introducing a further refinement of the mesh and finite element space, etc.. We denote the resulting nested subregions and subspaces by Ω_i and V^{h_i} respectively. Throughout, we have $\Omega_i \subset \Omega_{i-1}$ and $V^{h_{i-1}} \cap H_0^1(\Omega_i) \subset V^{h_i} \subset H_0^1(\Omega_i)$, $i = 2, \cdots, k$. The composite finite element space on the repeatedly refined mesh, is

$$V^h = V^{h_1} + V^{h_2} + \cdots + V^{h_k}.$$

We assume that all the elements are shape regular in the sense that there is a uniform bound on h_K/ρ_K. Here h_K and ρ_K are the diameter and the radius of the largest inscribed sphere of any element K, respectively. Our theoretical bounds, developed in this chapter, also depend on the shapes of the subregions Ω_i.

The finite element problem is defined by equation (3) and the corresponding stiffness matrix can conveniently be computed by using a process of subassembly. Introducing subscripts to indicate the domain of integration, we write

$$a(u,v) = a_{\Omega_1 \setminus \Omega_2}(u,v) + a_{\Omega_2 \setminus \Omega_3}(u,v) + \cdots + a_{\Omega_k}(u,v).$$

The stiffness matrices corresponding to the regions $\Omega_i \setminus \Omega_{i+1}, i \leq k-1$, and Ω_k are computed by working with basis functions related to the mesh size h_i. The quadratic form corresponding to the composite stiffness matrix is the sum of the quadratic forms corresponding to $\Omega_i \setminus \Omega_{i+1}, i \leq k-1$, and Ω_k. When we refine a finite element model locally, the modified stiffness matrix is obtained by replacing the quadratic form associated with the subregion in question by the one corresponding to the refined model on the same subregion. It is therefore relatively easy to design a method which systematically generates the stiffness matrices for all the standard problems necessary while, at the same time, the stiffness matrix of the composite model is computed.

We use the framework of multilevel additive Schwarz methods, which is described in Dryja and Widlund [7], and Zhang [18] to develop a new kind of algorithm for composite finite element problems. If we compare this kind of algorithm with the AFAC methods in [4], we can see that they are both additive Schwarz methods. In the new algorithms, we decompose the problems corresponding to the refined subregions with uniform mesh size, used in AFAC, into many much smaller problems which are much easier to solve. However, this is at the expense of slower convergence of the algorithms.

We will apply quasi-interpolants and some theoretical results from FAC and AFAC methods to prove that the iteration operators of these methods have a uniform lower bound under quite general assumptions. We remark that our proof can be applied to the case of refinement everywhere and is different from Zhang's which was obtained by considering a decomposition based on the Galerkin projection on a larger convex domain. Bornemann and Yserentant [1] have also obtained another proof based on the use of K-functionals.

We can also consider some multiplicative versions of above methods. These variants corresponding to the FAC algorithms with inexact solvers consisting of one Gauss-Seidel or damped Jacobi iteration. We can use the similar arguments of Zhang to show that these variants have an optimal rate of convergence.

In Section 2, we describe general multilevel additive Schwarz methods and mention some theoretical results about them

In Section 3, we describe general multilevel additive Schwarz methods with partial refinement and give the first proof of optimality purely based on quasi-interpolants under some restricted conditions.

In Section 4, we develop the second optimality proof based on some assumptions coming from iterative refinement methods and describe some multiplicative variants..

In Section 5, we present some numerical experiments to verify our theoretical results.

2. General Multilevel Additive Schwarz Methods.

2.1. Basic Two-Level Methods. In 1987, Dryja and Widlund developed the framework of additive Schwarz methods which makes it possible to solve all the subproblems in parallel. Following Dryja and Widlund [5], [6], let us define two levels of triangulations. We start with a coarse triangulation $\mathcal{T}^H = \{\Omega_i\}_{i=1}^N$. Each Ω_i is then further divided into smaller elements to obtain a fine triangulation $\mathcal{T}^h$. We assume that the Ω_i in the triangulation $\mathcal{T}^H$ have a quasi-uniform mesh size H and that the elements in the triangulation $\mathcal{T}^h$ a quasi-uniform mesh size h.

To get an overlapping decomposition of Ω, we extend each Ω_i to a larger region $\tilde{\Omega}_i$, such that $cH \le \mathrm{dist}\{\partial\Omega_i, \partial\tilde{\Omega}_i\} \le CH$, and the $\partial\tilde{\Omega}_i$ align with boundaries of elements in $\mathcal{T}^h$. We cut off the part of each $\tilde{\Omega}_i$ that is outside of Ω.

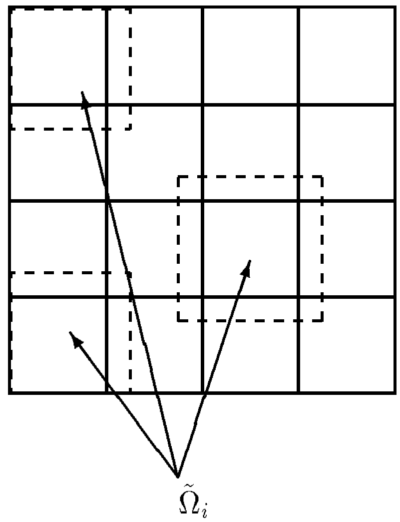

FIG. 1. *Two levels of triangulation*

It is easy to see that $\{\tilde{\Omega}_i\}$ forms a finite covering of domain Ω. We see that we can color $\{\tilde{\Omega}_i\}$, using at most N_c colors in such a way that the subdomains of the same color are disjoint. Here N_c is a constant which depends only on the shape regularity of $\mathcal{T}^H$. Due to the generous overlap of $\{\tilde{\Omega}_i\}$, we have a partition of unity $\{\theta_i\}$ satisfying

$$\sum_{i=1}^{N} \theta_i = 1 \text{ with } \theta_i \in W^{1,\infty}(\tilde{\Omega}_i), 0 \le \theta_i \le 1, \text{ and } |\theta_i|_{W^{1,\infty}} \le C/H.$$

Let V^h and V^H be the Lagrange finite elements of type 1 associated with the triangulations $\mathcal{T}^h$ and $\mathcal{T}^H$, respectively. Let $V_0 = V^H$ and $V_i = V^h(\tilde{\Omega}_i) = V^h \cap H_0^1(\tilde{\Omega}_i)$. We obtain a decomposition of the finite element space V^h

$$V^h = \sum_{i=0}^{N} V_i.$$

Let $P_{V_i} : V^h \to V_i$, be the Galerkin projection defined by

$$a(P_{V_i}u, v) = a(u, v), \qquad \forall v \in V_i.$$

The matrix form of P_{V_i}, after a permutation, is

$$P_{V_i} = \begin{pmatrix} K_i^{-1} & 0 \\ 0 & 0 \end{pmatrix} K = \begin{pmatrix} K_i & 0 \\ 0 & 0 \end{pmatrix}^{\dagger} K,$$

where K is the stiffness matrix associated with the domain Ω and K_i is the stiffness matrix associated with the Dirichlet problem on the subdomain $\tilde{\Omega}_i$. The additive

Schwarz operator, P is given by

$$P = \sum_{i=0}^{N} P_{V_i}.$$

The additive Schwarz algorithm for equation (3) is to solve an auxilary linear system which is equivalent to (3).

ALGORITHM 1. *Apply conjugate gradient method to the symmetric, positive definite system*

$$Pu_h = g_h, \tag{4}$$

with respect to the inner product $a(\cdot,\cdot)$ for an appropriate g_h such that the solution u_h is the same as that of (3).

We remark that we can compute $g = Pu_h = \sum_i g_i = \sum_i P_{V_i} u_h$ without knowing the solution of (3) by solving

$$a(g_i, v) = a(u, v) = f(v), \qquad \forall v \in V_i.$$

In this algorithm, we only need to compute Pv_h for a given $v_h \in V^h$ in each iteration, so the explicit representation of P is not needed. For the rate of convergence, we refer to the following theorem.

THEOREM 1 (DRYJA AND WIDLUND). *For the operator P defined above, there exists a constant C_0 such that*

$$C_0 a(u,u) \le a(Pu,u) \le (N_c+1)a(u,u), \qquad \forall u \in V^h.$$

Thus $\kappa(P) \le (N_c+1)/C_0$, and the rate of convergence of Algorithm 4.2.1 is independent of h and H.

The proof can be found in Dryja and Widlund [6]. We remark that if we do not use the coarse space, the condition number of the operator will grow like H^{-2}. This fact was pointed out and proved by Widlund in [15].

2.2. General Multilevel Additive Schwarz Methods. In basic two-level methods, we need to solve a coarse problem of size $O(1/H^2)$ and some local problems of size $(H/h)^2$. If h is small, we cannot both have $1/H$ and H/h small. Thus at least one of the subproblems is large. The computation can be made cheaper by recursively using the additive Schwarz method to solve the coarser problems.

We now give a description of general multilevel additive Schwarz methods which is developed in Dryja and Widlund [7]. We define a sequence of nested triangulations $\{\mathcal{T}^k_{l=1}\}$. We start with a coarse triangulation $\mathcal{T}^1 = \{\tau_i^1\}_{i=1}^{N_1}$ with quasi-uniform mesh size h_1, where τ_i^1 represent an individual triangle. The successively finer triangulations $\mathcal{T}^l = \{\tau_i^l\}(l = 2,\cdots,L)$ are defined by dividing each triangle in the triangulation $\mathcal{T}^{l-1}$ into several triangles, i.e.

$$\mathcal{T}^1 = \{\tau_i^1\}_1^{N_1} \overset{\text{refinement}}{\Longrightarrow} \mathcal{T}^2 = \{\tau_i^1\}_1^{N_2} \overset{\text{refinement}}{\Longrightarrow} \cdots \overset{\text{refinement}}{\Longrightarrow} \mathcal{T}^k = \{\tau_i^k\}_1^{N_k}.$$

We assume that the triangulations $\mathcal{T}^l$ have quasi-uniform mesh size h_{l-1} for each l.

Let $V^{h_l}, l = 1, \cdots, k$, be the space of continuous piecewise linear element associated with the triangulation $\mathcal{T}^l$. The finite element solution, $u_h \in V^h = V^{h_k}$, satisfies

$$a(u_h, \phi_h) = f(\phi_h), \quad \forall \phi_h \in V^h = V^{h_k}. \tag{5}$$

We assume that there are $k-1$ sets of overlapping subdomains $\{\tilde{\Omega}_i^l\}_{i=1}^{N_l}, l = 2, 3, \cdots, k$. On each level, we have an overlapping decomposition

$$\Omega = \cup_{i=1}^{N_l} \tilde{\Omega}_i^l.$$

We assume that the sets $\{\tilde{\Omega}_i^l\}$ satisfy the following assumption.

Assumption 1. *The decomposition $\Omega = \cup_{i=1}^{N_l} \tilde{\Omega}_i^l$ satisfies*

- *$\partial\tilde{\Omega}_i^l$ aligns with the boundaries of level l triangles, i.e. $\tilde{\Omega}_i^l$ is the union of level l triangles. Diameter$(\tilde{\Omega}_i^l) = O(h_{l-1})$.*
- *On each level, the subdomains $\{\tilde{\Omega}_i^l\}_{i=1}^{N_l}$ form a finite covering of Ω, with a covering constant N_c, i.e. we can color $\{\tilde{\Omega}_i^l\}_{i=1}^{N_l}$, using at most N_c colors in such a way that subdomains of the same color are disjoint.*
- *On each level, associated with $\{\tilde{\Omega}_i^l\}_{i=1}^{N_l}$, there exists a partition of unity $\{\theta_i^l\}$ satisfying*

$$\sum_i \theta_i^l = 1, \text{ with } \theta_i^l \in H_0^1(\tilde{\Omega}_i^l) \cap C^0(\tilde{\Omega}_i^l), 0 \le \theta_i^l \le 1 \text{ and } |\nabla\theta_i^l| \le C/h_{l-1}.$$

- *h_l/h_{l+1} is uniformly bounded.*

One way of constructing subdomains $\{\tilde{\Omega}_i^l\}_{i=1}^{N_l}, l = 2, \cdots, k$, with the above properties is described in Dryja and Widlund [5], [6]. Each triangle $\tau_i^{l-1}, i = 1, \cdots, N_l, l = 2, \cdots, k$, is extended to a larger region $\tilde{\tau}_i^{l-1}$ so that $ch_{l-1} \le \text{dist}(\partial\tilde{\tau}_i^{l-1}, \partial\tau_i^{l-1}) \le Ch_{l-1}$, aligning $\partial\tilde{\tau}_i^{l-1}$ with the boundaries of level l triangles. We cut off the part of $\tilde{\tau}_i^{l-1}$ that is outside Ω. We use $\tilde{\tau}_i^{l-1}$ as the subdomains $\tilde{\Omega}_i^l$. Another way of constructing $\{\tilde{\Omega}_i^l\}$ is given in the next section.

Let $N_1 = 1, V_1^{h_1} = V^{h_1}$ and $V_i^{h_l} = V^{h_l} \cap H_0^1(\tilde{\Omega}_i^l)$ for $i = 1, \cdots, N_l, l = 2, \cdots, k$. The finite element space $V^h = V^{h_k}$ is represented by

$$V^h = \sum_{l=1}^{k} V^{h_l} = \sum_{l=1}^{k}\sum_{i=1}^{N_l} V_i^{h_l}.$$

Let $P_i^l : V^h \to V_i^{h_l}$, be the projection defined by

$$a(P_i^l u, \phi) = a(u, \phi), \quad \forall \phi \in V_i^{h_l}.$$

The k-level additive Schwarz operator P is defined by

$$P = \sum_{l=1}^{k}\sum_{i=1}^{N_l} P_i^l. \tag{6}$$

Instead of solving the original finite element equation (5), we use the following algorithm.

ALGORITHM 2. *Let P be the operator defined by (6). Apply the conjugate gradient method to the following symmetric and positive definite system*

$$Pu_h = g_h,$$

with respect to the inner product $a(\cdot,\cdot)$ for an appropriate g_h such that the solution u_h is the same as that of (5).

The following theorem, which is given in Zhang [18], proves that this multilevel additive Schwarz method has an optimal rate of convergence.

THEOREM 2. *For P defined above, the following inequalities hold*

$$C_1 a(u_h, u_h) \le a(Pu_h, u_h) \le C_2 a(u_h, u_h) \qquad \forall u_h \in V^h.$$

Thus $\kappa(P) \le C_2 C_1^{-1}$. Here the constants C_1 and C_2 are independent of the mesh sizes $\{h_l\}$ and k.

3. Description of Multilevel Additive Schwarz Methods with Partial Refinement and the First Proof of Optimality. We can modify the general multilevel additive Schwarz methods such that they can handle the finite element problems (3) with composite mesh sizes.

We now give a description of multilevel additive Schwarz methods with partial refinement. Like the procedure in last section, we define a sequence of nested triangulations $\{\mathcal{T}_{l=1}^k\}$. We start with a coarse triangulation $\mathcal{T}^1 = \{\tau_i^1\}_{i=1}^{N_1}$ with quasi-uniform mesh size h_1, where τ_i^1 represent an individual triangle. The successively finer triangulations $\mathcal{T}^l = \{\tau_i^l\}(l = 2, \cdots, k)$ are defined by dividing each triangle in the triangulation $\mathcal{T}^{l-1}$ into several triangles, i.e.

$$\mathcal{T}^1 = \{\tau_i^1\}_1^{N_1} \overset{\text{refinement}}{\Longrightarrow} \mathcal{T}^2 = \{\tau_i^1\}_1^{N_2} \overset{\text{refinement}}{\Longrightarrow} \cdots \overset{\text{refinement}}{\Longrightarrow} \mathcal{T}^k = \{\tau_i^k\}_1^{N_k}.$$

We assume that the triangulations $\mathcal{T}^l$ have quasi-uniform mesh sizes h_{l-1} for each l.

Let us define $\Omega_1 = \Omega$. Then for each $2 \le l \le k$, we choose Ω_l, which is a subregion of Ω_{l-1}, such that $\partial\Omega_l$ aligns with boundaries of level $l-1$ triangles. Let $\tilde{V}^{h_k}, l = 1, \cdots, k$, be the subspace of continuous piecewise linear element associated with the triangulation $\mathcal{T}^l$ of $H_0^1(\Omega)$. We also set V^{h_l} to be $\tilde{V}^{h_l} \cap H_0^1(\Omega_i)$. The finite element problem is to find $u_h \in V^h = V^{h_1} + \cdots + V^{h_k}$ satisfying

$$a(u_h, \phi_h) = f(\phi_h), \qquad \forall \phi_h \in V^h = V^{h_1} + \cdots + V^{h_k}. \tag{7}$$

We assume that there are $k-1$ sets of overlapping subdomain $\{\tilde{\Omega}_i^l\}_{i=1}^{N_l}, l = 2, 3, \cdots, k$. On each level, we have an overlapping decomposition

$$\Omega_l = \cup_{i=1}^{N_l} \tilde{\Omega}_i^l.$$

We also assume that there are another $k-1$ sets of overlapping subdomains $\{\tilde{\Omega}_i^l\}_{i=N_l+1}^{N_l+M_l}$ such that we have

$$\Omega = \cup_{i=1}^{N_l+M_l} \tilde{\Omega}_i^l.$$

We can now make the following assumptions similar to Assumption 1.

ASSUMPTION 2. *Let us assume that*

- *The mesh sizes h_l are bounded from above and below by const.q^l uniformly for all l. Here q is a positive constant less than 1.*
- $(\partial\Omega_{l-1} \cap \partial\Omega_l) \setminus \partial\Omega = \emptyset$ *for* $l = 2, 3, \cdots, k$.
- $\partial\tilde{\Omega}_i^l$ *aligns with boundaries of level l triangles, i.e. $\tilde{\Omega}_i^l$ is the union of level l triangles. Diameter*$(\tilde{\Omega}_i^l) = O(h_{l-1})$.
- *On each level, the subdomains $\{\tilde{\Omega}_i^l\}_{i=1}^{N_l+M_l}$ form a finite covering of Ω, with a covering constant N_c, i.e. we can color $\{\tilde{\Omega}_i^l\}_{i=1}^{N_l+M_l}$, using at most N_c colors in such a way that subdomains of the same color are disjoint.*
- *On each level, associated with $\{\tilde{\Omega}_i^l\}_{i=1}^{N_l}$, there exists a* partition of unity $\{\theta_i^l\}$ *satisfying*

$$\sum_i \theta_i^l = 1, \ \textit{with} \ \theta_i^l \in H_0^1(\tilde{\Omega}_i^l) \cap C^0(\tilde{\Omega}_i^l), 0 \le \theta_i^l \le 1 \ \textit{and} \ |\nabla\theta_i^l| \le C/h_{l-1}.$$

One way of constructing subdomains $\{\tilde{\Omega}_{i=1}^{N_l+M_l}\}$,$l = 2, \cdots, k$, with the above properties is mentioned in last section. Let $N_1 = 1$, $V_1^{h_1} = V^{h_1}$ and $V_i^{h_l} = V^{h_l} \cap H_0^1(\tilde{\Omega}_i^l)$ for $i = 1, \cdots, N_l + M_l$, $l = 2, \cdots, k$. The finite element space V^h is represented by

$$V^h = \sum_{l=1}^{k} V^{h_l} = \sum_{l=1}^{k}\sum_{i=1}^{N_l} V_i^{h_l}.$$

Let us define P_i^l as the orthogonal projection from V^h onto $V_i^{h_l}$ with respect to $a(\cdot,\cdot)$ which is the same as those in last section. The k-level additive Schwarz operator P is defined by

$$P = \sum_{l=1}^{k}\sum_{i=1}^{N_l} P_i^l. \tag{8}$$

Then we have the following algorithm.

ALGORITHM 3 (MAS WITH PARTIAL REFINEMENT). *Let P be the operator defined by (8). Apply the conjugate gradient method to the following symmetric and positive definite system*

$$Pu_h = g_h,$$

with respect to the inner product $a(\cdot,\cdot)$ for appropriate g_h such that the solution u_h is the same as that of (7).

Our main purpose of this section is to prove the following theorem.

THEOREM 3. *For P defined by (8), the following inequalities hold*

$$C_1 a(u_h, u_h) \le a(Pu_h, u_h) \le C_2 a(u_h, u_h) \qquad \forall u_h \in V^h.$$

Thus $\kappa(P) \le C_2 C_1^{-1}$. Here the constants C_1 and C_2 are independent of the mesh sizes $\{h_l\}$ and k.

In order to prove the above theorem, we need to introduce the concept of quasi-interpolants.

DEFINITION 1. *Given a triangulation $\mathcal{T}$ of Ω, we associate with $\mathcal{T}$ a finite element subspace $V(\mathcal{T})$ of $L^2(\Omega)$ which consists of piecewise polynomials of degree less than or equal to m. A linear mapping*

$$Q : L^2(\Omega) \to V(\mathcal{T})$$

is called a quasi-interpolant of order m if it satisfies the properties

$$Qu = u \qquad \forall u \in V(\mathcal{T}),$$

and for a constant C depending only on the shape regularity such that

$$\|Qu\|_{L^2(K)} \leq C \cdot \|u\|_{L^2(\overline{K})}, \qquad \forall K \in \mathcal{T}, \qquad \forall u \in L^2(\Omega).$$

Here $\overline{K}$ denotes the union of the neighbouring elements of K.

The following example is given in Oswald [12].

Example. We construct a quasi-interpolant for linear elements in two dimension. The procedure can be generalized to the cases of more general Lagrange elements and higher-dimensional spaces. Consider an arbitrary nodal point P_i(=vertex) of $\mathcal{T}$ and its adjacent triangles and define on the region a piecewise linear and continuous function with value 3 at P_i and -1 at the other vertices of this region. Extend this function by zero to Ω and scale it by the factor $3/|A_i|$, where A_i denotes the support of this function. Denote the nodal basis function corresponding to P_i by ϕ_i and this L^∞ function by ϕ_i^*. Now we may take

$$Qu = \sum_i \int_\Omega (u, \phi_i^*)_{L^2(\Omega)} \cdot \phi_i.$$

If we look at the quadrature rule for triangles which uses the side midpoints as integration points and is exact for polynomials of degree 2, it is easy to see that Q satisfies the first condition in Definition 1. There remains to verify the second condition. For any element K with area $|K|$, we have $|A_i| \geq |K|$, for each of its three vertices P_i, and

$$|(u, \phi_i^*)_{L^2(\Omega)}| \leq \|u\|_{L^2(A_i)} \cdot \|\phi_i^*\|_{L^2(A_i)} \leq C \cdot |A_i|^{-1/2} \cdot \|u\|_{L^2(\overline{K})}$$

for the corresponding coefficients. Therefore

$$\|Qu\|_{L^2(K)} \leq |K|^{1/2} \cdot \max |(u, \phi_i^*)_{L^2(\Omega)}| \leq C \cdot \|u\|_{L^2(\overline{K})}.$$

Finally we remark that the value of Qu at an arbitrary vertex P only depends upon the values of u in the elements which have P as a vertex.

Let $\tilde{Q}_l$ be a quasi-interpolant of order 1 from $L^2(\Omega)$ onto the space $\tilde{V}^{h_l}$, for $l = 1, 2, \cdots, k$. The proof of the following lemma is similar to one that appears in Xu [17] after replacing the L^2 projection by quasi-interpolants. However, in our proof, we do not need to use the fact that the quasi-interpolants are bounded linear mappings in the space $H_0^1(\Omega)$.

In order to prove the main lemmas in this section, we need the following four lemmas. The proof of the first lemma can be carried out using a standard duality argument; cf. [4]. The proof of the second lemma is based on using smooth functions to approximate elements in $H^1(\Omega)$ and applying the fundamental theorem of calculus to the region. Proof of Lemma 3 may be found in [10] and [11]. Proofs of Lemma 4 may be found in [9] and [16].

LEMMA 1. *Let $\tilde{P}_l$ be the orthogonal projection onto the space $\tilde{V}^{h_l}$ with respect to $a(\cdot,\cdot)$ and suppose that the coefficients $\{a_{ij}(x)\}$ of elliptic problem (1) are smooth enough. Then there exists a $s \in (1/2, 1]$ and a constant C such that*

$$\|(I-\tilde{P}_l)u\|_{H^{1-s}(\Omega)} \leq C h_l^s \|(I-\tilde{P}_l)u\|_{H^1(\Omega)}, \quad \forall u \in H_0^1(\Omega).$$

LEMMA 2. *Let Ω be a domain in R^2, which has the following special form: $\{(x_1,x_2)|a < x_1 < b, g(x_1) < x_2 < f(x_1)\}$. Here $f(x)$ and $g(x)$ are piecewise C^1, continuous functions on $[a,b]$ such that $g(x) < f(x)$ on (a,b). Then for all $u \in H^1(\Omega)$ vanishing on $\{(x_1,x_2)|a \leq x_1 \leq b, x_2 = g(x_1)\}$, we have*

$$\|u\|_{L^2(\Omega)} \leq \max_{a\leq x\leq b} |f(x)-g(x)| \cdot |u|_{H^1(\Omega)}.$$

We can also get similar inequalities in R^n for $n > 2$.

LEMMA 3 (POINCARÉ'S INEQUALITY). *Let*

$$\{u\}_\Omega = \frac{1}{|\Omega|}\int_\Omega u.$$

Then there exists a constant $C(\Omega)$, which depends only on the Lipschitz constant of $\partial\Omega$, such that for all $u \in H^1(\Omega)$ we have

$$\|u - \{u\}_\Omega\|_{L^2(\Omega)} \leq C(\Omega) H_\Omega |u|_{H^1(\Omega)}.$$

Here H_Ω is the diameter of Ω.

LEMMA 4. *Let V be a finite-dimensional Hilbert space with the inner product $a(\cdot,\cdot)$ and let V_i be subspaces of V so that $V = V_1 + \cdots + V_N$. We define P_i as the orthogonal projection onto V_i and $P = P_1 + \cdots + P_N$. If a decomposition of u, $u = \sum_i u_i$ where $u_i \in V_i$, can be found such that*

$$\sum_i a(u_i,u_i) \leq C_1 a(u,u), \quad \forall u \in V^h,$$

then

$$\lambda_{\min}(P) \geq C_1^{-1}.$$

Conversely if for all representations of u, $u = \sum_i u_i$, we have

$$a(u,u) \leq C_2 \sum_i a(u_i,u_i), \quad \forall u \in V^h,$$

then

$$\lambda_{\max}(P) \le C_2.$$

After formulating these lemmas, we can start to prove the following two lemmas which we also need.

LEMMA 5. *There exists a constant C, which depends only on the shape regularity, such that*

$$\sum_{l=2}^{k} \|(\tilde{Q}_l - \tilde{Q}_{l-1})u\|^2_{L^2(\Omega)} \cdot \frac{1}{h_l^2} \le C|u|^2_{H^1(\Omega)} \qquad \forall u \in H_0^1(\Omega).$$

Proof. Let us set

$$\hat{Q}_l = \tilde{Q}_l - \tilde{Q}_{l-1}$$

and

$$u_i = (\tilde{P}_i - \tilde{P}_{i-1})u.$$

Here $\tilde{P}_i$ is the orthogonal projection onto $\tilde{V}^{h_i}$ with respect to $a(\cdot,\cdot)$. We observe that

$$\|\hat{Q}_l u_i\|_{L^2(\Omega)} \le C\|u_i\|_{L^2(\Omega)}$$

by the shape regularity assumption and that

$$\|\hat{Q}_l u_i\|_{L^2(\Omega)} = \|\hat{Q}_l(u_i - Q_{l-1}u_i)\|_{L^2(\Omega)} \le C\|u_i - Q_{l-1}u_i\|_{L^2(\Omega)} \le Ch_l\|u_i\|_{H^1(\Omega)}.$$

Here Q_l is the L^2 projection onto the space $\tilde{V}^{h_l}$. By using an interpolation theorem of Hilbert scales; cf. [4] and [8], we have

$$\|\hat{Q}_l u_i\|_{L^2(\Omega)} \le Ch_l^{1-s} \cdot \|u_i\|_{H^{1-s}(\Omega)}, \qquad \forall s \in (0,1).$$

We choose s as in Lemma 1. Then

$$\|\hat{Q}_l u_i\|_{L^2(\Omega)} \le Ch_l^{1-s}\|u_i\|_{H^{1-s}(\Omega)} \le Ch_l^{1-s}h_i^s\|u_i\|_{H^1(\Omega)}.$$

With $i \wedge j = \min(i,j)$, we have

$$\sum_{l=1}^{k} \|\hat{Q}_l u\|^2_{L^2(\Omega)} \cdot \frac{1}{h_l^2} = \sum_{l=1}^{k}\sum_{i,j=l}^{k} (\hat{Q}_l u_i, \hat{Q}_l u_j)_{L^2(\Omega)} \cdot \frac{1}{h_l^2} = \sum_{i,j=1}^{k}\sum_{l=1}^{i\wedge j} (\hat{Q}_l u_i, \hat{Q}_l u_j)_{L^2(\Omega)} \cdot \frac{1}{h_l^2}$$

$$\le C\sum_{i,j=1}^{k}\sum_{l=1}^{i\wedge j} \frac{1}{h_l^2} h_l^{2(1-s)}\|u_i\|_{H^{1-s}(\Omega)}\|u_j\|_{H^{1-s}(\Omega)} \le C\sum_{i,j=1}^{k}\sum_{l=1}^{i\wedge j} \frac{1}{h_l^2} h_l^{2(1-s)} h_i^s h_j^s \|u_i\|_{H^1(\Omega)}\|u_j\|_{H^1(\Omega)}$$

$$= C\sum_{i,j=1}^{k} h_i^s h_j^s \|u_i\|_{H^1(\Omega)}\|u_j\|_{H^1(\Omega)} \sum_{l=1}^{i\wedge j} h_l^{-2s} \le C\sum_{i,j=1}^{k} h_{i\wedge j}^{-2s} h_i^s h_j^s \|u_i\|_{H^1(\Omega)}\|u_j\|_{H^1(\Omega)}$$

$$\le C\sum_{i,j=1}^{k} q^{s|i-j|}\|u_i\|_{H^1(\Omega)}\|u_j\|_{H^1(\Omega)} \le C\sum_{i=1}^{k}\|u_i\|^2_{H^1(\Omega)} \le C(1+C'(d))|u|^2_{H^1(\Omega)}.$$

The last step follows from Friedrichs' inequality. $C'(d)$ is a constant which only depends upon the diameter d of the domain Ω. By using a simple dilation argument, we can completely remove the dependence of this constant upon the diameter of Ω and complete the proof. $\Box$

Proofs of the boundness of the L^2 projection in $H_0^1(\Omega)$ are given in Scott and Zhang [13], and in Bramble and Xu [3]. However, the proof in [3] does not seem to apply to the elements which have nonempty intersection with $\partial\Omega$. We have the following similar result for quasi-interpolants.

LEMMA 6. *There exists a constant C, which depends only on the shape regularity, such that*

$$|\tilde{Q}_l u|_{H^1(\Omega)} \leq C|u|_{H^1(\Omega)}, \quad \forall u \in H_0^1(\Omega).$$

Proof. Let us first consider the elements K satisfying $\overline{K} \cap \partial\Omega = \emptyset$. Then we have

$$\begin{aligned} |\tilde{Q}_l u|_{H^1(K)} &= |\tilde{Q}_l u - c|_{H^1(K)} = |\tilde{Q}_l(u-c)|_{H^1(K)} \\ &\leq Ch_l^{-1}\|\tilde{Q}_l(u-c)\|_{L^2(K)} \leq Ch_l^{-1}\|u-c\|_{L^2(\overline{K})} \end{aligned}$$

for any constant c. Here $\overline{K}$ is the union of the neighbouring elements of K.

Take c to achieve the infinum. By Lemma 2, we have

$$|\tilde{Q}_l u|_{H^1(K)} \leq Ch_l^{-1} \cdot \inf_c \|u-c\|_{L^2(\overline{K})} \leq Ch_l^{-1} \cdot C'h_l|u|_{H^1(\overline{K})} \equiv C|u|_{H^1(\overline{K})}.$$

Let Ω_0 denote the union of such elements K. By shape regularity, we obtain

$$|\tilde{Q}_l u|_{H^1(\Omega_0)} \leq C|u|_{H^1(\Omega)}$$

for some constant C. Now it is sufficient to prove that

$$|\tilde{Q}_l u|_{H^1(\Omega\setminus\Omega_0)} \leq C|u|_{H^1(\Omega)}.$$

It is obvious that we can write $\overline{\Omega} \setminus \Omega_0 = \cup_{i=1}^{N_0}\overline{\Omega}_{0i}$ as a nonoverlapping union. Let $\Omega'_{0i} = \cup_{K\in\Omega_{0i}}\overline{K}$ where $\overline{K}$ is the union of the neighbouring elements of K. It is obvious that each region Ω'_{0i} and each function $u \in H_0^1(\Omega)$ satisfy the conditions of Lemma 3 and that the constant of this lemma is $O(h_l)$. Then we have

$$\begin{aligned} &|\tilde{Q}_l u|_{H^1(\Omega_{0i})} \leq Ch_l^{-1}\|\tilde{Q}_l u\|_{L^2(\Omega_{0i})} \\ \leq Ch_l^{-1}\|u\|_{L^2(\Omega'_{0i})} &\leq Ch_l^{-1} \cdot h_l|u|_{H^1(\Omega'_{0i})} = C|u|_{H^1(\Omega'_{0i})}. \end{aligned}$$

By combining the above results, the proof of the lemma follows easily. $\Box$

In order to prove the main theorem of this section, we need the following two lemmas. In these lemmas, $\tilde{I}_l$ denotes the usual nodal interpolants from $\tilde{V}^{h_k}$ onto $\tilde{V}^{h_l}$. The proof of the first lemma can be found in [2] and that of the second lemma can be established by replacing the L^2 norm with a discrete equivalent norm in the finite element space $\tilde{V}^{h_l}$.

LEMMA 7. *Assume that $i \le j$. Then there exists a constant C, which depends only on the shape regularity, such that*

$$|\tilde{I}_i u_j|^2_{H^1(\Omega)} \le C \log(h_i/h_j) \cdot |u_j|^2_{H^1(\Omega)}, \qquad \forall u_j \in \tilde{V}^{h_j}$$

in two-dimensional space and

$$|\tilde{I}_i u_j|^2_{H^1(\Omega)} \le C (h_i/h_j)^{n-2} \cdot |u_j|^2_{H^1(\Omega)}, \qquad \forall u_j \in \tilde{V}^{h_j}$$

in higher dimensional space.

LEMMA 8. *Assume that $i \le j$. Then there exists a constant C, which depends only on the shape regularity, such that*

$$\|\tilde{I}_i u_j\|^2_{L^2(\Omega)} \le C (h_i/h_j)^{n} \cdot \|u_j\|^2_{L^2(\Omega)}, \qquad \forall u_j \in \tilde{V}^{h_j}.$$

Proof of Theorem 3. Let us first prove that the operator P has an uniform upper bound. We define $\tilde{P}$ by

$$\tilde{P} = \sum_{l=1}^{k} \sum_{i=1}^{N_l+M_l} P_i^l.$$

We observe that $\tilde{P}$ has a uniform upper bound by Theorem 2 and $P \le \tilde{P}$. Therefore P has a uniform upper bound.

To establish the uniform lower bound, we will apply the first part of Lemma 4. We note that it is sufficient to find a good decomposition of $u \in V^h$ such that the constant is uniformly bounded from above. Let us first decompose u as

$$u = \tilde{Q}_1 u + \sum_{l=2}^{k} (\tilde{Q}_l - \tilde{Q}_{l-1}) u \equiv \sum_{l=1}^{k} v_l.$$

Although $(\tilde{Q}_l - \tilde{Q}_{l-1})u$ is not in the space $\tilde{V}^{h_l}$, since the value of the function $\tilde{Q}_{l-1}u(x)$ at the node x of $\tilde{V}^{h_{l-1}}$ on $\partial\Omega_l$ are not equal to $u(x)$ in general, this function nevertheless have their support in Ω_{l-1} by the assumption $(\partial\Omega_l \cap \partial\Omega_{l-1}) \setminus \partial\Omega = \emptyset$ and belongs to $\tilde{V}^{h_{l-1}}$ in $\Omega \backslash \Omega_l$. If we set $u_l = v_l - \tilde{I}_{l-1} v_l + \tilde{I}_l v_{l+1}$ for $l = 2, \cdots, k$ and $u_1 = v_1 + \tilde{I}_1 v_2$, then $u_l \in V^{h_l}$ for all l and $u = \sum_{l=1}^{k} u_l$. By Lemma 5 and Lemma 6, we have

$$|v_1|^2_{H^1(\Omega)} + \sum_{l=2}^{k} \|v_l\|^2_{L^2(\Omega)} \cdot \frac{1}{h_l^2} \le C |u|^2_{H^1(\Omega)}.$$

If we apply Lemmas 7 and 8 to the above equation, we will find that

$$|u_1|^2_{H^1(\Omega)} + \sum_{l=2}^{k} \|u_l\|^2_{L^2(\Omega)} \cdot \frac{1}{h_l^2} \le C |u|^2_{H^1(\Omega)}. \tag{9}$$

We need to further decompose u^l, for $l \ge 2$, as

$$u^l = \sum_{i=1}^{N_l} u_i^l, \text{ with } u_i^l \equiv \tilde{I}_l(\theta_i^l u^l) \in V_i^{h_l}.$$

Here $\{\theta_i^l\}$ is a partition of unity as in Assumption 2. It can be shown that

$$\begin{aligned} |u_i^l|^2_{H^1(\tilde\Omega_i^l)} &= |\tilde I_l(\theta_i^l u^l)|^2_{H^1(\tilde\Omega_i^l)} \\ \le C(|\theta_i^l|^2_{L^\infty(\Omega)}|u^l|^2_{H^1(\tilde\Omega_i^l)} &+ |\theta_i^l|^2_{W^{1,\infty}(\Omega)}\|u^l\|^2_{L^2(\tilde\Omega_i^l)}) \\ &\le C(|u^l|^2_{H^1(\tilde\Omega_i^l)} + (1/h_{l-1}^2)\|u^l\|^2_{L^2(\tilde\Omega_i^l)}). \end{aligned}$$

Summing over i and using the finite covering property of $\{\tilde\Omega_i^l\}$, we obtain

$$\begin{aligned} \sum_i |u_i^l|^2_{H^1(\Omega)} = \sum_i |u_i^l|^2_{H^1(\tilde\Omega_i^l)} &\le C\sum_i(|u^l|^2_{H^1(\tilde\Omega_i^l)} + \frac{1}{h_{l-1}^2}\cdot\|u^l\|^2_{L^2(\tilde\Omega_i^l)}) \\ &\le C(|u^l|^2_{H^1(\Omega)} + \frac{1}{h_l^2}\|u^l\|^2_{L^2(\Omega)}) \le C\frac{1}{h_l^2}\|u^l\|^2_{L^2(\Omega)}. \end{aligned}$$

Summing over l, for $1 \le l \le k$, and using inequality (9), we get

$$\sum_{l=1}^k\sum_i |u_i^l|^2_{H^1(\Omega)} \le C(|u^l|^2_{H^1(\Omega)} + \sum_{l=2}^k \frac{1}{h_l^2}\|u^l\|^2_{L^2(\Omega)}) \le C|u|^2_{H^1(\Omega)}.$$

The lower bound of P now follows. □

We end this section by mentioning a special decomposition of the domain Ω in Zhang [18]. It is called the multilevel diagonal scaling. Let ϕ_i^l be a nodal basis function of V^{h_l}, and associate with each ϕ_i^l the subdomain $\tilde\Omega_i^l = supp\{\phi_i^l\}$. We may choose $V_i^l = span\{\phi_i^l\} = V^{h_l} \cap H_0^1(\tilde\Omega_i^l)$ and obtain the decomposition

$$V^{h_k} = \sum_{l=1}^k\sum_{i=1}^{N_l} V_i^{h_l}$$

and the Galerkin projection P_i^l corresponding to $V_i^{h_l}$. Let $P' = \sum_{l=1}^k\sum_{i=1}^{N_l} P_i^l$. It is easy to see that the above construction satisfies Assumption 2. Therefore we have another variant of Algorithm 3 whose optimality follows from Theorem 3.

ALGORITHM 4 (MDS WITH PARTIAL REFINEMENT). *Let P' be the operator defined above. Apply the conjugate gradient method to the following symmetric and positive definite system*

$$P'u_h = g_h,$$

with respect to the inner product $a(\cdot,\cdot)$ for an appropriate g_h such that the solution u_h is the same as that of (7).

Let K_l be the stiffness matrix associated with V^{h_l}, let K_h be the stiffness matrix associated with V^h and let $D_l = diag(K_l)$. Let $\mathcal{I}_l : V^{h_l} \to V^h$, $1 \le l \le k$, be the standard inclusion operator, and let $\mathcal{I}_l^t : V^h \to V^{h_l}$ be an operator related to $\mathcal{I}_l$ in the following way:

$$(\mathcal{I}_l^t u_h, v^l)_l = (u_h, \mathcal{I}_l v^l)_{L^2(\Omega)} \qquad \forall v^l \in V^{h_l}.$$

Here $(\cdot,\cdot)_l$ is the discrete inner product in V^{h_l}, which is equivalent to $L^2(\Omega)$, defined by

$$(u^l, v^l)_l = h_l^n \sum_{x\in\mathcal{N}_l} u^l(x)v^l(x) \qquad \forall u^l, v^l \in V^{h_l}.$$

Here $\mathcal{N}_l$ is the set of nodes of the degrees of freedom in V^{h_l}. Algorithm 4 can then be written as: Find the solution of $K_h x = b$ by solving the preconditioned system

$$B_h^{-1}K_h x = B_h^{-1}b,$$

where

$$B_h^{-1} = h_1^n \cdot \mathcal{I}_1 K_1^{-1}\mathcal{I}_1^t + \cdots + h_{k-1}^n \cdot \mathcal{I}_{k-1}D_{k-1}^{-1}\mathcal{I}_{k-1}^t + h_k^n \cdot D_k^{-1}.$$

We remark that if we replace the matrices D_l by identity matrices, we obtain the BPX algorithm with partial refinement.

4. The Second Proof of Optimality and Some Multiplicative Variants. In this section, we will construct another proof of our main theorem by using the approach of iterative refinement methods. For convenience, we use the same notations as in last section. The typical assumption for iterative refinement methods is related to the extension theorem for finite element functions with respect to the $a(\cdot,\cdot)$.

ASSUMPTION 3. *For each j, there exists a bounded Lipschitz polyhedral region $\tilde{\Omega}_j$ such that $\Omega_j \subset \tilde{\Omega}_j$, $(\tilde{\Omega}_j \setminus \Omega_j)\cap\Omega = \emptyset$, $\partial\tilde{\Omega}_j \cap \partial\Omega_{j+1} = \emptyset$ and the Lipschitz constants of $\tilde{\Omega}_j \setminus \Omega_{j+1}$ are uniformly bounded.*

The above assumption can usually be weakened to Assumption 4

ASSUMPTION 4. *For each j, either $\Omega_j = \Omega_{j+1}$ or there exists a bounded Lipschitz polyhedral region $\tilde{\Omega}_j$ such that $\Omega_j \subset \tilde{\Omega}_j$, $(\tilde{\Omega}_j \setminus \Omega_j)\cap\Omega = \emptyset$, $\partial\tilde{\Omega}_j \cap \partial\Omega_{j+1} = \emptyset$ and the Lipschitz constants of $\tilde{\Omega}_j \setminus \Omega_{j+1}$ are uniformly bounded.*

Let us define P_j^i, $i \le j$, as the orthogonal projections onto the spaces $V_{h_i} \cap H_0^1(\Omega_j)$ with respect to the inner product $a(\cdot,\cdot)$. Now we can recall a result from Cheng [4].

LEMMA 9. *Under Assumption 4, there is an absolute constant C which depends on the Lipschitz constant in Assumption 4 and shape regularity such that for any $u \in V^h$ we can decompose u into $u = \sum_{i=1}^k u_i$, where $u_i \in Range(P_i^i - P_i^{i-1})$, and*

$$\sum_{i=1}^k a(u_i, u_i) \le C a(u,u).$$

We remark that the proof of this Lemma can be done by first considering the case under Assumption 3 and then doing a further decomposition of u under Assumption 4.

Now let us make the remaining assumption used in the main theorem in this section and then state the main theorem.

ASSUMPTION 5. *Let us assume that*

- *These mesh sizes h_l are bounded from above and below by const.q^l uniformly for all l. Here q is a positive constant less than 1.*

- $\partial\tilde{\Omega}_i^l$ *aligns with boundaries of level* l *triangles, i.e.* $\tilde{\Omega}_i^l$ *is the union of level* l *triangles. Diameter*$(\tilde{\Omega}_i^l) = O(h_{l-1})$.
- *On each level, the subdomains* $\{\tilde{\Omega}_i^l\}_{i=1}^{N_l+M_l}$ *form a finite covering of* Ω, *with a covering constant* N_c, *i.e. we can color* $\{\tilde{\Omega}_i^l\}_{i=1}^{N_l+M_l}$, *using at most* N_c *colors in such a way that subdomains of the same color are disjoint.*
- *On each level, associated with* $\{\tilde{\Omega}_i^l\}_{i=1}^{N_l}$, *there exists a* partition of unity $\{\theta_i^l\}$ *satisfying*

$$\sum_i \theta_i^l = 1, \ \textit{with } \theta_i^l \in H_0^1(\tilde{\Omega}_i^l) \cap C^0(\tilde{\Omega}_i^l), 0 \leq \theta_i^l \leq 1 \ \textit{and } |\nabla \theta_i^l| \leq C/h_{l-1}.$$

THEOREM 4. *Under Assumptions 4 and 5, there exist absolute constants* C_1 *and* C_2 *such that*

$$C_1 a(u_h, u_h) \leq a(Pu_h, u_h) \leq C_2 a(u_h, u_h) \qquad \forall u_h \in V^h.$$

Here P *is defined by (8). Thus* $\kappa(P) \leq C_2 C_1^{-1}$. *Here the constants* C_1 *and* C_2 *are independent of the mesh sizes* $\{h_l\}$ *and* k.

The main idea of proving Theorem 4 is that constructing a good decomposition of $u \in V^h$ satisfies the condition of Lemma 4. Let us define the operators $R_l : V^h \to \tilde{V}^{h_l}$ by

$$R_l u(x) = \begin{cases} u(x) & \text{if } x \in \overline{\Omega} \backslash \Omega_{l+1} \\ \tilde{Q}_l(x) & \text{if } x \in \Omega_{l+1} \end{cases}$$

for $l = 1, 2, \cdots, k-1$ and $R_k u = u$. It is obvious that $R_m R_n = R_n$ for $1 \leq n \leq m \leq k$. It is also clear that there exists an absolute constant C such that

$$\|R_l u\|_{L^2(\Omega)} \leq C\|u\|_{L^2(\Omega)}.$$

There are some other important properties of R_l which we need. They will be stated below.

LEMMA 10. *There is an absolute constant* C *such that*

$$\|u - R_l u\|_{L^2(\Omega)} \leq C h_l |u|_{H^1(\Omega)}.$$

Proof. Let us denote the union of the element K of level l in Ω_{l+1} which satisfies $K \cap (\Omega \backslash \Omega_{l+1}) \neq \emptyset$ by Ω_{0l}. Then

$$\begin{aligned} \|u - R_l u\|_{L^2(\Omega)} &\leq \|u - \tilde{Q}_l u\|_{L^2(\Omega)} + \|\tilde{Q}_l u - R_l u\|_{L^2(\Omega)} \\ &\leq C h_l |u|_{H^1(\Omega)} + \|u - \tilde{Q}_l u\|_{L^2(\Omega \backslash \Omega_{l+1})} + \|w\|_{L^2(\Omega_{0l})}. \end{aligned}$$

Here $w \in \tilde{V}^{h_l}$ in Ω_{l+1} and

$$w(x) = \begin{cases} (u - \tilde{Q}_l u)(x) & \text{if } x \in \partial\Omega_{l+1} \\ 0 & \text{if } x \text{ is a node in } \Omega_{l+1}. \end{cases}$$

By considering a discrete norm of w as in Lemma 8, it is easy to see that

$$\|w\|_{L^2(\Omega_{0l})} \le C\|u - \tilde{Q}_l u\|_{L^2(\Omega\backslash\Omega_{l+1})}.$$

Therefore

$$\|u - R_l u\|_{L^2(\Omega)} \le Ch_l|u|_{H^1(\Omega)} + C\|u - \tilde{Q}_l u\|_{L^2(\Omega)} \le Ch_l|u|_{H^1(\Omega)}. \square$$

In order to proceed with the proof of the next lemma, we need to introduce the operators $H_l : V^h \to \tilde{V}^{h_l}$ by

$$\begin{aligned} H_l u(x) &\in V^{h_1} + \cdots + V^{h_l}, \\ H_l u(x) &= u(x), \quad x \in \Omega\backslash\Omega_{l+1}, \\ a(H_l u, w_h) &= 0, \quad \forall w_h \in V^{h_l} \cap H_0^1(\Omega_{l+1}). \end{aligned}$$

It is natural to call $H_l u$ the h_l-harmonic extension of u to Ω_{l+1}. Let us recall a result from Cheng [4]; cf. Widlund [14]. There exists an absolute constant C such that

$$a(H_l u, H_l u) \le Ca(u,u), \quad \forall u \in V^h. \tag{10}$$

By using this inequality, we can prove that R_l is a bounded operator from V^h into $\tilde{V}^{h_l}$ in the H_0^1-norm.

LEMMA 11. *There exists an absolute constant C such that*

$$|R_l u|_{H^1(\Omega)} \le C|u|_{H^1(\Omega)}, \quad \forall u \in V^h.$$

Proof. We observe that

$$\begin{aligned} |R_l u|^2_{H^1(\Omega)} &= |u|^2_{H^1(\Omega\backslash\Omega_{l+1})} + |R_l u|^2_{H^1(\Omega_{l+1})} \\ &\le |u|^2_{H^1(\Omega)} + C(|R_l u - H_l u|^2_{H^1(\Omega_{l+1})} + |H_l u|^2_{H^1(\Omega_{l+1})}) \\ &= |u|^2_{H^1(\Omega)} + C(|\tilde{Q}_l(u - H_l u)|^2_{H^1(\Omega_{l+1})} + |H_l u|^2_{H^1(\Omega_{l+1})}) \end{aligned}$$

and that $u - H_l u = 0$ on $\partial\Omega_{l+1}$. Therefore we can apply Lemma 6 to conclude that

$$|R_l u|^2_{H^1(\Omega)} \le |u|^2_{H^1(\Omega)} + C(|u - H_l u|^2_{H^1(\Omega_{l+1})} + |H_l u|^2_{H^1(\Omega_{l+1})}) \le C|u|^2_{H^1(\Omega)}.$$

The last step follows from equation (10). $\square$

We next prove an analog of Lemma 5 for the operators R_l.

LEMMA 12. *There exists an absolute constant C, which depends only on these Lipschitz constants that appear in Assumption 4 and the shape regularity, such that*

$$\sum_{l=2}^{k} \|(R_l - R_{l-1})u\|^2_{L^2(\Omega)} \cdot \frac{1}{h_l^2} \le C|u|^2_{H^1(\Omega)} \qquad \forall u \in V^h.$$

Proof. Let us first decompose u into $u = \sum_{i=1}^{k} u_i$, where $u_i \in Range(P_i^i - P_i^{i-1})$ is the same as in Lemma 9. We observe that

$$\|(R_l - R_{l-1})u_i\|_{L^2(\Omega)} \leq C\|u_i\|_{L^2(\Omega)},$$

by the shape regularity assumption, and that

$$\|(R_l - R_{l-1})u_i\|_{L^2(\Omega)} = \|R_l(u_i - R_{l-1}u_i)\|_{L^2(\Omega)} \leq C\|u_i - R_{l-1}u_i\|_{L^2(\Omega)} \leq Ch_l\|u_i\|_{H^1(\Omega)},$$

by using Lemma 10. By using an interpolation theorem of Hilbert scales; cf. [4] and [8], we have

$$\|(R_l - R_{l-1})u_i\|_{L^2(\Omega)} \leq Ch_l^{1-s} \cdot \|u_i\|_{H^{1-s}(\Omega)} \qquad \forall s \in (0,1).$$

We choose s as in Lemma 1. Then

$$\|(R_l - R_{l-1})u_i\|_{L^2(\Omega)} \leq Ch_l^{1-s}\|u_i\|_{H^{1-s}(\Omega)} \leq Ch_l^{1-s}h_i^s\|u_i\|_{H^1(\Omega)}.$$

With $i \wedge j = \min(i,j)$ and the observation that $(R_l - R_{l-1})u_i = 0$ for $i < l$, we have

$$\sum_{l=1}^{k} \|(R_l - R_{l-1})u\|_{L^2(\Omega)}^2 \cdot \frac{1}{h_l^2} = \sum_{l=1}^{k} \sum_{i,j=l}^{k} ((R_l - R_{l-1})u_i, (R_l - R_{l-1})u_j)_{L^2(\Omega)} \cdot \frac{1}{h_l^2}$$

$$= \sum_{i,j=1}^{k} \sum_{l=1}^{i\wedge j} ((R_l - R_{l-1})u_i, (R_l - R_{l-1})u_j)_{L^2(\Omega)} \cdot \frac{1}{h_l^2}$$

$$\leq C \sum_{i,j=1}^{k} \sum_{l=1}^{i\wedge j} \frac{1}{h_l^2} h_l^{2(1-s)} \|u_i\|_{H^{1-s}(\Omega)} \|u_j\|_{H^{1-s}(\Omega)} \leq C \sum_{i,j=1}^{k} \sum_{l=1}^{i\wedge j} \frac{1}{h_l^2} h_l^{2(1-s)} h_i^s h_j^s \|u_i\|_{H^1(\Omega)} \|u_j\|_{H^1(\Omega)}$$

$$= C \sum_{i,j=1}^{k} h_i^s h_j^s \|u_i\|_{H^1(\Omega)} \|u_j\|_{H^1(\Omega)} \sum_{l=1}^{i\wedge j} h_l^{-2s} \leq C \sum_{i,j=1}^{k} h_{i\wedge j}^{-2s} h_i^s h_j^s \|u_i\|_{H^1(\Omega)} \|u_j\|_{H^1(\Omega)}$$

$$\leq C \sum_{i,j=1}^{k} q^{s|i-j|} \|u_i\|_{H^1(\Omega)} \|u_j\|_{H^1(\Omega)} \leq C \sum_{i=1}^{k} \|u_i\|_{H^1(\Omega)}^2 \leq C(1 + C'(d))|u|_{H^1(\Omega)}^2$$

by using Lemma 9 and Friedrichs' inequality. Here $C'(d)$ is a constant which only depends upon the diameter d of the domain Ω. By using a simple dilation argument, we can completely remove the dependence of this constant upon the diameter of Ω and complete the proof. □

We now return to the proof of Theorem 4 by using the previous lemmas.

Proof of Theorem 4. The proof that P has an uniform upper bound is the same as in Theorem 3. To establish the uniform lower bound, we again apply the first part of Lemma 4. Let us first decompose u as

$$u = R_1 u + \sum_{l=2}^{k} (R_l - R_{l-1})u \equiv \sum_{l=1}^{k} u_l.$$

It is easy to see that $u_l \in V^{h_l}$. We need to further decompose u^l, for $l \geq 2$, as

$$u^l = \sum_{i=1}^{N_l} u_i^l, \text{ with } u_i^l \equiv \tilde{I}_l(\theta_i^l u^l) \in V_i^{h_l}.$$

Here $\{\theta_i^l\}$ is a partition of unity as in Assumption 5. The remaining part of the proof is essentially the same as in Theorem 3 using Lemmas 11 and 12 rather than Lemmas 5 and 6. □

Then we discuss some multiplicative variants of the MDS algorithm with partial refinement. In particular, we can estimate the energy norm of the following operators

$$E_G = \prod_{l=1}^{k} \prod_{i=1}^{N_l} (I - P_i^l),$$

$$E_J = \prod_{l=1}^{k} (I - T_l) \equiv \prod_{l=1}^{k} (I - \beta \sum_{i=1}^{N_l} P_i^l),$$

where β is a damping factor such that $\|T_l\|_a \leq w < 2$. The operators E_G and E_J correspond to the FAC algorithms with inexact solvers consisting of one Gauss-Seidel and damped Jacobi iteration, respectively, except for the coarsest space V^{h_1}. We can use the techniques in Zhang [18] and the fact that the multilevel additive Schwarz operator P has a uniform lower bound to prove the following theorem.

THEOREM 5. *There exist absolute constants η_G and η_J, which depend only on the Lipschitz constants appearing in Assumption 4 and the shape regularity, such that*

$$\|E_G\|_a \leq \eta_G < 1 \quad \textit{and} \quad \|E_J\|_a \leq \eta_J < 1.$$

In order to prove Theorem 5, we need the following lemma, which is given in Zhang [18].

LEMMA 13. *Let T_i, $i = 1, \cdots, N$, be symmetric, semi-positive definite operators with respect to the $a(\cdot,\cdot)$ and let $\|T_i\|_a \leq \omega < 2$. Let $T = \sum_{i=1}^{N} T_i$ and $E = (I - T_1)(I - T_2)\cdots(I - T_N)$. Then*

$$\|E\|_a \leq \sqrt{1 - (2-\omega)\frac{\lambda_{\min}(T)}{\|\Theta_T\|_2^2}}.$$

Here $\Theta_T = \{\theta_T^{ij}\}$, where $\theta_T^{ii} = 1$ and θ_T^{ij}, $i \neq j$, are given by

$$\theta_T^{ij} = sup_{u,v} \frac{a(T_i u, T_j v)}{a(T_i u, u)^{1/2} a(T_j v, v)^{1/2}}.$$

Proof of Theorem 5. We first estimate E_G. In this case, $T_i = P_i^l$ for each subspace $V_i^{h_l}$. Let us denote by $\tilde{T}$ the operator corresponding to the case of refinement everywhere. In [18], Zhang established that $\|\Theta_{\tilde{T}}\|_2^2$ is uniformly bounded. We note that each space corresponding to T_i is a space corresponding to a $\tilde{T}_j$ for some j. Therefore $\|\Theta_T\|_2^2$ is uniformly bounded. By Lemma 13 and Theorem 4, the first part follows easily.

As for the case of E_J, we take $T_l = \beta \sum_i P_i^l$ and use an argument similar to the above one used. □

In the next section, we will report on some numerical experiments to evaluate the η_G and η_J for some model problems

Finally we discuss the extension to the cases of general mixed type boundary condition and more general Lagrange elements. For general Lagrange elements in higher-dimensional space, it is possible to construct high order quasi-interpolants by looking at quadrature rules preserving high order polynomials which are similar to the example given in Section 3. For general boundary condition, it is sufficient to prove counterparts of Lemmas 1, 6, and 9. The modification of Lemmas 1 and 9 have been discussed for the work of iterative refinement methods in [4]. However, we can construct the counterpart of Lemma 6 by separately considering two cases of the elements K satisfying $\overline{K} \cap \partial\Omega_D = \emptyset$ and those who do not.

5. Numerical Results. In this section, we report on some numerical experiments which verify our theoretical results. We take the differential operator to be the Laplacian and the subdomains are triangulated using right triangles of equal size. In these experiments, we choose arbitrary right hand sides and the iterations are stopped when the residual size with respect to the energy norm has been reduced by a factor 10^{-6}. In all experiments, we choose $h_{l+1} = h_l/2$ and the ratio of the diameters of Ω_{l+1} and Ω_l to be 1/2 for $l = 1, 2, \cdots, k-1$ as shown in Fig. 2 and Fig 3. We will call the problem of Fig. 2 Model Problem 1 and that of Fig. 3 Model Problem 2.

Tables 1 and 2 show results for the MAS method with partial refinement. To each node x_i in $\mathcal{N}_{l-1}$, for $l \geq 2$, we associate a subdomain $\tilde{\Omega}_i^l$ which is a rectangle with center at x_i and side length $2h_{l-1}$. It is easy to see that this construction of $\{\tilde{\Omega}_i^l\}$ satisfies our previous assumption.

In Tables 3 and 4, we consider the MDS method with partial refinement. We see that the condition numbers of Table 3 and 4 are usually larger than those of Table 1 and 2 because the overlap between the subdomains in MDS is smaller than those in MAS and that more subspaces are used in MDS than in MAS.

In Tables 5 and 6, we consider the symmetrized FAC algorithms with the inexact solvers of one Gauss-Seidel iteration for model problem 1 and 2. We can see that the spectral radius η_G is about 0.30 which is comparable to the FAC algorithm with exact solver whose spectral radius is about 0.18.

In Tables 7 and 8, we consider the symmetrized FAC algorithm with one damped Jacobi iteration as an inexact solver and with $\beta = 0.5$. In Tables 9 and 10, we consider the same methods as in Table 7 and 8, but choose $\beta = 0.8$.

In Table 11, we try to determine the optimal β for damped Jacobi iteration such that we can get the best rate of convergence. We choose $h_1 = 1/8$ and $k = 2$ for model problem 1 and compare β between 0.10 and 0.95. From this table, we see that

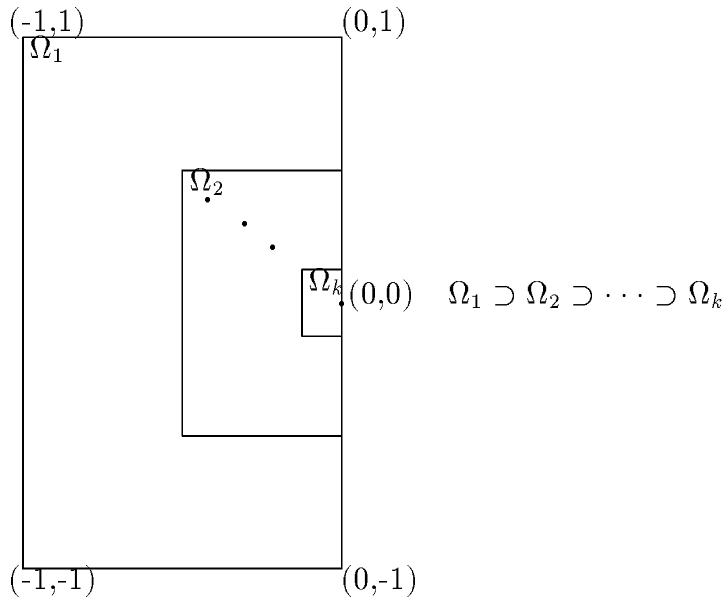

FIG. 2. *Model Problem 1*

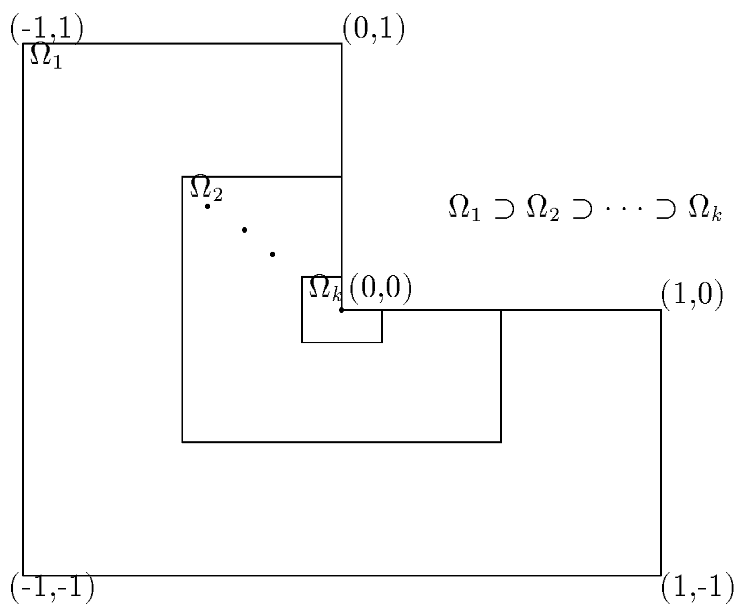

FIG. 3. *Model Problem 2*

$h_1 = 1/8$									
k	2	3	4	5	6	7	8	9	10
min. ev.	0.6660	0.6615	0.6589	0.6576	0.6570	0.6567	0.6564	0.6562	0.6561
max. ev.	4.462	6.329	6.623	6.717	6.843	6.983	7.100	7.194	7.270
cond. no.	6.700	9.568	10.05	10.21	10.42	10.63	10.82	10.96	11.08
no. of iter.	16	20	21	22	22	22	22	22	22
$h_1 = 1/16$									
k	2	3	4	5	6	7	8	9	10
min. ev.	0.6675	0.6645	0.6639	0.6635	0.6633	0.6632	0.6632	0.6631	0.6630
max. ev.	4.529	6.756	8.400	8.635	8.683	8.758	8.836	8.903	8.958
cond. no.	6.785	10.17	12.65	13.01	13.09	13.21	13.32	13.43	13.51
no. of iter.	16	21	23	24	24	24	24	25	25

TABLE 1
MAS with partial refinement for Model problem 1

$h_1 = 1/8$									
k	2	3	4	5	6	7	8	9	10
min. ev.	0.6685	0.6618	0.6589	0.6577	0.6571	0.6567	0.6564	0.6562	0.6560
max. ev.	4.481	6.335	7.301	7.757	7.943	8.032	8.080	8.109	8.128
cond. no.	6.703	9.572	11.08	11.79	12.09	12.23	12.31	12.36	12.39
no. of iter.	16	20	21	21	22	22	23	23	23
$h_1 = 1/16$									
k	2	3	4	5	6	7	8	9	10
min. ev.	0.6703	0.6650	0.6642	0.6637	0.6635	0.6634	0.6633	0.6633	0.6633
max. ev.	4.525	6.818	8.412	9.128	9.459	9.599	9.667	9.725	9.754
cond. no.	6.751	10.25	12.66	13.75	14.26	14.47	14.57	14.66	14.71
no. of iter.	16	21	23	24	25	25	25	25	25

TABLE 2
MAS with partial refinement for Model problem 2

$h_1 = 1/8$									
k	2	3	4	5	6	7	8	9	10
min. ev.	0.7830	0.7836	0.7833	0.7835	0.7848	0.7834	0.7842	0.7844	0.7849
max. ev.	7.815	11.26	12.63	13.10	13.50	13.91	14.24	14.50	14.70
cond. no.	9.981	14.37	16.12	16.72	17.20	17.76	18.16	18.49	18.73
no. of iter.	21	25	27	28	28	29	29	29	29
$h_1 = 1/16$									
k	2	3	4	5	6	7	8	9	10
min. ev.	0.7880	0.7890	0.7891	0.7894	0.7914	0.7924	0.7933	0.7937	0.7940
max. ev.	9.903	11.78	13.84	14.92	15.26	15.55	15.87	16.13	16.35
cond. no.	10.03	14.93	17.54	18.90	19.28	19.62	20.01	20.32	20.59
no. of iter.	21	26	28	29	29	29	29	29	29

TABLE 3
MDS with partial refinement for Model problem 1

$h_1 = 1/8$									
k	2	3	4	5	6	7	8	9	10
min. ev.	0.7852	0.7846	0.7847	0.7851	0.7872	0.7885	0.7872	0.7880	0.7883
max. ev.	7.855	11.40	12.95	13.72	14.01	14.16	14.24	14.30	14.43
cond. no.	10.00	14.53	16.50	17.48	17.80	17.96	18.09	18.15	18.30
no. of iter.	21	25	27	28	28	28	29	29	29
$h_1 = 1/16$									
k	2	3	4	5	6	7	8	9	10
min. ev.	0.7884	0.7889	0.7898	0.7908	0.7923	0.7940	0.7949	0.7955	0.7959
max. ev.	7.949	11.84	13.96	15.20	15.82	16.05	16.17	16.24	16.30
cond. no.	10.08	15.01	17.68	19.22	19.97	20.21	20.34	20.41	20.48
no. of iter.	21	26	28	29	29	29	29	29	29

TABLE 4
MDS with partial refinement for Model problem 2

$h_1 = 1/8$									
k	2	3	4	5	6	7	8	9	10
min. ev.	0.7134	0.7059	0.7033	0.7011	0.6993	0.6981	0.6973	0.6966	0.6962
max. ev.	0.9998	0.9997	0.9996	0.9996	0.9995	0.9995	0.9995	0.9994	0.9994
cond. no.	1.401	1.416	1.421	1.426	1.429	1.432	1.433	1.435	1.436
no. of iter.	6	6	6	6	6	6	6	6	6
$h_1 = 1/16$									
k	2	3	4	5	6	7	8	9	10
min. ev.	0.7149	0.7089	0.7080	0.7080	0.7077	0.7074	0.7072	0.7071	0.7071
max. ev.	0.9997	0.9997	0.9996	0.9996	0.9996	0.9996	0.9996	0.9995	0.995
cond. no.	1.398	1.410	1.412	1.412	1.412	1.413	1.414	1.414	1.414
no. of iter.	6	6	6	6	6	6	6	6	6

TABLE 5
FAC with one Gauss-Seidel iteration for Model problem 1

$h_1 = 1/8$									
k	2	3	4	5	6	7	8	9	10
min. ev.	0.7287	0.6961	0.6947	0.6922	0.6803	0.6782	0.6760	0.6738	0.6716
max. ev.	0.9997	0.9994	0.9993	0.9992	0.9994	0.9993	0.9993	0.9993	0.9992
cond. no.	1.372	1.436	1.438	1.443	1.469	1.474	1.478	1.483	1.488
no. of iter.	6	6	6	6	7	7	7	7	7
$h_1 = 1/16$									
k	2	3	4	5	6	7	8	9	10
min. ev.	0.7297	0.7089	0.7066	0.7010	0.6953	0.6901	0.6814	0.6779	0.6751
max. ev.	0.9998	0.9997	0.9996	0.9996	0.9995	0.9995	0.9996	0.9996	0.9996
cond. no.	1.370	1.410	1.415	1.426	1.438	1.448	1.467	1.474	1.481
no. of iter.	6	6	6	6	6	6	7	7	7

TABLE 6
FAC with one Gauss-Seidel iteration for Model problem 2

$h_1 = 1/8$									
k	2	3	4	5	6	7	8	9	10
min. ev.	0.4426	0.4339	0.4241	0.4198	0.4178	0.4167	0.4159	0.4155	0.4151
max. ev.	0.9758	0.9869	0.9930	0.9932	0.9934	0.9935	0.9936	0.9937	0.9937
cond. no.	2.205	2.275	2.341	2.366	2.378	2.384	2.389	2.392	2.394
no. of iter.	9	9	10	10	10	10	10	10	10
$h_1 = 1/16$									
k	2	3	4	5	6	7	8	9	10
min. ev.	0.4426	0.4370	0.4330	0.4286	0.4219	0.4194	0.4180	0.4172	0.4166
max. ev.	0.9739	0.9842	0.9874	0.9887	0.9916	0.9919	0.9922	0.9923	0.9923
cond. no.	2.201	2.252	2.280	2.307	2.350	2.365	2.374	2.379	2.382
no. of iter.	9	9	9	9	10	10	10	10	10

TABLE 7

FAC with one damped Jacobi iteration of $\beta = 0.5$ for Model problem 1

$h_1 = 1/8$									
k	2	3	4	5	6	7	8	9	10
min. ev.	0.4368	0.4060	0.3904	0.3789	0.3697	0.3623	0.3563	0.3514	0.3473
max. ev.	0.9891	0.9939	0.9966	0.9972	0.9976	0.9980	0.9982	0.9983	0.9987
cond. no.	2.264	2.448	2.553	2.632	2.698	2.755	2.802	2.841	2.875
no. of iter.	9	10	10	10	10	10	10	10	11
$h_1 = 1/16$									
k	2	3	4	5	6	7	8	9	10
min. ev.	0.4386	0.4061	0.3890	0.3774	0.3682	0.3609	0.3559	0.3504	0.3472
max. ev.	0.9886	0.9934	0.9970	0.9976	0.9980	0.9984	0.9988	0.9991	0.9993
cond. no.	2.254	2.446	2.563	2.644	2.710	2.767	2.806	2.851	2.878
no. of iter.	9	10	10	10	10	10	10	10	11

TABLE 8

FAC with one damped Jacobi iteration of $\beta = 0.5$ for Model problem 2

$h_1 = 1/8$									
k	2	3	4	5	6	7	8	9	10
min. ev.	0.6463	0.6200	0.6012	0.5930	0.5896	0.5879	0.5868	0.5831	0.5856
max. ev.	0.9852	0.9901	0.9900	0.9899	0.9900	0.9901	0.9903	0.9929	0.9906
cond. no.	1.524	1.597	1.647	1.669	1.679	1.684	1.688	1.703	1.692
no. of iter.	7	7	7	7	7	7	7	8	7
$h_1 = 1/16$									
k	2	3	4	5	6	7	8	9	10
min. ev.	0.6459	0.6286	0.6175	0.6088	0.6005	0.5958	0.5901	0.5874	0.5857
max. ev.	0.9832	0.9896	0.9915	0.9914	0.9913	0.9912	0.9912	0.9912	0.9912
cond. no.	1.522	1.574	1.606	1.629	1.651	1.664	1.680	1.687	1.692
no. of iter.	7	7	7	7	7	7	7	7	7

TABLE 9
FAC with one damped Jacobi iteration of $\beta = 0.8$ for Model problem 1

$h_1 = 1/8$									
k	2	3	4	5	6	7	8	9	10
min. ev.	0.6364	0.5846	0.5605	0.5467	0.5359	0.5272	0.5202	0.5145	0.5099
max. ev.	0.9892	0.9940	0.9963	0.9966	0.9968	0.9969	0.9970	0.9970	0.9970
cond. no.	1.554	1.700	1.777	1.823	1.860	1.891	1.917	1.938	1.956
no. of iter.	7	7	8	8	8	8	8	8	8
$h_1 = 1/16$									
k	2	3	4	5	6	7	8	9	10
min. ev.	0.6382	0.5809	0.5598	0.5464	0.5356	0.5270	0.5199	0.5140	0.5097
max. ev.	0.9865	0.9937	0.9952	0.9957	0.9960	0.9961	0.9961	0.9961	0.9961
cond. no.	1.546	1.710	1.778	1.822	1.859	1.890	1.916	1.938	1.954
no. of iter.	7	8	8	8	8	8	8	8	8

TABLE 10
FAC with one damped Jacobi iteration of $\beta = 0.8$ for Model problem 2

β	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
η_J	0.9017	0.8543	0.8081	0.7630	0.7194	0.6768	0.6358	0.5956	0.5574
cond. no.	3.552	3.404	3.252	3.053	2.889	2.699	2.540	2.360	2.205
β	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
η_J	0.5207	0.4841	0.4500	0.4156	0.3840	0.3537	0.3920	0.5496	0.7294
cond. no.	2.058	1.915	1.803	1.670	1.588	1.524	1.626	2.202	3.688

TABLE 11

FAC with one damped Jacobi iteration with respect to different values β

the optimal β is about 0.80. However, with this optimal value β, it is still not better than using the Gauss-Seidel iteration.

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