The Complexity of Resolvent Resolved

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Abstract

The concept of a resolvent *of a prime ideal was originally introduced by J.F. Ritt along with the notion of a characteristic set. The motivation for studying* resolvents *comes from its connections with the birational isomorphisms that describe structures of irreducible algebraic varieties by means of an equivalent hypersurface and a one-to-one rational map. As a result, these ideas have a wide range of applications in such areas as solid modeling, computer aided design and manufacturing. An algorithm to compute the resolvent by means of characteristic sets was first proposed by Ritt. This and some related algorithms have resurfaced as interest in resolvent structures have grown, spurred by its applicability.*

Unfortunately, the algebraic complexity of the resolvent and the computational complexity of the associated algorithms have never been satisfactorily explored. In this paper, we give single exponential upper and lower bounds for the degrees of the resolvent and its associated parametrizing polynomials. We also show that the resolvent can be computed deterministically in single exponential sequential and polynomial parallel time complexity. All previous algorithms for resolvent had relied on a random choice of certain extraneous parameters.

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1 Introduction

We begin with a self-contained exposition of the theory of *resolvents* for a system of algebraic equations in n unknowns and over an algebraically closed field k of characteristic zero. Ritt describes this theory in his books ([9,10]) both for systems of purely algebraic and of differential algebraic equations and we follow his presentation closely.

The motivation for studying *resolvents* comes from its connections with the birational isomorphisms that describe structures of irreducible algebraic varieties by means of an equivalent hypersurface and a one-to-one rational map. This connection will be elaborated further below. Furthermore, the structure provided by resolvents can also be fruitfully exploited in many applications involving description of and operations on algebraic surfaces.

Definition 1.1 (Rational Maps and Birational Isomorphisms) A *rational map* $\phi : X \to Y \subseteq A^n$ is a collection of n rational functions ϕ_1, ..., $\phi_n \in k(X)$, the field of rational functions over the variety X, such that $(\phi_1(x)$, ..., $\phi_n(x)) \in Y$ for every point $x \in X$ at which all the functions ϕ_i are regular.

A rational map $\phi : X \to Y$ is called a *birational isomorphism* if there is a rational map $\psi : Y \to X$ such that ψ is the inverse of ϕ. In this case X and Y are birationally isomorphic or equivalent. $\square$

A birational map between varieties is, set-theoretically, a one-to-one correspondence between open sets (in the sense of Zariski topology) of one variety with another.

This definition leads naturally to a classification, modulo birational maps, of all the irreducible varieties: Two varieties are in the same class if they are birationally equivalent. Such classification is quite rough. For example, it does not preserve the singularities of a variety, while instead it preserves the genus and the dimension of a variety. In fact it is a fundamental problem in algebraic geometry to look, inside a birational class of varieties, for representatives (also called *models*) with some special properties, for example *smoothness* (see [12] pp. 105–108, for a summary of the known results about the existence of non-singular models).

It is sometimes important for applications in graphics and solid modeling to find a birational model of a given variety that can be described with a minimal number of indeterminates. This model may be easier to parametrize; its points may be easier to construct, thus allowing it to be used, together with the rational map that gives the birational equivalence, in investigating the original variety or at least some open subsets of it. The existence of such a rational model is a consequence of the following basic theorem of Algebraic Geometry:

Theorem 1.1 *Every irreducible closed set X is birationally isomorphic to a hypersurface in some affine space A^n (see [12] pp. 29).* $\square$

The proof of this result is an application of Abel's "Primitive Element Theorem." It was well known already in the last century (see [2] pp. 28–29) and relies on the construction of the minimal polynomial for an algebraic element over a field.

The resolvent construction due to Ritt (but probably known to Kronecker for the algebraic case), gives another constructive proof of this result. Ritt, moreover, generalized the above theorem to algebraic systems of differential equations and it is one of the main tools in the investigation of such systems.

The resolvent computation relies on an elimination procedure: Ritt used his 'characteristic sets,' but it is easily seen that other techniques, for example Gröbner bases, can be used as well. Here we will use characteristic sets, and the bounds on their complexity in [3] and [4], to get bounds on the complexity of the resolvent.

For the definition of characteristic sets and algorithms to compute them the reader may refer to [4], [5], [8], [9] or [13].

2 The Resolvent

Through out this paper, $I = (f_1, \ldots, f_m)$ denotes a *prime ideal* generated by the polynomials $f_1, \ldots, f_m$ in the ring of $k[x_1, \ldots, x_n]$, where k is an *algebraically closed field of characteristic zero.*

Let $u_1, \ldots, u_q$ be a maximal set of algebraically independent variables, with respect to I, and let the other $p = (n - q)$ variables be renamed as $y_1, \ldots, y_p$.

Notice that it is possible to compute such a maximal set of independent variables using $O(m^{O(1)} d^{O(n^2)})$ time on a sequential computer or $O(n^4 \log^2(m + d + 1))$ time on a parallel computer ([1,7]). Following Ritt, we shall call this maximal set of algebraically independent variables a *parametric set*, since, in fact, they allow a rational parametrization of the variety, as shown below.

Consider the following ordering on the variables:

$$u_1 < u_2 < \cdots < u_q < y_1 < y_2 < \cdots < y_p.$$

A characteristic set for I, with respect to this ordering will be of the form $g_1, \ldots, g_p$ where g_i is a polynomial in $k[u_1, \ldots, u_q, y_1, \ldots, y_i]$. Moreover, since I is prime, $I = (g_1, \ldots, g_p)$.

Lemma 2.1 *If $h \in k[x_1, \ldots, x_n]$ is a polynomial not in I, then J, the ideal generated by $(g_1, \ldots, g_p, h)$, contains a polynomial only in the u_i's.*

PROOF.

Since I is prime, the variety $V(I)$ defined by the ideal

$$I = (g_1, \ldots, g_p)$$

is irreducible. Further, this variety $V(I)$ is not contained in the hypersurface defined by h, since, otherwise, this would imply $h \in I$. Thus each irreducible component of the variety defined by J has dimension at most $q-1$ and u_1, ..., u_q must be algebraically dependent with respect to J, i.e., there must be a polynomial in J only in the u_i's. This polynomial can be effectively computed using an elimination procedure. $\square$

The resolvent computation requires existence (but not direct construction) of an open subset (in the sense of Zariski topology) of the variety defined by I where there is a one-to-one correspondence between the values taken by the parameters and the values of some linear function in the y_i's. The existence of such an open set is proved in the following lemma (see [9] pp. 26–31), and is defined by $h \neq 0$.

Lemma 2.2 *Let I be a prime ideal generated by its characteristic set $(g_1, \ldots, g_p)$ constructed as above. Then there exist a polynomial h only in the u_i's and a polynomial $c = a_1y_1 + a_2y_2 + \cdots + a_py_p$, with a_j's constant, such that if*

$$(\overline{u_1}, \ldots, \overline{u_q}, y'_1, \ldots, y'_p) \quad \textit{and} \quad (\overline{u_1}, \ldots, \overline{u_q}, y''_1, \ldots, y''_p)$$

are two points of the variety defined by I then

$$h(\overline{u_1}, \ldots, \overline{u_q}) \neq 0 \quad \textit{and} \quad (y'_1, \ldots, y'_p) \neq (y''_1, \ldots, y''_p)$$

implies that

$$c(\overline{u_1}, \ldots, \overline{u_q}, y'_1, \ldots, y'_p) \neq c(\overline{u_1}, \ldots, \overline{u_q}, y''_1, \ldots, y''_p).$$

PROOF.
Introduce new variables $z_1, \ldots, z_p$ and consider the polynomials $s_1, \ldots, s_p$ obtained by substituting the y_i's with corresponding z_i's in the polynomials g_1, ..., g_p. Now, consider the variety V defined in a $q+2p$ dimensional affine space by the ideal J generated by $(g_1, \ldots, g_p, s_1, \ldots, s_p)$.

It is then easy to verify that V has still dimension q and that $u_1, \ldots, u_q$ form a maximal set of algebraically independent variables with respect to J. Let

$$V = V_1 \cup V_2 \cup \cdots \cup V_t$$

be an irreducible decomposition of V. For each irreducible component V_i of V, one of the following three cases holds:

1. *V_i is of dimension lower than q.* Thus, the prime ideal associated with this component in the irreducible decomposition of J must contain a polynomial only in the u_i's. Call this polynomial h_i.
2. *V_i is of dimension* q and it is contained in the q-dimensional linear subspace determined by the equations

$$z_1 = y_1, \quad z_2 = y_2, \quad \ldots, \quad z_p = y_p.$$

In this case, put $h'_i = 1$.

3. V_i *is of dimension* q and it is not contained in the q-dimensional linear subspace determined by the equations

$$z_1 = y_1, \quad z_2 = y_2, \quad \ldots, \quad z_p = y_p.$$

In this case V_i will not be contained in the hyperplane $z_j = y_j$ for some index j. Hence, the ideal generated by the prime ideal associated with V_i and by the polynomial $z_j - y_j$ must contain, by the preceding lemma, a polynomial only in the u_i's. Call this polynomial h'_i.

Let now $\overline{c}$ be the hyperplane of the linear equation

$$a_1(y_1 - z_1) + a_2(y_2 - z_2) + \cdots + a_p(y_p - z_p).$$

For a generic choice of a_i's, $\overline{c}$ intersects each irreducible component of V of dimension q in a lower dimensional variety, if the component is not completely contained in the diagonal

$$z_1 = y_1, \quad z_2 = y_2, \quad \ldots, \quad z_p = y_p.$$

Then, by the preceding lemma, the ideal generated by $\overline{c}$ and by the prime ideals associated with these irreducible components of V must contain a polynomial only in u_i's, say h''_i.

Define h to be the product of all of the h_i's, the h'_i's and the h''_i's. It is now simple to verify that the polynomials

$$c = a_1 y_1 + a_2 y_2 + \cdots + a_p y_p$$

and h, by construction, have the properties desired in the statement of the lemma; in fact, $\overline{c}$ is never zero in the open subset of V where $h \neq 0$. $\square$

Now, we are ready to define a *resolvent*:

Definition 2.1 (Resolvent) Let I be a prime ideal and let $g_1, \ldots, g_p$ a characteristic set for I with respect to an ordering of the variables in which the independent variables precede the dependent variables:

$$u_1 < u_2 < \ldots < u_q < y_1 < y_2 < \ldots < y_p,$$

u_i's form a maximal set of independent variables.

Let w be a new variable and

$$w - c \;=\; w - a_1 y_1 - \cdots - a_p y_p,$$

a linear polynomial with the choices of a_i's as in the preceding lemma.

Let $l, l_1, \ldots, l_p$ be a characteristic set of $L \subseteq k[x_1, \ldots, x_n, w]$, generated by $(g_1, \ldots, g_p, w - c)$, computed with respect to the following ordering:

$$u_1 < u_2 < \ldots < u_q < w < y_1 < y_2 < \ldots < y_p.$$

The first polynomial (i.e. of the smallest rank) of the characteristic set, $l = l(u_1, u_2, \ldots, u_q, w)$, is called a *resolvent* of I. $\square$

The following theorem is the main result for the theory of the resolvent:

Theorem 2.3 *Let I be a prime ideal with a characteristic set, $g_1, \ldots, g_p$, as in the preceding definition. Let w be a new indeterminate and let h and c be two polynomials satisfying the properties of lemma 2.2.*

Consider the ideal $L \subseteq k[x_1, \ldots, x_n, w]$,

$$L = (g_1, \ldots, g_p, w - c).$$

1. *L is prime.*
2. *Let l, $l_1, \ldots, l_p$ be a characteristic set of L, computed as in the preceding definition. Then $l =$ resolvent of I is only in the variables u_i's and in w, and each l_i is of degree 1 in y_i, i.e. is of the form*

$$l_i = l_{i1} y_i + l_{i2},$$

where l_{ik}'s are polynomials free from y_j's.

PROOF.
(1) Assume to the contrary, that is L is not prime. Consider two polynomials, f and g, such that $f\,g \in L$ while neither f nor g belongs to L. Next, observe that $L \cap k[x_1, \ldots, x_n] = I$ and that using the polynomial $w - c$, it is possible to eliminate w from a polynomial f to obtain a polynomial $\widehat{f} \in k[x_1, \ldots, x_n]$ such that $f \in L$ if and only if $\widehat{f} \in I$. Now, using the polynomials $\widehat{f}$ and $\widehat{g}$, resulting from the elimination process, it is easily seen that

$$\widehat{f} \notin I, \ \ \widehat{g} \notin I, \quad \text{but} \quad \widehat{f}\,\widehat{g} \in I.$$

But this contradicts our original assumption that I is prime.

(2) Since L is prime it is easy to verify that the dimension of the variety it determines is still q and that its characteristic sets contain $p + 1$ polynomials. Since the assumed ordering is

$$u_1 < \cdots < u_q \ < w < \ y_1 < \cdots < y_p,$$

l, the polynomial of the lowest rank is free from y_j's. Moreover it has to be irreducible as L is prime.

To prove that each l_i is of degree 1 in y_i, we begin by showing that l_1 is linear in y_1. Suppose the contrary holds and let $(\overline{u}, \overline{w})$ be a solution to the resolvent equation $l = 0$.

Then $l_1(\overline{u}, \overline{w}, y_1)$ has the same degree in y_1 as the polynomial $l_1(u, w, y_1)$, since l does not divide the initials of any of the l_i's. If $l_1(\overline{u}, \overline{w}, y_1)$ is of degree more than one, then the system of equations defining L has at least two distinct solutions, $(\overline{u}, \overline{w}, \overline{y})$ and $(\overline{u}, \overline{w}, \overline{y}')$ with $\overline{y} \neq \overline{y}'$. The polynomial h

(as in lemma 2.2) is not in L, because the u_i's are algebraically independent with respect to L. Hence, $h(\overline{u}) \neq 0$. But, then as a direct consequence of the lemma 2.2, we have

$$c(\overline{y}) \neq c(\overline{y}'),$$

and that not both $\overline{w} - c(\overline{y}) = 0$ and $\overline{w} - c(\overline{y}') = 0$. But then this contradicts our initial assumption that the system of equations defining L has at least two distinct solutions and that l_1 is of degree more than 1.

Since the polynomial l_1 is linear in y_1 it can be used to eliminate y_1 from l_2, …, l_p. And the arguments of the earlier paragraph can be repeatedly used to show that l_2 and the successive polynomials are all linear in the corresponding y_i's. $\square$

Note that the resolvent defines a hypersurface H in the $q+1$-dimensional space which is birational to the variety V defined by I. The equations of the rational map from H to V are obtained by resolving the polynomials l_1, …, l_p for the y_i's.

$$y_1 = \frac{l_{11}(u_1, \ldots, u_q, w)}{l_{12}(u_1, \ldots, u_q, w)}, \quad \ldots, \quad y_p = \frac{l_{p1}(u_1, \ldots, u_q, w)}{l_{p2}(u_1, \ldots, u_q, w)},$$

where $(u_1, \ldots, u_q, w) \in H$ ranges over the solutions of the resolvent $l(u_1, \ldots, u_q, w)$. Thus, the rational map from V to H is given by the projection on the u_i's of the points (u, y) of V and by the equation $w = c(y)$. As a result, we shall also say that the equations l_1, …, l_p provide a *parametrization* of the variety V defined by I.

The above results lead to a very straightforward algorithm to compute the resolvent of an irreducible variety defined by a prime ideal I via characteristic set computations. However, a direct implementation of the constructions given in the proof leads to the computation of characteristic sets twice and consequently, fails to provide a tight upper bound for the degree of the resolvent and the parametrization. A sketch of the algorithm may be as follows:

Algorithm RESOLVENT: (first version)

- **Input:** A set of generators for the prime ideal I: $f_1, f_2, \ldots, f_m$.
- **Output:** Resolvent, l and a rational parametrization defined by: $l_1, l_2, \ldots, l_p$.

1. Compute a maximal set of independent variables u_1, …, u_q with respect to I. Rename the other variables as y_1, …, y_p.

2. Compute a characteristic set g_1, …, g_p for I with respect to the ordering

$$u_1 < \ldots < u_q < y_1 < \ldots < y_p.$$

3. repeat the following steps

step1. Choose at random a_1, …, a_p elements in k.

step2. Compute a characteristic set l, l_1, …, l_p for the ideal generated by g_1, …, g_p, $w - a_1 y_1 - \cdots - a_p y_p$ with respect to the ordering

$$u_1 < \ldots < u_q < w < y_1 < \ldots < y_p.$$

step3. If the polynomials l_1, …, l_p are linear in the variables $y_1, \ldots, y_p$
then output l as the resolvent and the l_i's as a rational parametrization of an open set of the original variety and terminate.

end{RESOLVENT.} $\square$

There is no need to compute a characteristic set twice. It is straightforward to verify that the following algorithm correctly computes the resolvent. It will be used in the next section to provide a better upper bounds for the degree of the resolvent.

Algorithm RESOLVENT: (second version)

- **Input:** A set of generators for the prime ideal I: $f_1, f_2, \ldots, f_m$.
- **Output:** Resolvent, l and a rational parametrization defined by: $l_1, l_2, \ldots, l_p$.

1. Compute a maximal set of independent variables u_1, …, u_q with respect to I. Rename the other variables as y_1, …, y_p.

2. repeat the following steps

step1. Choose at random a_1, …, a_p elements in k.

step2. Compute a characteristic set l, l_1, …, l_p for the ideal generated by f_1, …, f_m, $w - a_1 y_1 - \cdots - a_p y_p$ with respect to the ordering

$$u_1 < \ldots < u_q < w < y_1 < \ldots < y_p.$$

step3. If the polynomials l_1, …, l_p are linear in the variables $y_1, \ldots, y_p$
then output l as the resolvent and the l_i's as a rational parametrization of an open set of the original variety and terminate.

end{RESOLVENT.} $\square$

As stated in the proof of the previous theorem the probability that the condition in step 3 of the previous algorithm will be satisfied approaches 1 if the choice of the a_i is done over a sufficiently large subset of k. To prove this observe that, if one treats the a_i's as variables in the computation of Step 4 each polynomial l_i is of the form $l_{i1}(a, u, w)y_i + l_{i2}(a, u, w)$, where l_{ij} is a polynomial in u_i's and a_j's whose degree can be bounded a priori, knowing the degree of the f_k's. The 'good' a_j's hence should not be such that they make the l_{i1}'s identically zeros. We shall show how using the techniques similar to the ones in [11], it is possible to devise deterministic algorithms for resolvent computation without any additional complexity penalty.

3 Bounds on the Degree of the Resolvent

In this section we will present upper and lower bounds on the degree of the resolvent for a prime ideal $I \subseteq k[x_1, \ldots, x_n]$ generated by m polynomials f_1, …, f_m each of degree at most d in the x_i's.

3.1 Lower bound

Example 3.1 In the ring of polynomials $k[u_1, \ldots, u_n, y_1, \ldots, y_n]$ consider the ideal I generated by the polynomials

$$y_1^d - u_1,\ y_2^d - u_2,\ \ldots,\ y_n^d - u_n.$$

I is prime. The variables $u_1, \ldots, u_n$ are independent with respect to I. Thus the variety V defined by I in a $2n$-dimensional affine space is irreducible and of dimension n.

The above set of generators is a characteristic set for I with respect to the ordering

$$u_1 < u_2 < \cdots < u_n < y_1 < y_2 < \cdots < y_n.$$

From the results in the previous section, V is birational to a hypersurface in a $n+1$-dimensional affine space. This is equivalent to saying that the field of rational functions over V, $k(V)$, obtained by taking the quotient field of $k[u_1, \ldots, u_n, y_1, \ldots, y_n]/I$ is isomorphic to the field of rational functions $k(u_1, u_2, \ldots, u_n, \overline{w})$ where u_i's are algebraically independent over k and $\overline{w}$ is algebraic over $k(u_1, u_2, \ldots, u_n)$.

It is clear, then, that the degree in w of the resolvent cannot be less than the degree of the minimal polynomial g for $\overline{w}$ over $k(u_1, u_2, \ldots, u_n)$. From field theory the degree of g is equal to the dimension of $k(u_1, u_2, \ldots, u_n, \overline{w})$ as a vector space over $k(u_1, u_2, \ldots, u_n)$. This is of course equal to the dimension of $k(V)$ as a vector space over $k(u_1, u_2, \ldots, u_n)$. Now it is easy to observe that $k(V)$ is generated, as a vector space over $k(u_1, u_2, \ldots, u_n)$ by the monomials in y_i's with degree in each y_i less than d. Since there are d^n of such monomials the degree of the resolvent must be at least of degree d^n. □

Observe that the degree of the resolvent depends on the particular ordering chosen for the variables. In fact V is a rational variety and with the choice of y_i's as parameters, it is immediately seen that V is birational to a hyperplane in the $n+1$-dimensional affine space. This phenomenon was already known to Ritt (see [9] pp. 44).

In the example above we were concerned only with the degree of the resolvent. The following (rather classical) example shows that the degree of the expressions involved in the rational map from the variety to its birationally equivalent surface may necessarily be of high degree also.

Example 3.2 In the ring $k[u, y_1, \ldots, y_{n-1}]$, consider the ideal I generated by the polynomials

$$y_1 - y_2^d,\ y_2 - y_3^d,\ \ldots,\ y_{n-1} - u^d.$$

The variety V described by I is irreducible and it is a rational curve, parametrized by u. The resolvent is then $w = 0$. However, the parametric equations for the curve are

$$y_1 - u^{d^n},\ y_2 - u^{d^{n-1}},\ \ldots,\ y_{n-1} - u^d. \quad \square$$

3.2 Upper bound

We begin by looking at the degree bounds for the resolvent and its associated parametrizing polynomials (i.e. l_i's). For a polynomial $f \in k[u_1, \ldots, u_q, w, y_1, \ldots, y_p]$, we define $\deg_{u_i}(f)$ (degree of f with respect to u_i) as the maximum degree of the variable u_i appearing in f. We define $\deg_w(f)$ and $\deg_{y_i}(f)$ analogously. We also use the notations $\deg_U(f)$ and $\deg_Y(f)$ to imply

$$\deg_U(f) = \sum_{i=1}^{q} \deg_{u_i}(f), \quad \text{and} \quad \deg_Y(f) = \sum_{i=1}^{p} \deg_{y_i}(f).$$

Finally, we write $\deg(f)$ to mean

$$\deg(f) = \deg_U(f) + \deg_w(f) + \deg_Y(f).$$

The following theorem follows from the bounds on the degrees of the polynomials of a characteristic set given in [4].

Theorem 3.1 (Resolvent Upper Bound Theorem) *Let $I = (f_1, \ldots, f_m)$ be a prime ideal in $k[x_1, \ldots, x_n]$ (k is a field of characteristic zero) and $\deg(f_i) \leq d$. Assume that $x_1 = u_1, \ldots, x_q = u_q$ are the independent variables with respect to I and the remaining $p = n - q$ variables, $x_{q+1} = y_1, \ldots, x_n = y_p$, are dependent. Let w be the new variable introduced as in the definition of the resolvent.*

Let l be the resolvent of I and $l_1, \ldots, l_p$, its associated parametrizing polynomials. Then

$$\begin{aligned}\deg(l) &= O\Big(m\ d^{O(p^2)}\Big), \quad \textit{and}\\ \deg(l_i) &= O\Big(m\ d^{O(p^2)}\Big), \quad \textit{for all } i = 1, \ldots, p.\end{aligned}$$

PROOF.
The proof follows from the GENERAL UPPER BOUND THEOREM of [4] and the algorithm for Resolvent as in the second version. □

In fact a careful examination of the proof of the GENERAL UPPER BOUND THEOREM of [4] reveals more.

$$\begin{aligned}\deg_U(l) &\leq 4(m+1)(18p)^{4p} d(d+1)^{16p^2},\\ \deg_w(l) &\leq 2(d+1)^{2p+2},\\ \deg_Y(l) &= 0, \quad \text{and}\end{aligned}$$

$$\begin{aligned}\deg_U(l_i) &\leq 4(m+1)(18p)^{4p} d(d+1)^{16p^2},\\ \deg_w(l_i) &\leq 2(d+1)^{2p+2},\\ \deg_Y(l) &= 1, \quad \text{for all } i = 1, \ldots, p.\end{aligned}$$

Furthermore, l and each of the l_i's can be expressed as a linear combination of the f_j's and $w - c = w - a_1 y_1 - \cdots - a_p y_p$ as follows

$$\begin{aligned}l &= b_0(w-c) + \sum_{j=1}^{m} b_j f_j,\\ l_i &= a_{i0}(w-c) + \sum_{j=1}^{m} a_{ij} f_j,\end{aligned}$$

where b's and a's are polynomials in $k[u_1, \ldots, u_q, w, y_1, \ldots, y_p]$ and

$$\begin{aligned}\deg(b_0),\ \deg(b_j f_j) &\leq 11(m+1)(18p)^{4p} d(d+1)^{16p^2}, \quad \text{and}\\ \deg(a_{i0}),\ \deg(a_{ij} f_j) &\leq 11(m+1)(18p)^{4p} d(d+1)^{16p^2}.\end{aligned}$$

These bounds lead to straightforward algorithms to compute the resolvent and correspondingly, a single exponential sequential time bound and a polynomial parallel time bound. (See [4].)

Theorem 3.2 (Randomized Complexity of Resolvent) *Let $I = (f_1, \ldots, f_m)$ be a prime ideal in $k[x_1, \ldots, x_n]$ (k is a field of characteristic zero) and $\deg(f_i) \le d$. Assume that $x_1 = u_1, \ldots, x_q = u_q$ are the independent variables with respect to I and the remaining $p = n - q$ variables, $x_{q+1} = y_1, \ldots, x_n = y_p$, are dependent.*

Then assuming a suitable choice of the a_i's, one can compute the resolvent of I and its associated parametrizing polynomials, in $O\Big(m^{O(n)}(d+1)^{O(n^3)}\Big)$ sequential time or $O(n^7 \log^2(m+d+1)$ parallel time. $\square$

Since a random choice of a_i's satisfies the requirements with probability 1, the above theorem gives a probabilistic complexity analysis for our Resolvent algorithm in its second version.

However, the algorithm can be made deterministic since it is possible to search for appropriate a_i's over large subsets of k, where the search process is guaranteed to succeed.

Let
$$K = \{c_1, c_2, c_3, \ldots\} \subseteq k$$
be a countably infinite subset of k. For instance, we could have chosen
$$K = \{1, 2, 3, \ldots\}$$
where
$$1 = 1, \;\; 2 = 1+1, \;\; 3 = 1+1+1, \;\; \text{etc.}$$
Our a_i's will be chosen from some large but finite subsets $S_i \subset K$.

Let $f(a_1, \ldots, a_p) \in k[a_1, \ldots, a_p]$ be a nontrivial multivariate polynomial, i.e.
$$f(a_1, \ldots, a_p) \not\equiv 0.$$
We would like to count the number of elements of $S_1 \times S_2 \times \cdots \times S_p \subset K^p$ where f vanishes, i.e. the cardinality of the set
$$Z(f) = \{(\overline{a_1}, \ldots, \overline{a_p}) \;:\; \overline{a_1} \in S_1, \ldots, \overline{a_p} \in S_p \& f(\overline{a_1}, \ldots, \overline{a_p}) = 0\}.$$
Let $d_1 = \deg_{a_1}(f), \ldots, d_p = \deg_{a_p}(f)$.

$$f = f_{d_1}(a_2, \ldots, a_p)a_1^{d_1} + \cdots + f_0(a_2, \ldots, a_p).$$
At a point $(\overline{a_1}, \ldots, \overline{a_p})$, $f(\overline{a_1}, \ldots, \overline{a_p})$ can vanish for one of two reasons:

1. $\overline{a_1}$ is a root of the univariate polynomial in a_1 with coefficients $f_j(\overline{a_2}, \ldots, \overline{a_p})$,
2. for each j $(0 \le j \le d_1)$, $f_j(\overline{a_2}, \ldots, \overline{a_p}) = 0$.

Proceeding as in [11], we have

$$|Z(f)| \leq d_1 \prod_{j=2}^{p} |S_j| + |Z(f_{d_1})|\, |S_1|.$$

Thus

$$\begin{aligned}
\frac{|Z(f)|}{\prod_{j=1}^{p} |S_j|} &\leq \frac{d_1}{|S_1|} + \frac{|Z(f_{d_1})|}{\prod_{j=2}^{p} |S_j|} \\
&\leq \sum_{j=1}^{p} \frac{d_j}{|S_j|} \\
&\leq \frac{\sum_{j=1}^{p} d_j}{\min |S_j|} = \frac{\deg(f)}{\min |S_j|}.
\end{aligned}$$

Now choosing

$$S_1 = S_2 = \cdots = S_p = \ S \subset K, \quad \text{and} \quad |S| = 2\deg(f),$$

we are guaranteed to have at least one elment in S^p at which f assumes a nonzero value. Actually, f does not vanish for at least half the points of S^p; that is, if we choose a point uniformly randomly from S^p then, after two draws on the average, we would have found a point where f does not vanish.

Now going back to our original problem, let us perform our resolvent computation with a_i's as symbolic variables. (In the ordering of the variables, a_i's are assumed to occur before all other variables.) Then our parametrizing polynomials are of the following form:

$$\begin{aligned}
l_1 &= l_{11}(a_1, \ldots, a_p, u_1, \ldots, u_q, w) y_1 + l_{12}(a_1, \ldots, a_p, u_1, \ldots, u_q, w) \\
&\vdots \\
l_p &= l_{p1}(a_1, \ldots, a_p, u_1, \ldots, u_q, w) y_p + l_{p2}(a_1, \ldots, a_p, u_1, \ldots, u_q, w).
\end{aligned}$$

Now, we would like to choose $(\overline{a_1}, \ldots, \overline{a_p})$ such that

$$\begin{aligned}
l_{11}(\overline{a_1}, \ldots, \overline{a_p}, u_1, \ldots, u_q, w) &\not\equiv 0, \\
&\vdots \\
l_{p1}(\overline{a_1}, \ldots, \overline{a_p}, u_1, \ldots, u_q, w) &\not\equiv 0.
\end{aligned}$$

Equivalently, if

$$L(a_1, \ldots, a_p, u_1, \ldots, u_q, w) = \prod_{j=1}^{p} l_{j1}(a_1, \ldots, a_p, u_1, \ldots, u_q, w),$$

our choice of $(\overline{a_1}, \ldots, \overline{a_p})$ must satisfy

$$L(\overline{a_1}, \ldots, \overline{a_p}, u_1, \ldots, u_q, w) \not\equiv 0.$$

Now, using the general upper bound theorem of [4], we see that for all j $(1 \leq j \leq p)$,

$$\deg_A(l_{j1}) \leq 4(m+1)(18p)^{4p} d(d+1)^{16p^2},$$

and

$$\deg_A(L) \leq 4(m+1)p(18p)^{4p} d(d+1)^{16p^2}.$$

Thus, if we choose an $I \subset K$ such that

$$|S| = 8(m+1)p(18p)^{4p} d(d+1)^{16p^2}$$

then there is an $(\overline{a_1}, \ldots, \overline{a_p}) \in S^p$ for which the resolvent algorithm is guaranteed to produce appropriate parametrizing polynomials. Since we need to search only over a space of cardinality

$$|S^p| = O\left(m^{O(p)} d^{O(p^3)}\right),$$

the time complexities of our algorithms (both under sequential as well as parallel models) remain unaffected. The deterministic version of our algorithm is as follows:

Algorithm Resolvent: (third version)

- **Input:** A set of generators for the prime ideal I: $f_1, f_2, \ldots, f_m$.
- **Output:** Resolvent, l and a rational parametrization defined by: $l_1, l_2, \ldots, l_p$.

1. Compute a maximal set of independent variables $u_1, \ldots, u_q$ with respect to I. Rename the other variables as $y_1, \ldots, y_p$.

2. Let $S = \{c_1, c_2, \ldots, c_N\}$, $c_i \in k$, c_i's distinct.

$$N = 8(m+1)p(18p)^{4p} d(d+1)^{16p^2}$$

3. foreach $a_1 \in S, \ldots, a_p \in S$
repeat the following steps

step1. Compute a characteristic set $l, l_1, \ldots, l_p$ for the ideal generated by $f_1, \ldots, f_m, w - a_1 y_1 - \cdots - a_p y_p$ with respect to the ordering

$$u_1 < \ldots < u_q < w < y_1 < \ldots < y_p.$$

step2. If the polynomials $l_1, \ldots, l_p$ are linear in the variables $y_1, \ldots, y_p$
then output l as the resolvent and the l_i's as a rational parametrization of an open set of the original variety and terminate.

end{RESOLVENT.} $\square$

Thus, in summary, we have the following:

Theorem 3.3 (Complexity of Resolvent) *Let $I = (f_1, \ldots, f_m)$ be a prime ideal in $k[x_1, \ldots, x_n]$ (k is a field of characteristic zero) and $\deg(f_i) \leq d$. Assume that $x_1 = u_1, \ldots, x_q = u_q$ are the independent variables with respect to I and the remaining $p = n - q$ variables, $x_{q+1} = y_1, \ldots, x_n = y_p$, are dependent.*

Then one can compute the resolvent of I and its associated parametrizing polynomials, deterministically in $O\left(m^{O(n)}(d+1)^{O(n^3)}\right)$ sequential time or $O(n^7 \log^2(m+d+1)$ parallel time. $\square$

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