SCHWARZ METHODS OF NEUMANN-NEUMANN TYPE FOR THREE-DIMENSIONAL ELLIPTIC FINITE ELEMENT PROBLEMS

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Abstract. Several domain decomposition methods of Neumann-Neumann type are considered for solving the large linear systems of algebraic equations that arise from discretizations of elliptic problems by finite elements. We will only consider problems in three dimensions. Several new variants of the basic algorithm are introduced in a Schwarz method framework that provides tools which have already proven very useful in the design and analysis of other domain decomposition and multi-level methods.

The Neumann-Neumann algorithms have several advantages over other domain decomposition methods. The subregions, which define the subproblems, only share the boundary degrees of freedom with their neighbors. The subregions can also be of quite arbitrary shape and many of the major components of the preconditioner can be constructed from subprograms available in standard finite element program libraries. However, in its original form, the algorithm lacks a mechanism for global transportation of information and its performance therefore suffers when the number of subregions increases. In the new variants of the algorithms, considered in this paper, the preconditioners include global components, of low rank, to overcome this difficulty. Bounds are established for the condition number of the iteration operator, which are independent of the number of subregions, and depend only polylogarithmically on the number of degrees of freedom of individual local subproblems. Results are also given for problems with arbitrarily large jumps in the coefficients across the interfaces separating the subregions.

Key words. domain decomposition, elliptic finite elements, parallel computing, preconditioners, conjugate gradient methods, Schwarz methods

AMS(MOS) subject classifications. 65N22, 65N30, 65N55

1. Introduction.

The Neumann-Neumann algorithms are domain decomposition methods for solving large linear systems of algebraic equations arising from elliptic partial differential equations. Their origin can be traced to the work of Dihn, Glowinski, and Périaux [12]. The algorithms have been developed further by a number of French scientists in particular Bourgat, Glowinski, Le Tallec, and Vidrascu [3], De Roeck and Le Tallec [11], Le Tallec, De Roeck, and Vidrascu [25]; see in particular the thesis of Yann-Hervé De Roeck [10].

We have previously discussed this method, and variants with preconditioners that include a coarse space component, in two conference papers; cf. Dryja and Widlund [18,15]. The purpose of this paper is to provide a fuller discussion, to introduce additional variants of the algorithm, and to give detailed proofs. We also again address the interesting issue of finding preconditioners with a performance insensitive

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to the jumps in the coefficients that e.g. arise in simulation of composite materials; cf. De Roeck and Le Tallec [11]. Previous results that are independent of the size of such discontinuities have primarily been given for preconditioners that can be constructed from a direct sum of subspaces. Examples of such methods are given in Bramble, Pasciak, and Schatz [5], Mandel [28,27,26], and Smith [38]; see also the discussion in Dryja and Widlund [18] or section 2.5 of Smith [37]. For other recent work on Neumann-Neumann preconditioners that incorporates a coarse solver, see Mandel [30,31].

It has been known since 1958, cf. Hestenes [24], that the rate of convergence of a preconditioned conjugate gradient method can be estimated in terms of the condition number of a generalized eigenvalue problem; see also Golub and Van Loan [22]. The results for all good algorithms of the Neumann-Neumann class show that the condition number of the operator, which is relevant for the iterative method, is bounded polylogarithmically in H/h and that the bound is independent of the number of subregions. Here H and h denote the typical diameter of a subregion and an element, respectively. We note that $\log(H/h)$ is a measure of the number of refinement steps if the final triangulation is obtained by successive refinement, by a fixed factor, starting from an original, coarse decomposition into subregions. Typically, if a loosely coupled parallel or distributed computer system is used, each subregion is assigned to one or a pair of processors of the system. Feasible values of H/h are then dictated by the size of the local memory of each processor. We can therefore expect that a doubling of H/h would require at least an eightfold increase in the size of the local memory. If, on the other hand, the number of subregions and processors grows, it becomes increasingly important that the rate of convergence should remain close to constant; for a discussion of actual numerical performance of various domain decomposition methods for three-dimensional problems, see Gropp and Smith [23], Mandel [28,27,29], and Smith [39].

We focus exclusively on three dimensional problems, and assume that the elliptic problem is of second order, and that it is defined on a bounded polygonal region Ω. Two model problems are introduced in Section 2 where we also show how the large linear system of equations can be reduced to a system for the degrees of freedom on Γ, the union of the interfaces that separate the subregions. As an introduction, we also discuss the algorithm in the case of two subregions. In Section 3, we develop basic tools for the analysis of Schwarz methods. We have written on this subject before but have found that the current more powerful version of the theory provides improved bounds in several cases. In Section 4, we prove a result for the original Neumann-Neumann algorithm. As shown already in Widlund [40], we must introduce a coarse space in order to make the rate of convergence independent of the number of substructures. A variety of possibilities are explored in the remainder of the paper where we establish polylogarithmic bounds, i.e. establish that the algorithms are almost optimal in a certain sense. Several of these results have previously been announced without proof in Dryja and Widlund [20].

At times our analysis is quite technical. We have found subtle differences between

methods sometimes succeeding in establishing a strong result for one method while failing in the case of another, seemingly quite similar algorithm.

It is known from numerical experiments, as well as theory, that the rate of convergence of many domain decomposition algorithms is adversely affected by high aspect ratios of the subregions; see e.g. Mandel and Lett [33] and Smith [37]. It is believed that the Neumann-Neumann algorithms are less sensitive to extreme geometry than the iterative substructuring methods of Bramble, Pasciak, and Schatz [5], Dryja [13], and Smith [38]. We know of no systematic experimental study comparing the performance of the Neumann-Neumann and other main contending algorithms for difficult problems in three dimensions, but recent results by Mandel and Brezina [32] indicate that the Neumann-Neumann algorithms indeed are quite promising in this respect. We note, however, that the Neumann-Neumann algorithm can be up to twice as expensive per step as the Neumann-Dirichlet algorithms introduced in Dryja, Proskurowski, and Widlund [14], Dryja [13], and Widlund [40]; cf. discussion in Section 2. At this time, it is not clear how much more robust the Neumann-Neumann algorithms, which treat the subregions in a fully symmetric way, are than the Neumann-Dirichlet methods if equal care is exercised in designing a proper scaling of the variables.

2. The Finite Element Model Problems. To simplify our presentation, we consider only two model problems: a standard Poisson equation and a special second order problem with discontinuous, piecewise constant coefficients.

Model Problem I is of the form:

Find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = f(v), \qquad \forall v \in H_0^1(\Omega) \tag{1}$$

where

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx, \quad f(v) = \int_\Omega f v \, dx \tag{2}$$

and $H_0^1(\Omega)$ is the subspace of $H^1(\Omega)$ with zero trace on $\partial\Omega$, the boundary of Ω. It is well known that this bilinear form defines the standard seminorm of $H^1(\Omega)$. We also consider the case where an essential boundary condition is imposed only on a subset $\partial\Omega_D \subset \partial\Omega$ of positive measure with a natural boundary condition on its complement $\partial\Omega_N = \partial\Omega \setminus \partial\Omega_D$. By Friedrichs' inequality, $a(\cdot,\cdot)$ is still positive definite; cf. Nečas [35].

The region Ω is a bounded, polyhedral region in three dimensions. A coarse triangulation is introduced by dividing Ω into nonoverlapping simplices Ω_i, $i = 1, \ldots, N$, also called substructures. We assume that $\partial\Omega_D$ is the union of the closure of faces of some, or all, of the boundary substructures. The substructures Ω_i are further divided into elements in such a way that a conforming triangulation of all of Ω is obtained. We associate parameters H and h with these coarse and fine triangulations and assume that all the substructures and elements are shape regular in the sense common in finite element theory. All our results, except those in Section 5, can be extended to

non-simplicial substructures. The sets of nodes in $\Omega_i, \partial\Omega_i$ are denoted by $\Omega_{i,h}, \partial\Omega_{i,h}$, etc.

Let $V^h(\Omega)$ be the finite element space of continuous, piecewise linear functions, defined on the fine triangulation, which vanish on $\partial\Omega_D$. The discrete Model Problem I is of the form:

Find $u_h \in V^h$ such that

$$a(u_h, v_h) = f(v_h), \qquad \forall v_h \in V^h \tag{3}$$

or alternatively, find x such that

$$Kx = b \tag{4}$$

Here K is the stiffness matrix with elements $k_{ij} = a(\phi_i, \phi_j)$ and ϕ_i and ϕ_j are standard finite element basis functions. x is the vector of nodal values and b the load vector with elements $f(\phi_i)$.

Our results can be extended to general conforming finite element approximations and self-adjoint, second order, elliptic problems. This includes cases where the coefficients are discontinuous and vary greatly from substructure to substructure. For some of our algorithms, we are able to obtain bounds that are independent of the size of the jumps. In particular, we consider *Model Problem II*, in which case

$$a^{(\rho)}(u, v) = \int_\Omega \rho(x)\, \nabla u \cdot \nabla v \, dx \tag{5}$$

with $\rho(x) > 0$. We assume that the jumps of $\rho(x)$ occur only at substructure interfaces and, for simplicity, that the coefficient takes on a constant value $\rho(x) = \rho_i, x \in \Omega_i$, in each substructure. We can easily generalize our results to the case when the relative variation of $\rho(x)$ over each subregion is modest.

In a first step of many domain decomposition methods, all unknowns except those common to at least two substructures are eliminated, by Gaussian elimination, reducing the system (4) to a system of linear algebraic equations for the nodal values on the interface Γ only. We now describe this procedure.

Let $K^{(i)}$ be the stiffness matrix of the bilinear form $a_i(u_h, v_h) \equiv a_{\Omega_i}(u_h, v_h)$ which represents the contribution of the substructure Ω_i to the integral $a(u_h, v_h) \equiv a_\Omega(u_h, v_h)$. Let x and y be the vectors of nodal values that correspond to two arbitrary finite element functions u_h and v_h, respectively. Then the stiffness matrix K of the entire problem can be obtained by using the *method of subassembly,*

$$x^T K y = \sum_i x^{(i)^T} K^{(i)} y^{(i)} \tag{6}$$

Here $x^{(i)}$ is the subvector of all nodal parameters of $\Omega_i \cup \partial\Omega_i$. We represent $K^{(i)}$ as

$$\begin{pmatrix} K_{II}^{(i)} & K_{IB}^{(i)} \\ K_{IB}^{(i)T} & K_{BB}^{(i)} \end{pmatrix} \tag{7}$$

dividing the subvector $x^{(i)}$ into two, $x_I^{(i)}$ and $x_B^{(i)}$, corresponding to the variables that are interior to the substructure and those on Γ, respectively. All of the latter are shared with at least one other substructure. In the case when $\partial\Omega_N$ is not empty, we allocate the degrees of freedom on $\partial\Omega_N \setminus \Gamma$ to the appropriate sets of interior variables. We note that given the $K^{(i)}$ for Model Problem I, we can, after introducing weights ρ_i, easily assemble the stiffness matrix for Model Problem II in the same way as in the formula (6).

Since the interior variables are associated with only one of the substructures, they can be eliminated locally and in parallel. The resulting reduced matrices are Schur complements and have the form

$$S^{(i)} = K_{BB}^{(i)} - K_{IB}^{(i)T} K_{II}^{(i)-1} K_{IB}^{(i)} \tag{8}$$

It now follows straightforwardly that the Schur complement corresponding to the global stiffness matrix K and the nodal values of Γ_h is given by S where

$$x_B^T S y_B = \sum_i x_B^{(i)T} S^{(i)} y_B^{(i)} \tag{9}$$

We note that this is also a process of subassembly and that we can obtain the Schur complement for Model Problem II from the weights ρ_i and the $S^{(i)}$ of Model Problem I. If the local problems are solved exactly, what remains is to find a sufficiently accurate approximation of the solution of the linear system

$$S x_B = \tilde{b}_B \tag{10}$$

The new right hand side $\tilde{b}_B$ is obtained from the elimination of all the sets of interior variables. We note that the elimination of the interior variables of a substructure can be viewed in terms of an orthogonal projection, with respect to the bilinear form, of the solution u_h of (3) onto the subspace $H_0^1(\Omega_i) \cap V^h$. (We will always tacitly assume that $H_0^1(\Omega_i)$ is extended by zero outside of Ω_i.) These subspaces are orthogonal, in the sense of $a(\cdot,\cdot)$, to the *piecewise discrete harmonic* functions defined by

$$a(u_h, v_h) = 0\ , \qquad \forall v_h \in H_0^1(\Omega_i) \cap V^h, \quad i = 1, \ldots, N \tag{11}$$

or alternatively by

$$K_{II}^{(i)} x_I^{(i)} + K_{IB}^{(i)} x_B^{(i)} = 0, \quad i = 1, \ldots, N \tag{12}$$

From now on, we will only consider finite element functions and drop the subscript h from our functions.

It is convenient to rewrite (10) in variational form. Let $s_i(u,v)$ and $s(u,v)$ be the forms defined by (9), i.e.

$$s_i(u,v) = x_B^{(i)} S^{(i)} y_B^{(i)} \quad \text{and} \quad s(u,v) = x_B^T S y_B \tag{13}$$

In what follows, it is convenient to associate the bilinear form $s_i(\cdot,\cdot)$ with Model Problem I and write those of Model Problem II as $\rho_i s_i(\cdot,\cdot)$.

Equation (10) can be rewritten as

$$y_B^T S x_B = y_B^T \tilde{b}_B, \quad \forall y_B \quad \text{or} \quad s(u,v) = \tilde{f}(v), \quad \forall v \in V^h(\Gamma) \tag{14}$$

For Model Problem II, we obtain the bilinear form $s^{(\rho)}(\cdot,\cdot)$, by subassembly, from the forms $\rho_i s_i(\cdot,\cdot)$.

Here, and in what follows, u is the discrete harmonic part of the solution and $V^h(\Gamma)$ the subspace of $V^h(\Omega)$ spanned by the nodal basis functions associated with the nodes on Γ. It is easy to see that since u is discrete harmonic, we can instead use discrete harmonic functions, uniquely defined in terms of their values on Γ, as test functions. In what follows, we will therefore primarily regard $V^h(\Gamma)$ as a space of discrete harmonic functions.

Problem (10) can be solved by a variety of Neumann-Neumann methods, some of which will be described as Schwarz type methods in this paper. As an introduction, we now consider the case of two substructures; see Bjørstad and Widlund [1,2] for early work on a variety of algorithms. In the two substructure case, it is natural to denote the subvectors of nodal values associated with the two substructures Ω_1, Ω_2, and their common interface Γ by $x^{(1)}, x^{(2)}$, and $x^{(3)}$, respectively. Then, the construction of an iterative substructuring method amounts to finding a preconditioner for

$$S x^{(3)} = (S^{(1)} + S^{(2)}) x^{(3)} = b^{(3)}$$

The well-known Neumann-Dirichlet method corresponds to multiplying this equation by $(S^{(1)})^{-1}$, or $(S^{(2)})^{-1}$, and solving the resulting equation by a conjugate gradient method. The Schur complements need not be explicitly computed and the action of $(S^{(1)})^{-1}$ on a vector can be found at the expense of solving a problem on Ω_1, with appropriate Neumann data on Γ, and then extending this solution continuously to Ω_2 by solving a Dirichlet problem; cf. Bjørstad and Widlund [2]. For the Neumann-Neumann method, we instead use $(S^{(1)})^{-1} + (S^{(2)})^{-1}$ treating the subregions in a symmetric way. We then have to solve one Neumann and one Dirichlet problem on each subregion in each step of the iteration.

In the case of many regions, a Neumann-Neumann method can be derived from a special Neumann-Dirichlet algorithm if there is a red-black ordering of the subregions; cf. Dryja [13], Dryja and Widlund [18], and Widlund [40]. Good bounds can therefore be obtained, for certain variants of the Neumann-Neumann method, from older results on the Neumann-Dirichlet algorithm by noting that the operator of the preconditioned system of equations is the sum of two Neumann-Dirichlet operators corresponding to a red-black and the corresponding black-red ordering. We also note that the preconditioner for the original Neumann-Neumann algorithm can be constructed from the $(S^{(i)})^{-1}$, and certain diagonal scaling matrices, by a subassembly process; cf. De Roeck [10]. Here we have ignored the technical complication stemming from the fact that the Schur complements, corresponding to interior subregions, can be singular. This is no longer an issue for the algorithms, considered in this paper.

3. Abstract Schwarz Methods. In this section, we describe and analyze the convergence of abstract Schwarz methods. This is a modification of a theory that

has been developed previously for the additive case by Dryja and Widlund [16,17,19] and Nepomnyaschikh [34]. A difference is that we now include the effects of inexact solvers from the beginning.

Let V be a finite dimensional space with the scalar product $b(u,v)$. We consider the abstract problem

$$b(u,v) = f(v), \qquad \forall v \in V \tag{15}$$

and let

$$V = V_0 + V_1 + \cdots + V_N$$

This is not necessarily a direct sum of spaces; in many cases of interest, the representation of an element of V in terms of components of the V_i is not unique. The first space V_0 represents a special coarse subspace. If no such space is used, it is just left out of the consideration.

Let $b_i(u,v),\ i = 0, \cdots, N$, be symmetric, positive definite bilinear forms on $V_i \times V_i$. We introduce operators $T_i : V \to V_i$, by

$$b_i(T_i u, v) = b(u,v), \qquad \forall v \in V_i \tag{16}$$

and put

$$T = T_0 + T_1 + \cdots + T_N$$

A possible choice is $b_i(u,v) = b(u,v)$. In that case, the operator $T_i = P_i$, the projection which is orthogonal with respect to the inner product $b(\cdot,\cdot)$.

We replace (15) by

$$Tu = g, \quad g = \sum_{i=0}^{N} g_i, \quad g_i = T_i u \tag{17}$$

The right hand side g is obtained by solving

$$b_i(g_i, v_i) = b(u, v_i) = f(v_i), \quad \forall v_i \in V_i$$

Equation (17) is typically solved by a conjugate gradient method, without further preconditioning, using $b(\cdot,\cdot)$ as the inner product.

THEOREM 1. *Let*

(i) there exist a constant C_0 such that for all $u \in V$ there exists a decomposition $u = \sum_{i=0}^{N} u_i$, $u_i \in V_i$, such that

$$\sum_{i=0}^{N} b_i(u_i, u_i) \le C_0^2 b(u,u)$$

(ii) there exist a constant ω such that for $i = 0, \ldots, N$,

$$b(u,u) \le \omega\, b_i(u,u), \qquad \forall u \in V_i$$

(iii) there exist constants ϵ_{ij}, $i,j = 1,\ldots,N$, *such that*

$$b(u_i,u_j) \le \epsilon_{ij} b(u_i,u_i)^{1/2}\, b(u_j,u_j)^{1/2}, \qquad \forall u_i \in V_i, \qquad \forall u_j \in V_j$$

Then, T *is invertible and*

$$C_0^{-2}\, b(u,u) \le b(Tu,u) \le (\rho(\mathcal{E})+1)\,\omega\, b(u,u), \quad \forall u \in V \tag{18}$$

Here $\rho(\mathcal{E})$ *is the spectral radius of the matrix* $\mathcal{E} = \{\epsilon_{ij}\}_{i,j=1}^N$.

Proof. The left inequality: By (16) and assumption (i), we have

$$\begin{aligned} b(u,u) &= \textstyle\sum_{i=0}^N b(u,u_i) = \sum_{i=0}^N b_i(T_iu,u_i) \\ &\le (\textstyle\sum_{i=0}^N b_i(T_iu,T_iu))^{1/2} (\sum_{i=0}^N b_i(u_i,u_i))^{1/2} \\ &\le (\textstyle\sum_{i=0}^N b(u,T_iu))^{1/2}\, C_0 (b(u,u))^{1/2} \end{aligned}$$

Hence, T is invertible and

$$b(u,u) \le C_0^2 b(Tu,u)$$

The right inequality: We first note that it is easy to prove, using the definition of the T_i given in (16) and assumption (ii), that $\|T_i\|_b \le \omega$. Indeed,

$$\begin{aligned} b(T_iu,T_iu) &\le \omega b_i(T_iu,T_iu) = \omega\, b(u,T_iu) \\ &\le \omega b(u,u)^{1/2}\, b(T_iu,T_iu)^{1/2} \end{aligned}$$

Thus,

$$b(T_iu,T_iu) \le \omega^2 b(u,u)$$

which implies $\|T_i\|_b \le \omega$.

Using assumptions (iii) and (ii) and the definition of the T_i, we obtain

$$\begin{aligned} b(\textstyle\sum_{i=1}^N T_iu,\, \sum_{i=1}^N T_iu) &= \textstyle\sum_{i,j=1}^N b(T_iu,T_ju) \\ &\le \textstyle\sum_{i,j=1}^N \epsilon_{ij} b(T_iu,T_iu)^{1/2} b(T_ju,T_ju)^{1/2} \\ &\le \textstyle\sum_{i,j=1}^N \epsilon_{ij} (\|T_i\|_b \|T_j\|_b)^{1/2} b(T_iu,u)^{1/2} b(T_ju,u)^{1/2} \\ &\le \rho(\mathcal{E})\omega \textstyle\sum_{i=1}^N b(T_iu,u) \\ &\le \rho(\mathcal{E})\omega b(u,u)^{1/2} b(\textstyle\sum_{i=1}^N T_iu, \sum_{i=1}^N T_iu)^{1/2} \end{aligned}$$

Hence,

$$b(\sum_{i=1}^N T_iu, \sum_{i=1}^N T_iu) \le \rho(\mathcal{E})^2\omega^2 b(u,u)$$

and therefore

$$b(\sum_{i=1}^N T_iu, u) \le \rho(\mathcal{E})\omega\, b(u,u) \tag{19}$$

Finally, we note that

$$b(T_0 u, u) \leq \omega b(u, u)$$

and complete the proof of the right inequality of (18) by adding the two last inequalities. □

We remark that the theory easily can be extended, with minor modifications, to cases where there are several special spaces. Thus, if there are two such spaces, we exclude them both when considering the strengthened Cauchy-Schwarz inequalities of (iii) of Theorem 1 and the factor $(\rho(\mathcal{E}) + 1)$ is replaced by $(\rho(\mathcal{E}) + 2)$. We also remark that an examination of the proof shows that the upper bound of T can be replaced by $(\rho(\hat{\mathcal{E}}) + \|T_0\|_b)\, b(u, u)$. Here the elements of $\hat{\mathcal{E}}$ are given by $\|T_i\|_b^{1/2} \epsilon_{ij} \|T_j\|_b^{1/2}$. We note that the rate of convergence of an additive Schwarz method is affected by the scaling of the bilinear forms $b_i(\cdot, \cdot)$, relative to each other, and hence the scaling of the T_i. Not only the upper but also the lower bound on T is affected. It is also easy to show that if we normalize the operators T_i so that they all have norm 1, then the bound for the condition number of the additive algorithm will be at least as good as before.

There is a corresponding theory for the *multiplicative* Schwarz methods. The principal contributors are Bramble, Pasciak, Wang, and Xu [6], and Xu [42]; cf. also Cai and Widlund [9] for a variant of the theory for nonsymmetric and indefinite problems. In the multiplicative case, we need to provide an upper bound for the spectral radius, or norm, of the error propagation operator

$$E_N = (I - T_N) \cdots (I - T_0) \tag{20}$$

Examining the factors of this product, we note that $\|I - T_i\|_b > 1$ if $\|T_i\|_b > 2$. Therefore an assumption that $\omega < 2$ is most natural. If ω is too large, we can scale the bilinear forms $b_i(\cdot, \cdot)$ to decrease $\|T_i\|_b$ appropriately. As previously noted, the parameter C_0 also changes if the T_i, and hence the $b_i(\cdot, \cdot)$, are rescaled. It is also clear that a multiplicative method might be very slow if some of the $\|T_i\|_b$ are very small.

The result that we are going to state in Theorem 2 will, for technical reasons, be given in terms of $\hat{\omega} = \max(1, \omega)$ rather than ω. This bound is of interest only when the parameter $\omega < 2$ and is bounded from below away from zero. The result is expressed in terms of $\hat{\omega}$ and the two other parameters of Theorem 1 and is a variant of results due to Bramble, Pasciak, Wang, and Xu [6].

THEOREM 2. *The error propagation operator of the multiplicative Schwarz algorithm satisfies*

$$\|E_N\|_b \leq \sqrt{1 - \frac{(2 - \hat{\omega})}{(1 + 2\hat{\omega}^2 \rho(\mathcal{E})^2) C_0^2}}$$

Here $\hat{\omega} = \max(1, \omega)$.

Proof. Our task is to estimate the norm of the error propagation operator E_N of the multiplicative Schwarz method. We begin by observing that with

$$E_j = (I - T_j) \cdots (I - T_0), \qquad E_{-1} = I,$$

$$\text{and} \qquad R_j = 2T_j - T_j^2 = (2 - T_j)T_j$$

we have

$$E_j^T E_j - E_{j+1}^T E_{j+1} = E_j^T R_{j+1} E_j$$

Here, and in what follows, the transpose is with respect to the bilinear form $b(\cdot,\cdot)$. This leads to the identity

$$I - E_N^T E_N = \sum_{j=0}^{N} E_{j-1}^T R_j E_{j-1} \tag{21}$$

It is easy to see that a satisfactory upper bound for $\|E_N\|_b$ can be obtained by showing that the operator on the right hand side of (21) is sufficiently positive definite. We note that, for $\omega = \max \|T_j\|_b < 2$, the operators R_j are positive semidefinite and that

$$R_j \geq (2 - \omega)T_j \geq (2 - \hat{\omega})T_j$$

Therefore,

$$I - E_N^T E_N \geq (2 - \hat{\omega}) \sum_{j=0}^{N} E_{j-1}^T T_j E_{j-1} \tag{22}$$

A direct consequence of the definition of the operator E_j is that

$$I = E_{j-1} + \sum_{k=0}^{j-1} T_k E_{k-1} = E_{j-1} + T_0 + \sum_{k=1}^{j-1} T_k E_{k-1} \tag{23}$$

For $j > 0$, we therefore obtain

$$b(T_j u, u) = b(T_j u, E_{j-1} u) + b(T_j u, T_0 u) + \sum_{k=1}^{j-1} b(T_j u, T_k E_{k-1} u)$$

This expression can, by using Schwarz's inequality, the upper bound on $\|T_i\|_b$, and the definition of the ε_{ij}, be bounded from above by

$$b(T_j u, u)^{1/2} (b(T_j E_{j-1} u, E_{j-1} u)^{1/2} + b(T_j T_0 u, T_0 u)^{1/2} + \omega \sum_{k=1}^{j-1} \varepsilon_{jk} b(T_k E_{k-1} u, E_{k-1} u)^{1/2})$$

Since $\varepsilon_{jj} = 1$, we can combine the first and third terms. Denote by c a vector with the components

$$c_k = b(T_k E_{k-1} u, E_{k-1} u)^{1/2}, \quad k = 1, \cdots, N$$

Cancelling a common factor and using elementary arguments, we find that

$$b(T_j u, u) \leq 2\hat{\omega}^2 (\mathcal{E} c)_j^2 + 2b(T_j T_0 u, T_0 u)$$

We now sum from $j = 1$ to N, use (19), and add the term $b(T_0 u, u)$ to both sides and obtain

$$b(Tu, u) \leq 2\hat{\omega}^2 \rho(\mathcal{E})^2 |c|^2_{\ell^2} + (1 + 2\hat{\omega}^2 \rho(\mathcal{E})) b(T_0 u, u)$$

and finally,

$$b(Tu, u) \leq (1 + 2\hat{\omega}^2 \rho(\mathcal{E})^2) \sum_{j=0}^{N} b(E_{j-1}^T T_j E_{j-1} u, u)$$

The proof can now be completed by using (22) and the lower bound given in Theorem 1. □

We will only formulate our results for additive Neumann-Neumann algorithms. However, all our proofs rely only on bounds on the three parameters C_0, ω, and $\rho(\mathcal{E})$. Therefore there are no difficulties in formulating results for the corresponding multiplicative Schwarz algorithms.

We note that there are many other variants of the basic algorithms that can be analyzed similarly. We note that one of the most powerful of the algorithms is obtained by solving the equation

$$(I - E_N)u = \hat{g}$$

by the GMRES method or another conjugate gradient type method for nonsymmetric problems; cf. Saad and Schultz [36]. A preconditioner for the standard conjugate gradient method can also be obtained from a symmetrized version of the multiplicative method; see Cai [8] and Dryja, Smith, and Widlund [15] for further discussion.

4. A Method Without a Coarse Space. We only consider Model Problem I, i.e. when $s(u, v)$ is derived from the bilinear form given by (2). Let $V_i(\Gamma) \subset V^h(\Gamma)$ be the subspace of discrete harmonic functions that vanish on $\Gamma_h \setminus \partial\Omega_{i,h}$. It is then easy to verify that

$$V^h(\Gamma) = V_1(\Gamma) + \cdots + V_N(\Gamma) \tag{24}$$

Using the notations of Section 3, we choose $V = V^h(\Gamma)$, $V_i = V_i(\Gamma)$, $i = 1, 2, \ldots, N$, and $b(u, v) = s(u, v)$. There is no coarse space V_0.

The auxiliary bilinear forms $b_i(u, v)$: $V_i(\Gamma) \times V_i(\Gamma) \rightarrow R$, are introduced in terms of a set of bilinear forms $\hat{s}_i(\cdot, \cdot)$ and counting functions $\nu_i \in V^h(\Gamma)$, which are associated with the $\partial\Omega_i$. The bilinear forms $\hat{s}_i(\cdot, \cdot)$ on $V_i \times V_i$ are defined similarly to the $s_i(\cdot, \cdot)$ of Section 2, but in terms of the bilinear form

$$\hat{a}_i(u, v) = \int_{\Omega_i} \nabla u \cdot \nabla v \, dx + \frac{1}{H^2} \int_{\Omega_i} uv \, dx \tag{25}$$

Thus, $\hat{s}_i(u, v)$ is the bilinear form given by the Schur complement with respect to $\hat{a}_i(u, v)$. The second term in $\hat{a}_i(u, v)$ is introduced to make the local problems, related to the $b_i(u, v)$, nonsingular. We remark that for the algorithm considered in this

section, we can replace $1/H^2$ in the last term in (25) by a positive constant and still obtain as strong a result on the condition number. However, we have chosen a scaling that makes the smallest eigenvalue of $\hat{a}_i(\cdot,\cdot)$ of the same order of magnitude as the smallest nonzero eigenvalue of the Neumann problem, of the Laplace operator, on the same subregion.

The auxiliary bilinear forms, which complete the definition of this Schwarz algorithm, are given by

$$b_i(u,v) = \hat{a}_i(\hat{\mathcal{H}}_i(\nu_i u), \hat{\mathcal{H}}_i(\nu_i v)) \equiv \hat{s}_i(I_h(\nu_i u), I_h(\nu_i v)) \tag{26}$$

Here I_h is the interpolation operator onto $V^h(\Gamma)$. $\hat{\mathcal{H}}_i w$ is the discrete harmonic extension, with respect to the bilinear form $\hat{a}_i(\cdot,\cdot)$, of the finite element interpolant $I_h w$ of the values of a continuous function w defined on $\partial\Omega_i$, i.e. the solution of a Dirichlet problem

$$\hat{a}_i(\hat{\mathcal{H}}_i w, v) = 0, \quad \forall v \in V^h \cap H_0^1(\Omega_i)$$

with $\hat{\mathcal{H}}_i w(x) = w(x), \ x \in \partial\Omega_{i,h}$. (Similarly, the discrete harmonic extension, in the sense of $a_i(\cdot,\cdot)$, of w is denoted by $\mathcal{H}_i w$.) The counting functions ν_i are defined by

$$\begin{aligned} &\nu_i(x) = \text{ number of } \partial\Omega_{j,h} \quad \text{ to which } x \in \partial\Omega_{i,h} \text{ belongs} \\ &\nu_i(x) = 0, \quad x \in \Gamma_h \setminus \partial\Omega_{i,h} \end{aligned}$$

Thus at the nodes of a *face* $\mathcal{F}_{ij}$ of Ω_i, i.e. the interior of the intersection of the closures of two adjacent substructures Ω_i and Ω_j, $\nu_i(x) = 2$, while $\nu_i(x) > 2$ for any nodal point on the *wire basket* $\mathcal{W}_i$, formed by the union of the edges and vertices of the substructure Ω_i. If $\partial\Omega_i$ intersects $\partial\Omega$ then we set

$$\nu_i(x) = 1, \ x \in (\partial\Omega_{i,h} \cap \partial\Omega_h) \setminus \Gamma_h. \tag{27}$$

The pseudo inverses $\nu_i^\dagger$ of the ν_i define a partition of unity, i.e.

$$\sum_i \nu_i^\dagger(x) \equiv 1, \ x \in \Gamma \cup \partial\Omega$$

Here $\nu_i^\dagger(x) \in V_i(\Gamma)$ is defined by

$$\nu_i^\dagger(x) = \nu_i(x)^{-1}, \ x \in \partial\Omega_{i,h}, \quad \nu_i^\dagger(x) = 0, \ x \in (\Gamma_h \cup \partial\Omega_h) \setminus \partial\Omega_{i,h} \tag{28}$$

The operators T_i: $V^h \to V_i$, are introduced by

$$b_i(T_i u, v) = s(u,v), \qquad \forall v \in V_i \tag{29}$$

and the operator T by

$$T = T_1 + \cdots + T_N$$

THEOREM 3. *For all $u \in V^h(\Gamma)$, with the T_i defined by (29), and for Model Problem I,*

$$\gamma_0 H^2 s(u,u) \le s(Tu,u) \le \gamma_1 (1+\log(H/h))^2 s(u,u) \tag{30}$$

Here γ_0 and γ_1 are constants independent of H and h.

In preparation for the proof of Theorem 3, we formulate several auxiliary results. We use the weighted norm defined by (25),

$$\|u\|^2_{H^1(\Omega_i)} = |u|^2_{H^1(\Omega_i)} + \frac{1}{H^2}\|u\|^2_{L^2(\Omega_i)}.$$

LEMMA 1. *For all $u \in V^h(\Omega_i)$*

$$h \sum_{x \in \mathcal{W}_{i,h}} |u(x)|^2 \le C(1+\log(H/h))\|u\|^2_{H^1(\Omega_i)}$$

Here $\mathcal{W}_{i,h}$ is the set of nodes on $\mathcal{W}_i$, the wire basket of Ω_i.

This lemma is virtually identical to Lemma 4 of Dryja [13]; cf. also Lemma 2.4 in Bramble and Xu [7].

We need an estimate of the discrete harmonic extension of boundary values obtained by replacing the values on $\partial\Omega_{i,h}$ by zero except on one face $\mathcal{F}_{ij}$. Let $\theta_{ij} \in V_i(\Gamma) + V_j(\Gamma)$ be defined by its boundary values $\theta_{ij}(x) = 1,\ x \in \mathcal{F}_{ij,h}$, and $\theta_{ij}(x) = 0,\ x \in \Gamma_h \setminus \mathcal{F}_{ij,h}$. An estimate of the norm of this discrete harmonic function is given in Dryja, Smith, and Widlund [15].

LEMMA 2. *Let θ_{ij} be defined as above. Then*

$$|\theta_{ij}|^2_{H^1(\Omega_i)} \le CH(1+\log(H/h))$$

The proof of the next lemma differs only slightly from that of Lemma 3 in Dryja [13]; see also Dryja, Smith, and Widlund [15].

LEMMA 3. *Let θ_{ij} be defined as above and let $u \in V^h(\Omega_i)$. Then,*

$$|\mathcal{H}_i(\theta_{ij}u)|^2_{H^1(\Omega_i)} \le C(1+\log(H/h))^2\|u\|^2_{H^1(\Omega_i)}$$

Before we prove Theorem 3, we establish an additional auxiliary result, which is closely related to part (ii) of Theorem 1.

LEMMA 4. *Consider Model Problem I and let the bilinear forms $b_i(\cdot,\cdot)$ and $s(\cdot,\cdot)$ be defined as in (26) and (13), respectively. Then, for $i > 0$,*

$$s(u,u) \le C(1+\log(H/h))^2 b_i(u,u),\ \forall u \in V_i$$

Proof. For $u \in V_i$, $i = 1,\dots,N$,

$$s(u,u) = s_i(u,u) + \sum_j s_j(u,u) \tag{31}$$

where the summation is over the Ω_j that have a face $\mathcal{F}_{ij}$, edge $\mathcal{E}_{ij}$, or vertex $\mathcal{V}_{ij}$ in common with Ω_i. We first show that

$$s_i(u,u) \leq C(1+\log(H/h))\, b_i(u,u), \ \forall u \in V_i \tag{32}$$

We decompose u into two terms, $u = \frac{1}{2} I_h(\nu_i u) + \frac{1}{2} I_h((2-\nu_i)u)$, using the counting function ν_i. We note that the second term vanishes on the faces of Ω_i. It is elementary to show that

$$s_i(u,u) \leq \frac{1}{2} s_i(I_h(\nu_i u), I_h(\nu_i u)) + \frac{1}{2} s_i(I_h((2-\nu_i)u), I_h((2-\nu_i)u)) \tag{33}$$

The first term is bounded by $\frac{1}{2}\hat{s}_i(I_h(\nu_i u), I_h(\nu_i u)) = \frac{1}{2} b_i(u,u)$. This follows from basic properties of discrete harmonic functions and the definition of $b_i(\cdot,\cdot)$. To estimate the second, we use Lemma 1. We note that the energy of a discrete harmonic function can be bounded from above by the function obtained by extending its boundary values by zero. We combine this observation with a simple computation of the energy of a function which vanishes at all nodes except those on the wire basket. We obtain

$$s_i(I_h((2-\nu_i)u), I_h((2-\nu_i)u)) = a_i(\mathcal{H}_i((2-\nu_i)u), \mathcal{H}_i(2-\nu_i)u) \leq Ch \sum_{x \in \mathcal{W}_{i,h}} |(2-\nu_i)u(x)|^2$$

Since $|(2-\nu_i(x))/\nu_i(x)| \leq C, x \in \mathcal{W}_{i,h}$, we can use Lemma 1 to estimate the right hand side of this expression by $C(1+\log(H/h))b_i(u,u)$ as required. Inequality (32) has thus been established.

We now consider $s_j(u,u)$ of a substructure Ω_j that has only a vertex $\mathcal{V}_{ij}$ in common with $\partial\Omega_i$. It is easy to see, using an inverse inequality and Lemma 1, that

$$\begin{aligned} s_j(u,u) &\leq Ch\,|u(\mathcal{V}_{ij})|^2 \leq Ch \textstyle\sum_{x \in \mathcal{W}_{i,h}} |\nu_i(x)u(x)|^2 \\ &\leq C(1+\log(H/h))\, b_i(u,u) \end{aligned} \tag{34}$$

Here we have used the fact that ν_i is bounded from below on $\partial\Omega_i$ and the definition of $b_i(u,u)$.

We next estimate $s_j(u,u)$ of a substructure that has an edge $\mathcal{E}_{ij}$ in common with $\partial\Omega_i$. Proceeding as in (34), we obtain by the same argument,

$$\begin{aligned} s_j(u,u) &\leq Ch \textstyle\sum_{x \in \bar{\mathcal{E}}_{ij,h}} |u(x)|^2 \leq Ch \sum_{x \in \mathcal{W}_{i,h}} |\nu_i(x)u(x)|^2 \\ &\leq C(1+\log(H/h))\, b_i(u,u) \end{aligned} \tag{35}$$

We finally estimate a term $s_j(u,u)$ of a substructure that has a face $\mathcal{F}_{ij}$ in common with $\partial\Omega_i$. We use the decomposition $u = I_h(\theta_{ij}u) + I_h((1-\theta_{ij})u)$. Since $\nu_i(x) = 2, x \in \mathcal{F}_{ij,h}$, we obtain

$$s_j(u,u) \leq \frac{1}{2} s_j(I_h(\theta_{ij}\nu_i u), I_h(\theta_{ij}\nu_i u)) + 2 s_j(I_h((1-\theta_{ij})u), I_h((1-\theta_{ij})u)) \tag{36}$$

Using Lemma 3 and the definition of $b_i(u,u)$, the first term can be estimated by $C(1+\log(H/h))^2\, b_i(u,u)$. The common argument of the second term of (36) vanishes

at all nodes except those on the wire basket $\mathcal{W}_i$ and the term can be estimated using the techniques already developed in this proof.

Substituting these bounds into (36), we obtain

$$s_j(u,u) \leq C(1+\log(H/h))^2 b_i(u,u)$$

The proof of the lemma is now completed by using formula (31) and the estimates of the individual terms. □

We now return to Theorem 3. The proof consists of estimating the three parameters of Theorem 1.

Proof. Assumption (i): We use the partition of unity defined by the $\nu_i^\dagger$ and choose $u_i = I_h(\nu_i^\dagger u)$, $i = 1,\ldots,N$, where $u \in V^h(\Gamma)$; see (28). It is easy to see that $u_i \in V_i$, that $u = \sum_{i=1}^N u_i$, and that

$$b_i(u_i,u_i) = \hat{a}_i(\hat{\mathcal{H}}_i u, \hat{\mathcal{H}}_i u) \leq \hat{a}_i(u,u)$$

cf. (26). Hence, by subassembly and Friedrichs' inequality, we obtain

$$\sum_{i=1}^N b_i(u_i,u_i) \leq a(u,u) + \frac{1}{H^2}(u,u)_{L^2(\Omega)} \leq C\frac{1}{H^2}a(u,u) = C\frac{1}{H^2}s(u,u)$$

Here C can be chosen as $H^2 + \lambda_1^{-1}$ with λ_1 the smallest eigenvalue of the Laplace operator on Ω with the boundary conditions of the original problem. This estimate shows that the parameter C_0^2 of assumption (i) is bounded by C/H^2, as required.

Assumption (ii): A bound of the form $\omega \leq C(1+\log(H/h))^2$ is obtained directly from Lemma 4.

Assumption (iii): It is easy to see that $\rho(\mathcal{E})$ is bounded by a constant that is independent of the number of substructures since the number of substructures, to which any $x \in \Omega$ belongs, is uniformly bounded. □

A variant of Theorem 3 has previously been established in De Roeck and Le Tallec [11]. We believe that our proof, based on Theorem 1, is simpler. We also note that an estimate of the condition number as in Theorem 3, but with a factor $H^{-2}(1+\log(H/h))^4$, i.e. with an additional factor $(1+\log(H/h))^2$, was given already in Theorem 4 of Dryja and Widlund [18]. In that paper, we also discussed the special case when there is a red-black ordering of the substructures and indicated how certain estimates for algorithms with coarse spaces could be derived from earlier work on the Neumann-Dirichlet algorithm. In such a case a bound with a factor $(1+\log(H/h))^3$ suffices; cf. e.g. Dryja [13].

4.1. Remarks on Implementation. We now briefly discuss how the method can be implemented. We certainly recommend the use of a preconditioned conjugate gradient method, but for simplicity, we consider only the first Richardson method for solving the equation

$$Tu = g, \qquad g = \sum_{i=1}^N T_i u \tag{37}$$

Here the T_i are defined by (29). The Richardson iteration is defined by

$$u^{n+1} = u^n - \tau T(u^n - u)$$

The best choice of the acceleration parameter is $\tau = 2/(\lambda_{min}(T) + \lambda_{max}(T))$; some information on the spectrum is necessary for a good choice of the parameter τ. Let $r^n = T(u^n - u) = \sum_{i=1}^N r_i^n$ where $r_i^n = T_i(u^n - u)$. To find $r_i^n \in V_i$, we solve

$$b_i(r_i^n, v) = s(u^n, v) - \tilde{f}(v) \equiv F(v), \quad \forall v \in V_i \tag{38}$$

We use $\tilde{\phi}_k = I_h(\nu_i^\dagger \phi_k)$ as test functions where ϕ_k is the standard nodal basis function associated with the nodal point $x_k \in \partial\Omega_i$. We can rewrite (38) as

$$\hat{a}_i(\hat{\mathcal{H}}_i \tilde{r}_i^n, \phi_k) = \widetilde{F}(\phi_k), \quad \forall \phi_k \in V^h(\Omega_i)$$

cf. (26). Here $\tilde{r}_i^n = I_h(\nu_i r_i^n)$ and $\widetilde{F}(\phi_k) = F(I_h(\nu_i^\dagger \phi_k))$ for any ϕ_k associated with a node of $\partial\Omega_{i,h}$. $\widetilde{F}(\phi_k)$ vanishes for any ϕ_k associated with an interior node. To compute the values of $s(u^n, \tilde{\phi}_k)$, we must solve

$$a_i(\mathcal{H}_i u^n, v) = 0\ , \quad \forall v \in H_0^1(\Omega_i) \cap V^h(\Omega), \quad i = 1, \ldots, N$$

with Dirichlet data given by u^n on $\partial\Omega_i \setminus \partial\Omega_N$.

In summary, in each step and for each substructure, we must solve a Neumann problem for the local problem defined by the positive definite form $\hat{a}_i(\cdot,\cdot)$ and a Dirichlet problem for the original elliptic problem. In addition, we have to compute and assemble the contributions to the residual from neighboring substructures and carry out some basic linear algebra operations. The same observations are valid for the conjugate gradient version of the algorithm.

5. A Method with a Standard Coarse Subspace. We now describe a variant of the Neumann-Neumann method with a preconditioner which incorporates a mechanism for the global transportation of information. The factor $1/H^2$ can now be removed from the estimate of the condition number that follows from Theorem 3. We consider Model Problem I only; we have not been able to provide a good bound which is independent of the values of the coefficients of Model Problem II; cf. the discussion of this issue in Dryja, Smith, and Widlund [15]. We use the decomposition

$$V^h(\Gamma) = V_0 + V_1 + \cdots + V_N \tag{39}$$

$V_0(\Gamma)$ is the restriction of $V^H(\Omega)$ to Γ where $V^H(\Omega) \subset V^h(\Omega)$ is the space of piecewise linear, continuous functions, defined on the coarse mesh, and which vanish on $\partial\Omega$. (In its present form, this algorithm is limited to the case where the set of substructures form a regular finite element triangulation of Ω.) We use

$$b_0(u, v) = s(u, v).$$

The spaces V_i, $i = 1, \ldots, N$, are defined as in Section 4, i.e. V_i is the subspace of discrete harmonic functions that vanish on $\Gamma_h \setminus \partial\Omega_{i,h}$. The bilinear forms $b_i(\cdot,\cdot)$ are defined by equation (26).

The operators T_i and T are defined as before in terms of the bilinear forms and subspaces, i.e. by

$$b_i(T_i u, v) = s(u, v), \quad v \in V_i \tag{40}$$

and by

$$T = T_0 + T_1 + \cdots + T_N$$

THEOREM 4. *For all $u \in V^h(\Gamma)$, with the T_i defined by (40), and for Model Problem I,*

$$\gamma_0 \, s(u,u) \leq s(Tu,u) \leq \gamma_1 (1 + \log(H/h))^2 \, s(u,u) \tag{41}$$

Here γ_0 and γ_1 are constants independent of H and h.

Proof. Assumption (i): Let $u_0 = Q_0 u$ where $Q_0 : V^h(\Omega) \to V_0$, is the L_2-projection; cf. Bramble and Xu [7] and Xu [41] for a detailed discussion of the properties of this operator. In particular, Q_0 is bounded in the energy norm and $w = u - u_0$ satisfies

$$\|w\|^2_{L_2(\Omega)} \leq C H^2 |u|^2_{H^1(\Omega)}$$

w is decomposed, as in Section 4:

$$w = \sum_{i=1}^{N} u_i = \sum_{i=1}^{N} I_h(\nu_i^\dagger w).$$

Using the H^1-stability of Q_0, we see that

$$b_0(u_0, u_0) = s(u_0, u_0) \leq C s(u,u)$$

By using the definitions of the u_i and $b_i(\cdot,\cdot)$, we find

$$\sum_{i=1}^{N} b_i(u_i, u_i) = \sum_{i=1}^{N} \hat{a}_i(\hat{\mathcal{H}}_i(w), \hat{\mathcal{H}}_i(w)) \leq |w|^2_{H^1(\Omega)} + \frac{1}{H^2} \|w\|^2_{L_2(\Omega)}$$

The two terms on the right hand side can be estimated by $Cs(u,u)$ by using the H^1-stability of Q_0, and the bound on the L_2-norm of w, respectively. We thus obtain an estimate of the parameter of Assumption (i): $C_0^2 = C$.

Assumption (ii): Since we use an exact solver for the coarse space, $\|T_0\|_s = 1$. The bound for the other subspaces follows directly from Lemma 4.

Assumption (iii): By exactly the same argument as in Theorem 3, we see that $\rho(\mathcal{E}) \leq C$. □

The implementation of this method is similar to that of Section 4 . We only note that the problem defined in $V_0 = V^H$ reduces to a regular finite element problem defined on the coarse mesh.

6. A Method with a Coarse Space of Minimal Dimension. The algorithm in this section has much in common with a method developed and analyzed recently by Mandel and Brezina, cf. [30,32,31]. Essentially the same coarse space is used, but our results and algorithms differ in several ways. Thus, we are able to design algorithms with good bounds without imposing extra restrictions on the intersection of the boundaries of the individual substructures and that of the original region. We consider both Model Problem I and II and show that the condition number is bounded by $C(1 + \log(H/h))^2$.

The coarse spaces used in this Neumann-Neumann method is of minimal dimension with only one degree of freedom per substructure. We note that the basis functions of this space have been used by Bramble, Pasciak, and Schatz [4,5] in the design of an algorithm to solve the coarse problem, which in their case also involves a number of additional degrees of freedom.

Before we introduce the subspaces and bilinear forms that define the method, we partition the set of substructures into two sets $\mathcal{N}_I$ and $\mathcal{N}_B$, respectively. The boundary of a substructure in $\mathcal{N}_B$ intersects $\partial\Omega_D$ in at least one point while those of the other, the interior set $\mathcal{N}_I$, do not. The set $\mathcal{N}_B$ is further divided into three disjoint subsets $\mathcal{N}_{B,F}$, $\mathcal{N}_{B,E}$, and $\mathcal{N}_{B,V}$. The boundaries of these substructures have a face, only edges, and only single vertices in common with $\partial\Omega_D$, respectively.

We first consider Model Problem I. The coarse space is defined by

$$V_0 = \text{span}\{\nu_i^\dagger\}_{i\in\mathcal{N}_I}$$

where the generalized inverses $\nu_i^\dagger \in V^h(\Gamma)$ of the counting functions are given by (28). It follows from the definition of the counting functions that an element of V_0 takes on constant values at the nodes of each face and of each edge. We note that when Model Problem II is considered, later in this section, we will modify these basis functions and enrich the coarse space by adding similar functions for the boundary substructures in the set $\mathcal{N}_{B,V}$. In fact, we could use a larger coarse space with an additional degree of freedom for each of the boundary substructures without any adverse effect on our spectral bounds. Later we will also modify the bilinear forms for the boundary substructures in the set $\mathcal{N}_{B,E}$.

The bilinear form for V_0 is chosen as

$$b_0(u,v) = (1 + \log(H/h))^{-2}\, s(u,v) \tag{42}$$

The spaces $V_i(\Gamma)$, $i = 1,\ldots,N$, are the same as in Sections 4 and 5 and the bilinear forms are also chosen as before; cf. (26). The T_i: $V^h \to V_i$, are defined by

$$b_i(T_i u, v) = s(u,v), \qquad \forall v \in V_i \tag{43}$$

and

$$T = T_0 + T_1 + \cdots + T_N$$

THEOREM 5. *For all $u \in V^h(\Gamma)$, with the T_i defined by (26), (42), and (43), and for Model Problem I,*

$$\gamma_0 s(u,u) \leq s(Tu,u) \leq \gamma_1 (1+\log(H/h))^2 \, s(u,u)$$

Here γ_0 and γ_1 are constants independent of H and h.

Proof. Assumption (i): For $u \in V^h(\Gamma)$, let $u_0 = \sum_{i \in \mathcal{N}_I} \bar{u}_i \nu_i^\dagger$ where

$$\bar{u}_i = \frac{1}{m_i} \sum_{x \in \partial\Omega_{i,h}} u(x) \quad \text{with} \quad m_i = \sum_{x \in \partial\Omega_{i,h}} 1. \tag{44}$$

For convenience, we set $\bar{u}_i = 0, \ i \in \mathcal{N}_B$. We can then write $u_0 = \sum_i \bar{u}_i \nu_i^\dagger$. Let

$$u_i = I_h(\nu_i^\dagger (u - \bar{u}_i))$$

It is easy to see that $u_i \in V_i$ and that

$$u = u_0 + \sum_{i=1}^N u_i$$

It follows from the definitions of the u_i and $b_i(\cdot,\cdot)$ that

$$b_i(u_i,u_i) \leq \hat{a}_i(u-\bar{u}_i, u-\bar{u}_i) = \|\nabla u\|^2_{L_2(\Omega_i)} + \frac{1}{H^2}\|u-\bar{u}_i\|^2_{L_2(\Omega_i)} \tag{45}$$

We have to estimate $b_i(u_i,u_i)$ in terms of $s_i(u,u)$. Let $i \in \mathcal{N}_I$. We first show, by a simple computation, that $\|\bar{u}_i\|^2_{L_2(\Omega_i)} \leq CH\|u\|^2_{L_2(\partial\Omega_i)}$. By a simple trace theorem, $\|u\|^2_{L_2(\partial\Omega_i)} \leq CH\hat{a}_i(u,u)$. It then follows that

$$\|\bar{u}_i\|^2_{L_2(\Omega_i)} \leq CH^2 \hat{a}_i(u,u)$$

Therefore,

$$\frac{1}{H^2}\|u-\bar{u}_i\|^2_{L_2(\Omega_i)} \leq C\hat{a}_i(u,u) \tag{46}$$

Since the left hand side of (46) does not change if we add a constant to u, we can use Poincaré's inequality and find that

$$b_i(u_i,u_i) \leq C s_i(u,u)$$

For $i \in \mathcal{N}_B$, we instead use Friedrichs' inequality. We recall that we have set $\bar{u}_i = 0$ for these values of i. We extend the region of integration in the last term of (45), including the substructures which are next neighbors. It follows from the assumptions on $\partial\Omega_D$ that one of these has a face which is part of $\partial\Omega_D$. We can then use Friedrichs' inequality to remove the L_2-term of the weighted H^1-norm and we obtain

$$\sum_{i=1}^N b_i(u_i,u_i) \leq C \sum_{i=1}^N s_i(u,u) = Cs(u,u) \tag{47}$$

To obtain a uniform bound on the parameter C_0^2, we must also show that

$$b_0(u_0,u_0) = (1+\log(H/h))^{-2}\, s(u_0,u_0) \le C s(u,u)\ ,\forall u \in V_0 \tag{48}$$

Let $u_0 = u - w$, where $w = \sum_i u_i$. We can reduce our task to estimating $s(w,w)$ since

$$s(u_0,u_0) \le 2s(w,w) + 2s(u,u) \tag{49}$$

Most pairs of subspaces are mutually orthogonal. Therefore,

$$s(w,w) \le C \sum_{i=1}^{N} s(u_i,u_i) \tag{50}$$

We can now use Lemma 4 and reduce the problem to estimating $\sum_{i=1}^N b_i(u_i,u_i)$, a problem which has already been solved in this proof. We obtain

$$s(u_0,u_0) \le C(1+\log(H/h))^2\, s(u,u)$$

which proves (48).

Assumption (ii): For $i = 0$, we find that $\|T_0\|_s = (1+\log(H/h))^2$, since, by definition,

$$s(u,u) = (1+\log(H/h))^2\, b_0(u,u)\ ,\ \forall u \in V_0$$

The estimate

$$s(u,u) \le C(1+\log(H/h))^2\, b_i(u,u)\ ,\ \forall u \in V_i,\ i = 1,\ldots,N \tag{51}$$

is a direct consequence of Lemma 4.

Assumption (iii): By standard arguments, we show that $\rho(\mathcal{E}) \le$ C; cf. the proof of Theorem 3. □

We now turn to Model Problem II. A crucial change in the algorithm is that the counting functions ν_i are replaced by weighted counting functions $\mu_i \in V_i$ defined in terms of the coefficients of equation (5). They are introduced by

$$\mu_i(x) = \sum_j \rho_j^{1/2},\ x \in \partial\Omega_{i,h}, \qquad \mu_i(x) = 0,\ x \in \Gamma_h \setminus \partial\Omega_{i,h}$$

For each $x \in \Gamma_h$, the sum is taken over the values of j for which $x \in \partial\Omega_j$. We note that $\mu_i \equiv \nu_i$ for Model Problem I, where $\rho_i = 1,\ \forall i$. The pseudo inverse $\mu_i^\dagger$ of μ_i is introduced in the same way as $\nu_i^\dagger$; cf. (28).

We can now define the coarse space V_0 by

$$V_0 = \mathrm{span}\{\rho_i^{1/2}\mu_i^\dagger\}_{i\in\mathcal{N}_I}$$

We note that the set of functions $\{\rho_i^{1/2}\mu_i^\dagger\}$ also form a partition of unity. For the coarse space V_0, we use a bilinear form $b_0(\cdot,\cdot)$, obtained from (42) by replacing $s(\cdot,\cdot)$ by $s^{(\rho)}(\cdot,\cdot)$. The other bilinear forms are defined by

$$b_i(u,v) = \hat{a}_i(\hat{\mathcal{H}}_i(\mu_i u), \hat{\mathcal{H}}_i(\mu_i v)) \equiv \hat{s}_i(I_h(\mu_i u), I_h(\mu_i v)) \tag{52}$$

We next prove an analog of Lemma 4. Later on, we will introduce a different bilinear form for the boundary substructures in $\mathcal{N}_{B,E}$ and establish a bound similar to that of this lemma; cf. Lemma 7.

LEMMA 5. *Consider Model Problem II and let the bilinear forms $b_i(\cdot,\cdot)$ and $s^{(\rho)}(\cdot,\cdot)$ be defined as in (52) and (14), respectively. Then,*

$$s^{(\rho)}(u,u) \leq C(1+\log(H/h))^2 b_i(u,u), \ \forall u \in V_i$$

Proof. This proof is quite similar to that of Lemma 4. We begin by establishing a bound for $\rho_i s_i(u,u)$; cf. (32). We partition $u \in V_i$ as

$$u = \sum_j I_h(\theta_{ij} u) + \hat{w}$$

and estimate the terms separately. Since, by assumption, ρ_i and $\mu_i(x)$ are constant on each face and $\rho_i < \mu_i(x)^2$, we obtain, by using Lemma 3,

$$\begin{aligned} \rho_i s_i(I_h(\theta_{ij}u), I_h(\theta_{ij}u)) = \ & \rho_i |\mathcal{H}_i(\theta_{ij}u)|^2_{H^1(\Omega_i)} < \mu_{ij}^2 |\mathcal{H}_i(\theta_{ij}u)|^2_{H^1(\Omega_i)} = \\ & |\mathcal{H}_i(\theta_{ij}\mu_i u)|^2_{H^1(\Omega_i)} \leq C(1+\log(H/h))^2 b_i(u,u) \end{aligned}$$

Here μ_{ij} is the constant value of μ_i on the face $\mathcal{F}_{ij}$.

We can use virtually the same argument as in the proof of Lemma 4 to estimate $\rho_i s_i(\hat{w},\hat{w})$.

We also have to estimate $\rho_j s_j(u,u)$, in terms of $b_i(u,u)$, for the values of j for which $\partial\Omega_j$ has at least one point in common with $\partial\Omega_i$. There are no surprises; the tools used here and in the proof of Lemma 4 are sufficient to obtain all the necessary bounds. □

THEOREM 6. *Assume that all the boundary substructures $\Omega_i, i \in \mathcal{N}_B$, have a face which is a subset of $\partial\Omega_D$, i.e. $\mathcal{N}_B = \mathcal{N}_{B,F}$. Then, for all $u \in V^h(\Gamma)$, with the T_i defined by (42), (43), after replacing $s(\cdot,\cdot)$ by $s^{(\rho)}(\cdot,\cdot)$, and (52), and for Model Problem II,*

$$\gamma_0 s^{(\rho)}(u,u) \leq s^{(\rho)}(Tu,u) \leq \gamma_1(1+\log(H/h))^2 \, s^{(\rho)}(u,u)$$

Here γ_0 and γ_1 are constants independent of h, H, and the jumps of $\rho(x)$.

We will return to the general case after the proof of this theorem.

Proof. Assumption (i): The coarse space component of an arbitrary $u \in V^h(\Gamma)$ is chosen as

$$u_0 = \sum_{i \in \mathcal{N}_I} \bar{u}_i \rho_i^{1/2} \mu_i^\dagger \tag{53}$$

where $\bar{u}_i$ is defined in (44). We again set $\bar{u}_i = 0$ for $i \in \mathcal{N}_B$. The rest of the decomposition is given by

$$u_i = I_h(\rho_i^{1/2} \mu_i^\dagger (u - \bar{u}_i)), \ i > 0 \tag{54}$$

We first consider $i \in \mathcal{N}_I$ and obtain,

$$b_i(u_i,u_i) \leq \rho_i \hat{a}_i(u-\bar{u}_i, u-\bar{u}_i)$$

which can be estimated by $C\rho_i s_i(u,u)$; cf. the proof of Theorem 5. For a boundary substructure, which by assumption has a face belonging to $\partial\Omega_D$, we can simplify the arguments given in that proof; we work with only one substructure avoiding any complications stemming from the different values of the coefficient of equation (5).

The bound $b_0(u,u) \leq C s^{(\rho)}(u,u)$ can also be established as before, without relying on any new ideas.

Assumption (ii): We use the same arguments as in the proof of Theorem 5, replacing $s(\cdot,\cdot)$ by $s^{(\rho)}(\cdot,\cdot)$ and Lemma 4 by Lemma 5, and obtain the bound $\omega \leq C(1+\log(H/h))^2$.

Assumption (iii): A constant bound for $\rho(\mathcal{E})$ follows just as in the other proofs. □

To complete our discussion, we now consider the case when the intersection of $\partial\Omega_i$ and $\partial\Omega_D$ consists of only single points or edges, i.e. substructures in $\mathcal{N}_{B,V}$ and $\mathcal{N}_{B,E}$, respectively. For those in $\mathcal{N}_{B,V}$, we use the same bilinear form as for $i \in \mathcal{N}_I$, cf. (52), while for those in $\mathcal{N}_{B,E}$, we use the Schur complement of the original problem (1), i.e.

$$b_i(u,v) = s_i(I_h(\mu_i u), I_h(\mu_i v)), \quad i \in \mathcal{N}_{B,E} \tag{55}$$

In addition, for each boundary substructure in $\mathcal{N}_{B,V}$, we add a basis function $\hat{\mu}_i^\dagger$ to the coarse space V_0; we treat such substructures as if they belong to $\mathcal{N}_I$ except that we truncate $\mu_i^\dagger$ replacing its value at $x \in \partial\Omega_{i,h} \cap \partial\Omega_{D,h}$ by zero. For details, see the proof of Theorem 7.

We are now ready to formulate a more general theorem.

THEOREM 7. *For all $u \in V^h(\Gamma)$, with the T_i defined by (42), (43), after replacing $s(\cdot,\cdot)$ by $s^{(\rho)}(\cdot,\cdot)$, (52), and (55) and for Model Problem II,*

$$\gamma_0 s^{(\rho)}(u,u) \leq s^{(\rho)}(Tu,u) \leq \gamma_1(1+\log(H/h))^2\, s^{(\rho)}(u,u)$$

Here γ_0 and γ_1 are constants independent of h, H, and the jumps of $\rho(x)$.

Before we turn to the proof of this theorem, we establish two additional auxiliary results. We first establish a variant of Friedrichs' inequality.

LEMMA 6. *For any $u \in V_i$ which vanishes only on a single edge of Ω_i,*

$$\|u\|^2_{L^2(\Omega_i)} \leq C H^2(1+\log(H/h))|u|^2_{H^1(\Omega_i)}$$

Proof. The proof of this result has much in common with that of Lemma 1; cf. Bramble and Xu [7]. We limit ourselves to the case of a cubic subregion $(0,H)^3$ assuming that the function vanishes on the edge between $(0,0,0)$ and $(0,0,H)$. We can trivially bound $\|u\|^2_{L^2((0,H)^3)}$ by $H^2 \max_{x_1,x_2} \|u(x_1,x_2,\cdot)\|^2_{L_2(0,H)}$. For each value of x_3, we can use the well known estimate

$$\max_{x_1,x_2} |u(x_1,x_2,x_3)|^2 \leq C(1+\log(H/h))\|u(\cdot,\cdot,x_3)\|^2_{H^1((0,H)^2)}$$

see Lemma 5 in Dryja and Widlund [21] or Lemma 2.3 of Bramble and Xu [7]. Since any function $v(x)$, which vanishes at a point satisfies,

$$\max|v(x)| \leq \max|v(x)-c| + |c| \leq 2\max|v(x)-c| \tag{56}$$

for any constant c, we can use Poincaré's inequality and replace the H^1-norm by the corresponding seminorm. The proof is completed by integrating with respect to x_3. □

The next Lemma complements Lemma 5.

LEMMA 7. *Consider Model Problem II and a boundary substructure in the set $\mathcal{N}_{B,E}$. Let the bilinear forms $b_i(\cdot,\cdot)$ and $s^{(\rho)}(\cdot,\cdot)$ be defined as in (55) and (14), respectively. Then,*

$$s^{(\rho)}(u,u) \leq C(1+\log(H/h))^2 b_i(u,u), \ \forall u \in V_i$$

The same bound holds for any subspace V_i with elements constrained to be zero along an edge.

Proof. The proof of this result has much in common with those of Lemmas 4 and 5. The main differences stem from the fact that we are using $s_i(\cdot,\cdot)$ instead of $\hat{s}_i(\cdot,\cdot)$ when defining the bilinear form $b_i(\cdot,\cdot)$. We also have to show that we can use Lemma 6 instead of the standard Friedrichs' inequality and still obtain a bound that is quadratic rather than cubic in $(1+\log(H/h))$.

We begin by estimating the contribution of $\rho_i s_i(u,u)$ to the bilinear form $s^{(\rho)}(u,u)$. The proofs of Lemmas 4 and 5 show that we can obtain an upper bound of the form $C(1+\log(H/h))\hat{s}_i(I_h(\mu_i u), I_h(\mu_i u))$. By using Lemma 6, we can replace $\hat{s}_i(\cdot,\cdot)$ by $s_i(\cdot,\cdot)$ at the expense of a second logarithmic factor.

We now consider the remaining contributions $\rho_j s_j(u,u)$ to the left hand side $s^{(\rho)}(u,u)$. No new ideas are required for those j which correspond to a neighboring substructure Ω_j which has only a vertex or an edge in common with Ω_i. Similarly, when two substructures have a face $\mathcal{F}_{ij}$ in common, the terms related to the edges common to Ω_j and Ω_i can be handled in the same way as before. What remains is to consider $\rho_j s_j(I_h(\theta_{ij}u), I_h(\theta_{ij}u))$. We write $\theta_{ij}u = \bar{u}_{ij}\theta_{ij} + \theta_{ij}(u-\bar{u}_{ij})$ where

$$\bar{u}_{ij} = \frac{1}{m_{ij}} \sum_{x \in \mathcal{F}_{ij,h}} u(x) \quad \text{with} \quad m_{ij} = \sum_{x \in \mathcal{F}_{ij,h}} 1 \tag{57}$$

and estimate the resulting two terms separately. We find that

$$\rho_j s_j(\bar{u}_{ij}\theta_{ij}, \bar{u}_{ij}\theta_{ij}) \leq C(1+\log(H/h))\hat{s}_i(\mu_i u, \mu_i u)$$

We obtain this bound by using Lemma 2 and the estimate

$$H\bar{u}_{ij}^2 \leq C\|u\|^2_{H^1(\Omega_i)} \tag{58}$$

which in turn follows from a trace theorem and elementary considerations. Using Lemma 6, we can replace $\hat{s}_i(\mu_i u, \mu_i u)$ by $s_i(\mu_i u, \mu_i u)$ at the expense of a second logarithmic factor. The remaining term

$$\rho_j s_j(I_h(\theta_{ij}(u-\bar{u}_{ij}), I_h(\theta_{ij}(u-\bar{u}_{ij}))$$

can be estimated using Lemma 3 and Poincaré's inequality. □

We now turn to the proof of Theorem 7.

Proof. We will only consider the changes necessary in the proofs of Theorems 5 and 6; we have only to consider the substructures in the sets $\mathcal{N}_{B,E}$ and $\mathcal{N}_{B,V}$. The only new terms, $b_i(u_i, u_i)$, that we need to bound from above are those corresponding to $\mathcal{N}_{B,V}$. They are very similar to those of the interior substructures but must nevertheless be examined carefully. This is the only new technical work required; the bounds related to Assumption (ii) follow from Lemmas 7 and 5 and the observation that Lemma 5 is valid for the boundary substructures in the set $\mathcal{N}_{B,V}$; the boundary condition only imposes a harmless constraint on the subspace for which the required estimate already has been established.

We now turn to the estimate

$$b_i(u_i, u_i) \leq C \rho_i s_i(u, u)$$

for $i \in \mathcal{N}_{B,V}$. Here $u_i = I_h(\rho_i^{1/2} \hat{\mu}_i^\dagger (u - \bar{u}_i))$ where $\hat{\mu}_i^\dagger$ is obtained from $\mu_i^\dagger$ by setting its value at the node (or nodes) of $\partial\Omega_{i,h} \cap \partial\Omega_{D,h}$ to zero. We assume that there is only one such node. The idea is quite simple. We write $\hat{\mu}_i^\dagger = \mu_i^\dagger - \mu_i^\dagger(x_k)\phi_k$ and note that $\rho_i b_i(I_h(\mu_i^\dagger(u - \bar{u}_i)), I_h(\mu_i^\dagger(u - \bar{u}_i)))$ can be estimated as if the substructure were interior. We have thus reduced the problem to estimating

$$b_i(I_h(\rho_i^{1/2}\mu_i^\dagger(x_k)\phi_k(u - \bar{u}_i)), I_h(\rho_i^{1/2}\mu_i^\dagger(x_k)\phi_k(u - \bar{u}_i)))$$

This expression, which is equal to $\rho_i \hat{s}_i(I_h(\phi_k(u - \bar{u}_i)), I_h(\phi_k(u - \bar{u}_i)))$, can be bounded by

$$C\rho_i \max_{x \in \partial\Omega_i} |u(x)|^2 \|\phi_k\|^2_{H^1(\Omega_i)}$$

Since $u(x_k) = 0$, we can again use inequality (56). We finally use Lemma 1, Poincaré's inequality and the simple fact that $\|\phi_k\|^2_{H^1(\Omega_i)} \leq Ch$. □

7. Methods with an Alternative Coarse Space. In this section, we add to the diversity of our family of algorithms by describing and analyzing two additional Neumann-Neumann methods for Model Problem II. We choose the same coarse space, and $b_0(\cdot,\cdot)$, for both, but the local spaces V_i and the bilinear forms $b_i(\cdot,\cdot)$, $i > 0$, are different in the two cases. We obtain bounds which are quadratic and cubic in $(1 + \log(H/h))$, respectively.

We use a coarse space that is similar but somewhat larger than a space introduced by Smith [37,38]. An element in Smith's space is fully defined by the values on the wire baskets of the substructures. We extend Smith's space by introducing an additional degree of freedom for each substructure face that does not belong to $\partial\Omega_D$. We note that a less satisfactory result, than those of this section, is given as Theorem 5.3 in Dryja and Widlund [20] where we used Smith's original coarse space.

It is convenient to describe the space V_0 as the range of a nonstandard interpolation operator $\hat{I}_h$ given locally for $x \in \partial\Omega_i$ by

$$\hat{I}_h u(x) = \sum_{x_k \in \mathcal{W}_{i,h}} u(x_k)\, \phi_k(x) + \sum_j \bar{u}_{ij} \theta_{ij}(x) \tag{59}$$

Here the $\phi_k(x)$ are the standard nodal basis functions, the $\theta_{ij}(x)$ the functions used in Lemmas 2 and 3, and the $\bar{u}_{ij}$ the average values defined by (57).

The associated bilinear form is given by

$$\begin{aligned} b_0(u,v) = \quad & (1+\log(H/h))^{-1} \textstyle\sum_{i\in\mathcal{N}} \rho_i (h \sum_{x\in\mathcal{W}_{i,h}} (u(x)-\bar{u}_i)(v(x)-\bar{v}_i) + \\ & H \textstyle\sum_{\mathcal{F}_{ij}\subset\partial\Omega_i} (\bar{u}_{ij}-\bar{u}_i)(\bar{v}_{ij}-\bar{v}_i)) \end{aligned}$$

where $\bar{u}_i$ is the average value defined in formula (44) and $\mathcal{N}$ is the set of all substructures.

For the first of the methods considered in this section, we choose the same local spaces V_i, $i>0$, and bilinear forms $b_i(\cdot,\cdot)$, as in Sections 4, 5, and 6; cf. (26). The operator T is given in terms of the bilinear forms as before.

THEOREM 8. *For all $u \in V^h(\Gamma)$, with the T_i defined by the bilinear forms and subspaces just introduced, and for Model Problem II,*

$$\gamma_0\, s^{(\rho)}(u,u) \le s^{(\rho)}(Tu,u) \le \gamma_1 (1+\log(H/h))^2\, s^{(\rho)}(u,u) \tag{60}$$

Here γ_0 and γ_1 are constants independent of h, H, and the jumps of $\rho(x)$.

Proof. We can use many of the arguments developed in Theorem 7, but we must also consider the effect of changing the coarse subspace V_0 and bilinear form $b_0(\cdot,\cdot)$. When considering Assumption (i), we use the same decomposition of u as in Theorems 6 and 7; cf. (53) and (54). This is possible since the basis functions $\mu_i^\dagger$ belong to the coarse space V_0 currently being considered. We have to prove that

$$b_0(u_0,u_0) \le C s^{(\rho)}(u,u)$$

for the new bilinear form. Let $w = u - u_0$. Then

$$b_0(u_0,u_0) \le 2b_0(u,u) + 2b_0(w,w)$$

To estimate the first term of the right hand side, we consider the contributions to the sum from one substructure Ω_i. We use Lemma 1, the estimate (58), and Poincaré's inequality and arrive at the bound

$$b_0(u,u) \le C s^{(\rho)}(u,u)$$

As in the previous section, $u_i = I_h(\rho_i^{1/2}\mu_i^\dagger(u-\bar{u}_i))$. If we can prove that

$$b_0(u_i,u_i) \le C \rho_i s_i(u,u)$$

then we can estimate $b_0(w,w)$ by $s^{(\rho)}(u,u)$ and can conclude that $C_0^2 \le C$. Essentially no new ideas are required and details are therefore not provided.

Assumption (ii): The estimate of $\|T_i\|_s$, $i>0$, follows as in the proof of Theorems 6 and 7. Therefore, what remains is only to establish that

$$s^{(\rho)}(u,u) \le C(1+\log(H/h))^2 b_0(u,u), \ \forall u \in V_0$$

We recall that any $u_0 \in V_0$ is given by equation (59). We estimate the two terms separately. Knowing that the energy of a discrete harmonic function, which differs from zero only on the wire basket, can be bounded from above by the energy of its trivial extension, we find that

$$\rho_i s_i(u,u) = \rho_i s_i(u-\bar{u}_i, u-\bar{u}_i) \leq C\rho_i(h \sum_{x\in\mathcal{W}_{i,h}} |u(x)-\bar{u}_i|^2 + \sum_j |\bar{u}_{ij}-\bar{u}_i|^2 \|\theta_{ij}\|^2_{H^1(\Omega_i)})$$

The proof is completed by using Lemma 2.

Assumption (iii): The constant bound required can be obtained in a completely routine way. □

We finally consider the second variant, which employs different local spaces. The local space V_i, $i = 1,\ldots,N$, associated with $\partial\Omega_i$, is now chosen as the space of functions $v \in V^h(\Gamma)$ which vanish on the wire basket $\mathcal{W}_i$ as well as at the nodes of $\Gamma_h \setminus \partial\Omega_{i,h}$. We note that the bilinear form $s_i(u,v)$ is positive definite on this space since we use Dirichlet conditions on the wire basket $\mathcal{W}_i$. It is easy to see that

$$V^h(\Gamma) = V_0 + V_1 + \cdots + V_N \tag{61}$$

For the local spaces, we now work with the Schur complements of Model Problem I, i.e.

$$b_i(u,v) = s_i(I_h(\mu_i u), I_h(\mu_i v)), \quad i = 1,\cdots,N \tag{62}$$

THEOREM 9. *For all $u \in V^h(\Gamma)$, with the T_i defined by the bilinear forms and subspaces just introduced, and for Model Problem II,*

$$\gamma_0(1+\log(H/h))^{-1}\, s^{(\rho)}(u,u) \leq s^{(\rho)}(Tu,u) \leq \gamma_1(1+\log(H/h))^2\, s^{(\rho)}(u,u)\ . \tag{63}$$

Here γ_0 and γ_1 are constants independent of h, H, and the jumps of $\rho(x)$.

Proof. Assumption (i): Let $u_0 = \hat{I}_h u$, as in (59), and let $u = u_0 + w$. Let $u_i = I_h(\rho_i^{1/2}\mu_i^\dagger w)$. By using the definition (62), we easily obtain

$$\sum_{i>0} b_i(u_i,u_i) = \sum_i s_i(I_h(\mu_i u_i), I_h(\mu_i u_i)) = \sum_i \rho_i s_i(w,w) = s^{(\rho)}(w,w)$$

By using Lemmas 1 and 2, and the formula for u_0, we find that

$$s^{(\rho)}(u_0,u_0) \leq C(1+\log(H/h))s^{(\rho)}(u,u).$$

Finally, we estimate $b_0(u_0,u_0)$ by $s^{(\rho)}(u,u)$ by using (58), Lemmas 1 and 2, and Poincaré's inequality. The bound $C_0^2 \leq C(1+\log(H/h))$ follows.

Assumption (ii): The bound $\|T_0\|_s \leq C(1+\log(H/h))^2$ has been established in the proof of Theorem 8. Bounds of the same type follows for $\|T_i\|_s$, $i > 0$, from Lemma 7. We just have to recall that the elements of V_i vanish on all the edges of Ω_i.

Assumption (iii): The constant bound required can be obtained in a completely routine way. □

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