On a Conjecture of Micha Perles

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March 17, 1994

\(^1\)Supported in part by NSA grant MDA 904-89-H-2030, ONR grant N00014-85-K-0046 and by NSF grants CCR-8901484, CCR-8902221 and CCR-8906949.
Abstract

We prove a conjecture of Micha Perles concerning simple polytopes, for a subclass that properly contains the duals of stacked and crosspolytopes. As a consequence of a special property of this subclass it also follows that, the entire combinatorial structure of a polytope in the subclass can be recovered from its graph, by applying our results recursively.
1 Introduction

Let $P$ be a simple $d$-polytope and $G(P)$ the graph (1-skeleton) of $P$. Perles conjectured that every $(d - 1)$-regular, induced, connected and non-separating subgraph of $G(P)$ determines a facet of $P$ [2]. In this paper we prove the conjecture for a proper subclass of simple polytopes.

The motivation for our results comes from two subclasses of simplicial polytopes, namely the \textit{stacked polytopes} and the \textit{crosspolytopes}. Polytopes obtained from a simplex by successive addition of pyramids over facets are called \textit{stacked polytopes}. Stacked polytopes form an important subclass of simplicial polytopes in that, only they attain equality in the lower bound theorem [3]. Secondly, if $\{e_1, \ldots, e_d\}$ is a set of linearly independent vectors in $\mathbb{R}^d$ then $X = \text{conv}\{\pm e_1, \ldots, \pm e_d\}$ is called a \textit{d-crosspolytope}. $d$-crosspolytopes can be formed by successively building bipyramids $d - 1$ times starting with a 1-simplex.

Consider the dual analogues of the operations of stacking and forming bipyramids. It can be shown [3] that, if $P$ is a simplicial polytope and $P^*$ a simple polytope dual to $P$, then a polytope obtained by forming a pyramid over a facet of $P$ is dual to a polytope obtained by truncating the corresponding vertex of $P^*$. Also, [4] any bipyramid with basis $P$ is dual to any prism with basis $P^*$.

Against this background, we show that if $P$ is any polytope for which Perles’ conjecture is true, then the conjecture is also true for a polytope obtained by truncating a vertex of $P$ and also for any prism with basis $P$. Since Perles’ conjecture is trivially true for a simplex, one concludes that it is true for any polytope obtained from a simplex by building prisms and truncating vertices finitely many times, in any arbitrary order. The class $\mathcal{C}$ of polytopes so generated, properly contains duals of crosspolytopes and duals of stacked polytopes. We also show that $\mathcal{C}$ is a proper subclass of simple polytopes. Moreover the class $\mathcal{C}$ has the interesting property that every face of a polytope in $\mathcal{C}$ also belongs to $\mathcal{C}$. This allows one to recover the entire combinatorial structure of a polytope in $\mathcal{C}$ from its graph, by applying our results.
recursively.

2 Notation

Please refer to [3] for a discussion on polytopes and for related terminology. Here we merely indicate the convention that we will follow in the sequel.

If \( P \) is a polytope then its vertex set will be denoted \( V(P) \) and its graph \( G(P) \). The vertex set and the edge set of a graph \( \Gamma \) will be denoted \( V(\Gamma) \) and \( E(\Gamma) \) respectively. Given a \( d \)-polytope \( P \), let \( \bar{P} = P + z \) be a translate of \( P \) where \( z \) is a non-zero vector in \( \mathbb{R}^{d+1} \). Then the convex hull of \( P \) and \( \bar{P} \) is called a prism with basis \( P \) denoted \( Pr(P) \). A hyperplane \( H \) in \( \mathbb{R}^d \) is said to truncate a vertex \( x \) of \( P \), if \( x \) and \( V(P) \setminus \{x\} \) lie in different open half-spaces of \( H \). We denote by \( Tr(P) \) the intersection of \( P \) with the closed half-space containing \( V(P) \setminus \{x\} \). For our purposes, it does not matter which vertex of \( P \) is truncated to obtain \( Tr(P) \). \( Tr(P) \) represents a (not necessarily unique) polytope obtained by truncating some vertex of \( P \).

Define a subclass \( C \) of simple polytopes as follows: A polytope \( P \) belongs to \( C \) iff there is a sequence of polytopes

\[
P_0, P_1, \ldots, P_n = P
\]

where \( P_0 \) is a \( k \)-simplex \( (k \geq 1) \) and for \( 1 \leq i \leq n \) either \( P_i = Tr(P_{i-1}) \) or \( P_i = Pr(P_{i-1}) \).

From the definition of the subclass \( C \) it follows that it contains the duals of stacked and crosspolytopes.

3 Perles’ Conjecture for the Subclass \( C \)

Theorem 1 If Perles’ conjecture is true for a simple \( d \)-polytope \( P \) then it is also true for any prism \( Pr(P) \) with basis \( P \).
Proof: Let $Pr(P)$ be the convex hull of $P$ and its translate $\bar{P} = P + z$. Then every vertex $v \in V(P)$ is adjacent to the vertex $\tilde{v} = v + z$ of $\bar{P}$. If $X \subseteq V(P)$ then $\tilde{X}$ will denote the corresponding subset of $V(\bar{P})$. If $\Lambda$ is any induced subgraph of $P$ then $\tilde{\Lambda}$ will denote the subgraph of $G(\bar{P})$ induced by the corresponding vertices of $\bar{P}$.

Let $\Gamma$ be a $d$-regular, induced, connected and non-separating subgraph of $Pr(P)$. We show that $\Gamma$ must determine a facet of $Pr(P)$.

If $\Gamma$ is a subgraph of $G(P)$ (resp. $G(\bar{P})$), since both $\Gamma$ and $G(P)$ (resp. $G(\bar{P})$) are $d$-regular graphs, $\Gamma = G(P)$ (resp. $\Gamma = G(\bar{P})$). Hence $\Gamma$ determines a facet of $Pr(P)$. So we may assume that $V(\Gamma) \cap V(P) \neq \emptyset$ and $V(\Gamma) \cap V(\bar{P}) \neq \emptyset$.

Let $\Gamma_P$ and $\Gamma_{\bar{P}}$ be the restrictions of $\Gamma$ to $P$ and $\bar{P}$ respectively. Consider any vertex $v$ of $\Gamma_P$. $v$ is adjacent to only one vertex in $\bar{P}$. Also, $\Gamma$ is $d$-regular. Hence $v$ has at least $d - 1$ neighbors in $\Gamma_P$. We consider two cases.

Case 1: Each vertex in $\Gamma_P$ has exactly $d - 1$ neighbors in $\Gamma_P$.

Observe that by symmetry each vertex of $\Gamma_{\bar{P}}$ is also $(d - 1)$-valent in $\Gamma_P$ and that the two subgraphs $\Gamma_P$ and $\Gamma_{\bar{P}}$ are copies of each other. We also observe that:

1. $\Gamma_P$ is $(d - 1)$-regular.

2. If $\Gamma_P$ has more than one connected component, pick one and call it $C$. Then the subgraph $\Gamma_C$ of $\Gamma$ induced by $V(C) \cup V(\bar{C})$ is $d$-regular and hence not connected to $\Gamma \setminus \Gamma_C$ contrary to the assumption that $\Gamma$ is connected. Hence $\Gamma_P$ must be connected.

3. Suppose $x, y \in V(P)$ are separated by $\Gamma_P$. Let $C(x)$ and $C(y)$ be the connected components of $G(P) \setminus \Gamma_P$ containing $x$ and $y$ respectively. It is easy to see that $\tilde{C}(x)$ and $\tilde{C}(y)$ are separated by $\Gamma_{\bar{P}}$ in $G(\bar{P})$. Then $\Gamma$ would separate $C(x)$ and $C(y)$, contrary to our assumption. Hence $\Gamma_P$ cannot separate $G(P)$.

Since Perles’ conjecture is true for $P$, using 1, 2 and 3 we conclude that $\Gamma_P$ determines a facet $F$ of $P$. Since $\Gamma_{\bar{P}}$ is the image of $\Gamma_P$ it also determines the facet $\tilde{F}$ of $\bar{P}$. So $\Gamma$ determines a facet of $Pr(P)$.  

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Case 2: At least one vertex in $\Gamma_P$ has $d$ neighbors in $\Gamma_P$.

Let $X$ be the set of all the $d$-valent vertices in $\Gamma_P$, i.e.,

$$X = \{ w \mid w \in V(\Gamma_P) \text{ and } w \text{ has } d \text{ neighbors in } \Gamma_P \}$$

Let $Y$ be the set of all vertices in $\Gamma_P$ that are adjacent to at least one vertex in $X$, i.e.,

$$Y = \{ w \mid w \notin X; \exists x \in X, (w, x) \in E(\Gamma_P) \}$$

Since vertices in $Y$ are $(d - 1)$-valent in $\Gamma_P$, $\tilde{Y} \subseteq V(\Gamma_{\tilde{P}})$. In $G(\tilde{P})$, all edges coming out of $\tilde{X}$ terminate in $\tilde{Y}$. In other words, any edge path in $G(Pr(\tilde{P}))$ between a vertex $x \in \tilde{X}$ and a vertex $v \notin \tilde{X}$ must contain a vertex in $\tilde{Y}$. We know that there is a vertex $v \in (G(\tilde{P}) \setminus \Gamma)$ because $G(\tilde{P})$ being $d$-regular cannot be a proper subgraph of $\Gamma$ which is also $d$-regular (recall that we assumed $\Gamma \cap V(\tilde{P}) \neq \emptyset$.) So $\tilde{Y} \subseteq V(\Gamma)$ separates $v$ and $\tilde{X}$ contrary to the assumption that $\Gamma$ does not separate $G(Pr(\tilde{P}))$. Hence Case 2 is impossible.

The above argument, shows that $Pr(\tilde{P})$ satisfies Perles’ conjecture if $\tilde{P}$ does. ♦

**Theorem 2** If Perles’ conjecture is true for a simple $d$-polytope $P$, then it is also true for the $d$-polytope $Tr(P)$ obtained by truncating a vertex of $P$.

**Proof**: Assume that vertex $v \in V(P)$ was truncated to obtain $Tr(P)$. Suppose $(v, w_1), \ldots, (v, w_d)$ are the $d$ edges incident on $v$ in $P$. Then $z_1, \ldots, z_d$ are the new vertices in $Tr(P)$ where $z_i$ is the intersection of $(v, w_i)$ and the truncating hyperplane $H$. Also, the new facet of $Tr(P)$ (namely $H \cap Tr(P)$) is a $(d - 1)$-simplex determined by the vertex set $Z = \{ z_1, \ldots, z_d \}$.

Let $\Gamma$ be a $(d - 1)$-regular, connected, induced and non-separating subgraph of $G(Tr(P))$.

If $Z \cap V(\Gamma) = \emptyset$, there is nothing to prove. Also, if $Z \subseteq V(\Gamma)$ then since $Z$ induces a $(d - 1)$-regular subgraph $Z = V(\Gamma)$ and hence $\Gamma$ determines a facet of
$Tr(P)$. So the only case left to consider is where $\Gamma$ contains a proper nonempty subset of $Z$. Since $\Gamma$ is $(d - 1)$-regular, if it contains a vertex of $Z$, it must contain at least $d - 2$ of its neighbors in $Z$. Therefore at most one vertex of $Z$ can be left out and without loss of generality we assume $z_d \notin V(\Gamma)$. Hence, $w_i \in V(\Gamma)$ for $1 \leq i \leq d - 1$.

Consider the subgraph $\Gamma'$ of $G(P)$ induced by the vertex set $(V(\Gamma) \setminus Z) \cup \{v\}$. $\Gamma'$ is a $(d - 1)$-regular, connected, induced subgraph. $z_d$ has only one neighbour in $V(Tr(P)) \setminus V(\Gamma)$. Therefore $V(\Gamma) \cup \{z_d\}$ does not separate $G(Tr(P))$ which means $\Gamma'$ does not separate $G(P)$. So $\Gamma'$ determines a facet $F$ of $P$ and $w_1, \ldots, w_{d-1}$ are the neighbors of $v$ in $F$. Let $H$ be the supporting hyperplane for $F$ in $P$. $H \cap Tr(P)$ is a facet of $Tr(P)$ and the graph of this facet is $\Gamma$; that completes the proof. ♦

As an immediate consequence of theorems 1 and 2 we have,

**Corollary 1** Perles’ conjecture is true for every polytope in the subclass $C$.

The subclass $C$ has the property that any face of a polytope in $C$ also belongs to $C$. We prove this property in the following lemma.

**Lemma 1** If $Q \in C$ and $F$ is a facet of $Q$ then $F \in C$.

**Proof:** Let $Q \in C$ be a polytope for which the lemma is true. Let $X$ be a facet of $Pr(Q)$. If $X = Q$ or $X = \bar{Q}$ then by assumption $X \in C$. If however $X = Pr(F)$ where $F$ is a facet of $Q$, then since the lemma is true for $Q$, $F \in C$ and hence $Pr(F) = X \in C$. So, if the lemma is true for a $Q \in C$ then it is also true for $Pr(Q)$.

Now we consider $Tr(Q)$. Let $v \in V(Q)$ be truncated to obtain $Tr(Q)$ and let $H$ be the truncating hyperplane. Assume that $H^+$ contains $Tr(Q)$ ($H^+$ denotes one of the closed half-spaces of $H$). Let $Y$ be a facet of $Tr(Q)$.

If $Y = H \cap Tr(Q)$ then $Y$ is a simplex and hence $Y \in C$. So assume $Y = F \cap H^+$ where $F$ is a facet of $Q$. Since the lemma is true for $Q$, $F \in C$. Hence $Tr(F) = Y \in C$. The only case left to consider is when $Y = F$ where $F$ is a facet of $Q$. Once again $Y \in C$.
Therefore if the lemma is true for a $Q \in \mathcal{C}$ it is also true for $Tr(Q)$; that completes the proof.\)

As an immediate consequence of this lemma we obtain

**Corollary 2** The entire combinatorial structure of a polytope $P \in \mathcal{C}$ can be determined from $G(P)$ by repeated application of theorems 1 and 2 and lemma 1.

The following lemma shows that $\mathcal{C}$ is properly contained in the class of simple polytopes.

**Lemma 2** $\mathcal{C}$ is a proper subclass of simple polytopes.

**Proof:** We show that $\mathcal{C}$ does not contain a simple 4-polytope with 9 vertices. Suppose it did. Then, to construct the polytope we can either start with a simple 3-polytope or a 4-simplex. Suppose we started with a 4-simplex which has 5 vertices. In this case we may only truncate vertices. But each truncation (when $d=4$) increases the vertex count by 3; so we get 4-polytopes with 5, 8, 11, \cdots vertices but not with 9 vertices. On the contrary suppose we started with a 3-polytope. Since constructing a prism doubles the vertex count we can only construct a prism over a 3-polytope with 4 vertices. The same argument as before shows that again we cannot obtain a 4-polytope with 9 vertices.

Now consider $\mathcal{C}(6,4)$ - the cyclic 4-polytope with 6 vertices has 9 facets. (refer to [3] for details). It is a simplicial polytope. Its dual which is simple has 9 vertices and is hence not in $\mathcal{C}$.\)

Also, it is easy to show that the dual-stacked and the dual-crosspolytopes form proper subclasses of $\mathcal{C}$.

**4 Remarks**

Perles’ conjecture is true for any simple 3-polytope[5]. So we could as well start with any simple 3-polytope and build prisms and truncate vertices finitely many
times. The foregoing results would still be valid without any modification for a polytope so obtained.

References


