Cutting a Polytope

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Abstract

We show that given two vertices of a polytope one cannot in general find a hyperplane containing the vertices, that has two or more facets of the polytope in one closed half-space. Our result refutes a long-standing conjecture.

We prove the result by constructing a 4-dimensional polytope that provides the counter-example. Also, we show that such a cutting hyperplane can be found for each pair of vertices, if the polytope is either simplicial or 3-dimensional.
1 Introduction

In many problems concerning polyhedra (eg. Simplex algorithm) one is required to find paths (having certain properties) between a pair of vertices. Another example (of such problems) is the following unresolved conjecture: Let $v$ and $w$ be any two vertices of a d-polytope $P$. Does the 1-skeleton of $P$ contain a refinement of $K_{d+1}$ in which $v$ and $w$ are d-valent [1]? For yet another example please refer [2].

One would have a powerful inductive tool for solving all such problems, if the following question had an affirmative answer: Given any two vertices $v$ and $w$ of a polytope $P$, does there exist a hyperplane containing $v$ and $w$, that has at least 2 facets of $P$ in one of its closed half-spaces? (For terminology, please refer [3].) In this paper we answer the following more general question: Given a subset of vertices of a polytope $P$, can one find a hyperplane containing the chosen subset, that has two or more facets of $P$ in one closed half-space?

The answer to the question is clear if either $P$ is a simplex or if $|W| = 1$. If $|W| = d$, then the cutting hyperplane might be fully determined by $W$. Also many polytopes admit hyperplanes that intersect the relative interiors of all the facets. Hence it is not surprising that when $|W| = d$, the answer to our question is in the negative in general; the following example elaborates.

**Ex.**: Consider the $d$-cube: $0 \leq x_i \leq 1$, $1 \leq i \leq d$, $d > 3$. One can verify that the hyperplane $x_1 + x_2 + \ldots + x_d = 2$ intersects all the facets of the cube. Also one can easily find $d$ vertices of the cube that affinely span the hyperplane.

So we restrict attention to the range $2 \leq |W| \leq d - 1$. The rather surprising result we prove is that the answer to our question is in the negative in general, even when $|W| = 2$. However, if $d = 3$ or if the given polytope is simplicial the question has an affirmative answer for $|W| = 2$. 

1
2 Results

In Theorem 1, we describe the construction of a 4-polytope $P$ and pick a pair of vertices in $P$, such that no hyperplane containing the pair can have more than one facet of $P$ in either closed half-space. For the construction we need:

**Lemma 1** There exist tetrahedra $T$ and $T'$ in $R^3$ such that:

1. The origin is in the interior of both the tetrahedra and

2. No closed half-space whose boundary plane passes through the origin contains more than one of the eight facets of the two tetrahedra.

**Proof:** Note that a half-space contains a facet of a tetrahedron whenever the half-space contains three vertices of the tetrahedron. Furthermore, condition 1 ensures that no half-space whose boundary contains the origin contains all four vertices of either tetrahedron. Therefore, condition 2 is equivalent to the assertion that no closed half-space whose boundary plane passes through the origin contains 3 vertices from each tetrahedron.

We now dualize the problem. Let $v_1, \ldots, v_4$ and $v'_1, \ldots, v'_4$ denote the vertices of $T$ and $T'$. For a point $p \neq 0$, let $\bar{p}$ denote the closed half-space containing $p$ whose boundary plane passes through the origin and is normal to the position vector of $p$. Note that any closed half-space whose boundary contains the origin can be written as $\bar{q}$ for some point $q$, and furthermore $p \in \bar{q}$ iff $q \in \bar{p}$. Hence conditions 1 and 2 are equivalent to the following:

There are eight closed half-spaces $\bar{v}_1, \ldots, \bar{v}_4, \bar{v}'_1, \ldots, \bar{v}'_4$ in $R^3$ with boundary planes passing through the origin such that:

(a) $\bar{v}_1 \cup \cdots \cup \bar{v}_4 = \bar{v}'_1 \cup \cdots \cup \bar{v}'_4 = R^3$ and

(b) No point of $R^3$ other than the origin lies in more than five of the $\bar{v}_i$ and $\bar{v}'_i$. 

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Observe that there are four nonintersecting great semicircles on the unit 2-sphere $S^2$ (figure 1). They can be widened to four nonintersecting crescents which determine the eight half-spaces; each crescent is the intersection of $S^2$ and two half-spaces corresponding to a pair of vertices from the same tetrahedron. Any point other than the origin lying in six of the half-spaces would (when projected radially onto $S^2$) lie in two of the crescents, which is impossible, proving (b). Similarly, (a) follows from the statement that the two crescents corresponding to each tetrahedron do not intersect. □

We use the foregoing lemma to prove

**Theorem 1** There is a convex 4-polytope $P \subset R^4$ with vertices $v$ and $w$ for which no hyperplane containing $v$ and $w$ has more than one facet of $P$ in either closed half-space.

**Proof :** Let $T$ and $T'$ be the tetrahedra from Lemma 1 sitting in $R^3$. Coordinatize $R^4$ with $t, x, y$ and $z$ axes and identify $R^3$ with the hyperplane $z = 0$. Translate $T$ along the $z$-axis to the hyperplane $z = 1$ and similarly $T'$ to the hyperplane $z = -1$. Let $\epsilon$ be small enough that every line containing a point of $T'$ and the point $v = (0, 0, 0, 1 + \epsilon)$ intersects $T$, and every line containing a point of $T$ and the point $w = (0, 0, 0, -1 - \epsilon)$ intersects $T'$. (Such an $\epsilon$ exists because of condition 1 in Lemma 1.) Let $C$ denote the cone with vertex $v$ and cross-section $T$; let $C'$ denote the cone with vertex $w$ and cross-section $T'$; then $P = C \cap C'$ is a convex 4-polytope with eight facets — four in the star of $v$ and the other four in the star of $w$. If possible, let $H$ be a hyperplane containing $\{v, w\}$, that has two facets of $P$ on one side. If the star of $v$ (resp. $w$) contains both the facets then all the four edges incident on $v$ (resp. $w$) would lie in one closed half-space of $H$, hence $H \cap P \nsubseteq \{v, w\}$. On the other hand, if the stars of $v$ and $w$ contribute one facet each then the orthogonal projection of $H$ onto the hyperplane $z = 0$ would produce a plane in $R^3$ with a facet of $T$ and a facet of $T'$ in one closed half-space (w.r.t. $R^3$), contradicting Lemma 1. □
We can counteract this negative result by restricting our attention to subclasses of polytopes. We illustrate two such instances below. By restricting our attention to 3-polytopes or to simplicial polytopes, we prove the existence of the hyperplane of Theorem 1.

**Theorem 2** Let $P \subset \mathbb{R}^3$ be any convex 3-polytope; let $v$ and $w$ be two vertices of $P$. Then there is a plane $H$ containing $v$ and $w$ such that at least two facets of $P$ lie in one of the closed half-spaces of $H$.

**Proof:** If $v$ and $w$ lie on the same facet of $P$, the theorem is clear. Otherwise, let $G$ be a plane normal to the segment $[v, w]$. Let $f_v$ and $f_w$ be vertex-figures of $v$ and $w$; they can be projected orthogonally onto $G$ to give polygons $\bar{f}_v$ and $\bar{f}_w$. Clearly the point $0 = [v, w] \cap G$ lies in the relative interiors of both $\bar{f}_v$ and $\bar{f}_w$. Let $u$ be any vertex of $\bar{f}_v$; the line $L = \overline{0u}$ intersects the relative interior of at most one side of $\bar{f}_v$ and at most two sides of $\bar{f}_w$; hence at least three sides of the two polygons are contained entirely in the closed half-planes of $L$. So three facets of $P$ are contained entirely in the two closed half-spaces of the plane $H = \text{aff}(u, v, w)$; two of them must lie on the same side of $H$. □

**Theorem 3** Let $P \subset \mathbb{R}^d$ be a simplicial $d$-polytope. Let $v$ and $w$ be vertices of $P$. Then there is a hyperplane containing $v$ and $w$ with two facets of $P$ in one closed half-space.

**Proof:** We can assume that $v$ and $w$ do not lie on the same facet of $P$. If every $(d - 3)$-face in the link of $v$ lies in only three facets of $P$, it is easily checked that $P$ must be a simplex, in which case the theorem is trivial. So we can assume that there is at least one $(d - 3)$-face $G$ in the link of $v$ which lies in at least four facets of $P$. Then $lk(G)$ is a topological circle; let $lk(G) = l_0l_1 \ldots l_n$, where $l_0 = l_n = v$, all the other $l_i$ are distinct, the line segments $[l_i, l_{i+1}]$ are in the link and $n \geq 4$. Let $F = G \ast v$ (* denotes join) and let $H$ be the hyperplane containing $v$, $w$ and $G$. The facets $F \ast l_1$ and $F \ast l_{n-1}$ are in opposite closed half-spaces of $H$; if we can find one other facet of $P$ not intersecting $H$ we are done. Let us call the two half-spaces of $H$ ‘above’ and ‘below’; say $l_1$ lies above $H$ and $l_{n-1}$ lies below $H$. Then we are done unless $l_2$ lies below $H$ and $l_{n-1}$ above. (For example, if $l_2$ lies above $H$, then the facet $G \ast l_1 \ast l_2$ lies entirely in
and above $H$.) Hence the line segments $[l_1, l_2]$ and $[l_{n-2}, l_{n-1}]$ both intersect $H$ at points which
we call $p$ and $q$. Now the $(d - 2)$-flats $G * p, G * q$ and $G * v$ are distinct and all lie in $H$ and
intersect the relative boundary of $P$ but not its relative interior. Hence each divides $H$ into two parts – a part
that contains points of $P$ and a part that does not. In particular, $p$ and $q$ are
on the same side of $G * v$, $p$ and $v$ are on the same side of $G * q$ and $q$ and $v$ are on the same
side of $G * p$. Projecting $R^d$ onto a plane in such a way that $G$ goes to a point, we obtain four
points $G', p', q'$ and $v'$ in the plane such that each pair of $p', q'$ and $v'$ is on the same side of the
line passing through the third and $G'$. But this is impossible. Hence $H$ fails to intersect at least
three facets of $P$ and we are done. □

3 Related Work

[4] shows that for any hyperplane $H$ intersecting a $d$-polytope $P$, there exist two faces of $P$ lying
in different closed half-spaces of $H$ with total dimension at least $d - 1$. [5] proves a stronger
version of the above assertion. [4] and [6] also show that it is not possible to cut all the $j$-faces
of a $d$-polytope $P$, $j < \lfloor \frac{d}{2} \rfloor$, by a hyperplane that does not pass through any vertex of $P$.

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References


[2] Problem 57, page 269; Contributions to Geometry - Proceedings of the Geometry Sympo-

