Cutting a Polytope

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Abstract

We show that given two vertices of a polytope one cannot in general find a hyperplane containing the vertices, that has two or more facets of the polytope in one closed half-space. Our result refutes a long-standing conjecture.

We prove the result by constructing a 4-dimensional polytope that provides the counter-example. Also, we show that such a cutting hyperplane can be found for each pair of vertices, if the polytope is either simplicial or 3-dimensional.

1 Introduction

In many problems concerning polyhedra (eg. Simplex algorithm) one is required to find *paths* (having certain properties) *between a pair of vertices.* Another example (of such problems) is the following unresolved conjecture: *Let v and w be any two vertices of a d-polytope P. Does the 1-skeleton of P contain a refinement of K_{d+1} in which v and w are d-valent [1]?* For yet another example please refer [2].

One would have a powerful inductive tool for solving all such problems, if the following question had an affirmative answer: *Given any two vertices v and w of a polytope P, does there exist a hyperplane containing v and w, that has at least 2 facets of P in one of its closed half-spaces?* (For terminology, please refer [3].) In this paper we answer the following more general question: *Given a subset of vertices of a polytope P, can one find a hyperplane containing the chosen subset, that has two or more facets of P in one closed half-space?*

The answer to the question is clear if either P is a simplex or if $\mid W \mid = 1$. If $\mid W \mid = d$, then the cutting hyperplane might be fully determined by W. Also many polytopes admit hyperplanes that intersect the relative interiors of all the facets. Hence it is not surprising that when $|W| = d$, the answer to our question is in the negative in general; the following example elaborates.

Ex. : Consider the d-cube : $0 \leq x_i \leq 1$, $1 \leq i \leq d$, $d > 3$. One can verify that the hyperplane $x_1 + x_2 + \ldots + x_d = 2$ intersects all the facets of the cube. Also one can easily find d vertices of the cube that affinely span the hyperplane.

So we restrict attention to the range $2 \leq |W| \leq d-1$. The rather surprising result we prove is that the answer to our question is in the negative in general, even when $|W| = 2$. However, if $d = 3$ or if the given polytope is simplicial the question has an affirmative answer for $|W| = 2$.

2 Results

In Theorem 1, we describe the construction of a 4-polytope P and pick a pair of vertices in P, such that no hyperplane containing the pair can have more than one facet of P in either closed half-space. For the construction we need:

Lemma 1 *There exist tetrahedra* $\mathbf{T}$ *and* $\mathbf{T}'$ *in* $\mathbf{R}^3$ *such that:*

1. *The origin is in the interior of both the tetrahedra and*
2. *No closed half-space whose boundary plane passes through the origin contains more than one of the eight facets of the two tetrahedra.*

Proof : Note that a half-space contains a facet of a tetrahedron whenever the half-space contains three vertices of the tetrahedron. Furthermore, condition 1 ensures that no half-space whose boundary contains the origin contains all four vertices of either tetrahedron. Therefore, condition 2 is equivalent to the assertion that no closed half-space whose boundary plane passes through the origin contains 3 vertices from each tetrahedron.

We now dualize the problem. Let $v_1, \ldots, v_4$ and $v_1', \ldots, v_4'$ denote the vertices of $\mathbf{T}$ and $\mathbf{T}'$. For a point $p \neq \mathbf{0}$, let $\bar{p}$ denote the closed half-space containing p whose boundary plane passes through the origin and is normal to the position vector of p. Note that any closed half-space whose boundary contains the origin can be written as $\bar{q}$ for some point q, and furthermore $p \in \bar{q}$ iff $q \in \bar{p}$. Hence conditions 1 and 2 are equivalent to the following:

There are eight closed half-spaces $\bar{v}_1, \ldots, \bar{v}_4, \bar{v}_1', \ldots, \bar{v}_4'$ *in* R^3 *with boundary planes passing through the origin such that:*

(a) $\bar{v}_1 \cup \ldots \cup \bar{v}_4 = \bar{v}_1' \cup \ldots \cup \bar{v}_4' = R^3$ *and*

(b) No point of R^3 *other than the origin lies in more than five of the* $\bar{v}_i$ *and* $\bar{v}_i'$*.*

Observe that there are four nonintersecting great semicircles on the unit 2-sphere S^2 (figure 1). They can be widened to four nonintersecting crescents which determine the eight half-spaces; each crescent is the intersection of S^2 and two half-spaces corresponding to a pair of vertices from the same tetrahedron. Any point other than the origin lying in six of the half-spaces would (when projected radially onto S^2) lie in two of the crescents, which is impossible, proving (b). Similarly, (a) follows from the statement that the two crescents corresponding to each tetrahedron do not intersect. $\square$

We use the foregoing lemma to prove

Theorem 1 *There is a convex 4-polytope $P \subset R^4$ with vertices v and w for which no hyperplane containing v and w has more than one facet of P in either closed half-space.*

Proof : Let $\mathbf{T}$ and $\mathbf{T}'$ be the tetrahedra from Lemma 1 sitting in R^3. Coordinatize R^4 with t, x, y and z axes and identify R^3 with the hyperplane $z = 0$. Translate $\mathbf{T}$ along the z-axis to the hyperplane $z = 1$ and similarly $\mathbf{T}'$ to the hyperplane $z = -1$. Let ϵ be small enough that every line containing a point of $\mathbf{T}'$ and the point $v = (0, 0, 0, 1 + \epsilon)$ intersects $\mathbf{T}$, and every line containing a point of $\mathbf{T}$ and the point $w = (0, 0, 0, -1 - \epsilon)$ intersects $\mathbf{T}'$. (Such an ϵ exists because of condition 1 in Lemma 1.) Let C denote the cone with vertex v and cross-section $\mathbf{T}$; let C' denote the cone with vertex w and cross-section $\mathbf{T}'$; then $P = C \cap C'$ is a convex 4-polytope with eight facets $-$ four in the star of v and the other four in the star of w. If possible, let H be a hyperplane containing $\{v, w\}$, that has two facets of P on one side. If the star of v (resp. w) contains both the facets then all the four edges incident on v (resp. w) would lie in one closed half-space of H, hence $H \cap P \not\supseteq \{v, w\}$. On the other hand, if the stars of v and w contribute one facet each then the orthogonal projection of H onto the hyperplane $z = 0$ would produce a plane in R^3 with a facet of $\mathbf{T}$ and a facet of $\mathbf{T}'$ in one closed half-space (w.r.t. R^3), contradicting Lemma 1. $\square$

We can counteract this negative result by restricting our attention to subclasses of polytopes. We illustrate two such instances below. By restricting our attention to 3-polytopes or to simplicial polytopes, we prove the existence of the hyperplane of Theorem 1.

Theorem 2 *Let $P \subset R^3$ be any convex 3-polytope; let v and w be two vertices of P. Then there is a plane H containing v and w such that at least two facets of P lie in one of the closed half-spaces of H.*

Proof : If v and w lie on the same facet of P, the theorem is clear. Otherwise, let G be a plane normal to the segment $[v, w]$. Let f_v and f_w be vertex-figures of v and w; they can be projected orthogonally onto G to give polygons $\bar{f}_v$ and $\bar{f}_w$. Clearly the point $\mathbf{0} = [v, w] \cap G$ lies in the relative interiors of both $\bar{f}_v$ and $\bar{f}_w$. Let u be any vertex of $\bar{f}_v$; the line $L = \overleftrightarrow{\mathbf{0}u}$ intersects the relative interior of at most one side of $\bar{f}_v$ and at most two sides of $\bar{f}_w$; hence at least three sides of the two polygons are contained entirely in the closed half-planes of L. So three facets of P are contained entirely in the two closed half-spaces of the plane $H = aff(u, v, w)$; two of them must lie on the same side of H. $\Box$

Theorem 3 *Let $P \subset R^d$ be a simplicial d-polytope. Let v and w be vertices of P. Then there is a hyperplane containing v and w with two facets of P in one closed half-space.*

Proof : We can assume that v and w do not lie on the same facet of P. If every $(d-3)$-face in the link of v lies in only three facets of P, it is easily checked that P must be a simplex, in which case the theorem is trivial. So we can assume that there is at least one $(d-3)$-face G in the link of v which lies in at least four facets of P. Then $lk(G)$ is a topological circle; let $lk(G) = l_0 l_1 \dots l_n$, where $l_0 = l_n = v$, all the other l_i are distinct, the line segments $[l_i, l_{i+1}]$ are in the link and $n \geq 4$. Let $F = G * v$ (* denotes join) and let H be the hyperplane containing v, w and G. The facets $F * l_1$ and $F * l_{n-1}$ are in opposite closed half-spaces of H; if we can find one other facet of P not intersecting H we are done. Let us call the two half-spaces of H 'above' and 'below'; say l_1 lies above H and l_{n-1} lies below H. Then we are done unless l_2 lies below H and l_{n-1} above. (For example, if l_2 lies above H, then the facet $G * l_1 * l_2$ lies entirely in and above H.) Hence the line segments $[l_1, l_2]$ and $[l_{n-2}, l_{n-1}]$ both intersect H at points which we call p and q. Now the $(d-2)$-flats $G * p, G * q$ and $G * v$ are distinct and all lie in H and intersect the relative boundary of P but not its relative interior. Hence each divides H into two parts – a part that contains points of P and a part that does not. In particular, p and q are on the same side of $G * v$, p and v are on the same side of $G * q$ and q and v are on the same side of $G * p$. Projecting R^d onto a plane in such a way that G goes to a point, we obtain four points G', p', q' and v' in the plane such that each pair of p', q' and v' is on the same side of the line passing through the third and G'. But this is impossible. Hence H fails to intersect at least three facets of P and we are done. $\square$

3 Related Work

[4] shows that for any hyperplane H intersecting a d-polytope P, there exist two faces of P lying in different closed half-spaces of H with total dimension at least $d-1$. [5] proves a stronger version of the above assertion. [4] and [6] also show that it is not possible to cut all the j-faces of a d-polytope P, $j < \lfloor \frac{d}{2} \rfloor$, by a hyperplane that does not pass through any vertex of P.

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