SUBSTRUCTURING METHODS FOR PARABOLIC PROBLEMS

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Abstract

Domain decomposition methods without overlapping for the approximation of parabolic problems are considered. Two kinds of methods are discussed. In the first method systems of algebraic equations resulting from the approximation on each time level are solved iteratively with a Neumann-Dirichlet preconditioner. The second method is direct and similar to certain iterative methods with a Neumann-Neumann preconditioner. An analysis of convergence of the methods is presented.

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1. Introduction. In this paper domain decmposition methods for solving discrete parabolic problems are discussed. The discrete problems result from finite element and finite difference approximations of parabolic problems with respect to the space and time variables, respectively. For simplicity of presentation only piecewise linear, continuous approximation on triangular elements and the backward Euler and Crank-Nicolson schemes are considered.

Two kinds of substructuring methods, i.e. domain decomposition methods which do not use overlapping subregions, are constructed and analyzed. In the first method systems of algebraic equations resulting from the approximation on each time level are solved iteratively by a conjugate gradient method with a Neumann-Dirichlet preconditioner. As in the elliptic case, it is shown that the rate of convergence is almost optimal with respect to h, the parameter of triangulation, and that it is independent of the time step τ; see Section 2.

Iterative substructuring methods with a Neumann-Dirichlet preconditioner for elliptic finite element problems have been analyzed in [3], [4] and [5].

The second method is direct and similar to certain iterative substructuring methods for elliptic finite element problems using a Neumann-Neumann preconditioner; see [1], [5]. A solution at each time level is obtained in two fractional steps. The rate of convergence of the method is of the order $\tau^{1/2}+h$ provided that τ is proportional to h; see Sections 3 and 4.

Most papers on domain decomposition methods are devoted to elliptic problems, see for example [5] and the literature cited therein. Extensions of the methods to parabolic problems have been considered in [2] and [6].

2. The differential and discrete problems. We consider the parabolic problem. Find $u \in L^2(0,T;H_0^1(\Omega)) \cap C^0(0,T;L^2(\Omega))$ such that

$$(2.1a) \qquad (\frac{\partial u}{\partial t}, \phi) + a(u,\phi) = (f,\phi)\ , \qquad \phi \in H_0^1(\Omega)\ , \quad \text{a.e.} \quad t \in (0,T)$$

$$(2.1b) \qquad (u,\phi) = (u_0,\phi)\ , \qquad \phi \in L^2(\Omega)$$

Here $(\cdot,\cdot)$ is the L^2-scalar product, $a(u,v) = (\nabla u, \nabla v) + (u,v)$ and Ω is a bounded polygonal region in R^2. We assume that $f \in L^2(0,T;L^2(\Omega))$ and $u_0 \in H_0^1(\Omega)$. The problem (2.1) has a unique solution and is stable, see for example [7].

We solve the problem (2.1) using a finite difference method (FDM) and a finite element method (FEM) for the t and x variables, respectively. For simplicity only the backward Euler and Crank-Nicolson and piecewise linear approximations are discussed. A triangulation of Ω is constructed as follows: We first divide Ω into triangular substructures Ω_i, with a parameter H, and we then divide each of the Ω_i into triangles e_j, with a parameter h. These form a coarse and a fine triangulation of Ω. We suppose that they are shape regular in the sense common to finite element theory. $V^h(\Omega)$ is the space of continuous, piecewise linear functions on the fine triangulation, which are zero on $\partial\Omega$. The interval $[0,T]$ is partitioned uniformly, $t_n = n\tau$, $n = 0,\ldots,N$, $N\tau = T$.

The discrete problem is of the form: For $n = 0,1,\ldots,N-1$,

$$(2.2a) \qquad (U_t^n,\phi) + a(U^{n+1},\phi) = (f^{n+1},\phi)\ , \quad \phi \in V^h(\Omega)\ ,$$

$$(2.2b) \qquad (U^0,\phi) = (u_0,\phi)\ , \quad \phi \in V^h(\Omega)\ ,$$

where $U_t^n = (U^{n+1} - U^n)/\tau$, $U^n(x) = U(x,t_n)$. The problem has a unique solution and is stable.

We rewrite (2.2) as the system of linear algebraic equations: For $n = 0, \ldots, N-1$,

$$A\underline{U}^{n+1} \equiv (M + \tau K)\underline{U}^{n+1} = \underline{F}^{n+1} \tag{2.3}$$

where $(M\underline{u}, \underline{v})_{R_m} = (u, v)$, $(K\underline{u}, \underline{v})_{R_m} = a(u, v)$, $u, v \in V^h$ and m is the number of nodal points in Ω. It is easy to derive a formula for $\underline{F}^{n+1}$ from formula (2.2a). We first describe an iterative method with a Neumann-Dirichlet preconditioner for solving (2.3).

3. The Neumann-Dirichlet iterative method. In order to describe the method, we first assume that there is a red-black ordering of the substructures Ω_i. We will refer to the red and black subregions as Neumann (N) and Dirichlet (D) substructures, respectively. Thus, no two substructures of the same type share an edge. For the case when there is no such ordering, see the remark below. Let Ω_N and Ω_D denote the union of the N-type and D-type substructures, respectively and let $\Gamma = \partial\Omega_N \setminus \partial\Omega$ and $\overline{\Omega}_N = \Omega_N \cup \Gamma$.

We now stop underlining vectors and represent the system (2.3) as

$$AU^{n+1} = \begin{pmatrix} A_{11} & 0 & A_{13} \\ 0 & A_{22} & A_{23} \\ A_{13}^T & A_{23}^T & A_{33} \end{pmatrix} \begin{pmatrix} U_1^{n+1} \\ U_2^{n+1} \\ U_3^{n+1} \end{pmatrix} = \begin{pmatrix} F_1^{n+1} \\ F_2^{n+1} \\ F_3^{n+1} \end{pmatrix} \tag{3.1}$$

The matrix A_{11} represents the couplings between pairs of degrees of freedom associated with the set Ω_D, A_{13} the couplings between pairs associated with Ω_D and Γ and so on. M and K are represented in the same way i.e. $A_{ij} = M_{ij} + \tau K_{ij}$. Let S be the Schur complement of A with respect to A_{33}. We obtain

$$S = S^{(N)} + S^{(D)} , \tag{3.2}$$

where $S^{(N)} = A_{33}^{(2)} - A_{23}^T A_{22}^{-1} A_{23}$ and $S^{(D)} = A_{33}^{(1)} - A_{13}^T A_{11}^{-1} A_{13}$. To define $A_{33}^{(1)}$, we introduce the matrix $A^{(D)}$ which corresponds to the bilinear form $(u, v) + \tau a(u, v)$ restricted to $\Omega_D \cup \Gamma$. This matrix is represented as

$$A^{(D)} = \begin{pmatrix} A_{11} & A_{13} \\ A_{13}^T & A_{33}^{(1)} \end{pmatrix}$$

where $A_{33}^{(1)}$ corresponds to the couplings between pairs of variables on Γ. $A^{(N)}$ and $A_{33}^{(2)}$ are defined in a similar way. We note that A_{33} is the sum of $A_{33}^{(1)}$ and $A_{33}^{(2)}$.

Let $V^h(\Gamma)$ be the restriction of $V^h(\Omega)$ to Γ.

Theorem 3.1. *For all* $v_3 \in V^h(\Gamma)$

$$(S^{(N)} v_3, v_3) \le (S v_3, v_3) \le \gamma (1 + \log \frac{H}{h})^2 (S^{(N)} v_3, v_3) , \tag{3.3}$$

where γ is a constant independent of h, H and τ, $(\cdot,\cdot) = (\cdot,\cdot)_{R^{m_3}}$ and m_3 is the number of nodal points on Γ.

The left inequality of (3.3) is obvious. To prove the right one, we need two lemmas.

Lemma 3.1. *For functions* $v \in V^h(\Omega)$, *which are solutions of*

$$A_{11} v_1 + A_{13} v_3 = 0 , \qquad A_{22} v_2 + A_{23} v_3 = 0 \tag{3.4}$$

the following hold

$$(Av, v)_{R^m} = (S v_3, v_3)_{R^{m_3}} , \qquad (A^{(D)}(v_1, v_3)^T, (v_1, v_3)^T)_{R^{m_1}} = (S^{(D)} v_3, v_3)_{R^{m_3}} ,$$

$$(A^{(N)}(v_2, v_3)^T, (v_2, v_3)^T)_{R^{m_2}} = (S^{(N)}v_3, v_3)_{R^{m_3}}$$

where $v = (v_1, v_2, v_3)^T$, $v_i \in R^{m_i}$.

Corollary.

$$(A^{(D)}(v_1, v_3)^T, (v_1, v_3)^T) \le (A^{(D)}(\tilde{v}_1, v_3)^T, (\tilde{v}_1, v_3)^T)$$

with $\tilde{v}_1 \in R^{m_1}$ *arbitrary.*

Lemma 3.1 and its Corollary are proved straightforwardly doing some manipulations.

Let

$$b_i(u, v) \equiv (u, v)_{L^2(\Omega_i)} + \tau a_i(u, v)$$

where $a_i(u, v)$ is the restriction of $a(u, v)$ to Ω_i.

Lemma 3.2. *For the solution* u *of* (3.4) *the following holds*

$$b_i(u, u) \le C(1 + \log \frac{H}{h})^2 \sum_j b_j(u, u) \tag{3.5}$$

where the summation is over the Ω_j *which are the* N*-type neighbors of the* D*-type subregion* Ω_i *and* C *is a constant independent of* h, H, and τ.

Proof: In view of the Corollary to Lemma 3.1, we have

$$b_i(u, u) \le b_i(w, w) \tag{3.6}$$

where w is arbitrary at the nodal points of Ω_i and equals u on $\partial\Omega_i$. We represent w as $w = w_1 + w_2 + w_3$ where w_j is equal to w on side Γ_{ij} of Ω_i including one of its end points x_{ij} and zero on $\partial\Omega_i \setminus (\Gamma_{ij} \cup \{x_{ij}\})$. Below, we construct w_j such that

$$b_i(w_j, w_j) \le C(1 + \log \frac{H}{h})^2 b_j(u, u) \ , j = 1, 2, 3 \ , \tag{3.7}$$

where $b_j(u, u)$ is defined for the N-type Ω_j with side Γ_{ij}, i.e. $\overline{\Gamma}_{ij} = \overline{\Omega}_i \cap \overline{\Omega}_j$. Summing this with respect to j and using (3.6) we obtain (3.5).

To construct the w_j, we first extend the function u, given in the N-type substructures Ω_j, to a function $\hat{w}_j$ defined in a larger region G containing the D-type substructure Ω_i and the N-type substructure Ω_j, such that $\hat{w}_j = u$ in $\overline{\Omega}_j$, $\hat{w}_j \in H_0^1(G)$ and

$$\|\hat{w}_j\|_{L^2(G)} \le C\|u\|_{L^2(\Omega_j)} \ , \qquad |\hat{w}_j|_{H^1(G)} \le C|u|_{H^1(\Omega_j)} \ .$$

The function $\hat{w}_j$ is not a finite element function in general. Using the extension theorem from [8], we construct a finite element function $\hat{w}_{jh}$ such that

$$\|\hat{w}_{jh}\|_{L^2(G)} \le C\|u\|_{L^2(\Omega_j)} \ , \quad |\hat{w}_{jh}|_{H^1(G)} \le C|u|_{H^1(\Omega_j)} \ . \tag{3.8}$$

Let ξ_{ij} be a harmonic function in Ω_i defined by its values on $\partial\Omega_i$. At the nodal points of $\Gamma_{ij} \cup \{x_{ij}\}$, it is equal to 1, it vanishes at the remaining nodes and is defined by linear interpolation between the nodes. The function $\xi_{ij} \in H^{3/2}(\Omega_i)$ and it can be shown that

$$|\xi_{ij}|^2_{H^1(\Omega_i)} \le C(1 + \log \frac{H}{h})$$

We now choose $w_j = I_h(\xi_{ij}\hat{w}_{jh})$, where I_h is the standard linear interpolation operator. We note that, in view of (3.8),

$$\|w_j\|^2_{L^2(\Omega_i)} \le C\|\xi_{ij}\|^2_{L^\infty(\Omega_i)}\|\hat{w}_{jh}\|^2_{L^2(\Omega_i)} \le C\|u\|^2_{L^2(\Omega_j)} \ .$$

We now estimate $\|\nabla w_j\|_{L^2(\Omega_i)}$. Let $\bar{\xi}_{ij}^k$ be the average value of ξ_{ij} over the element e_k. We have

$$\sum_k \|\nabla w_j\|^2_{L^2(e_k)} \le 2\sum_k \{\|\nabla(I_h(\xi_{ij}-\bar{\xi}^k_{ij})\hat{w}_{jh})\|^2_{L^2(e_k)} + \|\bar{\xi}^k_{ij}\nabla\hat{w}_{jh}\|^2_{L^2(e_k)}\}$$

and

$$\sum_k \|\bar{\xi}^k_{ij}\nabla\hat{w}_{jh}\|^2_{L^2(e_k)} \le C\|\nabla\hat{w}_{jh}\|^2_{L^2(\Omega_i)} \le C|u|^2_{H^1(\Omega_j)}$$

in view of (3.8). Using a standard error bound and the fact that $\xi_{ij} \in H^{3/2}(\Omega_i)$, we obtain

$$\|\nabla(I_h(\xi_{ij}-\bar{\xi}^k_{ij})\hat{w}_{jh}\|^2_{L^2(e_k)} \le C\|\hat{w}_{jh}\|^2_{L^\infty(e_k)}\|\nabla\xi_{ij}\|^2_{L^2(e_k)} \ .$$

Summing with respect to k and using (3.8) and a discrete Sobolev inequality, see e.g. [5], we obtain

$$\sum_k \|\nabla(I_h(\xi_{ij}-\bar{\xi}^k_{ij})\hat{w}_{jh}\|^2_{L^2(e_k)} \le C(1+\log\frac{H}{h})^2\|\hat{w}_{jh}\|^2_{H^1(\Omega_i)} \le C(1+\log\frac{H}{h})^2|u|_{H^1(\Omega_j)} \ .$$

Hence

$$\|\nabla w_j\|^2_{L^2(\Omega_i)} \le C(1+\log\frac{H}{h})^2|u|^2_{H^1(\Omega_j)} \ .$$

Combining this with the estimate for $\|w_j\|^2_{L^2(\Omega_i)}$, we obtain (3.7).

Proof of Theorem 3.1: The right inequality: It is enough to show that

$$(S^{(D)}v_3, v_3) \le C(1+\log\frac{H}{h})^2(S^{(N)}v_3, v_3) \tag{3.9}$$

Using Lemma 3.1, we have

$$(S^{(D)}v_3, v_3) = (A^{(D)}v_D, v_D) \le \|\tilde{v}_D\|^2_{L^2(\Omega_D)} + \tau\|\tilde{v}_D\|^2_{H^1(\Omega_D)}$$

where $v_D = (v_1, v_3)^T$ is the solution of (3.4) and $\tilde{v}_D = (\tilde{v}_1, v_3)^T$ with an arbitrary $\tilde{v}_1$. Applying Lemma 3.2 and then again Lemma 3.1, we obtain (3.9). The proof is complete.

We now define a preconditioner $\widehat{A}$ for A of the form

$$\widehat{A} = \begin{pmatrix} A_{11} & 0 & A_{13} \\ 0 & A_{22} & A_{23} \\ A_{13}^T & A_{23}^T & A_{33}^{(2)} + A_{13}^T A_{11}^{-1} A_{13} \end{pmatrix} \ .$$

Theorem 3.2. *For any $v \in V^h(\Omega)$*

$$(\widehat{A}v, v) \le (Av, v) \le \gamma(1+\log\frac{H}{h})^2(\widehat{A}v, v) \tag{3.10}$$

where γ is a positive constant independent of h, H and τ.

Proof: It is easy to see that

$$\widehat{A} = L^T\widehat{D}L \quad \text{and} \quad A = L^T D L$$

where

$$\widehat{D} = \begin{pmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & 0 & S^{(N)} \end{pmatrix}, \qquad D = \begin{pmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & 0 & S \end{pmatrix} \tag{3.11}$$
$$L = \begin{pmatrix} I_1 & 0 & A_{11}^{-1}A_{13} \\ 0 & I_2 & A_{22}^{-1}A_{23} \\ 0 & 0 & I_3 \end{pmatrix}$$

The generalized eigenvalue problem

$$Az = \lambda \widehat{A} z \ , \qquad z = (z_1, z_2, z_3)^T$$

reduces to

$$(S^{(N)} + S^{(D)}) z_3 = \lambda S^{(N)} z_3$$

since the matrices D and $\widehat{D}$ have the same first two block rows. Applying Theorem 3.1 we obtain (3.10) and the proof is complete.

¿From Theorem 3.2 follows that the systems (2.3) with the matrix A can be solved iteratively using $\widehat{A}$ as preconditioner in a conjugate gradient method. The number of iterations to obtain the solution with accuracy ϵ is on the order of $\log \frac{1}{\epsilon}(1 + \log \frac{H}{h})$. In each iteration step a system with the matrix $\widehat{A}$ is solved. For that the factorization $A = (L^T \widehat{D}) L$ is used. In each step, this invovles solving two sets of the Dirichlet subproblems with the bilinear form $b_i(u, v)$ for the D-type subregions and a set of the Neumann problems for the N-type subregions.

The D-type subproblems are independent and can be solved in parallel. The Neumann subproblems are coupled at the cross points, i.e. the vertices of N-type subregions. We can use a block Gaussian elimination or a capacitance matrix method for solving the systems corresponding to these subproblems, cf. [4].

The connections at the cross points provide a mechanism for the global transportation of information similar to that of a finite element approximation of an elliptic problem. As was shown in [9], the condition number of any method without such a mechanism deteriorates at least as fast as H^{-2}. For parabolic problems the situation is different. As we will see below this mechanism can be dropped if τ / H^2 is bounded cf. [2]. In particular, the set of Neumann subproblems can be split into independent subproblems and solved in parallel. We now study this case.

Let $A^{(N)}$ denote the matrix

$$A^{(N)} = \begin{pmatrix} A_{22} & A_{23} \\ A_{23}^T & A_{33}^{(2)} \end{pmatrix} \ .$$

Note that for $u, v \in V^h(\Omega)$, which are the solutions of (3.4),

$$(S^{(N)} u_3, v_3) = (A^{(N)} (v_2, v_3)^T, (u_2, u_3)^T = (1 + \tau)(v, u)_{L^2(\Omega_N)} + \tau(\nabla u, \nabla u)_{L^2(\Omega_N)} \ ;$$

cf. Lemma 3.1. To get the preconditioner $\widetilde{A}$ for A, we replace $A^{(N)}$ by $\widetilde{A}^{(N)}$ which is defined as follows: Let $\overline{\Omega}_{N,h}$ and Ω_H denote the sets of nodal points of the fine mesh in $\overline{\Omega}_N$ and the coarse mesh in Ω, respectively. Let $I_0 u \in V^h(\Omega)$ be zero for $x \in \overline{\Omega}_{N,h} \setminus \Omega_H$ and equal to $u(x)$ for $x \in \Omega_H$. We introduce a bilinear form $c(u, v)$ by

$$c(u, v) = (1 + \tau) h^2 \sum_{x \in \overline{\Omega}_{N,h}} u(x) v(x) + \tau \sum_{x \in \Omega_H} u(x) v(x) \qquad (3.12)$$

$$+ \tau (\nabla (u - I_0 u) \ , \ \nabla (v - I_0 v) \,)_{L^2(\Omega_N)} \ ,$$

and the matrix $\widetilde{A}^{(N)}$ by

$$(\widetilde{A}^{(N)} (u_2, u_3)^T \ , \ (v_2, v_3)^T)_{R^{m_2}} = c(u, v) \ ,$$

Let

$$\widetilde{A}^{(N)} = \begin{pmatrix} \widetilde{A}_{22} & \widetilde{A}_{23} \\ \widetilde{A}_{23}^T & \widetilde{A}_{33}^{(2)} \end{pmatrix} \ , \qquad \widetilde{S}^{(N)} = \widetilde{A}_{33}^{(2)} - \widetilde{A}_{23}^T \widetilde{A}_{22}^{-1} \widetilde{A}_{23} \ .$$

By using (3.11), we find that

$$\widetilde{A} = L^T \widetilde{D} L \ , \qquad \widetilde{D} = \begin{pmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & 0 & \widetilde{S}^{(N)} \end{pmatrix} .$$

Lemma 3.3. *For all $v \in V^h(\Omega)$, $v = (v_1, v_2, v_3)^T$*

$$\gamma_0 (1 + \frac{\tau}{H^2})^{-1} (1 + \log \frac{H}{h})^{-1} (\widetilde{A}^{(N)} v_N, v_N) \le (A^{(N)} v_N, v_N) \le \gamma_1 (\widetilde{A}^{(N)} v_N, v_N) \tag{3.13}$$

where $v_N = (v_2, v_3)^T$ and the γ_i are constants independent of h, H and τ.

Proof: We first prove the right inequality. It is easy to show

$$(v, v)_{L^2(\Omega_N)} \le C h^2 \sum_{x \in \overline{\Omega}_{N,h}} v^2(x) \ .$$

Using the triangle and inverse inequalities, we get

$$(\nabla v, \nabla v)_{L^2(\Omega_N)} \le 2 (\nabla (v - I_0 v), \nabla (v - I_0 v))_{L^2(\Omega_N)} + C \sum_{x \in \Omega_H} v^2(x) \ .$$

The right inequality of (3.13) now follows.

We have

$$(\nabla (v - I_0 v), \nabla (v - I_0 v))_{L^2(\Omega_N)} \le 2 (\nabla v, \nabla v)_{L^2(\Omega_N)} + C \sum_{x \in \Omega_H} v^2(x) \ . \tag{3.14}$$

It is known that

$$\|v\|^2_{L^\infty(\Omega_i)} \le C (H^{-2} \|v\|^2_{L^2(\Omega_i)} + (1 + \log \frac{H}{h}) \|\nabla v\|^2_{L^2(\Omega_i)}) \ .$$

¿From this we obtain

$$\tau \sum_{x \in \Omega_H} v^2(x) \le C (1 + \log \frac{H}{h}) (1 + \frac{\tau}{H^2}) (\|v\|^2_{L^2(\Omega_N)} + \tau \|\nabla v\|^2_{L^2(\Omega_N)}) \ . \tag{3.15}$$

Substituting (3.15) into (3.14) and using the inequality

$$h^2 \sum_{x \in \overline{\Omega}_{N,h}} v^2(x) \le C \|v\|^2_{L^2(\Omega_N)} \ ,$$

we obtain the left inequality of (3.13).

Theorem 3.3. *For all $v \in V^h(\Omega)$*

$$\gamma_2 (1 + \frac{\tau}{H^2})^{-1} (1 + \log \frac{H}{h})^{-1} (\widetilde{A} v, v) \le (A v, v) \le \gamma_3 (1 + \log \frac{H}{h})^2 (\widetilde{A} v, v) \ .$$

Here γ_i, $i = 2, 3$ are constants independent of h, H and τ.

Proof: This result follows from Lemma 3.3 and Theorem 3.2 and the representations of A, $\widehat{A}$ and $\widetilde{A}$, see (3.11).

Remark. In the method discussed, we have assumed that there exists a red-black ordering of the substructures i.e. no two substructures of the same type share an edge. To extend the method to the case when this condition is not satisfied, we introduce ND-type substructures which borders substructures of both N- and D-type. In this case, we construct a preconditioner as follows: On each Ω_i let u be represented as $u = P_i u + H_i u$ where $H_i u$ is the discrete harmonic extension of u from $\partial\Omega_i$ in the sense of $b(\cdot,\cdot) = (1+\tau)(u,v) + \tau(\nabla u, \nabla v)$. On a Ω_i of ND-type, denoted by $\Omega_{i,ND}$, we consider an extension $\widetilde{H}_i u$ which satisfies $b_{i,ND}(\widetilde{H}_i, \phi) = 0$ for all basis functions ϕ associated with the nodal points $x \in \overline{\Omega}_{i,ND} \setminus \Gamma_{ij}$ and with $\widetilde{H}_i u = u$ on the sides Γ_{ij} of $\Omega_{i,ND}$ which are shared with D-type substructures. On the remaining sides of $\Omega_{i,ND}$, we use homogeneous Neumann boundary conditions. Let

$$d(u,v) = \sum_i b_i(P_i u, P_i v) + \sum_i b_{i,N}(H_i u, H_i v) + \sum_i b_{i,ND}(\widetilde{H}_i u, \widetilde{H}_i v)$$

and let B be a matrix corresponding to this bilinear form. Here $b_{i,N}(\cdot,\cdot)$, $b_{i,ND}(\cdot,\cdot)$ are the restrictions of the bilinear form $b(\cdot,\cdot)$ to substructures of N-type and ND-type, respectively. It can be proved that B is spectrally equivalent to A except for a factor $(1+\log\frac{H}{h})^2$; cf. Theorem 3.2.

4. The Neumann-Neumann direct method. In this section, we discuss a direct method for problem (2.1), which can be interpreted as a generalization to the parabolic case of a method with a Neumann-Neumann preconditioner previously developed for elliptic problems; cf. [1], [5]. The problem (2.1) is approximated by the following scheme.

For $n = 0, 1, \ldots, N-1$,

$$(4.1a)\qquad \begin{cases} (U_t^n, \phi) + a_D((U^{n+1/2} + U^n)/2, \phi) = (f_1^{n+1}, \phi)\ , & \phi \in V^h(\Omega) \\ U^{n+1/2} = U^n\ , & x \in \Omega_{N,h} \end{cases}$$

and

$$(4.1b)\qquad \begin{cases} (U_t^{n+1/2}, \phi) + a_N((U^{n+1} + U^{n+1/2})/2, \phi) = (f_2^{n+1}, \phi)\ , & \phi \in V^h(\Omega) \\ U^{n+1} = U^{n+1/2}\ , & x \in \Omega_{D,h} \end{cases}$$

Here $U_t^n = (U^{n+1/2} - U^n)/\tau$, $U_t^{n+1/2} = (U^{n+1} - U^{n+1/2})/\tau$, $f^{n+1} = f_1^{n+1} + f_2^{n+1}$ and $f_1^{n+1} = 0$ and $f_2^{n+1} = 0$ for $x \in \Omega_{N,h}$ and $x \in \Omega_{D,h}$, respectively. a_D and a_N are the restrictions of the bilinear form $a(u,v)$ to Ω_D and Ω_N. $\Omega_{N,h}$ and $\Omega_{D,h}$ are the sets of nodal points in Ω_N and Ω_D, respectively, $U^{n+1/2} \in V^h(\Omega)$ is defined by the values of U^n at the nodal points $x \in \Omega_N$ and by the solution of (4.1a) in $\overline{\Omega}_D$. The function U^{n+1} is defined similarly. The schemes (4.1a) and (4.1b), taken separately, do not approximate the problem (2.1). In spite of this, we can show convergence with a rate $\tau^{1/2} + h$ provided that τ is proportional to h. We formulate the stability and convergence theorems without proofs.

Theorem 4.1. *The solutions of* (4.1) *satisfy the following inequality*

$$\max_n \|U^n\|^2_{L^2(\Omega)} + \sum_{n=0}^{N-1} (\|U^{n+1/2} + U^n\|^2_{H^1(\Omega_D)} + \|U^{n+1} + U^{n+1/2}\|^2_{H^1(\Omega_N)})$$

$$\leq C(\sum_{n=0}^{N-1} \tau(\|f_1^{n+1}\|^2 + \|f_2^{n+1}\|^2) + \|u^0\|^2)$$

where C is a constant independent of h, H and τ.

Theorem 4.2. *If the solution u of* (2.1) *is sufficiently smooth and τ is proportional to h then,*

$$\max_n \|z^n\|^2_{L^2(\Omega)} + \tau \sum_{n=0}^{N-1} (\|z^{n+1/2} + z^n\|^2_{H^1(\Omega_D)} + \|z^{n+1} + z^{n+1/2}\|^2_{H^1(\Omega_N)}) \leq C(\tau + h^2) \ ,$$

where $z^n = u^n - U^n$, $z^{n+1/2} = u^{n+1} - U^{n+1/2}$, U^n is the solution of (4.1) *and C is a constant independent of h, H and τ.*

Acknowledgements. The author is indebted to Olof Widlund for helpful comments and many suggestions to improve this paper.

REFERENCES

1 J.-F. Bourgat, R. Glowinski, P. le Tallec and M. Vidrascu, *Variational formulation and algorithm for trace operator in domain decomposition calculations*, in Domain Decomposition Methods, T. Chan et al., eds., SIAM, Philadelphia, 1989, pp. 3-16.

2 X. C. Cai, *Some domain decomposition algorithms for nonself-adjoint elliptic and parabolic partial differential equations*, Technical Report 461, Courant Inst. Math. Sci., Sept. 1989.

3 M. Dryja, *Iterative substructuring methods for elliptic problems*, in Computational Processes and Systems, G. I. Marchuk, ed., Nauka, Moscow, v. 6, 1988, pp. 110-120 (in Russian).

4 M. Dryja, W. Proskurowski and O. Widlund, *A method of domain decomposition with crosspoints for elliptic finite element problems*, in Optimal Algorithms, Bl. Sendov, ed., Bulgarian Academy of Sciences, 1986, pp. 97-111.

5 M. Dryja and O. Widlund, *Towards a unified theory of domain decomposition algorithms for elliptic problems*, in Third International Symposium on Domain Decomposition Method for Partial Differential Equations, T. Chan et al., eds., SIAM, Philadelphia, 1989, pp. 3-21.

6 Y. Kuzentsov, *Domain decomposition methods for unsteady convection-diffusion problems*, in Proceedings of the IX Int. Conf. on Computing Methods in Appl. Sci. and Eng., Paris, 1990.

7 J. L. Lions and E. Magenes, *Nonhomogeneous boundary value problems and applications I*, Springer, Berlin-Heidelberg-New York, 1972.

8 O. Widlund, *An extension theorem for finite element spaces with three applications*, in Proceedings of the Second GAMM-Seminar on Numerical Techniques in Continuum Mechanics, W. Hackbusch and K. Witsch, eds., Kiel, 1986, pp. 110-122.

9 O. Widlund, *Iterative substructuring methods: Algorithms and theory for elliptic problems in the plane*, in First Symposium on Domain Decomposition Methods for Partial Differential Equations, R. Glowinski et al., eds., SIAM, Philadelphia, 1988, pp. 113-128.