Written Qualifying Exam
Theory of Computing
Spring 2001
Friday, May 18, 2001

This is a four hour examination. All questions carry the same weight. Answer all of the following six questions.

- Please check to see that your name and address are correct as printed on your blue-card.
- Please print your name on each exam booklet. Answer each question in a separate booklet, and number each booklet according to the question.

Read the questions carefully. Keep your answers legible, and brief but precise. Assume standard results, except where asked to prove them.
Problem. 1  New booklet please. [10 points]

(a) (5 points) Let \( L_1 \) and \( L_2 \) be two languages. The shuffle of \( L_1 \) and \( L_2 \) is the language \( L_1 \# L_2 = \{ w_1 \ldots w_k \mid k \geq 0 \) and each string \( w_i \) belongs to \( L_1 \) or \( L_2 \)\}. Show that regular languages are closed under the shuffle operation. Moreover, given NFA’s for \( L_1 \) and \( L_2 \), construct an NFA for \( L_1 \# L_2 \).

(b) (5 points) Let the alphabet \( \Sigma = \{A, T, C, G\} \). Define the complementation operation on letters of \( \Sigma \) by \( A^C = T, T^C = A, C^C = G \), and \( G^C = C \). Given \( \sigma = x_1 \ldots x_k \), we let \( \sigma^C = x_1^C \ldots x_k^C \), while \( \sigma^R = x_k \ldots x_1 \). Then we let the language of reverse palindromes \( \text{RPAL}(\Sigma) \) over \( \{A, T, C, G\} \) to be

\[
\text{RPAL}(\{A, T, C, G\}) = \{ \sigma \in \{A, T, C, G\}^* \mid \sigma^R = \sigma^C \}.
\]

Show that the language \( \text{RPAL}(\Sigma) \) is context-free but not regular.

Problem. 2  New booklet please. [10 points]

Given a system of polynomial equations in many variables over integers, Hilbert’s 10th problem asks if the system has an integer solution. It was a celebrated result that Hilbert’s 10th problem is undecidable. Show a simpler result, namely that the problem is NP-hard. Moreover, show that NP-hardness holds even if all equations have degree at most 6 (if you cannot enforce degree at most 6, get partial credit for general NP-hardness). Is this problem NP-complete? Does your reduction show the NP-hardness of polynomial equations over the space of real and/or complex numbers instead of integers?

Problem. 3  New booklet please. [10 points] Show that it is undecidable if a Turing Machine with alphabet \( \{0, 1, B\} \) ever prints three consecutive 1’s on its tape.

Please turn over.
Problem 4  New booklet please. [10 points]

Let $T$ be a binary tree. For each node $v$ in $T$, let $h(v)$ be the length of the longest downward path from $v$ to a leaf, and let $d(v)$ be the length of the shortest downward path from $v$ to a leaf. (Thus at a leaf $v$, $d(v) = h(v) = 0$.) Define a whole binary tree to be one in which every internal node has two children, and in addition, $h(v) \leq 2 \cdot d(v)$ for all nodes $v$.

Let $T(\ell)$ be the smallest possible number of nodes in a whole binary tree with $h(root) = \ell$. Determine $T(\ell)$ exactly.

Hint: The form of the answer may depend on the parity of $\ell$, i.e. whether $\ell$ = even or odd.

Problem 5  New booklet please. [10 points]

The input is a sequence of $n$ elements $x_1, x_2, \ldots, x_n$ that we can read sequentially. We want to use a memory that can only store $O(k)$ (e.g., $\leq 4k$) elements at a time. Give a high level description of an algorithm that finds the $k$th smallest element in $O(n)$ time.

Hint: Use the linear-time median algorithm.

Problem 6  New booklet please. [10 points]

Define a common subsequence (not necessarily contiguous) of two strings $V = v_1 \cdots v_n$ and $W = w_1 \cdots w_m$ as a pair of sequence of indices:

$$1 \leq i_1 < \cdots < i_k \leq n \quad \text{and} \quad 1 \leq j_1 < \cdots < j_k \leq m,$$

such that

$$\forall 1 \leq t \leq k v_{i_t} = w_{j_t}.$$

Let $s(V, W) = k$ be the length of a longest common subsequence (LCS) of $V$ and $W$. For example, the LCS of two strings $V = ATCTGAT$ and $W = TGCAT$ is $TCTA$ and $s(V, W) = 4$. Devise an efficient algorithm to compute $s(V, W)$. 

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