Chapter 4

Quantities and Measurements

> I have figured out for you the distance between the horns of a dilemma, night and day, A and Z. I have computed how far is Up, how long it takes to get Away, and what becomes of Gone. I have discovered the length of the sea serpent, the price of the priceless, and the square of the hippopotamus. I know where you are when you are at Sixes and Sevens, how much Is you have to have to make an Are, and how many birds you can catch with the salt in the ocean — 187,796,132 if it would interest you to know.
>
> — James Thurber, *Many Moons*

Consider the following story:

> John's boss kept him at the office one evening, and he got home late. He had promised his wife that he would get supper ready in time to watch "Dallas," so he decided to make hamburgers instead of beef stew. He didn't have any ground meat in the house, so he took the car to go to the butcher. The supermarket was closer, but the butcher had superior meat. On the way, he noticed that he was almost out of gas, so he stopped at a gas station. When he got to the butcher's, he saw that the price of ground beef had gone up again, which the butcher explained was due to the rising price of cattle feed. John had been getting more and more frustrated, and he yelled at the butcher for being a thieving scoundrel.

This story has no numbers or mathematics in it. Nonetheless, it is replete with *quantities* — parameters which can take greater or lesser values — and it cannot be understood without some ability to reason about quantities. The quantities in this story include spatial quantities, such as the distance to the butcher and the supermarket; temporal quantities, such as the times at which these various events occur, and the length of time needed to cook hamburgers and stew; physical quantities, such as amount of ground beef and gasoline; psychological quantities, such as the degree of John's frustration; economic quantities, such as the price of hamburger; and quantities of other categories, such as the quality of meat. Some of these quantities are functions of other quantities; for example, the price of hamburger, the quantity of gasoline, and the distance to the butcher, are all functions of time. To assert that the gas will run out before the car reaches the butcher requires comparing the behavior of these two functions.

ORD.1. $X < Y \Rightarrow \neg(Y < X)$ (Anti-symmetry)

ORD.2. $X < Y \wedge Y < Z \Rightarrow X < Z$ (Transitivity)

ORD.3. $X < Y \vee Y < X \vee X = Y$ (Totality)

Table 4.1: Axioms of Ordering

It is fundamental to mathematics that many aspects of reasoning about quantities are independent of the particular domain involved. For that reason, we will study the representations and reasoning strategies associated with quantities as a whole before entering into their particular applications. Now, of course, mathematicians have been studying quantities for thousands of years, and numerical analysts have been studying computations with quantities for hundreds of years, so we in AI are hardly exploring new ground here. Much of our work has been done for us. Ontological issues have been largely settled for us by the work on foundations of mathematics, which has given solid answers to such questions as, "What is a number?" "What is a function?" "What is an infinite/infinitesimal quantity?" As regards representations, we have an embarrassment of riches in the wealth of mathematical symbols and theories which are available. The problem for the AI researcher is largely to determine which mathematical concepts and techniques are appropriate for his particular application. The common characteristic in AI reasoning about quantities is the focus on rapidly deriving partial, qualitative conclusions from partial input information, rather than deriving very detailed and precise information using lengthy calculations.

In this chapter, therefore, rather than systematically review the foundations of arithmetic, we will consider a series of examples of commonsense quantitative inferences, and examine the representations and inferences used in each.

4.1 Order

Examples:

> Brown is a better teacher than Crawford. Crawford is a better teacher than Gay. Infer that Brown is a better teacher than Gay.
>
> Any French wine is classier than any Rhode Island wine. Infer that Rhode Island Red '83 is not classier than Chateau Lafitte '23.

The most fundamental quantitative relation is the order relation "$X < Y$". (Note that, in mathematical parlance, an "inequality" is always an order relation on terms, though there are of course, many other possible relations besides equality and ordering.) In domains where assigning exact values is not very meaningful, such as quality of teaching or classiness of wines, order relations are often the only useful quantitative relations.

Table 4.1 shows the axioms of ordering. A partial ordering obeys axioms ORD.1 and ORD.2; a total ordering obeys axiom ORD.3 as well.

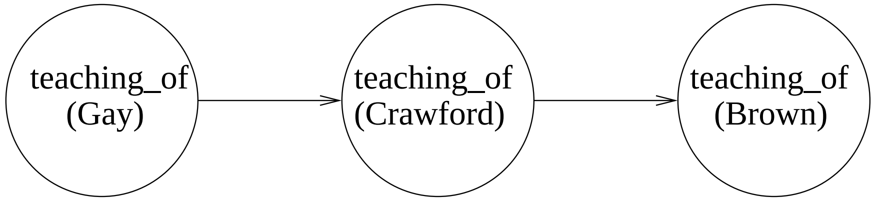


Figure 4.1: Inequalities as a DAG

A logical sort of entity that is partially or totally ordered is called a *measure space.*[1] For example, "lengths," "masses," or "dates" are measure spaces. It makes sense to ask which of two dates is later, but it makes no sense to ask whether a date is greater than a length. Thus, the relation "$X < Y$", like most of the quantitative relations we will introduce in this chapter, is polymorphously sorted. "$X < Y$" is meaningful just if X and Y are elements of the same measure space.

A collection of atomic ground inequalities can be implemented as a DAG (directed acyclic graph) whose nodes are the ground terms and whose arcs correspond to inequalities. In our first example above, the nodes would be labelled "teaching_of(brown)", "teaching_of(crawford)", and "teaching_of(gay)", with arcs from Crawford to Brown and from Gay to Crawford. (Figure 4.1.) The inequality "$X < Y$" is a consequence of the input if there is an arc from X to Y in the transitive closure of the DAG. This can be determined in time $O(n^2)$ in the worst case, using Dijkstra's algorithm.

4.2 Intervals

Example:

> The first Crusade occured during the Middle Ages. The Middle Ages predated the Enlightenment. Infer that the first Crusade predated the Enlightenment.

Many natural entities in commonsense reasoning correspond to intervals of a measure space, rather than single points. In the above example, the times of the first Crusade, of the Middle Ages, and of the Enlightenment are intervals of time. In reasoning about the temperature measure space, it might be natural to pick out the range of comfortable temperatures, the range of temperature in which water is liquid, and so on.

There are thirteen possible order relations between pairs of intervals [Allen, 1983]: Interval I is *before* interval J; I *meets* J; I *overlaps* J; I *starts* J; I *finishes* J; I occurs *during* J; I is *equal* to J; and the inverses of these (Figure 4.2). (The names of these relations were picked for a temporal measure space; however, the same basic relations apply to intervals in any measure space.) These relations can be combined according to rules of transitivity. For instance, the inference in the above example can be justified by the rule, "If I occurs during J and J precedes K, then I precedes K." (See exercise 2 for a discussion of the remaining rules.) It is possible to take as primitive just the relation "meets(I, J)" and define all the other relations in terms of it. (Table 4.2)

[1]The term was introduced with this meaning in [Hayes, 1978]. This usage differs from the meaning of "measure space" in mathematical analysis.

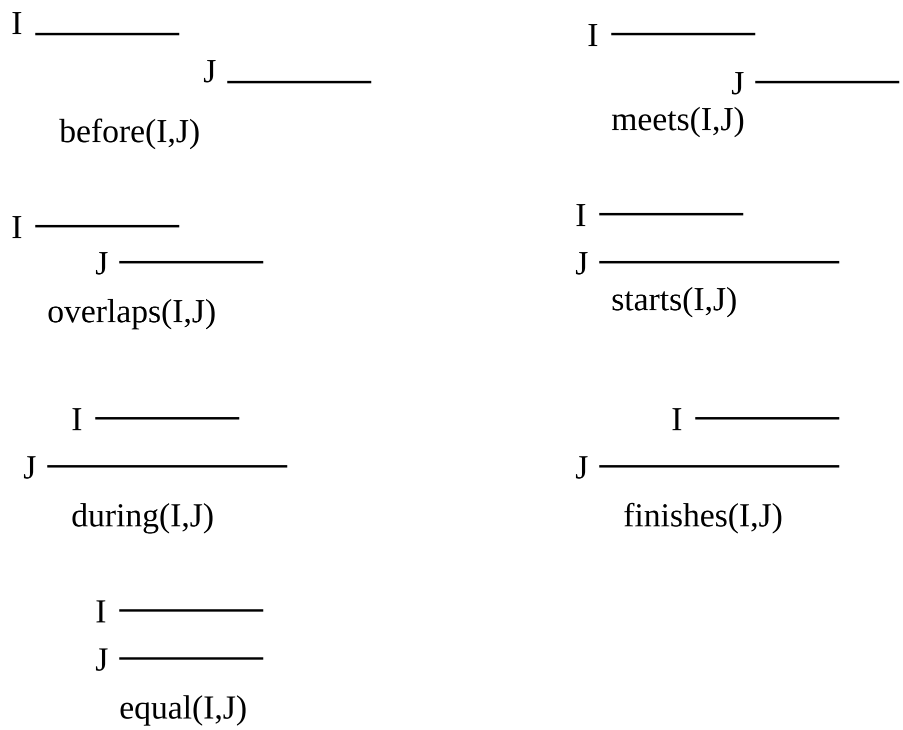


Figure 4.2: Interval relations

For many purposes, it is useful to combine "equals", "starts", "during", and "finishes" into a single relation "I is *contained in* J." It is also useful to introduce two partial functions on intervals: The *overlap* of I and J, written "overlap_of(I, J)" is the maximal interval that is contained in both I and J; it is defined only if I overlaps J, I is part of J, or vice versa. The *join* of I and J, written "join(I, J)" is the minimal interval containing I and J; it is defined for all I and J unless I is before or after J.

In some domains, it is reasonable to use both points and intervals. For example we might want to say that the death of Charlemagne (a point in time) occurred in the Middle Ages, or that the boiling point of alcohol is in the range of temperatures where water is liquid. One method to represent such statements is to treat a point just as an interval which does not properly contain any other interval. Another approach, which we will follow here, is to view an interval I just as the set of points that fall in I. Specifically, consider a totally ordered measure space $\mathcal{M}$. An interval I in $\mathcal{M}$ is any set of points in $\mathcal{M}$ such that if $X \in I$, $Y \in I$, and $X < Z < Y$, then $Z \in I$. In this view, the relation of a point falling inside an interval is just set membership. Table 4.2 shows how the other relations between intervals can be expressed in terms of points.

The definition of a interval over partial ordered measure spaces is trickier, but will be important in constructing branching models of time. Here, an interval is defined as a subset I of the measure space such that (i) I is totally ordered and (ii) no point Z between two points of I can be added to I without destroying the total ordering (Figure 4.3). The formulas below give a formal definition, using the predicate "ordered(X, Y)", meaning that X and Y are ordered in their measure space.

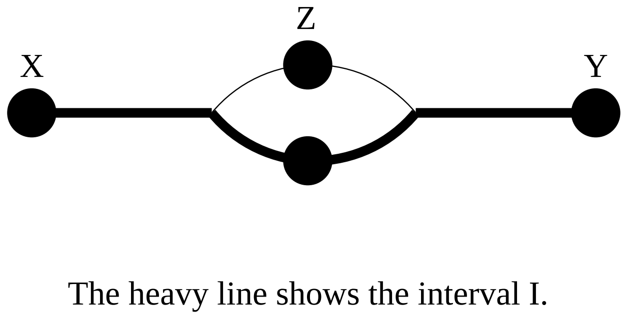


Figure 4.3: Interval in a Partial Ordering

ordered$(X, Y) \Leftrightarrow [\ X < Y \ \vee \ Y < X \ \vee \ X = Y\ ]$

interval$(I) \Leftrightarrow$
$[\ [\ \forall_{X,Y \in I}$ ordered$(X, Y)\ ] \wedge$
$[\ \forall_{Z \notin I} \forall_{X,Y \in I}\ X < Z < Y \Rightarrow \exists_{P \in I} \neg$ordered$(Z, P)\ ]]$

Another fundamental relation between points and intervals is that a point can be the starting point or ending point of an interval. We use the function symbols, "start(I)" and "end(I)" to represent the mappings from an interval to its start and end points. These can be defined as follows: We define X to be a *lower bound* of interval I if X is less than or equal to every element of I, and to be an *upper bound* of I if X is greater than or equal to every element of I. Then the start of I is the greatest lower bound for I, if this exists, and the end of I is the least upper bound, if that exists.

lower_bound$(X, I) \Leftrightarrow \forall_{Y \in I} Y \geq X$

upper_bound$(X, I) \Leftrightarrow \forall_{Y \in I} Y \leq X$

X=start$(I) \Leftrightarrow$ lower_bound$(X, I) \wedge \forall_Y$ lower_bound$(Y, I) \Rightarrow Y \leq X$.

X=end$(I) \Leftrightarrow$ upper_bound$(X, I) \wedge \forall_Y$ upper_bound$(Y, I) \Rightarrow Y \geq X$.

We can now distinguish four possible behaviors of an interval at each of its ends:

- Interval I is *closed* below if start(I) exists and is an element of I. I is closed above if end(I) exists and is an element of I. We use the standard notation '$[X, Y]$' to represent the closed interval from X to Y.

  In many applications, it is possible to restrict attention only to closed intervals. In a discrete measure space, all intervals are closed. In any measure space, closed intervals have the property that the overlap or join of two closed intervals is a closed interval. However, in order to perform complementation or set difference in a dense measure space, then it is necessary to go beyond closed intervals. For example, if the bathroom light is turned on during a time interval that is closed above, then it must be off during a time interval that is open below. Moreover, in order to make the standard interval calculus work on a dense interval space containing only closed intervals, it is necessary to disallow closed intervals consisting of a single point, and to define the relation between interval relations and the points they contain in a somewhat different way. See table 4.2.

- Interval I is *unbounded* below if it has no lower bound; it is unbounded above if it has no upper bound. A theory or algorithm that works correctly for closed intervals can, in many cases, be adapted to work for unbounded intervals by adding the mythic elements ∞ and $-\infty$ as the largest and smallest elements of the measure space, and treating an unbounded interval as a closed interval including these infinite elements. We use the predicates "infinite_on_right(I)" and "infinite_on_left(I)" to characterize intervals that are unbounded above (below).
- Interval I is *open* (but neither closed or unbounded) below if start(I) exists but is not an element of I; it is open above if end(I) exists but is not an element of I. The notation '(X, Y)' is standardly used for the open interval from X to Y. In a complete measure space, such as the real numbers, all intervals are either closed, open, or unbounded in each direction.
- Interval I is *gapped* below (above) if it has a lower (upper) bound, but no greatest lower bound (no least upper bound). Such intervals can exists in incomplete spaces. For example, the set of rational numbers whose square is less than 2 is gapped above, as is also the set of infinitesimals in a model of the reals with infinitesimals. (See section 4.10). Such intervals do not have an endpoint, and therefore cannot be denoted in terms of their endpoints; special purpose representations must be used. (Topologically, such intervals are both closed and open.)

A significant attraction of using just a pure language of intervals and avoiding references to points is precisely to avoid the hair-splitting issues involved in working out the behavior of the interval at its endpoint. Even where the structure of the measure space guarantees the existence of an endpoint, the concept definition may be vague enough to make the endpoints very questionable entities. For example, if the time line or the temperature scale is taken to have the structure of the real line, then, provably, every bounded interval has an endpoint. Nonetheless, concepts like "the ending instant of the Middle Ages," or "the lower endpoint of the range of comfortable temperatures" are rather dubious, and one would rather avoid formulating inferences in terms of them.

4.3 Addition and Subtraction

Examples:

> On Tuesday night, "Gone with the Wind" and "Duck Soup" are both showing on TV. "Gone with the Wind" starts later than "Duck Soup" and takes longer. Infer that "Gone with the Wind" will end later than "Duck Soup".
>
> Sophy removes a small quantity of flour from a flour bin, and then pours a much larger quantity into the bin. Infer that there is more flour in the bin at the end than at the beginning.

The basic concept in these two examples is that of the *difference* between two quantities, such as the difference between the ending and starting times of a movie, or the difference between two

The entries in this table have the following format:

a. Interval relation R(I, J)
b. Definition of R in terms of "meet" (from [Allen and Hayes, 1986]).
c. Definition of R in terms of points, valid for the space of all intervals over a measure space.
d. Definition of R in terms of points, valid for the space of closed intervals with more than one point.
e. Definition of R in terms of endpoints. Valid if endpoints exist.

a. before(I, J).
b. $\exists_K$ meets(I, K) $\wedge$ meets(K, J).
c. $\exists_X \forall_{W \in I, Z \in J} W < X < Z$
d. $\forall_{X \in I, Y \in J} X < Y$.
e. end(I) $<$ start(J).

a. meets(I, J).
b. meets(I, J).
c. $[\forall_{X \in I, Y \in J} X < Y] \wedge \neg \exists_X \forall_{W \in I, Z \in J} W < X < Z$
d. $[\ \forall_{X \in I, Y \in J} X \leq Y] \wedge [\exists_X X \in I \wedge X \in J]$
e. end(I) $=$ start(J).

a. overlaps(I, J)
b. $\exists_{A,B,C,D,E}$ meets(A, I) $\wedge$ meets(I, D) $\wedge$ meets(D, E) $\wedge$ meets(A, B) $\wedge$ meets(B, J) $\wedge$ meets(J, E).
c. $[\exists_{X \in I} \forall_{Y \in J} X < Y] \wedge [\exists_{Y \in J} \forall_{X \in I} X < Y] \wedge \exists_X X \in I \cap J$
d. $[\exists_{X \in I} \forall_{Y \in J} X < Y] \wedge [\exists_{Y \in J} \forall_{X \in I} X < Y] \wedge \exists_{X \in I, Y \in J} Y < X$
e. start(I) $<$ start(J) $<$ end(I) $<$ end(J).

a. starts(I, J).
b. $\exists_{A,B,C}$ meets(A, I) $\wedge$ meets(I, B) $\wedge$ meets(B, C) $\wedge$ meets(A, J) $\wedge$ meets(J, C).
c. $I \subseteq J \wedge I \neq J \wedge \forall_{Y \in J} \exists_{X \in I} X \leq Y$
d. $I \subseteq J \wedge I \neq J \wedge \forall_{Y \in J} \exists_{X \in I} X \leq Y$
e. start(I) $=$ start(J) $\wedge$ end(I) $<$ end(J)

a. equals(I, J).
b. $\exists_{A,B}$ meets(A, I) $\wedge$ meets(I, B) $\wedge$ meets(A, J) $\wedge$ meets(J, B).
c. $I = J$
d. $I = J$.
e. start(I) $=$ start(J) $\wedge$ end(I) $=$ end(J)

a. during(I, J)
b. $\exists_{A,B,C,D}$ meets(A, B) $\wedge$ meets(B, I) $\wedge$ meets(I, C) $\wedge$ meets(I, D) $\wedge$ meets(A, J) $\wedge$ meets(J, D).
c. $\exists_{P, Z \in J} \forall_{X \in I} P < X < Z$.
d. $\exists_{P, Z \in J} \forall_{X \in I} P < X < Z$.
e. start(J) $<$ start(I) $<$ end(I) $<$ end(J).

a. finishes(I, J).
b. $\exists_{A,B,C}$ meets(A, B) $\wedge$ meets(B, I) $\wedge$ meets(I, C).
c. $I \subseteq J \wedge I \neq J \wedge \forall_{Y \in J} \exists_{X \in I} X \geq Y$
c. $I \subseteq J \wedge I \neq J \wedge \forall_{Y \in J} \exists_{X \in I} X \geq Y$
e. start(J) $<$ start(I) $\wedge$ end(I) $=$ end(J).

Table 4.2: Relations among intervals

DIFF.1 $[X + P = Y] \Leftrightarrow [Y - P = X] \Leftrightarrow [Y - X = P]$. (Definition of addition.)

DIFF.2 $X + (-P) = X - P$. (Definition of unary negation.)

DIFF.3 $X + 0_{\mathcal{D}} = X$. (Identity element)

DIFF.4 $X + P = P + X$. (Commutativity.)

DIFF.5 $X + (P + Q) = (X + P) + Q$. (Associativity.)

DIFF.6 $X > Y \Leftrightarrow X - Y > 0_{\mathcal{D}}$. (Order preservation).

Table 4.3: Axioms for Differential Space

quantities of flour. If X and Y are elements of a measure space $\mathcal{M}$, then the difference between them, $Y - X$, is an element of a measure space $\mathcal{D}$, called the *difference space* of $\mathcal{M}$. The ordering on $\mathcal{M}$ defined a weak partial ordering on the elements of $\mathcal{D}$, as specified by the rule below.

- $X > Y \Rightarrow [X - Z > Y - Z] \wedge [Z - X < Z - Y]$.

For example, if Pooh is more cheerful than Kanga and Kanga is more cheerful than Eeyore, then the difference between Pooh's cheer and Eeyore's is greater than the difference between Pooh's cheer and Kanga's.

Two particularly important classes of measure space are *differential* and *integral* spaces.[2] A measure space $\mathcal{D}$ is a differential space if (a) $\mathcal{D}$ is totally ordered; (b) the difference space of $\mathcal{D}$ is $\mathcal{D}$ itself; (c) $\mathcal{D}$ satisfies the regularity axioms of table 4.3. A measure space $\mathcal{M}$ is an integral space if its difference space $\mathcal{D}$ is a differential space but is not equal to $\mathcal{M}$. For example, the space of masses, or of lengths of times, or of quantities of volume, are all differential spaces; the difference between two masses is a mass, the difference between two time lengths is a time length, the difference between two volumes is a volume. By contrast, the space of clock-times is an integral space; the difference between two clock-times is a length of time, which is not a clock-time.

Let $\mathcal{M}$ be an integral or differential measure space and let $\mathcal{D}$ be the difference space of $\mathcal{M}$. (Note that if $\mathcal{M}$ is differential, then $\mathcal{D} = \mathcal{M}$.) Let X, Y range over elements of $\mathcal{M}$ and let P, Q range over elements of $\mathcal{D}$. Let $0_{\mathcal{D}}$ be the zero point of $\mathcal{D}$. (Each different space has its own zero; usually, we will omit the subscript that distinguishes betwee them.) Then the axioms of table 4.3 are satisfied.

We can now formalize our sample inferences. The givens in the first inference can be stated in the following form: Let 'igone' and 'iduck' be the time intervals in which "Gone with the Wind" and "Duck Soup" are shown. (In chapter 5, we will introduce more general representations for these and for the constraints used in the second example.) The given constraints are then

$$\text{start(iduck)} < \text{start(igone)}$$
$$\text{end(iduck)} - \text{start(iduck)} < \text{end(igone)} - \text{start(igone)}$$

The conclusion, "end(iduck) < end(igone)", then follows directly from the givens and axioms DIFF.1 — DIFF.6. The second example, of the flour bin, is similar.

[2]These terms were introduced with these meanings in [McDermott, 80]

4.4 Real Valued Scales

Example:

> Sophy has 8-1/2 pounds of flour in a flour bin. She removes 1/4 pound, and adds 1-3/4 pounds. Infer that Sophy now has 10 pounds of flour in the bin.

In this example, we use numerical values (8-1/2, 1-3/4, 1/4, 10) together with the unit "pound" to denote different quantities of mass. The legitimacy of doing this, and of basic calculations in a quantity space on calculations over the reals, is established in the following theorem:

Theorem 4.1: Let $\mathcal{D}$ be any differential space satisfying the axioms DIFF.1 — DIFF.6 and also possessing the Archimedean property:

(Archimedes) For any $X, Y > 0$ there is an integer N such that $Y < X + X + \ldots X$ (N times).

Let U (a *unit* quantity) be any positive element of $\mathcal{D}$. Then there is a function $\text{scale}_U(X)$, mapping $\mathcal{D}$ into the real line, with the following properties:

1. $\text{scale}_U(U) = 1$
2. $\text{scale}_U(X + Y) = \text{scale}_U(X) + \text{scale}_U(Y)$
3. $X < Y \Leftrightarrow \text{scale}_U(X) < \text{scale}_U(Y)$

Thus, by fixing a standard unit, such as a pound, we can associate each element of $\mathcal{D}$ with a real number, and we can perform computations on elements of $\mathcal{D}$ by performing the same computations on the corresponding real numbers. (Note that the scale function may map $\mathcal{D}$ to some subset of the reals, such as the integers or the rationals. In that case, it may be possible to use a theory that is either computationally or ontologically simpler than real arithmetic.)

Similarly, given an integral space $\mathcal{M}$ whose difference space has the Archimedean property, it is possible to choose an arbitrary origin $O \in \mathcal{M}$ and an arbitrary unit $U \in \mathcal{D}$, and then define a scale in $\mathcal{M}$ where O corresponds to 0 and where U corresponds to a difference of 1. For example, the Centigrade scale associates temperatures with real numbers by choosing the origin to be the freezing point of water, and the unit to one one-hundredth of the difference between the boiling point and the freezing point of water.

4.5 More Arithmetic

Examples:

> Four quarts of water weigh twice as much as two quarts.
>
> If dinner at Chez Pierre costs more than \$50 per person, and there are at least 20 people in our dinner party, infer that the total bill will be more than \$1000.

MULT.1 $X \cdot Y = Y \cdot X$

MULT.2 $X \cdot (Y \cdot Z) = (X \cdot Y) \cdot Z$

MULT.3 $0 \cdot X = 0$

MULT.4 $1 \cdot X = X$

MULT.5 $X > 0 \land Y > 0 \Rightarrow X \cdot Y > 0$

Table 4.4: Axioms of Multiplication

The reasoning in these two examples requires introducing some further arithmetic relations. The first example involves the relation of multiplication. The multiplication function $X \cdot Y$ takes as arguments two quantities from any two differential quantity spaces, M and N, and returns a quantity from the product space of M and N. If M is the space of dimensionless quantities (pure numbers) then the product space of M and N is just N. Table 4.4 shows the well-known axioms of multiplication.

In axiom MULT.3, the '0' on the left side is the zero point of an arbitrary differential space $\mathcal{M}$; the '0' on the right is the zero point in the product space of $\mathcal{M}$ with the space of X. In axiom MULT.4, '1' is a dimensionless quantity.

We can now state the general rule that the weight of a quantity of pure stuff is equal to its volume times the density of the stuff (under standard conditions).

$$\text{pure}(Q, S) \Rightarrow \text{weight}(Q) = \text{volume}(Q) \cdot \text{density}(S)$$

Given this physical rule and the above axioms, we can make our desired inference.

[pure($Q1$,water) $\land$ pure($Q2$,water) $\land$
volume($Q1$) = 4·quart $\land$ volume($Q2$) = 2·quart] $\Rightarrow$
weight($Q1$) = weight($Q2$).

The second example requires reasoning about the arithmetic properties of an underspecified set. To represent this reasoning, we introduce two arithmetic functions over sets. The function "card(S)" or "$|S|$" gives the cardinality of a finite set S, which is a dimensionless integer. The function "sum_over(S, F)," usually written in the form

$$\sum_{X \in S} F(X)$$

gives the sum of function F over the set S. F must be a function mapping S into a differential space D. Axioms SSUM.1 — SSUM.3 define these functions:

SSUM.1 sum_over($\{X\}$,F) = $F(X)$

SSUM.2 $A \cap B = \emptyset \Rightarrow$ sum_over($A \cup B, F$) = sum_over(A, F) + sum_over(B, F).

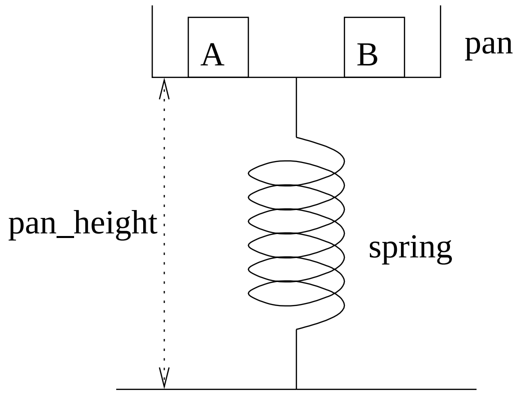


Figure 4.4: Scales

SSUM.3 card(S) = sum_over(S,$\lambda(X)(1)$)

(Note: These axioms involve a second-order logic, which allows quantifying over functions.)

From these axioms, we can deduce the basic result needed for our example: if $F(X) > C$ for each $X \in S$, then $\sum_{X \in S} F$ is greater than C times the cardinality of S.[3]

$$[\ \forall_{X \in S} \text{ value\_of}(F, X) > C\ ] \Rightarrow \text{sum\_over}(S, F) > C \cdot \text{card}(S)$$

Further arithmetic operators used in spatial reasoning, such as trigonometric functions, will be introduced in chapter 6.

4.6 Parameters; Signs; Monotonic Relations

Example:

> Consider the scales shown in figure 4.4. Suppose that the following constraints are specified:
>
> - The downward force exerted by each block on the pan is the mass of the block times the gravitational constant.
> - The force exerted by the spring upward on the pan is proportional to the stiffness of the spring times its compression from its rest length.
> - When the scales is at rest, the upward force exerted on the pan by the spring must exactly balance the sum of the downward forces exerted on the pan by the blocks.
> - The height of the pan is equal to the rest length of the spring minus its compression.
>
> Suppose that block A is made more massive, but the scales otherwise remain the same. Infer that the rest position of the pan is lowered, using the following line of reasoning:

[3]This proof requires a proof by induction over the cardinality of S, since it applies only to finite sets.

> Since the mass of A has increased, the force exerted by A must have increased. Since the mass of B remained the same, the force exerted by B must have remained the same. Therefore, the sum of the forces exerted by the two blocks must have increased. Therefore, the upward force exerted by the spring must have increased. Since the stiffness of the spring has not changed, the compression of the spring must have increased to create this new force. Since the compression of the spring has increased and its rest length has remained the same, the pan must be lower.

The form of the inference above is a common one in physical reasoning. We are comparing situations involving two systems with the same structure but with different values in the input parameters. We specify the direction of the change between the two situations but not its magnitude. The problem is to determine, if possible, the direction of change in the output parameters. Note that we are here only interested in the change in the equilibrium state, when the systems have come to rest; we are not asking how the system goes from one state to the other. (The dynamic behavior of this system will be discussed in section 4.9. The problem of deriving the above relations from a physical description of the system will be discussed in section 7.1.)

The above inference can be expressed and carried out without using any concepts or axioms beyond those already discussed in previous sections. However, since this type of inference is particularly common in commonsense reasoning about quantities, it is worthwhile developing a specialized notation and calculation method for dealing with it. In particular, we would like our formal inference method to resemble the above informal argument in focussing on the increase or decrease in specific quantities from one situation to the next, and to be able to calculate the direction of change in one quantity from the direction of change in related quantities.

To do this, we need to be able to refer to a entities such as "the mass of block A" which may correspond to a particular quantity in each different situation. We therefore introduce the ontological sort of a *parameter.* If $\mathcal{M}$ is a measure space, then an $\mathcal{M}$-valued parameter is an entity that associates each situation with a quantity from $\mathcal{M}$; extensionally, the parameter may be viewed as a function from situations to $\mathcal{M}$. (In chapter 5, we will generalize the concept of a parameter to that of a general fluent.)

We now introduce a number of functions on parameters. The most basic is the function "value_in(S, Q)", introduced in section 2.6, which maps a situation S and parameter Q to the value that Q takes in S. The function delta($Q, S1, S2$) gives the change in Q from $S1$ to $S2$; it is defined by axiom DEL.1 as the difference between Q's values in the two situations. We also introduce arithmetic operations such as addition, subtraction, multiplication, and so on as operators on parameters. These are defined by axiom schema DEL.2: the value of the combination of parameters is equal to the combination of their values.

DEL.1 delta($Q, S1, S2$) = value_in($S2, Q$) − value_in($S1, Q$).

DEL.2 value_in($S, \alpha(Q1, Q2)$) = α(value_in($S, Q1$), value_in($S, Q2$)) where α is any arithmetic operator.

We can now express statements like "A increases in mass" in the form "delta(mass(blocka),s1,s2) > 0." We can now simplify this form still further. We introduce the function symbol "sign(X)," also written "$[X]$", which maps a quantity X onto its sign. We also introduce three constants: "pos", the interval of positive quantities; "neg", the interval of negative quantities; and "ind", the set of all quantities. (As with the constant 0, we will use the same symbol to represent these intervals in all differential spaces.) Thus, if $X = 0$ then sign(X)=$[X] = 0$; if $X > 0$ then $[X]$ =pos; if $X < 0$ then $[X]$ =neg. We can then write the above statement in the form, "sign(delta(mass(blocka),s1,s2)) = pos". The sign function can be applied to a parameter in the same way as other arithmetic functions; that is, if Q is a parameter then $[Q]$ is the parameter whose value is always the sign of the value of the value of Q.

A further notational convenience,[4] used in circumstances where we are comparing two fixed situations s1 and s2, is to abbreviate "sign(delta(Q,s1,s2))" in the form ΔQ. Thus, we can express the statement "A increases in mass" in the form Δmass(blockA) = pos. Table 4.5 shows the complete specification for our example problem.

To carry out inferences from the specifications given in table 4.5, we need rules that will allow us to start with a parameter equation, such as "weight(blocka) = grav_acc · mass(blocka)," combine this with the values "Δmass(blocka) = pos," "Δgrav_acc = 0," and "[grav_acc] = pos," and to conclude "Δweight(blocka) = 0." These rules can be expressed elegantly by splitting them into two parts.

The first part of these inference rules defines the arithmetic operations on the signs pos, 0, and neg, together with the interval ind (indeterminate), which is the interval of all quantities, positive, negative, and zero. The arithmetic combination of two signs is all the possible signs of the combination of values in the signs. For example, pos + pos is pos, because the the sum of any two positive quantities is positive; pos + neg is ind, because the sum of a positive quantity and a negative quantity may be positive, zero, or negative, depending on the relative magnitudes of the two quantities. Table 4.6 shows the arithmetic relations on signs.

Two signs $SG1$ and $SG2$ are *compatible*, written "$SG1 \sim SG2$" if there is some quantity in both.[5] Specifically, any sign is compatible with itself, and any sign is compatible with the sign ind. Note that compatibility is not a transitive relation; ind is compatible with both pos and neg, but pos is not compatible with neg. Any equation on quantities can be converted into a corresponding compatibility relation on the signs of the quantities. For example, from the equation $X+Y = P \cdot Q$, it is legitimate to derive the compatibility relation $[X]+[Y] \sim [P]\cdot[Q]$. (See exercise 4). The converse is not the case; the truth of a compatibility relation does not imply the truth of the corresponding equation.

The second part of the inference rules defines the relations among the signs of parameters and the signs of their changes using the arithmetic on signs. Table 4.7 shows these rules. Combining these rules, it is straightforward to derive the desired result from the givens of our example of the scales. Table 4.8 shows the inference path.

The inference in our example that the pan will go down if the weight of A is increased does not, in fact, depend on the spring obeying the linear law, "spring_force = spring · compression." For the purposes of this inference, it would be sufficient to know that, for any fixed (positive) value of the

[4] In the literature, the notation ∂Q is often used for sign(delta(Q,s1,s2)). We will reserve this, however, to mean the sign of the derivative (see section 4.7).

[5] In the literature, this is usually written, "$SG1 = SG2$"; however, this notation is confusing.

Constants:

blocka, blockb	—	Two blocks.
s1, s2	—	Two situations.

Parameters:

mass(X)	—	The mass of block X.
weight(X)	—	Downward force exerted by block X.
blocks_weight	—	Total downward force exerted by blocks.
spring	—	Stiffness of the spring.
compression	—	Compression of the spring.
rest_length	—	Rest length of the spring.
spring_force	—	Upward force exerted by spring.
pan_height	—	Height of pan.
grav_acc	—	Gravitational accelleration.

Constraints:

weight(X) = grav_acc $\cdot$ mass(X)
blocks_weight = weight(blocka) + weight(blockb)
spring_force = spring $\cdot$ compression
spring_force = blocks_weight
pan_height = rest_length $-$ compression.

Parameter values:

[spring] = pos
[grav_acc] = pos

Given Comparisons:

Δmass(blocka) = pos.
Δmass(blockb) = 0
Δspring = 0.
Δrest_length = 0.
Δgrav_acc = 0.

Table 4.5: Problem Specification for Scales

$+$	neg	0	pos	ind
neg	neg	neg	ind	ind
0	neg	0	pos	ind
pos	ind	pos	pos	ind
ind	ind	ind	ind	ind

$\times$	neg	0	pos	ind
neg	pos	0	neg	ind
0	0	0	0	0
pos	neg	0	pos	ind
ind	ind	0	ind	ind

$-$	neg	0	pos	ind
	pos	0	neg	ind

$1/X$	neg	0	pos	ind
	neg	***	pos	ind

Table 4.6: Arithmetic of Signs

SGN.1 $\Delta(P+Q) \sim \Delta P + \Delta Q$

SGN.2 $\Delta(-P) = -\Delta P$

SGN.3a $\Delta(P \cdot Q) \sim \text{value\_in}(S1, [P]) \cdot \Delta Q + \text{value\_in}(S1, [Q]) \cdot \Delta P + \Delta P \cdot \Delta Q$

SGN.3b If either $[P]$ or $[Q]$ is constant over all situations, then
$\Delta(P \cdot Q) \sim \text{value\_in}(S1, [P]) \cdot \Delta Q + \text{value\_in}(S1, [Q]) \cdot \Delta P$.

Table 4.7: Differences of Arithmetic Functions

We can carry out our inference as follows:

From the givens: [grav_acc] = pos; Δgrav_acc = 0; Δmass(blocka) = pos;
and the rule: weight(X) = grav_acc $\cdot$ mass(X)
Infer: Δweight(blocka) = pos.

From the givens: [grav_acc] = pos; Δgrav_acc = 0; Δmass(blockb) = 0;
and the rule: weight(X) = grav_acc $\cdot$ mass(X)
Infer: Δweight(blockb) = 0.

From the calculated values: Δweight(blocka) = pos; Δweight(blockb) = 0;
and the rule: blocks_weight = weight(blocka) + weight(blockb)
Infer: Δblocks_weight = pos.

From the calculated value: Δblocks_weight = pos;
and the rule: spring_force = blocks_weight
Infer: Δspring_force = pos.

From the calculated value: Δspring_force = pos; and the givens: [spring]=pos; Δspring=0.
and the rule: spring_force = spring $\cdot$ compression
Infer: Δcompression = pos.

From the calculated value: Δcompression = pos; and the given: Δrest_length =0;
and the rule: pan_height = rest_length $-$ compression.
Infer: Δpan_height = neg.

Table 4.8: Applying the sign calculus

MON.1 monotonic$(QD, QI, QF, SG) \Leftrightarrow$
$[\ \forall_{S1,S2}$ value_in$(S1, QF) =$ value_in$(S2, QF) \Rightarrow$
$[$ value_in$(S1, QI) =$ value_in$(S2, QI) \Rightarrow$ value_in$(S1, QD) =$ value_in$(S2, QD)\ ] \wedge$
$[$ value_in$(S1, QI) <$ value_in$(S2, QI) \Rightarrow$ value_in$(S1, QD) -$ value_in$(S2, QD) \in SG\ ]$

MON.2 monotonic$(QD, QI, QF, SG1) \wedge$ monotonic$(QI, QD, QF, SG2) \Rightarrow SG1 = SG2$

MON.3 $[$ monotonic$(QD, QI, QF, SG) \wedge \Delta QF = 0\ ] \Rightarrow \Delta QD \sim \Delta QI \cdot SG$

MON.4 $[$ monotonic$(QD, QI, QF, SG1) \wedge$ monotonic$(QD, QF, QI, SG2)\ ] \Rightarrow$
$\Delta QD \sim \Delta QI \cdot SG1 + \Delta QF \cdot SG2$.

MON.5 [monotonic$(QA, QB, QC, SG1) \wedge$ monotonic$(QB, QC, QA, SG2) \wedge$
monotonic$(QC, QA, QB, SG3)\ ] \Rightarrow$
$SG1 \cdot SG2 \cdot SG3$=neg

Table 4.9: Axioms for Monotonic Dependence

spring constant, the force is a strictly increasing function of the compression. The inference in table 4.7 would go through exactly as before.

To express this formally, we introduce the predicate "monotonic(QD, QI, QF, SN)," meaning that, for any fixed value of the parameter QF, the dependent parameter QD depends monotonically on the independent parameter QI with sign SN. That is, if SN=pos, then QD is monotonically increasing in QI; if SN=neg, then QD is monotonically decreasing in QI. QD must depend functionally on QI and QF. (It is easy to extend the definition of this predicate to describe parameters that depend on more than two parameters.) We can now replace the constraint "spring_force = spring $\cdot$ compression" in the scales example with the weaker constraint "monotonic(spring_force, compression, spring, pos)", and still derive the same inference as before. Table 4.9 shows the formal definition and some basic properties of monotonic dependence.

Axiom MON.1 is the definition of monotonic dependence. Rules MON.2 — MON.5 are straightforward consequences of the definition which are useful for calculations.

4.7 Derivatives

Example:

> Water is pouring slowly into a tank at the top and draining out rapidly at the bottom. The height of water in a tank is an increasing function of the volume. Infer that both the volume and the height of the water in the tank is steadily decreasing.

This inference, like those in the previous section, centers around the concept of parameters such as the volume and height of water. However, it uses this concept in a rather different way. Rather than view a parameter as a function from separate situations representing different hypothetical states of the world, we must now view a parameter as a function from situations from a continuous time line to a quantity space. With this view, we can introduce the concept of the *derivative* of a

Parameters:

inpour	—	Rate that the water is pouring in at the top.
outdrain	—	Rate that the water is draining out at the bottom.
height_w	—	Height of water in the tank.
volume_w	—	Volume of water in the tank.

Constraints:
$\forall_S$ value_in(S,inpour) $<$ value_in(S,outdrain).
(The water always drains out faster than it pours in.)
deriv(volume_w) = inpour $-$ outdrain.
(The volume of water in the tank is increased by the pouring in and diminished by the draining.)
monotonic(height_w,volume_w,pos).
The height of water is an increasing function of the volume. (There is no other parameter to be fixed, assuming we are speaking of a tank of fixed shape.)

To infer:
$\forall_S$ value_in(S,deriv(height_w)) < 0.
(The height decreases over time.)

Table 4.10: Inference with derivatives

DRV.1 $\text{deriv}(P+Q) = \text{deriv}(P) + \text{deriv}(Q)$

DRV.2 $\text{deriv}(-P) = -\text{deriv}(P)$

DRV.3 $\text{deriv}(P \cdot Q) = Q \cdot \text{deriv}(P) + P \cdot \text{deriv}(Q)$

DRV.4 $\text{deriv}(1/P) = -\text{deriv}(P)/P^2$

DRV.5 $[\ \text{monotonic}(P, Q, R, SG1) \wedge \text{monotonic}(P, R, Q, SG2)\ ] \Rightarrow$
$\text{sign}(\text{deriv}(P)) \sim SG1 \cdot \text{sign}(\text{deriv}(Q)) + SG2 \cdot \text{sign}(\text{deriv}(R))$

Table 4.11: Axioms of derivatives

parameter. The function "deriv(P)," which we will often abbreviate as $\dot{P}$, maps a parameter over a measure space $\mathcal{M}$ to a parameter over the difference space of $\mathcal{M}$. Formally, it is defined in the usual way as the limit of the quotient.

$$\text{value\_in}(S, \text{deriv}(P)) = \lim_{S1 \to S} \frac{\text{value\_in}(S1, P) - \text{value\_in}(S, P)}{S1 - S}$$

Table 4.10 shows how our example problem can be formalized using parameters and the derivative function. Table 4.11 shows some of the basic properties of the derivative function, including those needed for our sample inference.

As this example suggests, the sign of the derivative of a parameter is often important in commonsense reasoning, since it indicates whether the parameter is increasing or decreasing at a given point in time. It is therefore common to designate it using the special notation ∂Q=sign(deriv(Q)).

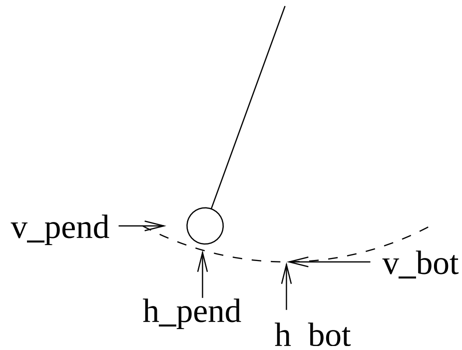


Figure 4.5: Pendulum

4.8 Mode Transition Networks

Example:

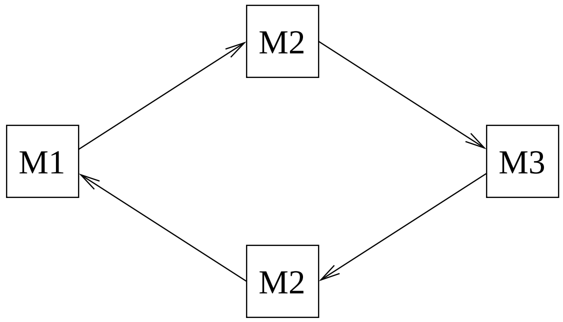
> A pendulum oscillates horizontally back and forth around a central lowest point. Infer that the height of the pendulum oscillates down to the lowest point, with two vertical oscillations for each horizontal oscillation. (Figure 4.5)

The major new representational problem in this example is how to describe the behavior of a collection of parameters, such as the horizontal position and the height of the pendulum, over time. A common solution to this problem proceeds along the following lines: Let Q be a parameter with values in the measure space $\mathcal{M}$. We partition $\mathcal{M}$ into a set of exclusive and exhaustive intervals according to some external criterion of significance, and we characterize the value of Q at any given instant by specifying which interval it falls into. If $\mathcal{M}$ is a differential space, it is generally divided into the intervals {neg, 0, pos}. If $\mathcal{M}$ is an integral space, it is common to choose a number of particularly significant *landmark* values, and to divide the space into the landmark values and the intervals between them. For example, if $\mathcal{M}$ is temperature, and Q is the temperature of a quantity of water, one might choose the landmark values to be 32° and 212°, and thus choose the intervals to be { (absolute_zero, 32°)$, 32°, (32°, 212°), 212°, (212°, \infty)$ }.

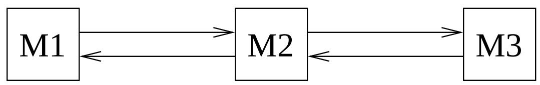
Our example of the pendulum uses two parameters: h_pend, the horizontal position of the pendulum, and v_pend, the vertical position of the pendulum. For characterizing h_pend, we use the landmark value of h_bot, the horizontal position of the bottom of the arc, and we partition the space into the three intervals { ($-\infty$, h_bot), h_bot, (h_bot,∞) }. For characterizing v_pend, we use the landmark value of v_bot, the height of the bottom of the arc, and we partition the space into the three intervals { ($-\infty$,v_bot), v_bot, (v_bot, ∞) }

Given a particular interval partitition of quantity space $\mathcal{M}$, and a quantity $X \in \mathcal{M}$, we extend the notation of section 4.6, so that $[X]$ now signifies the interval in the partition containing $\mathcal{M}$. It should be kept in mind that this significance of $[X]$ is now relative to some fixed partition, a dependence that is not explicitly shown in the representation.

Given a system of parameters, a *mode* of the system is a particular assignment of intervals to each parameter. For example, the pendulum system has three attainable modes:



Complete and correct network



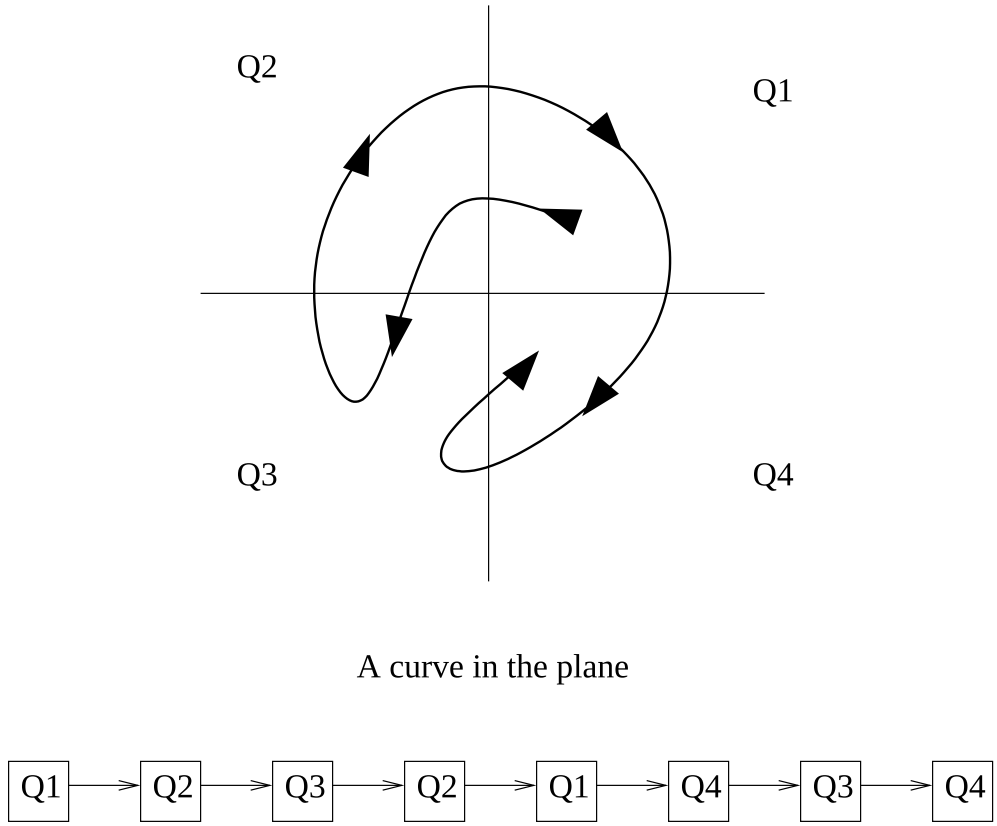
Correct but incomplete network

Figure 4.6: Mode Transition Networks

M1:	h_pend $\in (-\infty$,h_bot)	v_pend $\in$ (v_bot, ∞)
M2:	h_pend = h_bot	v_pend = v_bot
M3:	h_pend $\in$ (h_bot, ∞)	v_pend $\in$ (v_bot, ∞)

If a parameter system changes its mode only finitely many times in a given period, then we can represent its behavior by specifying the sequence of modes it goes through. For example, we can describe one full swing of the pendulum as an occurence of the mode sequence $<$ M1, M2, M3, M2, M1 $>$. To represent phenomena like oscillation, where the system changes modes, but does so in some systematic fashion, we can use a *mode transition network*, a finite-state transition network showing which modes can follow which other modes. A mode transition network is a valid representation of a parameter system over a period if the mode sequence executed by the system is a path through the network. Figure 4.6 shows two possible mode transition networks for the pendulum. Note that the second network is correct but not complete. The behavior of the pendulum is a path through the network, but there are paths through the network, such as $<$ M1,M2,M1 $>$, that do not correspond to possible behavior of the pendulum. The first network is both correct and complete; the paths through the network are just the possible behaviors of the pendulum. The first network is thus strictly stronger than the second.

We can now carry out our example inference. The givens to the inference are (i) the transition network for the horizontal position h_pend, corresponding to the statement that the pendulum oscillates; (ii) the physical constraints "h_pend=h_bot $\Rightarrow$ v_pend = v_bot" and "h_pend $\neq$ h_bot $\Rightarrow$ v_pend $>$ v_bot". Applying these constraints, it is trivial to add the mode of v_pend to the transition network. The new network directly expresses the fact that v_pend oscillates at twice the rate of h_pend. (Admittedly, it would not be easy to automate an inference such as "If h_pend goes through a mode cycle k times in a interval, then v_pend goes through a mode cycle $2k$ times in the same interval," from the network representation.)



B: Mode sequence

Figure 4.7: Mode Sequence for Spatial Curve

Mode sequences are used as representations for other functions than temporal sequences. For example, we may partially characterize the curve in figure 4.7a by considering it as a function from arc-length to a quadrant of the plane. The mode sequence for the curve would then be the cycle shown in figure 4.7b. (See section 6.2.6.)

Not all mode sequences are possible for well-behaved functions. If the time line is taken to be real-valued and all parameters are real-valued and continuous functions, then the mode sequence must obey certain constraints. These constraints are critical when mode transition networks are constructed. To express these constraints elegantly, we must add some additional structure to the representation of mode sequence, and consider their semantics in greater depth.

Let S be a system of parameters with a defined collection of modes. We define a time interval I to be *unimodal* with respect to S if S is in a single mode throughout I. I is maximal unimodal if I is unimodal and no interval properly containing I is unimodal. Clearly, any system of parameters partitions the time line into a collection of disjoint and exhaustive maximal unimodal intervals. We will impose the further condition on our parameter system that any finite duration of time contain only finitely many maximal unimodal intervals; that is, that no parameter changes its mode infinitely

Let M and N be successive augmented modes in a mode sequence. For any parameter Q let $[Q]_M$ and $[Q]_N$ be the modes of Q in M and N. Let I_M and I_N be the time intervals of M and N.

MODE.1 (Temporal topology) I_M is bounded above; I_N is bounded below. One of the following two possibilities must hold:

a. I_M is open above and I_N is closed below.

b. I_M is closed above and I_N is open below.

MODE.2 (Change) There is some parameter Q such that $[Q]_M \neq [Q]_N$.

MODE.3 (Continuity) For each parameter Q, either $[Q]_M = [Q]_N$ or the intervals $[Q]_M$ and $[Q]_N$ are adjacent in the quantity space of Q.

MODE.4 (Parameter topology) If $[Q]_M \neq [Q]_N$. then the boundary between $[Q]_M$ and $[Q]_N$ is topologically the same as the boundary between I_M and I_N. Specifically:

a. If $[Q]_M < [Q]_N$, I_M is open above, and I_N is closed below, then $[Q]_M$ is open above and $[Q]_N$ is closed below.

b. If $[Q]_M < [Q]_N$, I_M is closed above, and I_N is open below, then $[Q]_M$ is closed above and $[Q]_N$ is open below.

c. If $[Q]_M > [Q]_N$, I_M is open above, and I_N is closed below, then $[Q]_M$ is open below and $[Q]_N$ is closed above.

d. If $[Q]_M > [Q]_N$, I_M is closed above, and I_N is open below, then $[Q]_M$ is closed below and $[Q]_N$ is open above.

MODE.5 Let M, N, and P be three successive augmented modes and Q a parameter such that $[Q]_N$ is closed at both ends and has finite length (not a single point). If $[Q]_M$, $[Q]_N$, and $[Q]_P$ all have different values, then I_N has finite length.

Table 4.12: Rules for mode transitions

many times in any finite interval. In this case, the behavior of the system over any finite time interval may be characterized by a finite mode sequence.

To aid us in stating the restrictions that hold on continuous parameter systems, we augment our representation of mode sequences. An *augmented* mode of the parameter system is a specification, for a given maximal unimodal time interval I, of the following information: (i) the mode of the parameter system; (ii) the topology of I — is it unbounded, bounded and open, or bounded and closed above and below? If it is closed both above and below, is it a single point or an interval of finite length? An augmented mode sequence is then a sequence of augmented modes; an augmented mode transition network is a directed graph on augmented nodes.

The constraints in Table 4.12 govern any augmented mode sequence for real-valued continuous parameters:

If, as is common, the only closed intervals used in the partition parameter space are single landmark points, then axiom MODE.5 is vacuous.

Axioms MODE.1 — MODE.5 can be used to prune substantially the transitions that are possible in a mode transition graph. For example, consider a system of two identical independent pendulums,

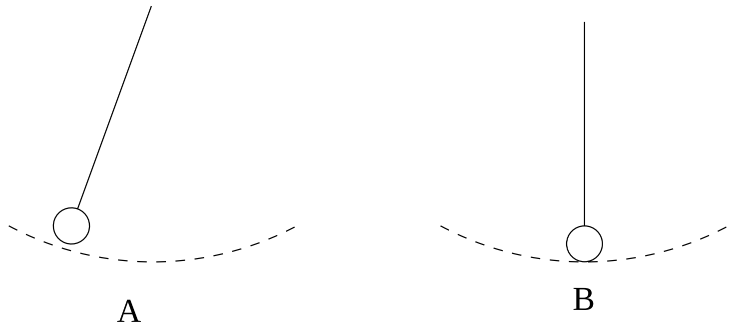


Figure 4.8: Two Pendulums

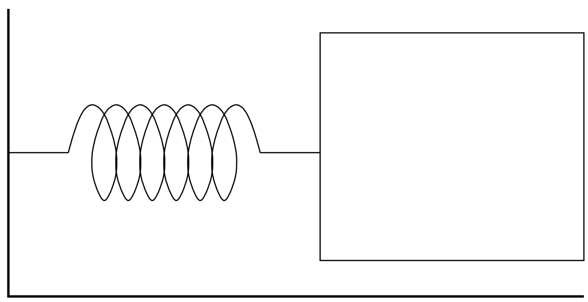


Figure 4.9: Block on a Spring

as shown in figure 4.8. Each pendulum can be in one of three modes; the system as a whole therefore has nine modes. We are given that, in the starting mode M, pendulum A is left of center, while pendulum B is at the center point. We can restrict the possibilities for the succeeding mode N as follows: By continuity [MODE.3], A cannot be to the right of center in N. This rules out three possible modes as succesors. By the axiom of change [MODE.2], N must be different from M. This rules out another possible successor. By parameter topology [MODE.4], if pendulum A moves from left of center to center, then $[I]_M$ must be open above, while if pendulum B moves out of the center then $[I]_M$ must be closed above. Therefore, these two types of transitions cannot both occur between M and N, ruling out two more possible successor modes. We are left with three possible successors: A and B are both at the center; A and B are both left of center; and A is left of center while B is right of center.

4.9 Qualitative Differential Equations

Example:

> Consider a block attached to a spring, as in figure 4.9. When the spring is extended, it exerts an inward force on the block; when it is compressed, it exerts an outward force. No other force acts on the block. Deduce that the block will oscillate back and forth around the rest point of the spring.

Parameters:
x — Position of the block
f — Force exerted on the block by the spring.

Atemporal constants:
m — Mass of the block.
compress — Interval of compressed spring positions.
rest — Rest length of the spring.
expand — Interval of expanded spring positions.

Constraints:
f = m $\ddot{x}$. (Newton's second law: The force is proportional to the acceleration.)
[f] = rest − [x].

Table 4.13: Dynamics of the spring system

The problem here is to derive the behavior of a system of parameters over time, given constraints obeyed by the parameters and their derivatives at each instant of time. These constraints derive in part from the problem specifications and in part from a background knowledge of physics. Specifically, the problem can be formulated as shown in table 4.12. (We shall discuss how this formulation can be derived from physical specifications in chapter 7.)

The givens thus form a system of differential relations; the problem is to solve them to derive a characterization of parameters over time. However, the problem differs from the differential equations in standard calculus courses in that the relation [f]=rest−[x] provide only a weak constraint between the parameters, rather than an exact functional relation. The solution of the problem can therefore at best come up with a partial characterization of the behavior, such as "the block oscillates," rather than a precise functional description. Specifically, the technique we will present will construct a mode transition network, called an *envisionment graph,* for the solutions to the equations.

The basic technique for constructing an envisionment graph for such a system of equations involves steps:

1. Translate any higher differential equations into a system of first order equations by introducing the intermediate derivatives as new variables. In our example, we would convert the equation "f = m·$\ddot{x}$" to the pair of equations "f = m · $\dot{v}$; v = $\dot{x}$", by introducing the velocity v as a new variable. All variable so introduced are derivatives, and so may be characterized in terms of the signs { neg, 0, pos }.

2. Change all equations to relations on the signs or intervals of the parameters. In our example, we would change the two equations introduced in (1) to the form "[f] = ∂v; [v] = ∂x." (Recall that $\partial Q = \text{sign}(\text{deriv}(Q))$. Note that we have used the fact that the mass m is positive). The constraints "[f] = rest−[x]" is already in the proper form. The resultant equations are called *qualitative differential equations* (QDE's). Often, the problem is presented in the form of a set of QDE's, steps (1) and (2) having been performed implicitly in the problem formulation.

3. Let S be a parameter system containing all the parameters and derivatives from the equations

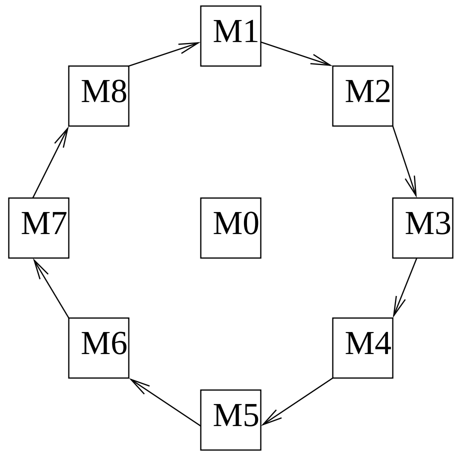


Figure 4.10: Envisionment Graph

QDE.1 (Mean value)
If $[Q]_M < [Q]_N$ and $[Q]_N$ is an open interval, then ∂Q_N=pos.
If $[Q]_M > [Q]_N$ and $[Q]_N$ is an open interval, then ∂Q_N=neg.
If $[Q]_M < [Q]_N$ and $[Q]_M$ is an open interval, then ∂Q_M=pos.
If $[Q]_M > [Q]_N$ and $[Q]_M$ is an open interval, then ∂Q_M=neg.

QDE.2 (Point transitions).
If $[Q]_N$ is a point interval and ∂Q_N=pos
then $[Q]_M < [Q]_N < [Q]_P$ and I_N is a point interval.
If $[Q]_N$ is a point interval and ∂Q_N=neg
then $[Q]_M > [Q]_N > [Q]_P$ and I_N is a point interval.

Table 4.14: Axioms for QDE's

in step (2). Construct all modes of the system consistent with the qualitative differential equations. In the spring example, there are nine such modes; they are listed in table 4.14.

4. Determine which transitions between modes are possible. The constraints on possible transitions are axioms MODE.1 — MODE.5, which apply to all continuous parameter systems, and axioms QDE.1 — QDE.2, which relate the derivative of a quantity to the change in the quantity. Table 5.15 shows how these rules apply to the spring example; figure 4.10 shows the final envisionment graph.

Let M, N, and P be successive augmented modes in a mode sequence. For any parameter Q and mode A, let $[Q]_A$ be the mode of Q in A; let ∂Q_A be the sign of the derivative of Q in A; and let I_A be the time interval in A. We will restrict attention in these axioms to partitions of measure spaces into landmark values and the open intervals between them. The analysis of half-open intervals and of closed intervals of finite magnitude is substantially messier.

One substantial limitation of the analysis, not obvious on cursory examination of table 4.15 or figure 4.10 is that it does not establish that the system does not end permanently in state M1, M3,

Mode	[x]	[v] = ∂x	[f] = ∂v
M0	rest	0	0
M1	expand	pos	neg
M2	expand	0	neg
M3	expand	neg	neg
M4	rest	neg	0
M5	compress	neg	pos
M6	compress	0	pos
M7	compress	pos	pos
M8	rest	pos	0

Table 4.15: Modes of the spring system

M0: No transitions into or out of M0 are consistent with the mean value axiom QDE.1.

M1: By continuity (MODE.3), M1 can be followed only by M0, M2 or M8. By mean value (QDE.1), M1 cannot be followed by M0 or M8. Hence M1 must be followed by M2.

M2: By point transition on parameter v (QDE.2) M2 must be followed by M3, M4, or M5, and must have a point time interval. By continuity (MODE.3), M2 cannot be followed by M5. By topology (MODE.4), M2 cannot be followed by M4. Hence M2 is followed by M3.

M3: By continuity (MODE.3), M3 can be followed only by M0, M2 or M4. By mean value (QDE.1), M3 cannot be followed by M0 or M2. Hence M3 must be followed by M4.

M4: By point transition on parameter x (QDE.2) M4 must be followed by M5, M6, or M7, and must have a point time interval. By continuity (MODE.3), M4 cannot be followed by M7. By topology (MODE.4), M4 cannot be followed by M6. Hence M4 is followed by M5.

M5: By continuity (MODE.3), M5 can be followed only by M0, M4 or M6. By mean value (QDE.1), M5 cannot be followed by M0 or M4. Hence M5 must be followed by M6.

M6: By point transition on parameter v (QDE.2) M6 must be followed by M7, M8, or M1, and must have a point time interval. By continuity (MODE.3), M6 cannot be followed by M1. By topology (MODE.4), M6 cannot be followed by M8. Hence M6 is followed by M7.

M7: By continuity (MODE.3), M7 can be followed only by M0, M6 or M8. By mean value (QDE.1), M7 cannot be followed by M0 or M6. Hence M7 must be followed by M8.

M8: By point transition on parameter x (QDE.2) M8 must be followed by M1, M2, or M3, and must have a point time interval. By continuity (MODE.3), M8 cannot be followed by M3. By topology (MODE.4), M8 cannot be followed by M2. Hence M8 is followed by M1.

Table 4.16: Constructing an envisionment graph for the scales

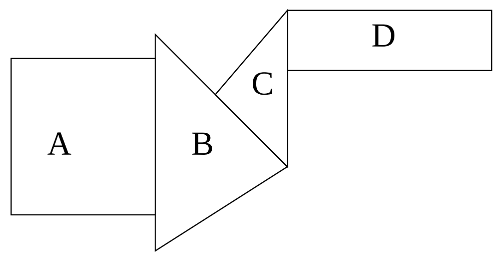


Figure 4.11: Instantaneous Sequence of Events

M5, or M7. The problem here is not an inadequacy of the axioms MODE.1 — MODE.5 and QDE.1 — QDE.2. Rather, it is a consequence of steps (1) and (2) of the solution process, in which the original equations are transformed to qualitative first-order equations. All solutions to the original equations oscillate. The tranformation, however, loses information so as to admit solutions that remain forever in a single state (Exercise 8). Strictly speaking, therefore, it is necessary to consider the higher-order QDE's to infer that the system oscillates [de Kleer and Bobrow, 1984], [Kuipers and Chiu, 1986]. In practice, however, most systems that use envisionment graphs make the assumption that, if a mode has any transitions out of it, then it will not last forever.

4.10 Orders of Magnitude

Examples:

> You weigh a letter and determine that it is less than an ounce and requires only a 25 cent stamp. You therefore plan to affix the stamp, but realize that the stamped letter will weigh more. However, reasoning that the weight of a stamp is negligible as compared to the weight of a letter, you conclude that it will not bring the weight of the letter over the limit.
>
> A number of blocks placed together are hit from the outside (Figure 4.11). Represent the statement that the shock travels from one block to another in sequence, but that the entire propagation is complete before any of the blocks can move any finite distance.

It is often useful to reason about one quantity being very much larger than another, without being precise about the numerical value of the ratio. Thus, rather than say that the shock wave moves one hundred or one thousand times faster than the blocks do, or that the letter is a hundred times heavier than the stamp, it may be easier, though inaccurate, to suppose that the larger quantity is actually infinite as compared to the smaller.

To make this notion coherent, we need a model of quantities that allow infinite ratios between quantities. Such a model has, in fact, been developed in the last thirty years; it is called the *non-standard* model of the reals with infinitesimals. It is beyond the scope of this book to describe the logical foundations of this theory. (See [Robinson, 1965], [Davis and Hersh, 1972].)

For the purposes of inferences such as those above, it suffices to introduce the concept of one quantity of a differential space being negligible as compared to another. We introduce the predicate

$X \ll Y$, which holds if X and Y are positive quantities and X is infinitesimal as compared to Y. The predicate observes the following axioms:

NEG.1 $X \ll Y \Rightarrow 0 < X < Y$.

NEG.2 $[0 < W \leq X \ll Y \leq Z] \Rightarrow W \ll Z$

NEG.3 $[W \ll Y \wedge X \ll Y] \Rightarrow (W + X) \ll Y$

NEG.4 $[W \ll X \wedge 0 < Y] \Rightarrow W \cdot Y \ll X \cdot Y$

NEG.5 $\exists_{X,Y} X \ll Y$

NEG.6 Any first-order statement which is true of the standard real numbers and does not involve the symbol $\ll$ is also true of the non-standard real numbers.

Axiom NEG.3 states that if both W and X are negligible compared to Y, then $W + X$ is likewise negligible. This directly contradicts the Archimedian property of the reals, that by adding any positive quantity X to itself sufficiently often, one can exceed any given quantity Y. Also, the completeness property of the reals, that every non-empty set with an upper bound has a least upper bound, does not hold on the non-standard line; for any $X > 0$, the set of numbers Y such that $Y \ll X$ does not have a least upper bound. (Neither of these properties can be fully axiomatized in a first-order statement; hence their failure does not contract the axiom schema NEG.5)

We can now formalize our example of the letter and stamp. We introduce the predicate "close(X, Y)" meaning that $Y - X$ is of negligible magnitude as compared to Y.

$$\text{close}(X, Y) \Leftrightarrow \text{abs}(Y - X) \ll \text{abs}(Y)$$

We now specify that the letter is less than an ounce and not close to an ounce, and that the weight of a stamp is negligible as compared to the weight of the letter. It then follows directly that the letter plus stamp is less than an ounce.

Given: letter $<$ ounce $\wedge$ $\neg$close(letter,ounce).
stamp $\ll$ letter.

Infer: stamp $+$ letter $<$ ounce.

4.11 References

General: [Hayes, 1978] contains a general discussion of the nature and structure of measure spaces used in commonsense reasoning, particularly physical reasoning. [Davis, 1987b] contains a survey of the different kinds of arithmetic primitives needed for various commonsense domains with real-valued measure spaces. [De Kleer and Weld, 1989] reprints many of the most significant papers on reasoning about quantities for physical reasoning, including many of the papers cited below on reasoning about collections of arithmetic relations, QDE's, and orders of magnitude.

Intervals: The interval calculus has been studied almost exclusively as a representation for temporal relations. In particular, [Van Benthem, 1983] contains a thorough study of the algebraic and topological properties and logical power of various sets of axioms on the ordering of points and intervals. The seminal AI paper on the interval calculus was [Allen, 1983], which introduced the 13 relations on intervals that we have used, and presented a transitivity table for combining them. Further studies of the logical and computational properties of the interval calculus include [Vilain and Kautz, 1986], [Allen and Hayes, 1985], [Ladkin, 1987].

Real-valued scales: The primary issue that has been in incorporating real arithmetic in AI systems has been the organization and maintenance of an efficient system for ground atomic relations. Propagation of exact values and of symbolic terms has been used to solve systems of equations in [Sutherland, 1963], [Borning, 1977], [Sussman and Steele, 1980]. Waltz propagation on real intervals has been applied to temporal reasoning in [Dean, 1984], and to spatial reasoning in [McDermott and Davis, 1984] and [Davis, 1986]; [Davis, 1987b] has a formal analysis of the power of Waltz propagation as applied to different classes of arithmetic relations. [Malik and Binford, 1983] advocates the use of the simplex algorithm for the analysis of linear inequalities that arise in AI systems. The ACRONYM system of [Brooks, 1981] has a powerful system for solving inequalities that may contain complex algebraic and trigonometric terms for spatial reasoning. The BOUNDER program [Sacks, 1987] uses a series of increasingly powerful and increasingly costly techniques for analyzing systems of inequalities; it extends a similar system of Simmons' [1986].

Sign calculus and QDE's: In the AI literature, these two techniques were developed in tandem for qualitative physical reasoning. QDE analysis was used implicitly in the NEWTON program [de Kleer, 1975] (see section 6.2.6.) QDE's and sign arithmetic were first developed as theories in their own right in [de Kleer and Brown, 1985], [Kuipers, 1985], and [Williams, 1985]. [Kuipers, 1986] is a careful and rigorous analyisis of the underlying theory. [Struss, 1989] studies the limits of sign and QDE analysis. Since then a number of papers have extended the basic theory by using richer information and more powerful analysis techniques. Many of these are collected in [de Kleer and Weld, 1989]. Particularly significant are [de Kleer and Bobrow, 1984] and [Kuipers and Chiu, 1988] which study higher-order QDE equations; [Weld, 1986], which shows how QDE techniques can be applied to certain discrete systems; and [Weld, 1988a] which analyses qualitatively how the solutions to a differential equations are affected by perturbations to the parameters of the equations. [Sacks, 1988] gives a more powerful qualitative analysis of exact differential equations, based on approximating the equations with piecewise linear equations, and categorizing the solutions in terms of transitions between regions of phase space.

Order of magnitude: The theory of non-standard real analysis was created by Abraham Robinson [1965]; for a popular account, see [Davis and Hersh, 1972]. In the AI literature, non-standard analysis has been applied to the automation of proofs in real analysis in [Ballantyne and Bledsoe, 1977]; to the semantics of robotic programming languages in [Davis, 1984]; and to physical reasoning in [Raiman, 1986]. [Davis, 1989b] and [Weld, 1988b] describe systems that combine order of magnitude reasoning with QDE's. [Mavrovouniotis and Stephanopoulos, 1989] gives a detailed account of an inference engine for order-of-magnitude reasoning, and its application to process engineering.

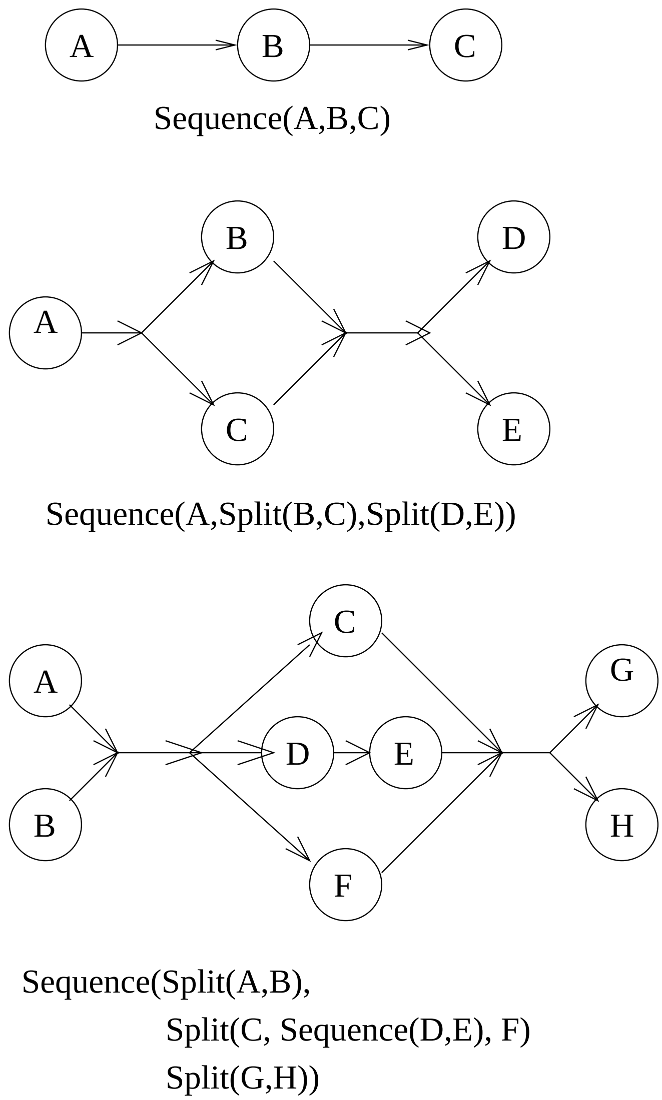


Figure 4.12: Partial Orders as Sequences and Splits

4.12 Exercises

1. Some AI programs (e.g. NOAH [Sacerdoti, 1975]) have represented partial orderings using the functions "sequence($P1, P2, \ldots Pk$)" and "split($P1, P2 \ldots Pk$)." The function "sequence($P1 \ldots Pk$) combines a collection of disjoint partial orderings $P1 \ldots Pk$ by placing them in increasing order; if $i < j$, $Xi \in Pi$ and $Xj \in Pj$ then $Xi < Xj$. The function "split($P1 \ldots Pk$)" combines disjoint partial orderings $P1 \ldots Pk$ by placing them in parallel; if $Xi \in Pi$, $Xj \in Pj$ and $i \neq j$, then Xi and Xj are unordered. Figure 4.12 shows some examples.

a. Give an example of a partial ordering that cannot be represented using "sequence" and "split".

b.* Define the functions "sequence" and "split" in a first-order theory that treats partial orderings as first-order entities. In order that these functions are applied only to partial ordering, and not to simple elements, you may use a function "unary_po(X)" which maps an element X to the partial ordering containing only X. You may also assume for simplicity that "sequence" and "split" take exactly two arguments. Thus the partial ordering with $X1$, $X2$, and $X3$ in that order could be written

sequence(sequence(unary_po($X1$),unary_po($X2$)),unary_po($X3$))

Use the predicates "ordered($X1, X2, P$)", meaning that $X1$ precedes $X2$ in ordering P, and "element(X, P)", meaning that X is an element of ordering P.

c. Another representation of partial orderings [Meehan, 1975] is to associate a real interval with each element of the partial ordering, and to define X as coming before Y in the ordering if the upper bound of the interval of X less than the lower bound of Y. For example the labels $X1 \rightarrow [0, 1]$, $X2 \rightarrow [2, 4]$, $X3 \rightarrow [3, 6]$, $X4 \rightarrow [5, 8]$ corresponds to the ordering $X1 < X2 < X4$, $X1 < X3$. Give an example of a partial ordering that cannot be represented in this way.

d.* The representations in (a) and (c) are "compact," in the sense that they require space linear in the number of elements of the partial ordering. Show that there are more than 2^{cn^2} different partial orderings on n elements, for some constant c. Argue that therefore no compact representation of partial orderings can represent all possible partial orderings.

2. Construct a *transitivity table* for the 13 interval relations discussed in section 4.2. This is a 13 by 13 table, in which each row is a relation between I and J, each column is a relation between J and K, and the entry is the possible relations between I and K. For example, in the row "during(I, J)" and the column "before(J, K)", the entry is "{ before(I, K) }". Note that some of the entries will not be single valued, if there is more than one possible relation between I and K. For example, in the row "starts(I, J)" and column "starts(K, J)", the entry is "{starts(I, K), $I = K$, starts(K, I)}".

3. Find an efficient (O(n^2)) algorithm to solve the following problem: Given a collection of atomic, ground interval constraints, determine whether the constraints are consistent. For example the set {before(a,b), meets(a,c), overlap(c,b)} is consistent; it is satisfied by the interval a=[0,1], b=[2,4], c=[1,3]. The set {before(a,b), meets(a,c), during(c,b)} is inconsistent.

4. Justify the statement in section 4.6, "Any equation on quantities can be converted into a corresponding compatibility relation on the signs of the quantities. For example, from the equation $X + Y = P \cdot Q$, it is legitimate to infer the compatibility relation, $[X] + [Y] \sim [P] \cdot [Q]$."

5.* Justify the axiom MON.5 from table 4.8.

MON.5 [monotonic($QA, QB, QC, SG1$) $\wedge$ monotonic($QB, QC, QA, SG2$) $\wedge$ monotonic($QC, QA, QB, SG3$)] $\Rightarrow$ $SG1 \cdot SG2 \cdot SG3$=neg

6. a. Find the mode transition network for the function $f(t) = \sin(t)\sin(2t)$ using the signs of the function and its derivative.

b. Find the mode transition network for the function $f(t) = \sin(t)\sin(2t)$ using the signs of the function and its first two derivatives.

7. If we modify the model of the block on the spring in section 4.9 by adding a damping force proportional to the velocity, then the equations become,

$f = m\,\ddot{x}$.
$v = \dot{x}$.
$[f] = \text{rest} - [x] - [v]$.

Use the technique of section 4.9 to construct an envisionment graph for this problem.

8.a.* Consider the qualitative differential equation "$[\ddot{x}] = -[x]$" with the initial conditions $[x(0)]$=pos, "$[\dot{x}[0]] = 0$. Show that x(t)=0 for some positive t.

b.* Show that there are solutions to the pair of QDE's "$\partial x=v$; $\partial v=-x$" with the initial values $[x(0)]$=pos, "$[\dot{x}[0]] = 0$ such that x(t) is positive for all t>0.