Pouring Liquids: 
A Study in Commonsense Physical Reasoning: 
Appendix: Verification of Pouring Scenario

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This appendix shows that the scenario specified in axioms PD.1-14 and PS.1–21 ends in a final state where some of the liquid l0 is in the pitcher and the remainder is in the pail.

The formal statements of the lemmas in this appendix, like the axioms in the main text of the paper, are (intended to be) written in a style that could be given directly to an automated theorem checker, after some straightforward syntactic desugaring. In the text of the proofs here, by contrast, I have followed the (God knows, rigid enough) comparatively informal style of normal mathematical writing. In particular:

• I will use partial functions, being careful only to use them only where defined.

• I assume standard results from Euclidean geometry and real analysis.

• I will use natural English for establishing the context of relations between fluents; I will often omit the hatch superscript for converting a function on atemporal entities to one on fluents; and I will often omit the place function on objects and liquids. For instance I will write “At time TA, L ⊂ Q,” rather than “holds(TA, ↑L ⊂# Q).”

• If RA and RB are regions then I will write RA−RB and RA∩RB for the regularized difference and intersection, and likewise for region-valued fluents. Also, I will allow these expressions even in cases where they evaluate to the null set, and sometimes omit consideration of the null set case where it is trivial.

• It will be convenient to apply Boolean operators to liquid chunks in the obvious way.

• Where M is a rigid mapping and G is a geometric entity, I have written the second-order formulation M(G) rather than the first-order expression “mappingImage(M,G)”. 

• From the point of view of automated theorem proving, a large part of the proof of these lemmas is definition hunting and elementary temporal and spatial argumentation. I have omitted most of this, not even citing the definitions or axioms involved, except where this is non-trivial or interesting.

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I have divided the lemmas into general results, which are independent of the particular problem specification, and problem specific results. Obviously, a result of either form can be rephrased into the other. The idea is that, for the most part, general results are of some general applicability, whereas problem specific results require so many of the particular problem specifications as to make the corresponding general theorem ludicrously long.

**General Results**

**Lemma 1:**

\[ \forall_{T, S, T', Q} \text{holds}(T, S, Q) \land \\
\left[ \forall_{T'} T < T' \land \text{throughout}(T, S, T', Q) \Rightarrow \text{holds}(T, Q) \right] \land \\
\left[ \forall_{T} \text{holds}(T, Q) \land T < T'E \Rightarrow \\
\exists_{T3} T < T3 \land \text{throughout}(T, T3, Q) \right] \Rightarrow \\
\text{throughout}(T, S, T', Q). \]

**Proof** by contradiction. Suppose not. Let \( T \) be the greatest lower bound of all times between after \( T \) when \( Q \) fails. If \( T = T \) then \( Q \) holds at \( T \) by the first condition of the lemma; if \( T > T \) then \( Q \) holds at \( T \) by the second condition of the lemma. In either case, by the third assumption of the lemma, \( Q \) continues to be true until some time \( T3 \) after \( T \), contradicting the choice of \( T \).

**Lemma 2:** (Conditionalized comprehension for fluents):

\[ [\forall_{T} \text{time } \Psi(T) \Rightarrow \exists_X \Theta(X, T)] \Rightarrow \exists_Q \forall_{T} \text{time } \Psi(T) \Rightarrow \Theta(\text{value}(T, Q)). \]

**Proof:** In axiom schema T.1, define \( \Phi(X, T) \equiv \Psi(T) \Rightarrow \Theta(X, T) \) (i.e. the value of \( X \) is arbitrary at times \( T \) where \( \Psi \) is not satisfied).

**Lemma 3:**

\( \text{region}(R) \land H > \text{bottom}(R) \Rightarrow \exists_{RB} \text{regionBelow}(R, H, RB). \)

**Proof:** Let \( P \) be a point in \( R \) for which \( \text{height}(P) < H \). Since \( R \) is regular, \( P \) is in the closure of the interior of \( R \); hence the interior of \( R \) includes points below \( H \). Therefore, we can define \( RB \) to be the closure of the part of the interior of \( R \) below \( H \).

**Definition 1:**

\( \text{disconnOpenBox}(RB, RI; \text{bregion}) \)

\( \forall_{P} P \in \text{boundary}(RI) \Rightarrow \text{height}(P) = \text{top}(RI) \lor P \in \text{boundary}(RB) \)

The definition of the relation \( \text{disconnOpenBox}(RB, RI) \), meaning “\( RB \) is an open box with disconnected interior \( RI \),” is the same as “\( \text{openBox}(RB, RI) \)” except that it does not require that \( RI \) be connected. Thus, \( RI \) can consist of a number of disjoint regions boxed by \( RB \), though all of these regions must have their tops at the same height.

**Lemma 4:**

\( \text{disconnOpenBox}(RB, RI) \land \text{topSurface}(PST, RI) \land RB2=\text{mappingImage}(M, RB) \land \\
\text{RIM}=\text{mappingImage}(M, RI) \land \text{PTM}=\text{mappingImage}(M, PST) \land \\
\text{regionBelow}(RIM, \text{bottom}(PSM), R12) \Rightarrow \\
\text{disconnOpenBox}(RB2, R12). \)

**Proof:** To establish that \( \text{disconnOpenBox}(RB2, R12) \), by definition 1 we must show that for any point \( P \) \( \in \text{boundary}(R12) \), either \( \text{height}(P) = \text{top}(R12) \) or \( P \in \text{boundary}(RB2) \). Let \( P \) be any point in \( \text{boundary}(R12) \), and let \( P1 = M^{-1}(P) \). By PD.6, \( \text{height}(P) \leq \text{bottom}(PTM) \). Let \( PT2 \) be the
top surface of \( RI_2 \). By construction either \( P \) is in \( PT_2 \) or \( P \) is in \( \text{boundary}(RI_1) \). If \( P \in PT_2 \) then \( \text{height}(P) = \text{top}(RI_2) \). Suppose that \( P \in \text{boundary}(RI_1) \). Then \( P_1 \in \text{boundary}(RI) \). Since \( \text{openBox}(RB, RI) \) either \( P_1 \in PST \) or \( P_1 \in \text{boundary}(RB) \). If \( P_1 \in PST \) then \( P \in PTM \), so \( \text{height}(P) \geq \text{bottom}(PTM) = \text{top}(RI) \). If \( P_1 \in \text{boundary}(RB) \) then \( P \in \text{boundary}(RB_2) \).

**Corollary 5:**
\[ \text{object}(O) \land O = \text{source}(BI) = \text{source}(BT) \land \]
holds(\( T_1, \text{openBox}^\#(\{O, \{BI\} \land \text{topSurface}^\#(\{BT, \{BI\}) \land \]
holds(\( T_2, \text{isolated}(\{BI, \{O\}, L\}) \land \text{holds}(T_2, \text{regionBelow}^\#(\{BI, \text{bottom}^\#(\{BT\}, RB)\} \Rightarrow \]
holds(\( T_2, \text{disconnOpenBox}^\#(O, RB)\).)

**Proof:** By PD.10 no objects enter into the interior of \( RI \). The result is then immediate from lemma 4.

**Lemma 6:**
\[ \text{openBox}(RB, R_1) \land \text{openBox}(RB, R_2) \land \text{rccO}(R_1, R_2) \land \]
top(\( R_1) \leq \text{top}(R_2) \Rightarrow \]
\( R_1 \subset R_2 \).

**Proof** by contradiction: Suppose that the left hand side holds, but that \( P_1 \) is a point in \( R_1 - R_2 \). Since \( \text{rccO}(R_1, R_2) \), let \( P_2 \) be a point in \( \text{interior}(R_1 \cap R_2) \). Since \( R_1 \) is regular and thickly connected, there exists an open connected pointset \( PSO \subset \text{interior}(R_1) \) such that \( P_2 \in PSO \), \( P_1 \in \text{closure}(PSO) \). Since \( P_1 \notin R_2 \), \( PSO \) must intersect \( \text{boundary}(R_2) \). Since all of \( PSO \) is below \( \text{top}(R_1) \), this intersection must be below \( \text{top}(R_1) \) and thus below \( \text{top}(R_2) \). But since \( \text{openBox}(RB, R_2) \), any boundary point of \( R_2 \) below \( \text{top}(R_2) \) is a boundary point of \( RB \); but there cannot be any boundary point of \( RB \) in the interior of \( R_1 \).

**Corollary 7:**
\[ \text{openBox}(RB, R_1) \land \text{openBox}(RB, R_2) \land \text{rccO}(R_1, R_2) \Rightarrow \]
\( R_1 \subset R_2 \lor R_2 \subset R_1 \).

**Proof:** By lemma 6, if \( \text{top}(R_1) \leq \text{top}(R_2) \) then \( R_1 \subset R_2 \), and vice versa.

**Lemma 8:**
\[ \text{openBox}(RB, RI) \Rightarrow \exists_{RM} RI \subset RM \land \text{maxBox}(RB, RI). \]

**Proof:** Let \( RM \) be the union of all regions \( R_1 \) such that \( R \subset R_1 \) and \( \text{openBox}(RB, RM) \). It is immediate from lemma 6 that \( RM \) satisfies the stated condition, and that no other region can satisfy the condition.

**Definition 2:**
\[ \text{maxCuppedRegion}(R; \text{region}) \rightarrow \text{fluent}[\text{Bool}] \]
\[ \text{maxCuppedRegion}(R) = \text{maxBox}^\#(\text{solidSpace}, R). \]

**Corollary 9:**
\[ \text{holds}(T, \text{cuppedRegion}(R)) \Rightarrow \]
\[ \exists_{RM} R \subset RM \land \text{holds}(T, \text{maxCuppedRegion}(RM)). \]

**Proof:** Immediate from lemma 8 and the definitions.

**Lemma 10:**
\[ \text{object}(OB) \land \text{holds}(T, \text{isolated}(RI, \{OB\}, L)) \Rightarrow \]
\[ \text{holds}(T, \text{openBox}^\#(\{OB, RI\} \leftrightarrow \# \text{cuppedRegion}(RI))). \]

**Proof:** By the isolation condition, no object other than \( OB \) borders \( RI \) (PD.10). The result is then immediate from the definitions of \( \text{cuppedRegion} \) (CUPD.1) and \( \text{solidSpace} \) (ONTD.4).
Lemma 11:
\[ \text{object}(OB) \land \text{holds}(T, \text{isolated}(RI, \{OB\}, L) \land \text{maxBox}^\#([OB, R]) \Rightarrow \text{holds}(T, \text{localMaxCup}(R)) \]

**Proof:** Using PD.10, let \( D > 0 \) be the minimal distance from \( RI \) to any object other than \( OB \). By lemma 10, \( R \) is a cupped region in \( S \). By the definition of maxBox, no superset of \( R \) is a cupped region cupped only by \( OB \), and if \( R1 \supset R \) is a cupped region involving some object in addition to \( OB \), then \( R1 \) must include points at least \( D \) from \( R \). Hence the conditions of SPILLD.1, SPILLD.2 are met.

Lemma 12:
\[ \text{holds}(T, \text{simpleBox}(OB)) \land \text{holds}(T, \text{isolated}(RI, \{OB\}, L) \land \text{holds}(T, \text{localMaxCup}(RI)) \Rightarrow \forall R \text{ holds}(T, \text{maxBox}^\#([OB, R]) \iff R = RI. \]

**Proof:** Since \( RI \) is a cupped region (SPILLD.2, SPILLD.1, CUPD.1), and is isolated from every object except \( OB \), it follows that holds(\( T, \text{openBox}(OB, RI) \)). It is easily shown that holds(\( T, \text{localMaxBox}(OB, RI) \)). From SPILLD.2, PD.9 it follows that \( RI \) is the only region for which holds(\( T, \text{localMaxBox}(OB, RI) \)). Trivially maxBox(\( OB, R \)) implies localMaxBox(\( OB, R \)); hence maxBox(\( OB, R \)) implies \( R = RI \).

Lemma 13:
\[ \forall R : \text{region}, E > 0 \exists D > 0 \text{ volumeOf}(\text{expand}([\text{boundary}(R), D] \cap R) < E. \]

**Proof:** By definition of volume, there exist \( \mu > 0 \) and a grid decomposition of space into cubical voxels of side \( \mu \) such that the total volume of the voxels that lie entirely inside \( R \) is at least \( \text{volumeOf}(R) - E/2 \). Thus, the number of grid voxels that lie entirely inside \( R \) is at least \( N = (\text{volumeOf}(R) - E/2)/\mu^3. \)

Now, choose \( D < E/12N\mu^2 \). For each grid voxel, the volume of the interior part of the voxel that lies at least \( D \) from the boundary of the voxel is \( (\mu - 2D)^3 > \mu^3 - 6D\mu^2 \); hence the union of these interior parts is at least \( N(\mu^3 - 6D\mu^2) = \text{volumeOf}(R) - E. \) But all these interior part of interior voxels are at least \( D \) from the boundary of \( R \); hence, the part of \( R \) that is within \( D \) of \( \text{boundary}(R) \) has volume less than \( E \).

Corollary 14:
\[ \forall R : \text{region}, E > 0 \exists D > 0 \text{ volumeOf}(\text{expand}([\text{boundary}(R), D] - R) < E. \]

**Proof:** Let \( R2 = \text{closure}(\text{expand}([R, 1] - R) \); thus boundary(\( R \) \( \subset \)) boundary(\( R2 \)). Therefore for any \( D < 1 \), \( \text{expand}(\text{boundary}(R), D] - R) \subset \text{expand}(\text{boundary}(R2), D] \cap R2 \). The result is then immediate from lemma 13.

Corollary 15:
\[ \forall R : \text{region}, E > 0 \exists D > 0 \text{ volumeOf}(\text{expand}([\text{boundary}(R), D]) < E. \]

**Proof:** Immediate from lemma 13 and corollary 14.

Lemma 16:
\( \text{openBox}(RB, RI) \Rightarrow RI \subset \text{convexHull}(RB). \)

**Proof** by contradiction. Suppose that \( P \) is a point in \( RI \) that is not in the convex hull of \( RB \). Then for any horizontal line \( L \) through \( P \), one side or the other of \( L \) does not meet \( RB \). Since \( RI \) is bounded, that ray of \( L \) from \( P \) must meet the boundary of \( RI \) at a point \( P2 \). Since openBox(\( RB, RI \)) and \( P2 \) is not in boundary(\( RB \)) we must have height(\( P2) = \text{top}(RI) \), so height(\( P) = \text{top}(RI) \). Thus, all points \( P \) that are not in the convex hull of \( RB \) have height exactly equal to top(\( RI \)); but since the convex hull of \( RB \) is topologically closed, this is impossible.

**Definition 3:**
boxedPoint(\( P \): point, \( R \): region) \( \Rightarrow \text{maxShift}(M: \text{rigidMapping}, R: \text{region}) \rightarrow \text{distance}. \)
boxedPoint(P, R) ≡
∃RI openBox(R, RI) ∧ P ∈ RI.

maxShift(M, R) = D ≡
[∃P ∈ R dist(P, mappingImage(M, P)) = D] ∧ [∀P ∈ R dist(P, mappingImage(M, P)) ≤ D]

Lemma 17:
maxShift(M, convexHull(RB)) = maxShift(M, RB).

Proof: It is easily shown that, if P is on a line between PA and PB then dist(P, M(P)) ≤ max(dist(PA, M(PA)), dist(PB, M(PB))). The result is then immediate.

Lemma 18:
maxShift(M, RB) ≤ E ∧ openBox(RB, RI) ∧ point(P) ∧ expand(P, 4E) ⊂ RI ⇒
boxedPoint(P, mappingImage(M, RB))

Proof: Let R2B = M(RB), RIX = M(RI). Using lemmas 16 and 17, maxShift(RI) ≤ E. Let PST be the topSurface of RI and let PT2 = M(PST). Since expand(P, 4E) ⊂ RI we have bottom(RI) ≤ height(P) − 4E and top(RI) ≥ height(P) + 4E. Hence bottom(PT2) ≥ bottom(PST) − E ≥ height(P) + 3E > height(M(P)). Since M(P) ∈ RIX, some of RIX is below bottom(PT2). Hence we can define RIY to be the part of RIX below bottom(PT2). By lemma 4, RIY is a disconnected open Box. Define RI2 to be the thickly connected component of RIY. Then openBox(RI2) and P ∈ RI2, satisfying the theorem.

Corollary 19:
maxShift(M, RB) ≤ E ∧ ¬boxedPoint(P, RB) ∧ boxedPoint(P, mappingImage(M, RB)) ⇒
∃PA dist(P, PA) ≤ 4E ∧ ¬boxedPoint(PA, RB)

Proof: Just a logical rearrangement of Lemma 18.

Lemma 20:
maxShift(M, RB) ≤ E ∧ ¬boxedPoint(P, RB) ∧ boxedPoint(P, mappingImage(M, RB)) ∧
dist(P, RB) > 4E ⇒
∃RI openBox(RB, RI) ∧ dist(P, RI) ≤ 4E.

Proof: Assume that M, P, RB, E meet the conditions of the lemma. Let R2B = M(RB) and let RI2 be such that P ∈ RI2, and openBox(RB2, RI2). By lemma 16, every point in RI2 is in the convex hull of RB; it follows easily that maxShift(M, RI) ≤ maxShift(M, RB) = E. Let RT2 = topSurface(RI2); thus every point in RT2 has height greater than that of P. Let RT1 = M −1(RT2); then bottom(RT1) ≥ height(RT2) − E ≥ height(P) − E.

Clearly dist(P, RB2) ≥ dist(P, RB) − maxShift(M, RB) ≥ 3E. Let P2A be the point directly below P at distance 3E from P; thus P2A ∈ RI2. Since all the points on the line from P to P2A are less than 3E from P, none of these points are in RB2; hence P2A ∈ RI2. Let PA = M −1(P2A). Since PA ∈ M −1(RI2) and height(PA) ≤ height(P) − 2E < bottom(RT1), it follows that some of M −1(RI2) is lower than bottom(RT1). Let RIX be the part of M −1(RI2) lower than bottom(RT1); by lemma 4, disconnectOpenBox(RB, RIX). Let RI be the thickly connected component of M −1(RI2) containing PA; thus openBox(RB, RI). Finally dist(PA, P) ≤ dist(PA, P2A) + dist(PA2, P) ≤ 4E, so dist(P, RI) ≤ 4E.

Definition 4:
allBoxes(RB, RI: region)
symDiff(RA, RB, RC: region)

allBoxes(RB, RI) ≡
∀P P ∈ RI ↔ boxedPoint(P, RB)
\[ \text{symDiff}(RA, RB, RC) \equiv \]
\[ [RA \subset RB \land \text{regDif}(RB, RA, RC)] \lor \]
\[ [RB \subset RA \land \text{regDif}(RA, RB, RC)] \lor \]
\[ \text{regDif}(RA, RB, RD) \land \text{regDif}(RB, RA, RE) \land RC = RD \cup RE. \]

In proofs, we will write \( RP \ominus RQ \) for the regularized symmetric difference of \( RP \) and \( RQ \). (We can’t write this in lemmas because it may be empty.)

**Corollary 21:**
\[
\text{maxShift}(M, RB) < E \land \text{allBoxes}(RB, RI) \land \text{allBoxes}(\text{mappingImage}(M, RB), RMI) \land \text{symDiff}(RI, RMI, RD) \Rightarrow \]
\[ RD \subset \text{expand}(\text{boundary}(RI, 4E) \cup \text{expand}(\text{boundary}(RB), 4E). \]

**Proof:** By corollary 19, if \( P \) is boxed in \( RB \) and not boxed in \( M(RB) \) then it is within \( 4E \) of \( \text{boundary}(RI) \). By lemma 20, if \( P \) is not boxed in \( RB \) and boxed in \( M(RB) \) then it is either within \( 4E \) of \( \text{boundary}(RI) \) or within \( 4E \) of \( \text{boundary}(RB) \).

**Lemma 22:**
\[
\text{simpleBox}(RB) \Rightarrow [\text{maxBox}(RB, RI) \Leftrightarrow \text{allBoxes}(RB, RI)]
\]

**Proof:** Immediate from the definitions.

**Definition 5:**
\[
\text{maxShift1}(T1, T2; time, O; object) \rightarrow \text{distance}.
\]
\[
\text{maxShift1}(T1, T2, O) = \]
\[
\text{maxShift}(\text{mappingImage}(\text{value}(T2, \text{placement}(O)), \text{inverse}(\text{value}(T1, \text{placement}(O))), \text{shape}(O)).
\]

**Lemma 23:**
\[
\text{throughout}(TS, TE, \text{simpleBox}^#(↑O) \land ^# \text{maxBox}^#(↑O, Q)) \Rightarrow \]
\[
\text{continuousVolume}(Q, TS, TE).
\]

**Proof:** Let \( T \) be any time between \( TS \) and \( TE \). Let \( E > 0 \). Using corollary 14, choose \( D1 > 0 \) such that
\[
\text{volumeOf}(\text{expand}(\text{value}(T, \text{boundary}^#(↑O \cup \text{boundary}^#(Q))), D1) < E.
\]
Since \( O \) moves continuously, choose \( D \) such that, for any time \( T1 \) between \( TS \) and \( TE \) and between \( T - D \) and \( T + D \), \( \text{maxShift1}(T1, T, O) < D1/4. \) Using corollary 21 and lemma 22,
\[
\text{volumeOf}(\text{value}(T1, Q) \ominus \text{value}(T, Q)) \leq \]
\[
\text{volumeOf}(\text{expand}(\text{value}(T, \text{boundary}^#(↑O \cup \text{boundary}^#(Q))), D1) < E.
\]

**Lemma 24:**
\[
\text{object}(O) \land \text{throughout}(TS, TE, \text{maxBox}^#(↑O, Q)) \land \text{continuousVolume}(Q, TS, TE) \Rightarrow \]
\[
\text{continuous}(\text{top}^#(Q), TS, TE).
\]

**Proof:** There are two cases:

Case 1: Every point \( P \) in \( Q \) such that \( \text{height}(P) = \text{top}(Q) \) is in the boundary of \( O \). In that case, \( O \) is a closed box and \( Q \) is always an entire thickly connected component of the complement of \( O \); that is, \( Q \) is a pseudo-object of constant shape moving with \( O \). Since the shape of \( Q \) is constant and its placement tracks the placement of \( O \) and is continuous, \( \text{top}(Q) \) is continuous.

Case 2: There exists a point \( P \) in the top surface of \( Q \) such that the ball of radius \( D > 0 \) does not intersect \( O \). Over a small enough time interval, \( O \) does not come inside that ball. A discontinuous change in \( \text{top}(Q) \) would cause the corresponding slice of that ball, of finite volume, to come in or out of \( Q \), leading to a volume discontinuity of \( Q \). (This is loosely worded, but can easily be made tight.)
We could weaken the condition “maxBox(O, Q)” in the preceding lemma to be just “openBox(O, Q)”, but the analysis of case 1 becomes a little trickier, and we do not need it.

**Corollary 25:**

\[ \text{object}(O) \wedge \text{throughout}(TS, TE, \text{simpleBox}#(\uparrow O) \wedge \# \text{maxBox}#(\uparrow O, Q)) \Rightarrow \text{continuous}(\text{top}#(Q), TS, TE) \]

**Proof:** Immediate from lemmas 23 and 24.

**Lemma 26:**

\[ \text{simpleBox}(RB) \wedge \text{openBox}(RB, RI1) \wedge \text{openBox}(RB, RI2) \Rightarrow RI1 \subset RI2 \lor RI2 \subset RI1. \]

**Proof** of the contrapositive. Suppose that neither RI1 nor RI2 is a subset of the other. Let P1 be a point in the interior of RI1 and let D1 be the distance from P1 to the boundary of RI1; thus RI1 contains a sphere of radius D1 centered at P1. Define P2 and D2 correspondingly for RI2. By lemma 7, rccDS(RI1, RI2). Let RA1 be the closure of the union of all regions RK1 such that RI1 ⊂ RK1, openBox(RB, RK1) and rccDS(RK1, RI2), and let RA2 be the closure of the union of all regions RK2 such that RI2 ⊂ RK2, openBox(RB, RK2) and rccDS(RK1, RI2). It is easily verified that openBox(RB, RA1), openBox(RB, RA2), and rccDS(RA1, RA2). Moreover suppose that RQ1 is any proper superset of RA1 such that openBox(RB, RQ). Then rccO(RQ1, RI2) by construction of RA1. Hence by lemma 7, RI2 ⊂ RQ1 (since RQ is clearly not a subset of RI2); so RQ contains P and thus contains a point that is at least D1 from any point in RA1. Thus RA1 is a localMaxBox for RB. Likewise RB1 is a localMaxBox for RB. But since RB has two localMaxBoxes, it does not satisfy simpleBox(RB).

**Lemma 27:**

\[ \text{openBox}(RB, RI) \wedge \text{regDif}((\text{convexHull}(RB), RB, RD)), \text{regionBelow}(RD, \text{top}(RI), RC) \Rightarrow \text{thicklyConnectedComponent}(RI, RC). \]

**Proof:** By lemma 16, RI is a subset of convexHull(RB). Since RI does not overlap RB, RI is a subset of RD. By assumption RI is thickly connected. Suppose that RO is a thickly connected set such that RI ⊂ RO ⊂ RC. If RO is a proper superset of RI, then some part of the boundary of RI must lie in the interior of RO; but this is impossible, since the interior of RO is entirely below top(RI) and entirely disjoint from RB. Hence RO = RI, so RI is a thickly connected component of RC.

**Definition 6:**

\[ \text{regInt}(R1, R2, R3: \text{region}). \]

\[ \text{regInt}(R1, R2, R3) = \forall R: \text{region} \ R \subset R2 \Leftrightarrow R \subset R1 \land R \subset R2. \]

**Lemma 28:**

\[ \text{continuousVolume}(QP, TS, TE) \wedge \text{continuousVolume}(QQ, TS, TE) \wedge \text{throughout}(TS, TE, \text{regInt}#(QP, QQ, Q)) \Rightarrow \text{continuousVolume}(Q, TS, TE). \]

**Proof:** Note that \( [RA \cap RB] \oplus [RC \cap RD] \subset [RA \oplus RC] \cup [RB \oplus RD] \) and therefore \( \text{volumeOf}([RA \cap RB] \oplus [RC \cap RD]) \leq \text{volumeOf}(RA \oplus RC) + \text{volumeOf}(RB \oplus RD) \). Let \( T1 \) and \( T2 \) be two times between TS and TE. Then taking \( RA = \text{value}(T1, QP), RB = \text{value}(T1, QQ), RC = \text{value}(T2, QP), RD = \text{value}(T2, QQ) \) gives

\[ \text{volumeOf}(\text{value}(T1, QP) \oplus \text{value}(T2, QP)) + \text{volumeOf}(\text{value}(T1, QQ) \oplus \text{value}(T2, QQ)). \]
Since $QP$ and $QQ$ are volume-continuous, the summands on the right-hand side of the inequality can be made arbitrarily small by requiring that $T_1$ and $T_2$ lie close enough; hence the term on the left-hand side, which is the definition of $Q$ being volume continuous.

**Corollary 29:**

continuousVolume($QP, TS, TE$) $\land$ continuousVolume($QQ, TS, TE$) $\land$

throughout($TS, TE, \text{regDif}^#(QP, QQ, QR)) \Rightarrow$

continuousVolume($QR, TS, TE$)

**Proof:** Immediate from lemma 28 using the fact that the regularized difference of $QP$ and $QQ$ is the regularized intersection of $QP$ with the complement of $QQ$.

**Lemma 30:**

continuousVolume($Q, TS, TE$) $\Rightarrow$ continuous(volumeOf($Q, TS, TE$))

**Proof:** Immediate.

**Definition 7:**

$\text{intersectVolume}(RA, RB; \text{region}) \rightarrow \text{volume}$

$\text{intersectVolume}(RA, RB) = V \iff$

$[\text{recDS}(RA, RB) \land V = 0] \lor [\exists_{RI} \text{regInt}(RA, RB, RI) \land V = \text{volumeOf}(RI)]$

**Corollary 31:**

continuousVolume($Q1, TS, TE$) $\land$ continuousVolume($Q2, TS, TE$) $\Rightarrow$

continuous($\text{intersectVolume}(Q1, Q2, TS, TE$).

**Proof:** The proof of Lemma 28 extends immediately to the case where either or both intersections involved are the null set. The result then follows from lemma 30.

**Definition 8:**

$\text{allLiquidIn}(R: \text{region}, L: \text{liquidChunk}) \rightarrow \text{fluent}[\text{Bool}]$

$\text{volumeOfLiquidIn}(R: \text{region}) \rightarrow \text{fluent}[\text{volume}]$

$\text{allLiquidIn}(R, L) = \text{regInt}^#(\text{liquidSpace}, R, L)$.

$value(T, \text{volumeOfLiquidIn}(R)) = V \iff$

$[\text{holds}(T, \text{allLiquidIn}(R, L)) \land \text{liqVolume}(L) = V] \lor [\text{emptyLiquid}(T, R) \land V = 0]$.

**Corollary 32:**

continuousVolume($Q, TS, TE$) $\Rightarrow$ continuous(volumeOfLiquidIn($ Q, TS, TE$))

**Proof:** Immediate by applying corollary 31 to the intersection of $Q$ with liquidSpace.

**Definition 9:**

$\text{netInflow}(L: \text{liquidChunk}, Q: \text{fluent}[\text{region}], TS, TE: \text{time})$

$\text{netOutflow}(L: \text{liquidChunk}, Q: \text{fluent}[\text{region}], TS, TE: \text{time})$

$\text{netInflowVolume}(Q: \text{fluent}[\text{region}], TS, TE: \text{time}) \rightarrow \text{volume}$.

$\text{netOutflowVolume}(Q: \text{fluent}[\text{region}], TS, TE: \text{time}) \rightarrow \text{volume}$.

$\text{netInflow}(L, Q, TS, TE) \equiv$

$\text{flowsIn}(L, Q, TS, TE) \lor [\forall_{L_1} \text{flowsIn}(L_1, Q, TS, TE) \Rightarrow \text{subChunk}(L_1, L)]$

$\text{netOutflow}(L, Q, TS, TE) \equiv$

$\text{flowsOut}(L, Q, TS, TE) \lor [\forall_{L_1} \text{flowsOut}(L_1, Q, TS, TE) \Rightarrow \text{subChunk}(L_1, L)]$

$\text{netInflowVolume}(Q, TS, TE) = V \iff$

$[\text{netInflow}(L, Q, TS, TE) \land \text{volumeOf}(L) = V] \lor$

$[\text{noInflow}(Q, TS, TE) \land V = 0]$. 

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$\text{netOutflowVolume}(Q, TS, TE) = V \iff$

$[\text{netOutflow}(L, Q, TS, TE) \land \text{volumeOf}(L) = V] \lor$

$[\text{noOutflow}(Q, TS, TE) \land V = 0].$

**Corollary 33:**
Let $TS$ and $TE$ be times and let $Q$ be a region-valued fluent such that $\text{continuousVolume}(Q, TS, TE)$. Define the functions of time $f(T) = \text{netInflowVolume}(Q, TS, T)$ and $g(T) = \text{netOutflowVolume}(Q, TS, T)$. Then $f$ and $g$ are continuous.

**Proof:** Let $L_1, L_2$ be such that holds($\text{start}(H, T), \text{allLiquidIn}(Q, L_1)$) and holds($\text{start}(H, \text{allLiquidIn}(Q, L_2)$). By definition $\text{netInflow}(Q, TS, T) = L_1 - L_2$ and $\text{netOutflow}(Q, TS, T) = L_2 - L_1$. (The minus signs here are set difference.) The result is then immediate from Corollary 32. (The case where either or both of $L_1, L_2$ are empty are trivial extensions.)

**Lemma 34:**
$\text{slowObjectsInContact}(Q, TS, TE) \land \text{continuousVolume}(Q, TS, TE) \land$

$\text{allLiquidIn}(TS, Q, L) \land$

$\text{throughout}(TS, TE, \text{cuppedRegion}^#(Q) \land^# \text{liqVolume}(L) =^# \text{volume}^#(Q) \land^# \text{noDrivenLiqIn}(Q))$

$\Rightarrow$

$\text{netOutflow}(Q, TS, TE) = \text{netInflow}(Q, TS, TE).$

**Proof:** by contradiction. Suppose $\text{netOutflow}(Q, TS, TE) < \text{netInflow}(Q, TS, TE)$, Then by lemma AK

$value(TE, \text{volumeOfLiqIn}(Q)) =$

$value(TS, \text{volumeOfLiqIn}(Q)) + \text{netInflowVolume}(Q, TS, TE) - \text{netOutflowVolume}(Q, TS, TE) >$

$value(TS, \text{volumeOfLiqIn}(Q)) = \text{liqVolume}(L) = value(TE, volume^#(Q)),

which is impossible.

Suppose $\text{netOutflowVolume}(Q, TS, TE) > \text{netInflowVolume}(Q, TS, TE)$. For $T$ between $TS$ and $TE$ let $h(T) = \text{netOutflowVolume}(Q, TS, T) - \text{netInflowVolume}(Q, TS, T)$.

Let $\epsilon = h(TE) > 0$. By corollary 29, $h$ is continuous; hence there exists a $T_1$ such that $h(T_1) = \epsilon/2$, and for all $T$ between $T_1$ and $TE$ $h(T) > \epsilon/2$. Then for all such $T,$

$value(T, \text{volumeOfLiqIn}(Q)) = value(TS, \text{volumeOfLiqIn}(Q)) + h(T) <

value(T, \text{volumeOfLiqIn}(Q)).$

Thus, if we bind $TS$ of CUP.2 to $T$ here and $TE$ of CUP.2 to $TE$ here, then the conditions of CUP.2 are satisfied. The conclusion of CUP.2 asserts that there is no outflow from $T$ to $TE$ which contradicts the fact the the volume of liquid in $Q$ decreases from $T$ to $TE.$

**Corollary 35:**
$\text{slowObjectsInContact}(Q, TS, TE) \land \text{continuousVolume}(Q, TS, TE) \land$

$\text{allLiquidIn}(TS, Q, L) \land$

$\text{throughout}(TS, TE, \text{cuppedRegion}^#(Q) \land^# \text{liqVolume}(L) \leq \text{volume}^#(Q) \land^# \text{noDrivenLiqIn}(Q))$

$\Rightarrow$

$\text{netOutflowVolume}(Q, TS, TE) \leq \text{netInflowVolume}(Q, TS, TE).$

**Proof:** Immediate from CUP.2 and Lemma 34.

**Lemma 36:**
$\text{slowObjectsInContact}(Q, TS, TE) \land \text{continuousVolume}(Q, TS, TE) \land$

$V_{MIN} \leq value(TS, \text{volumeOfLiqIn}(Q)) \land$

$\text{throughout}(TS, TE, \text{cuppedRegion}^#(Q) \land^# V_{MIN} \leq \text{volume}^#(Q) \land^# \text{noDrivenLiqIn}(Q))$

$\Rightarrow$

$V_{MIN} \leq value(TE, \text{volumeOfLiqIn}(Q)).$

**Proof:** Similar to the proof of lemma 34.
Suppose that \( \text{value}(TE, \text{volumeOfLiquidIn}(Q)) < VMIN \).
Let \( \epsilon = VMIN - \text{value}(TE, \text{volumeOfLiquidIn}(Q)) \).
By continuity there exists a time \( T1 \) such that
\( VMIN - \text{value}(T1, \text{volumeOfLiquidIn}(Q)) = \epsilon/2 \) and such that
\( VMIN - \text{value}(T1, \text{volumeOfLiquidIn}(Q)) > \epsilon/2 \) for all \( T \) between \( T1 \) and \( TE \). Axiom CUP.2 then applies over the subhistory between \( T1 \) and \( TE \), so there is no outflow from \( Q \) in that period; but that is inconsistent with the fact that the volume of liquid in \( Q \) decreases from \( T1 \) to \( TE \).

**Corollary 37:**
slowObjectsInContact(\( Q, TS, TE \)) \& continuousVolume(\( Q, TS, TE \)) \&
\( \text{simpleOverflows}(L, Q, TS, TE) \implies \)
throughout(\( TS, TE, \text{fullOfLiquid}(Q) \)).

**Proof:** Let \( T \) be any time between \( TS \) and \( TE \). Let \( VMIN \) be the volume of liquid in \( Q \) at time \( T \). The result is then immediate from lemma 36 and definition SPI LLD.4.

**Lemma 38:**
throughout(\( TS, TE, \text{bounded}(Q) \)) \& continuousVolume(\( Q, TS, TE \)) \&
continuous(\( QS, TS, TE \)) \&
\( \text{regionBelow}(\{Q, QZ, QB\}) \implies \)
continuousVolume(\( QB, TS, TE \)).

**Proof:** Let \( T1, T2 \) be two times between \( TS \) and \( TE \). \( RZ \) be the vertical column bounded below and above by \text{value}(\( T1, QZ \)) and \text{value}(\( T2, QZ \)) and whose horizontal cross-section is the union of the \( xy \)-projections of \text{value}(\( T1, Q \)) and \text{value}(\( T2, Q \)). Then it is easily seen that \text{value}(\( T1, QB \) \& \text{value}(\( T2, QB \) \subset [\text{value}(\( T1, Q \) \& \text{value}(\( T2, Q \)) \cup RZ). Since \( Q \) and \( QZ \) are continuous, the volumes of the terms on the right can be made arbitrarily small by requiring \( T1 \) and \( T2 \) to be sufficiently close. Thus, the same is true of the volume of \text{value}(\( T1, QB \) \& \text{value}(\( T2, QB \)), so \( QB \) is volume-continuous.

**Lemma 39:**
source(\( BSPOUT \))=OB \& DB > 0 \&
throughout(\( TS, TE, \text{spout1}(\{OB, QI1, QI2, BSPOUT, QOPEN, DB\}) \& \text{simpleBox}(\{OB \& \text{regDi}(QI1 \cup QI2, BSPOUT, QS)) \implies \)
continuousHausdorff(\( QS, TS, TE \)) \& continuousVolume(\( QS, TS, TE \)).

**Proof:** The boundaries of \( QS \) are formed by \( OB, BSPOUT, \) and the top of \( QI2 \), which is always \( DB \) above top(\( QI1 \), and is thus a continuous function of time. Continuity in the Hausdorff metric is immediate. Continuity in the volume metric follows directly from corollary 15 and lemma 38.

**Definition 10:** flatBottom(\( R: \text{region} \)).
\( \forall P \) bottomPoint(\( P, R \) \implies \text{height}(P)=\text{bottom}(R) \).

**Lemma 40:**
\text{thicklyConnected}(\( R \)) \& flatBottom(\( R \) \implies \text{connected}(\text{bottomSurface}(\( R \))).

**Proof** by contradiction. Suppose that the conditions hold but \text{bottomSurface}(\( R \)) is not connected. Let \( P1 \) and \( P2 \) be points in two different connected components of \text{bottomSurface}(\( R \)). Since \( R \) is thickly connected, there is a path \( PS \) from \( P1 \) to \( P2 \) through \( R \). Let \( PS1 \) be the projection of \( PS \) onto the horizontal plane at height(\( P1 \)). Since \( PS1 \) goes from \( P1 \) to \( P2 \) through \text{bottomSurface}(\( R \)), there must be a point \( PA \) in \( PS1 \) that is not a bottom point of \( R \). Let \( PB \) be a point in \( PS \) that is directly above \( PA \). There must be a bottom point \( PC \) of \( R \) directly below \( PB \). Since \( PC \neq PA \) and they are on the same vertical line, height(\( PC \) \neq height(\( PA \))=height(\( P1 \)), which contradicts
the assumption that connected(bottomSurface(R)).

**Lemma 41:**
flatBottom(R) ∧ openBox(RB, RI) ∧ rccO(RI, R) ∧ rccDS(RB, R) ⇒  
bottomSurface(R) ⊂ RI

**Proof:** Let P be a point in interior(RI) ∪ interior(R). Since openBox(RB, RI) there is a point PB on boundary RB directly below P. Since flatBottom(R) there is a point PC on bottomSurface(R) directly below P. Since R and RB are disjoint, height(PB) ≤ height(PC)

Suppose that there is a point P1 ∈ bottomSurface(R) which is not in RI. By lemma 40 there is a path form P1 to PC through bottomSurface(R). Since openBox(RB, RI), this path must meet boundary(RB) at a point PD. Since the path is not at top(RI), there is open set in RB above PD; this must overlap interior(R), which contradicts the assumptions.

**Problem Specific Results**

**Lemma 42:**
∀ T T ≥ t0 ⇒  
∃ RB holds(T, regionBelow#(↑bInsidePitcher, bottom#(↑bTopPitcher), RB)).

**Proof:** Immediate from PS.6 and lemma 3.

**Lemma 43:**
∀ T T ≥ t0 ⇒  
∃ R holds(T,cuppedRegion(R)) ∧ holds(T, R ⊂# ↑bInsidePitcher)

**Proof:** Let RB be as in Lemma 42. From corollary 5, PS.2, PS.3, PS.20, RB is a disconnected open box. If we choose R to be a thickly connected component of RB then by lemma 10 R is a cupped region.

**Lemma 44:**
∃ Q everAfter(t0,rccO#(Q, ↑bInsidePitcher) ∧# maxCuppedRegion#(Q)).

**Proof:** Immediate from lemmas 43 and corollary 9, with axiom T.2.

**Definition 11:**
Let qIn be the region-valued fluent satisfying lemma 44.

**Lemma 45:**
∀ TE t0 < TE ⇒ continuousVolume(qIn,t0,TE).

**Proof:** Immediate from lemma 23, PD.3, PD.4, PS.3.

**Lemma 46:**
everAfter(t0,noDrivenLiqIn(qIn)).

**Proof:** By SPILLD.5-7, a driven liquid L1 can only exist in upExpand of some liquid L2 in a localMaxCup that is overflowing. Since qIn is cupped by oPitcher, if L1 is in qIn then L2 must also be in qIn. Since there are no object other than OB that border any part of qIn (PS.22), the localMaxCup for L2 must be OB itself; but this is impossible since qIn contains it and OB is a simpleBox with only one localMaxBox.

**Lemma 47:**
t0 < TE ⇒ throughout(t0,TE, volumeOf#(qIn) ≥# liqVolume(l0)) ⇒  
throughout(t0,TE, l0 ⊂ qIn).

**Proof:** By PS.20, the only object in contact with qIn is oPitcher.
By PS.5, CUPD.2, slowObjectsInContact(qIn,t0,TE). By lemma 45, continuousVolume(qIn,t0,TE).
By construction, throughout(t0,TE,cuppedRegion(qIn)).
By hypothesis throughout(t0,TE,volumeOf#(qIn) ≥ #liqVolume(l0)).
By lemma 46, throughout(t0,TE,noDrivenLiqIn(qIn)).
Hence by corollary 35, netOutflowVolume(qIn,t0,TE) ≤ netInflowVolume(qIn,t0,TE).
However, since qIn is isolated from all liquids but l0 (PS.20), there is no inflow into qIn (FLOW.1, FLOW.3) so the next inflow volume is 0; hence the net outflow volume is 0. Since there is no outflow, l0 remains in qIn throughout t0, TE. (FLOW.2, FLOW.4).

**Corollary 48:**
throughout(t0,t1,↑l0 ⊂ qIn).

**Proof:** Immediate from lemma 47, PS.8, PD.7.

**Lemma 49:**
∃L subChunk(L,l0) ∧ everAfter(t0, L ⊂ bInsidePitcher)

**Proof:** By lemma 42, ever after t0 there is a region of bInsidePitcher below bottom(bTopPitcher). By lemma 4, this is a disconnected open box. By lemma 26, it must contain only one thickly connected component; thus it is a connected open box. Let q1 be the fluent whose value at a time is this region. By lemmas 38 and 30, volumeOf(q1) is a continuous function of time. Since volumeOf(q1) is always positive, it attains a positive minimum vMin over the closed time intervals [t0,t2]. Since the pitcher is motionless after t2, q1 and volumeOf(q1) are constant after t2, and thus volumeOf(q1) is at least vMin ever after t2. Thus, the conditions for lemma 36 are met, and there is always at least a volume vMin of liquid inside bInsidePitcher.

**Lemma 50:**
everAfter(t0, volumeOfLiquidIn(bInsidePail) < volume#(bInsidePail))

**Proof:** By PS.21, ever after t0 the liquid in the pail is a subchunk of l0. By PS.13 volumeOf(l0) < volumeOf(bInsidePail).

**Definition 12:** Using PS.13, let zp1 be the height such that the volume of the part of bInsidePail below zp1 is equal to liqVolume(l0). By PS.13, zp1 < top(bInsidePail)−maxOutflow. Let re0 be the part of pouringRegion above top(bInsidePail). It is easily seen that flatBottom(re0).

**Definition 13:**
horizExpand(PS:pointSet, D:distance) → pointSet.

\[ P \in \text{horizExpand}(PS, D) \iff \exists_{PC \in PS} \text{dist}(P, PC) \leq D \land \text{height}(P) = \text{height}(PC) \]
Let rccDC(R1, R2) be the RCC relation “R1 is disconnected from R2”.

**Lemma 51:**
t0 < T ∧ liquidChunk(L) ∧ holds(T,openBox(oPail,L)) ⇒ holds(T, top(L) < zp1) ∧ holds(T, rccDC(L,re0)).

**Proof:** By PS.14 L ⊂ bInsidePitcher in T. By lemma 50 L ⊂ l0. The result is immediate from definition 12.

**Lemma 52:**
¬∃T1,L simpleOverflows(L,bInsidePail,t0,T1).

**Proof:** Immediate from lemma 50 and SPILLD.5.

**Lemma 53:**
t1 ≤ T ∧ liquidChunk(L) ∧ holds(T,cuppedRegion#(↑L) ∧ # rccO(L,re0)) ⇒
holds\((T, L \subset qIn)\).

**Proof:** By lemma 41, there are two cases to consider: either \(L\) contains \(\text{bottomSurface}(re0)\) or \(\text{solidSpace}\) overlaps \(re0\).

If \(L\) contains \(\text{bottomSurface}(re0)\) then \(L\) overlaps with \(b\text{InsidePail}\). By corollary 7, either \(L\) is a subset of \(b\text{InsidePail}\) or vice versa. By lemma 51, if \(L\) is a subset of \(b\text{InsidePail}\), then \(L\) does not overlap \(re0\). If \(b\text{InsidePail}\) is a subset of \(L\) then \(\text{liqVolume}(L) > \text{volume}(b\text{InsidePail}) > \text{liqVolume}(l0)\) so \(L\) contains liquid other than \(l0\); but this is impossible by the isolation condition PS.21.

By the construction of \(\text{pouringRegion}\) PS.19 and the isolation condition PS.21, the only object that can overlap \(re0\) is \(o\text{Pitcher}\). By definition 11, \(qIn\) is the unique maximal cupped region formed by \(o\text{Pitcher}\). By the isolation condition PS.20, \(o\text{Pitcher}\) does not form any cupped region in combination with any other objects.

**Lemma 54:**
\[
\forall_T \ t_1 \leq T \implies \exists_{RI2, ROPEN} \text{holds}(T, \text{spout1}\#(o\text{Pitcher}, qIn, RI2, b\text{Spout}, ROPEN, \text{maxOutflow}))
\]

**Proof:** Immediate from PS.9, PD.5. It is easily shown that the value of the quantified variable \(RI1\) in PD.5 is uniquely determined and must be equal to \(qIn\).

**Definition 14:**
Using lemma 54, let \(q\text{Above}\), \(q\text{Open}\) be fluents whose value at every time \(T\) after \(t1\) satisfies
\[
\text{holds}(T, \text{spout1}\#(\uparrow o\text{Pitcher}, qIn, q\text{Above}, \uparrow b\text{Spout}, q\text{Open}, \text{maxOutflow}))
\]
Let \(q\text{Source} = qIn \cup\# q\text{Above} - \# \uparrow b\text{Spout}\).

**Definition 15:**
Let \(q\text{Expand}\) be the fluent equal to the union of all regions \(R\) such that \(\text{drivenReg}(R)\) (at times when some region is driven).

\[
\forall_T \ [t_1 \leq T \land \exists_R \text{holds}(T, \text{drivenReg}(R))] \implies \forall_P [\text{holds}(T, P \in\# q\text{Expand}) \iff \exists_R P \in R \land \text{holds}(T, \text{drivenReg}(R))].
\]

\(q\text{NearPitcher} = qIn \cup\# q\text{Above} \cup\# q\text{Expand}\).

**Lemma 55:**
\(t0 \leq T \implies \text{holds}(T, q\text{Expand} \subset\# q\text{Above} \cup\# \text{expand}\#(b\text{Spout}, \text{maxOutflow}))\).

**Proof:** Let \(P\) be a point in \(q\text{Expand}\). Let \(R\) satisfy the conditions of definition 15. By SPIILLD.6, \(R\) is a subset of some thickly connected component \(R1\) of \(\text{upExpand}(\text{topSurface}(qIn), \text{maxOutflow}, \text{solidFreeSpace})\). By definitions PD.4, PD.3, the boundaries of \(q\text{Above}\) are the top surface of \(qIn\), the boundary of \(b\text{Pitcher}\), the horizontal plane at height \(\text{top}(qIn) + \text{maxOutflow}\), and \(q\text{Open}\). \(R1\) does not penetrate into \(b\text{Pitcher}\), because it is disjoint from \text{solidSpace}; it does not penetrate into \(qIn\), because it is entirely above \(\text{top}(qIn)\); and it does not go above \(\text{top}(q\text{Above})\) because any point above \(\text{top}(q\text{Above})\) is more than \(\text{maxOutflow}\) from \(\text{topSurface}(qIn)\).

The boundary of \(qIn\) consists of the boundary of \(o\text{Pitcher}\) and the top surface of \(qIn\). As stated, \(R1\) does not overlap \(o\text{Pitcher}\) or \(qIn\); therefore, it meets the top surface of \(qIn\) from above. But the entire region immediately above \(\text{topSurface}(qIn)\) is either \(o\text{Pitcher}\) or \(q\text{Above}\).

Suppose that \(P\) is outside \(q\text{Above}\). By SPIILLD.6, there is a line \(PL\) of length at most \(\text{maxOutflow}\) from \(P\) to a point \(PB\) in \(\text{topSurface}(qIn)\) that goes through \(R1\). It is easily shown that for \(\epsilon > 0\) there
exist points $P_1$ and $PB_1$ within $\epsilon$ of $P_1$ and $PB_1$ respectively such that $PB_1$ is in the interior of $qAbove$ and such that the line from $P_1$ to $PB_1$ stays in the interior of $R1$. Since this line goes from inside to outside $qAbove$, it crosses the boundary of $qAbove$; since the crossing point $PC$ is in the interior of $R1$, it must be in $qOpen$. But then the distance from $PC$ to $P_1$ is at most maxOutflow, so the distance from $PC$ to $P$ is at most maxOutflow+.$\epsilon$. Since $\epsilon$ can be made arbitrarily small, and since $qOpen \subset bSpout$, $dist(P,bSpout) \leq dist(P,qOpen) \leq maxOutflow$.

Lemma 56:
$t1 \leq T \Rightarrow$ holds$(T,(\text{nonFlowingSpace} \cap \# \text{re0}) \subset \# (\uparrow \text{oPitcher} \cup \# qNearPitcher))$

Proof: By PS.17, PS.21 the only solid object that enters $\text{re0}$ is $\text{oPitcher}$. By construction, the only liquid cupped by $\text{oPitcher}$ is in $qIn$.

By SPILLD.6, SPILLD.7 any driven liquid must be within $\text{upExpand}$ of some overflowing cupped liquid. Any driven liquid associated with the overflow of $\text{oPitcher}$ is a subset of $qExpand$. Any weakly cupped liquid bounded by such a driven liquid together with $\text{oPitcher}$ is a subset of $qAbove$. Since $\text{oPitcher}$ is isolated, there cannot be any cupped region involving $\text{oPitcher}$ in combination with some other object.

By lemma 52, $bInsidePail$ does not overflow, and by 53, there is never a filled cupped region that overlaps $bInsidePail$.

Suppose that there is a cupped region $RC$ created by objects other than $bPitcher$. By the isolation condition PS.23, all such objects are at least maxOutflow from $\text{pouringRegion}$. Let $P$ be a point and let $PL$ be the shortest line from $P$ to $\text{re0}$. If $PL$ is horizontal or moves downward from $P$ to $\text{re0}$, then it must go through one of the solid objects that bounds $RC$; hence $dist(P,\text{re0}) > \text{maxOutflow}$. If $PL$ goes upward, then $PL$ must intersect a bottom point of $\text{re0}$; but these are all in $bInsidePail$. In either case, there is no way for a driven chunk of liquid that stays within maxOutflow of $RC$ to overlap the inside of $\text{re0}$. The same is trivially true of weakly cupped liquids associated with an overflow of some other object.

Corollary 57:
$t1 \leq T \Rightarrow$ holds$(T,\text{top}(\text{nonFlowingSpace} \cap \# \text{re0} - \# \text{oPitcher}) \leq \text{top}(\# qAbove))$

Proof: Immediate from lemma 56 plus the fact that $\text{top}(\text{qExpand}) \leq \text{top}(\text{qIn})+\text{maxOutflow} = \text{top}(\text{qAbove})$.

Lemma 58:
$t1 \leq T \Rightarrow$ holds$(T, R \subset \text{re0} \land \text{rccDC}(R, qNearPitcher) \Rightarrow \text{canFlowDown}(R))$

Proof: By lemma 56 the only non flowing space in $\text{RE0}$ is $\text{oPitcher} \cup \text{qExpand} \cup \text{qIn}$. By PS.7, PD.9, the only flow stopping points of $\text{oPitcher}$ are in $\text{qIn}$.

Lemma 59:
$t1 \leq T \Rightarrow$ holds$(T, \text{top}(\| l0 \|) \leq \text{top}(\# qAbove))$

Proof by contradiction: Suppose this is false. Then there is a time $TE$ after $t1$ and a liquid chunk $L1$ which is entirely above $\text{top}(\text{qAbove})$ at $TE$. Using KIN.5, KIND.1, let $L2$ be a subchunk of $L1$ that is continuous Hausdorff from $t1$ to $TE$. For any time $T$ between $t1$ and $TE$, let $f(T)=\text{value}(T,\text{top}(L2)-\text{top}(\text{qAbove}))$. Since $l0 \subset \text{qIn}$ at $t1$, we have $f(t1) < 0$ and $f(TE) > 0$. Since $f$ is continuous, there exists a time $TM$ in $H1$ such that $f(TM) = f(TE)/2$ and such that for all $T$ between $TM$ and $TE$, $f(T) > f(TM)$. Let $L3$ be a subchunk of $L2$ whose bottom is greater
than top(qAbove) in TM. By lemma 57, L3 is disconnected from nonFlowingSpace in TM; hence there is a finite time interval over which L3 can flow down (DOWND.8, DOWND.9); hence L3 does flow down (DOWN.2); however, this contradicts the choice of TM.

Lemma 60:

\[ \forall L, TS, TE \ subchunk(L, l0) \land t1 \leq TS \leq TE \land \text{flowsOut}(L, qSource, TS, TE) \land DA > 0 \Rightarrow \exists TM, L2 \ TS \leq TM \leq TE \land \text{subchunk}(L2, L) \land \text{holds}(TM, \{L2 \subset \# \text{expand}(bSpout, DA)\}) \land \forall T \ TM \leq T \leq TE \Rightarrow \text{holds}(T, rccDC(\{L2, qSource\})). \]

Proof: By KIN.5, KIND.1 there exists a subchunk L2 of L1 such that throughout(TS, TE, thicklyConnected(L2)), continuousHausdorff(L2, H), and throughout(TS, TE, diameter(L2) \leq DA).

Let TM be the greatest upper bound of all times when rccC(L2, qSource); that is, rccC(L2, qSource) at times prior to and arbitrarily close to T1 and rccDC(L2, qSource) from TM to TE.

By lemma 39 continuousVolume(qSource, TS, TE) and continuousHausdorff(qSource, TS, TE). By corollary 31, volumeOf(L2 \cap qSource) and distance(L2, qSource) are continuous functions of time. Since volumeOf(L2 \cap qSource)=0 arbitrarily soon after TM and dist(L2, qSource)=0 arbitrarily soon before TM, it follows that in TM, volumeOf(L2 \cap qSource)=0 and dist(L2 \cap qSource)=0; hence L2 is externally connected to qSource at TM. But the boundary of qSource consists of oPitcher, a top surface at height top(qAbove), and the current value of QOPEN, which is a surface inside bSpout. Since L2 cannot overlap with oPitcher or with the region above top(qExpand) (Lemma 59), it must meet qSource in QOPEN. Since QOPEN \subset bSpout and diameter(L2) < DA it follows that in TM, L2 \subset \text{expand}(bSpout, DA).

Definition 16:

d0Place: fluent[region].

d0Place = qSource \cup re0 \cup \# \text{bInsidePail}.

Lemma 61:

\[ t1 \leq T \land \text{holds}(T, \{l0 \subset \# d0Place\}) \Rightarrow \exists D, D > 0 \land \forall P \text{ holds}(T, P \in \# \{\text{nonFlowingSpace} \cup \# \text{flowDisruptedSpace}\} \land P \notin \# \text{oPitcher} \cup \# qSource) \Rightarrow \text{horizExpand}(P, D) \subset re0. \]

Proof: Let D=dist(boundary(pouringRegion), expand(bSpout, maxOutflow)). By PS.18 D > 0. Assume that P satisfies the conditions of the implication in S. There are two cases: Either P \in nonFlowingSpace or P \in flowDisruptedSpace.

By lemma 56, if P \in nonFlowingSpace \cap re0, then P is in oPitcher \cup qNearPitcher. By assumption P is not in oPitcher \cup qSource. By lemma 55, P is in expand(bSpout, maxOutflow). The result then follows from PS.18.

By DOWND.5, if P \in flowDisruptedSpace \cap re0, then P is in a thickly connected region R filled with liquid inside upExpand(P1, maxOutflow, solidFreeSpace) for some weak top point P1 of nonFlowingSpace. By PS.21, the liquid filling R is part of l0, so by lemma 59, R does not go higher than top(qAbove).

Since P1 is in l0, by the assumption l0 \subset d0Place, P1 is either in qSource, in re0, or in bInsidePail. By lemma 56, if P1 is in re0, then P1 is either in oPitcher, in qAbove, or in qExpand. There are
thus five possibilities, which we consider in turn.

1. \( P_1 \in qSource \). By the identical argument as in lemma 55, \( P \) is in \( qSource \cup \text{expand(bSpout,maxOutflow)} \) (because \( R \) can’t go out through the other boundaries of \( qSource \)). Since \( P \) is not in \( qSource \) by assumption, \( P \) in \( \text{expand(bSpout,maxOutflow)} \), so \( \text{expand}(P,\text{maxOutflow}) \subseteq \text{re0} \) by PS.20.

2. \( P_1 \in qPitcher \). By PS.21, PD.13, either \( P_1 \) is in \( bSpout \), \( \text{horizExpand}(P_1,\text{maxOutflow}) \) is in \( \text{re0} \), or \( P_1 \) is in \( qIn \). If \( P_1 \) is in \( bSpout \) then \( \text{horizExpand}(P_1,\text{maxOutflow}) \) is in \( \text{re0} \) by PS.16, PS.18. If \( P_1 \) is in \( qIn \), then \( P_1 \in qSource \), which is case 1.

3. \( P_1 \in qExpand \). By definition of \( qSource \), \( P_1 \) is either in \( qSource \), covered in case 1, or in \( bSpout \), covered in PS.18.

4. \( P_1 \in qAbove \). By Lemma 55, \( P_1 \) is either in \( qExpand \), covered in case 3, or in \( \text{expand(bSpout,maxOutflow)} \), covered in PS.18.

5. \( P_1 \) is in \( bInsidePail \). Impossible by lemma 51.

Lemma 62:
\[
\forall T E \ t_1 < T E \land \text{throughout}_{E}(t_1,TE,\text{q0InPlace}) \Rightarrow \text{hold}(TE,\text{q0InPlace}).
\]

Proof: Let \( QV = \text{volumeOf}(\text{l0} \cap (qSource \cup \text{re0} \cup \text{bInsidePail})) \). Since \( \text{l0} \) (KIN.4), \( qSource \) (lemma 39), \( \text{re0} \), and \( \text{bInsidePail} \) are all \text{continuousVolume}, \( QV \) is a continuous function of time (lemmas 28 and 30). Since \( QV \) is equal to \( \text{liqVolume(l0)} \) from \( t_1 \) up until \( T E \), it is still equal to \( \text{liqVolume(l0)} \) at \( T E \).

Lemma 63:
\[
t_1 \leq T S \land DX > 0 \Rightarrow \\
\exists T E \ \forall L.\text{liquidChunk}_{T A,T B} \ T S \leq T A \leq T E \land T S \leq T B \leq T E \land \\
\text{holds}(T A, \uparrow L \subseteq \# \text{re0} \land \# \text{flowUndisruptedSpace}) \Rightarrow \\
\text{hausdorff}(\text{value}(T A, \uparrow L),\text{value}(T B, \uparrow L)) < DX.
\]

Proof: Immediate from DOWN.4. Choose \( D \) of DOWN.4 to be \( D1/2 \) here, and observe that \( \text{hausdorff}(\text{value}(SA, \uparrow L)) \),\( \text{value}(SB, \uparrow L)) \leq \text{hausdorff}(\text{value}(SA, \uparrow L))\),\( \text{value}(\text{start}(H), \uparrow L)) + \text{hausdorff}(\text{value}(SB, \uparrow L))\),\( \text{value}(\text{start}(H), \uparrow L)) \) since the Hausdorff distance is a metric.

Lemma 64:
\[
t_1 \leq T S \land \text{holds}(T S,\text{q0InPlace}) \Rightarrow \\
\exists T Q \ T S < T Q \land \text{throughout}(T S,T Q,\text{q0InPlace}).
\]

Proof: Let \( D1 \) satisfy lemma 61. Let \( DX = \text{min}(D1,\text{maxOutflow}, \text{bottom(re0)}-(\text{zp1}+\text{maxOutflow})/3. \). (By lemma 51, \( \text{zp1} \) is an upper bound on the height of cupped liquid in the pail.) Let \( T E \) satisfy lemma 63 for \( DX \) and \( T S \).

We begin with three general observations:

Observation 1: Let \( T 2 \) be between \( T S \) and \( T E \). Suppose that \( L \) is in \( \text{re0} \) in \( T 2 \) and at least \( 3 \cdot DX \) from \( \text{boundary(re0)} \) and that \( L \) is disjoint from \( qSource \) throughout \([T 2, T E]\). Then \( L \) is inside \( \text{re0} \cup \text{bInsidePail} \) throughout \([T 2, T E]\). Proof: Suppose that \( L \) goes outside \( \text{re0} \cup \text{bInsidePail} \) some time between \( T 2 \) and \( T E \). By KIND.1, KIN.5 there is a subchunk \( L 2 \) of \( L \) such that \( L 2 \) is continuous Hausdorff from \( T 2 \) to \( T E \), and the diameter of \( L 2 \) is less than \( DX \). Let \( T 3 \) be the first time after \( T 2 \) where \( L 2 \) meets the complement of \( \text{re0} \). By continuity, \( L 2 \) is in \( \text{re0} \) in \( S3 \). Since \( L 2 \) is disjoint
from qSource in T3, and has diameter less than DX, by lemma 61 and construction of DX, L2 is in flowUndisruptedSpace in T3. But then the Hausdorff distance between the position of L2 in T3 and its position in T2 is at least DX, contradicting the definition of TE.

Observation 2: If L is a subset of qSource at some time T2 between TS and TE, then L is in qSource ∪ re0 ∪ bInsidePail throughout [TS, TE]. Proof by contradiction: Suppose that L1 is a subchunk of L that is outside qSource ∪ re0 ∪ bInsidePail at time T2. By lemma 60 there exists a subchunk L2 of L1 and a time TM between TS and T2 such that L2 is in expand(bSpout,maxOutflow) at TM and L2 is disjoint from qSource between TM and T2. But then by Observation 1, L2 remains in re0 ∪ bInsidePail throughout [TM, T2], which is a contradiction.

Observation 3: Suppose that L is in re0 in TS and at least 3·DX from boundary(re0). Then L is inside re0 ∪ bInsidePail from TS to TE. Proof by contradiction: Suppose that subchunk L1 of L is outside re0 ∪ bInsidePail between TS and TE. There are two cases:

- Case 3.A: Some of L1 goes inside qSource between TS and TE. Let L2 be a subchunk of L2 that is inside qSource at some time between TS and TE. Then L2 violates observation 2.
- Case 3.B: None of L1 goes inside qSource during between TS and TE. Then L1 violates observation 1.

We now divide l0 into the following parts by location at TS (these are exhaustive but not mutually exclusive).

LA is the part of l0 in bInsidePail.
LB is the part of l0 in flowUndisruptedSpace ∩ re0 below top(qIn).
LC is the part of l0 in qSource.
LD is the part of l0 in expand(bSpout,maxFlow).
LE is the part of l0 in (flowDisruptedSpace ∩ re0) − qSource.
LF is the part of l0 in flowUndisruptedSpace ∩ re0 between top(qIn) and top(qAbove).

Using lemmas 55 and 56 and the assumption that holds(TS,l0 ⊂ l0Place) it is immediate LA ∪ LB ∪ LC ∪ LD ∪ LE ∪ LF = l0.

We consider these 6 subchunks of l0 one by one:

- LA is the part of l0 in bInsidePail. By CUP.1, LA remains in bInsidePail.
- LB is the part of l0 in flowUndisruptedSpace ∩ re0 between top(bInsidePail) and top(qIn). Since LB is in flowUndisruptedSpace, the conditions of DOWN.3 are satisfied; hence there exists a TB and LX satisfying the conclusions of DOWN.3. Since LX is thickly connected and is below top(qIn) it does not overlap with qSource; to reach qSource it would have to go through the box OB. By PS.22 and PS.23, l0Place does not come into contact with any liquid other than l0; hence LX is a subchunk of l0 and is inside l0Place; thus it is in re0. By DOWN.3 LX flows straight down during [TS, TB] Since LX is in re0 in S, the region directly below LX is in re0 ∪ bInsidePail. Thus, LX is re0 ∪ bInsidePail throughout [TS, TB].
- LC is the part of l0 in qSource. By observation 2, LC remains in qSource ∪ re0 ∪ bInsidePail throughout [TS, TE].
- LD is the part of l0 in expand(bSpout,maxFlow). By PS.20 this satisfies observation 3.
- LE is the part of l0 in (flowDisruptedSpace ∩ re0)−qSource. By lemma 61, this satisfies observation 3.
\begin{itemize}
  \item $LF$ is the part of $l0$ in $flowUndisruptedSpace \cap re0$ between top($qIn$) and top($qAbove$). Since $LF$ is in $flowUndisruptedSpace$, the conditions of DOWN.3 are satisfied; hence there exists an $TF$ and $LX$ satisfying the conclusions of DOWN.3. By PS.22 and PS.23, $l0$Place does not come into contact with any liquid other than $l0$; hence $LX$ is a subchunk of $l0$ and is inside $l0$Place; thus it is in $re0 \cup qSource$. By DOWN.3 $LX$ flows straight down during $[TS, TF]$. Let $TF1 = \min(TF, TE)$.

  Suppose that there is a thickly connected subchunk $L2$ of $LX$ that is outside $q0$Place at time $T2$ between $TS$ and $TF1$. By DOWND.12 there exists a continuous fluent $Q2$ of constant $xy$ projection that coincides with $L2$ at $T2$ and throughout $[TS, T2]$ is thickly connected and inside $L$. Since $L2$ is outside $re0$ and $Q2$ moves vertically, $Q2$ is outside $re0$ throughout $[TS, T2]$. Therefore in $S$, $Q2$ is in $qSource$. Using KIND.2-4, let $Q3$ be any subchunk of $Q2$ and let $Q4$ be a subchunk of $Q3$ of diameter less than $maxOutflow$. Since $Q4$ is inside $qSource$ at the start and outside $qSource$ at the end, by continuity it must be partially inside $qSource$ in the middle. Since $Q4$ is thickly connected, it must cross the boundary of $qSource$. It can’t go above the top of $qSource$, because $l0$ does not go above $qSource$. It can’t go through $oPitcher$. Therefore it must go through $bSpout$ (not impossible, if $bSpout$ moves horizontally, while $Q4$ moves downward). But in that case $Q4$ must be in $re0$ while it crosses $bSpout$; but this is a contradiction.

  Therefore, if we choose $T1 = \min(TB, TF1)$, the lemma is satisfied. \[\]

**Corollary 65:** foreverAfter($t0$, $l0 \subset ^\# q0$Place)

**Proof:** Immediate from corollary 48 and lemmas 62, 64, and 1.

**Theorem 1:**

\[\exists L1, L2: liquidChunk
\]

\[\text{eventuallyForever} (\uparrow l0 = \# \uparrow L1 \cup \# \uparrow L2 \wedge \# \text{liqInContainer}(L1, oPitcher) \wedge \# \text{liqInContainer}(L2, oPail)).\]

**Proof:** By corollary 65, all of $l0$ is in $q0$Place throughout $j0$. By lemma O, some of $l0$ is always inside $oPitcher$. By PS.12, PS.13 the capacity of $oPitcher$ after $t2$ is less than the volume of $l0$; hence, not all $l0$ can be in $oPitcher$. By DOWN.5, $re0$ must eventually be empty sometime after $t2$. Hence, the part of $l0$ not in $oPitcher$ must be in $oPail$.\]