Pouring Liquids:
A Study in Commonsense Physical Reasoning:
Appendix: Verification of Pouring Scenario

Ernest Davis*
Dept. of Computer Science
New York University
davise@cs.nyu.edu

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This appendix shows that the scenario specified in axioms PD.1-14 and PS.1–21 ends in a final state where some of the liquid l0 is in the pitcher and the remainder is in the pail.

The formal statements of the lemmas in this appendix, like the axioms in the main text of the paper, are (intended to be) written in a style that could be given directly to an automated theorem checker, after some straightfoward syntactic desugaring. In the text of the proofs here, by contrast, I have followed the (God knows, rigid enough) comparatively informal style of normal mathematical writing. In particular:

- I will use partial functions, being careful only to use them only where defined.
- I assume standard results from Euclidean geometry and real analysis.
- I will use natural English for establishing the context of relations between fluents; I will often omit the hatch superscript for converting a function on atemporal entities to one on fluents; and I will often omit the place function on objects and liquids. For instance I will write "At time TA, $L \subset Q$," rather than "holds($TA, \uparrow L \subset^{\#} Q$)".
- If RA and RB are regions then I will write $RA - RB$ and $RA \cap RB$ for the regularized difference and intersection, and likewise for region-valued fluents. Also, I will allow these expressions even in cases where they evaluate to the null set, and sometimes omit consideration of the null set case where it is trivial.
- It will be convenient to apply Boolean operators to liquid chunks in the obvious way.
- Where M is a rigid mapping and G is a geometric entity, I have written the second-order formulation $M(G)$ rather than the first-order expression "mappingImage(M, G)".
- From the point of view of automated theorem proving, a large part of the proof of these lemmas is definition hunting and elementary temporal and spatial argumentation. I have omitted most of this, not even citing the definitions or axioms involved, except where this is non-trivial or interesting.

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I have divided the lemmas into general results, which are independent of the particular problem specification, and problem specific results. Obviously, a result of either form can be rephrased into the other. The idea is that, for the most part, general results are of some general applicability, whereas problem specific results require so many of the particular problem specifications as to make the corresponding general theorem ludicrously long.

General Results

Lemma 1:

$$\begin{array}{l} \forall_{TS,TE,Q} \text{ holds}(TS,Q) \wedge \\ \qquad [\forall_{T2}\ TS < T2 \wedge \text{throughoutxE}(TS,T2,Q) \Rightarrow \text{holds}(T2,Q)] \wedge \\ \qquad [\forall_{T} \text{ holds}(T,Q) \wedge T < TE \Rightarrow \\ \qquad\quad \exists_{T3}\ T < T3 \wedge \text{throughout}(T,T3,Q)] \\ \qquad \Rightarrow \\ \qquad \text{throughout}(TS,TE,Q). \end{array}$$

Proof by contradiction. Suppose not. Let T be the greatest lower bound of all times between after TS when Q fails. If $T = TS$ then Q holds at T by the first condition of the lemma; if $T > TS$ then Q holds at T by the second condition of the lemma. In either case, by the third assumption of the lemma, Q continues to be true until some time $T3$ after T, contradicting the choice of T.

Lemma 2: (Conditionalized comprehension for fluents):
$[\forall_{T:\text{time}}\ \Psi(T) \Rightarrow \exists_X\ \Theta(X,T)] \Rightarrow \exists_Q\ \forall_{T:\text{time}}\ \Psi(T) \Rightarrow \Theta(\text{value}(T,Q))$.

Proof: In axiom schema T.1, define $\Phi(X,T) \equiv \Psi(T) \Rightarrow \Theta(X,T)$ (i.e. the value of X is arbitrary at times T where Ψ is not satisfied).

Lemma 3:
region$(R) \wedge H >$bottom$(R) \Rightarrow \exists_{RB}$ regionBelow(R,H,RB).

Proof: Let P be a point in R for which height$(P) < H$. Since R is regular, P is in the closure of the interior of R; hence the interior of R includes points below H. Therefore, we can define RB to be the closure of the part of the interior of R below H.

Definition 1:
disconnOpenBox$(RB,RI$: bregion)

disconnOpenBox$(RB,RI) \equiv$
$\forall_P\ P \in$boundary$(RI) \Rightarrow$ height$(P) =$ top$(RI) \vee P \in$boundary(RB)

The definition of the relation disconnOpenBox(RB,RI), meaning "RB is an open box with disconnected interior RI," is the same as "openBox(RB,RI)" except that it does not require that RI be connected. Thus, RI can consist of a number of disjoint regions boxed by RB, though all of these regions must have their tops at the same height.

Lemma 4:
disconnOpenBox$(RB,RI) \wedge$ topSurface$(PST,RI) \wedge RB2$=mappingImage$(M,RB) \wedge$
RIM=mappingImage$(M,RI) \wedge PTM$=mappingImage$(M,PST) \wedge$
regionBelow$(RIM$,bottom(PSM),$RI2) \Rightarrow$
disconnOpenBox$(RB2,RI2)$.

Proof: To establish that disconnOpenBox$(RB2,RI2)$, by definition 1 we must show that for any point $P \in$boundary$(RI2)$, either height(P)=top$(RI2)$ or $P \in$boundary$(RB2)$. Let P be any point in boundary$(RI2)$, and let $P1 = M^{-1}(P)$. By PD.6, height$(P) \leq$ bottom(PTM). Let $PT2$ be the top surface of $RI2$. By construction either P is in $PT2$ or P is in boundary(RIM). If $P \in PT2$ then height(P)=top($RI2$). Suppose that $P \in$boundary(RIM). Then $P1 \in$boundary(RI). Since openBox(RB, RI) either $P1 \in PST$ or $P1 \in$boundary(RB). If $P1 \in PST$ then $P \in PTM$, so height(P) $\geq$ bottom(PTM) = top(RI). If $P1 \in$boundary(RB) then $P \in$boundary($RB2$). ∎

Corollary 5:
object(O) $\wedge$ O=source(BI)=source(BT) $\wedge$
holds($T1$, openBox$^\#$($\uparrow O, \uparrow BI$) $\wedge$ topSurface$^\#$($\uparrow BT, \uparrow BI$)) $\wedge$
holds($T2$,isolated($\uparrow BI, \{O\}, L$)) $\wedge$ holds($T2$, regionBelow$^\#$($\uparrow BI$, bottom$^\#$($\uparrow BT$), RB)) $\Rightarrow$
holds($T2$, disconnOpenBox$^\#$(O, RB)).

Proof: By PD.10 no objects enter into the interior of RI. The result is then immediate from lemma 4.

Lemma 6:
openBox($RB, R1$) $\wedge$ openBox($RB, R2$) $\wedge$ rccO($R1, R2$) $\wedge$
top($R1$) $\leq$ top($R2$) $\Rightarrow$
$R1 \subset R2$.

Proof by contradiction: Suppose that the left hand side holds, but that $P1$ is a point in $R1 - R2$. Since rccO($R1, R2$), let $P2$ be a point in interior($R1 \cap R2$). Since $R1$ is regular and thickly connected, there exists an open connected pointset $PSO \subset$interior($R1$) such that $P2 \in PSO$, $P1 \in$closure(PSO). Since $P1 \notin R2$, PSO must intersect boundary($R2$). Since all of PSO is below top($R1$), this intersection must be below top($R1$) and thus below top($R2$). But since openBox($RB, R2$), any boundary point of $R2$ below top($R2$) is a boundary point of RB; but there cannot be any boundary point of RB in the interior of $R1$.

Corollary 7:
openBox($RB, R1$) $\wedge$ openBox($RB, R2$) $\wedge$ rccO($R1, R2$) $\Rightarrow$
$R1 \subset R2 \vee R2 \subset R1$.

Proof: By lemma 6, if top($R1$) $\leq$ top($R2$) then $R1 \subset R2$, and vice versa.

Lemma 8:
openBox(RB, RI) $\Rightarrow \exists^1_{RM}\ RI \subset RM$ $\wedge$ maxBox(RB, RI).

Proof: Let RM be the union of all regions $R1$ such that $R \subset R1$ and openBox(RB, RM). It is immediate from lemma 6 that RM satisfies the stated condition, and that no other region can satisfy the condition.

Definition 2:
maxCuppedRegion(R:region) $\rightarrow$ fluent[Bool]

maxCuppedRegion(R) = maxBox$^\#$(solidSpace,R).

Corollary 9:
holds(T,cuppedRegion(R)) $\Rightarrow$
$\exists^1_{RM}\ R \subset RM$ $\wedge$ holds(T,maxCuppedRegion(RM)).

Proof: Immediate from lemma 8 and the definitions.

Lemma 10:
object(OB) $\wedge$ holds(T,isolated(RI,$\{OB\}$, L)) $\Rightarrow$
[holds(T,openBox$^\#$($\uparrow OB, RI$) $\Leftrightarrow^\#$ cuppedRegion(RI))].

Proof: By the isolation condition, no object other than OB borders RI (PD.10). The result is then immediate from the definitions of cuppedRegion (CUPD.1) and solidSpace (ONTD.4).

Lemma 11:
object(OB) $\wedge$ holds(T,isolated($RI, \{OB\}, L$) $\wedge^{\#}$ maxBox$^{\#}$($\uparrow OB, R$)) $\Rightarrow$
holds(T,localMaxCup(R))

Proof: Using PD.10, let $D > 0$ be the minimal distance from RI to any object other than OB. By lemma 10, R is a cupped region in S. By the definition of maxBox, no superset of R is a cupped region cupped only by OB, and if $R1 \supset R$ is a cupped region involving some object in addition to OB, then $R1$ must include points at least D from R. Hence the conditions of SPILLD.1, SPILLD.2 are met.

Lemma 12:
holds(T,simpleBox(OB)) $\wedge$ holds(T,isolated($RI, \{OB\}, L$) $\wedge$ holds(T,localMaxCup(RI)) $\Rightarrow$
$\forall_R$ holds(T,maxBox$^{\#}$($\uparrow OB, R$)) $\Leftrightarrow R = RI$.

Proof: Since RI is a cupped region (SPILLD.2,SPILLD.1,CUPD.1), and is isolated from every object except OB, it follows that holds(T,openBox(OB, RI)). It is easily shown that holds(T,localMaxBox(OB, RI)). From SPILLD.2, PD.9 it follows that RI is the only region for which holds(T,localMaxBox(OB, RI)). Trivially maxBox(OB, R) implies localMaxBox(OB, R); hence maxBox(OB, R) implies $R = RI$.

Lemma 13:
$\forall_{R:\text{region}, E>0}\ \exists_{D>0}$ volumeOf(expand(boundary(R),D) $\cap R$) $< E$.

Proof: By definition of volume, there exist $\mu > 0$ and a grid decomposition of space into cubical voxels of side μ such that the total volume of the voxels that lie entirely inside R is at least volumeOf(R)$-E/2$. Thus, the number of grid voxels that lie entirely inside R is at least $N =$(volumeOf(R)$-E/2)/\mu^3$.

Now, choose $D < E/12N\mu^2$. For each grid voxel, the volume of the interior part of the voxel that lies at least D from the boundary of the voxel is $(\mu - 2D)^3 > \mu^3 - 6D\mu^2$; hence the union of these interior parts is at least $N(\mu^3 - 6D\mu^2) =$ volumeOf(R)$-E$. But all these interior part of interior voxels are at least D from the boundary of R; hence, the part of R that is within D of boundary(R) has volume less than E.

Corollary 14:
$\forall_{R:\text{region}, E>0}\ \exists_{D>0}$ volumeOf(expand(boundary(R),D)$-R$) $< E$.

Proof: Let $R2$=closure(expand($R, 1$)$-R$); thus boundary(R) $\subset$ boundary($R2$). Therefore for any $D < 1$, expand(boundary(R),D)$-R$) $\subset$ expand(boundary($R2$),D) $\cap$ $R2$. The result is then immediate from lemma 13.

Corollary 15:
$\forall_{R:\text{region}, E>0}\ \exists_{D>0}$ volumeOf(expand(boundary(R),D)) $< E$.

Proof: Immediate from lemma 13 and corollary 14.

Lemma 16:
openBox(RB, RI) $\Rightarrow RI \subset$convexHull(RB).

Proof by contradiction. Suppose that P is a point in RI that is not in the convex hull of RB. Then for any horizontal line L through P, one side or the other of L does not meet RB. Since RI is bounded, that ray of L from P must meet the boundary of RI at a point $P2$. Since openBox(RB, RI) and $P2$ is not in boundary(RB) we must have height($P2$)=top(RI), so height(P)=top(RI). Thus, all points P that are not in the convex hull of RB have height exactly equal to top(RI); but since the convex hull of RB is topologically closed, this is impossible.

Definition 3:
boxedPoint(P: point, R:region)
maxShift(M:rigidMapping, R:region) $\rightarrow$ distance.

boxedPoint$(P, R) \equiv$
$\exists_{RI}$ openBox$(R, RI) \wedge P \in RI$.

maxShift(M, R)=$D \equiv$
$[\exists_{P \in R}$ dist$(P$,mappingImage$(M, P))$=$D] \wedge [\forall_{P \in R}$ dist$(P$,mappingImage$(M, P)) \leq D]$

Lemma 17:
maxShift$(M$,convexHull$(RB))$ = maxShift(M, RB).

Proof: It is easily shown that, if P is on a line between PA and PB then dist$(P, M(P)) \leq$ max(dist$(PA, M(PA))$, dist$(PB, M(PB))$). The result is then immediate.

Lemma 18:
maxShift$(M, RB) \leq E \wedge$ openBox$(RB, RI) \wedge$ point$(P) \wedge$ expand$(P, 4E) \subset RI \Rightarrow$
boxedPoint$(P$,mappingImage$(M, RB))$

Proof: Let $R2B$=$M(RB)$, RIX=$M(RI)$. Using lemmas 16 and 17, maxShift$(RI) \leq E$. Let PST be the topSurface of RI and let $PT2$=$M(PST)$. Since expand$(P, 4E) \subset RI$ we have bottom(RI) $\leq$ height$(P)-4E$ and top$(RI) \geq$ height$(P)+4E$. Hence bottom$(PT2) \geq$ bottom$(PST)-E \geq$ height$(P)+3E >$ height$(M(P))$. Since $M(P) \in RIX$, some of RIX is below bottom$(PT2)$. Hence we can define RIY to be the part of RIX below bottom$(PT2)$. By lemma 4, RIY is a disconnected open Box. Define $RI2$ to be the thickly connected component of RIY. Then openBox$(RI2)$ and $P \in RI2$, satisfying the theorem.

Corollary 19:
maxShift$(M, RB) \leq E \wedge \neg$boxedPoint$(P, RB) \wedge$ boxedPoint$(P$,mappingImage$(M, RB)) \Rightarrow$
$\exists_{PA}$ dist$(P, PA) \leq 4E \wedge \neg$boxedPoint$(PA, RB)$

Proof: Just a logical rearrangement of Lemma 18.

Lemma 20:
maxShift$(M, RB) \leq E \wedge \neg$boxedPoint$(P, RB) \wedge$ boxedPoint$(P$,mappingImage$(M, RB)) \wedge$
dist$(P, RB) > 4E \Rightarrow$
$\exists_{RI}$ openBox$(RB, RI) \wedge$ dist$(P, RI) \leq 4E$.

Proof: Assume that M, P, RB, E meet the conditions of the lemma. Let $RB2$=$M(RB)$ and let $RI2$ be such that $P \in RI2$, and openBox$(RB2, RI2)$. By lemma 16, every point in $RI2$ is in the convex hull of RB; it follows easily that maxShift$(M, RI) \leq$ maxShift$(M, RB) = E$. Let $RT2$=topSurface$(RI2)$; thus every point in $RT2$ has height greater than that of P. Let $RT1 = M^{-1}(RT2)$; then bottom$(RT1) \geq$ height$(RT2)-E \geq$ height$(P)-E$.

Clearly dist$(P, RB2) \geq$ dist$(P, RB)-$maxShift$(M, RB) \geq 3E$. Let $P2A$ be the point directly below P at distance $3E$ from P; thus $P2A \in RI2$. Since all the points on the line from P to $P2A$ are less than $3E$ from P, none of these points are in $RB2$; hence $P2A \in RI2$. Let $PA = M^{-1}(P2A)$. Since $PA \in M^{-1}(RI2)$ and height$(PA) \leq$ height$(P)-2E <$ bottom$(RT1)$, it follows that some of $M^{-1}(RI2)$ is lower than bottom$(RT1)$. Let RIX be the part of $M^{-1}(RI2)$ lower than bottom$(RT1)$; by lemma 4, disconnOpenBox(RB, RIX). Let RI be the thickly connected component of $M^{-1}(RI2)$ containing PA; thus openBox$(RB, RI1)$. Finally
dist$(PA, P) \leq$ dist$(PA, PA2)$ + dist$(PA2, P) \leq 4E$, so dist$(P, RI) \leq 4E$.

Definition 4:
allBoxes$(RB, RI$:region)
symDiff$(RA, RB, RC$: region)

allBoxes$(RB, RI) \equiv$
$\forall_P \; P \in RI \Leftrightarrow$ boxedPoint(P, RB)

symDiff(RA, RB, RC) $\equiv$
[[$RA \subset RB \wedge$ regDif(RB, RA, RC)] $\vee$
[[$RB \subset RA \wedge$ regDif(RA, RB, RC)] $\vee$
[regDif(RA, RB, RD) $\wedge$ regDif(RB, RA, RE) $\wedge$ $RC = RD \cup RE$.]

In proofs, we will write $RP \ominus RQ$ for the regularized symmetric difference of RP and RQ. (We can't write this in lemmas because it may be empty.)

Corollary 21:
maxShift(M, RB) $< E \wedge$ allBoxes(RB, RI) $\wedge$ allBoxes(mappingImage(M, RB),RMI) $\wedge$
symDiff(RI, RMI, RD) $\Rightarrow$
$RD \subset$ expand(boundary(RI, $4E$) $\cup$ expand(boundary(RB),$4E$).

Proof: By corollary 19, if P is boxed in RB and not boxed in $M(RB)$ then it is within $4E$ of boundary(RI). By lemma 20, if P is not boxed in RB and boxed in $M(RB)$ then it is either within $4E$ of boundary(RI) or within $4E$ of boundary(RB).

Lemma 22:
simpleBox(RB) $\Rightarrow$ [maxBox(RB, RI) $\Leftrightarrow$ allBoxes(RB, RI)]

Proof: Immediate from the definitions.

Definition 5:
maxShift1($T1, T2$:time, O:object) $\rightarrow$ distance.

maxShift1($T1, T2, O$) =
maxShift(mappingImage(value($T2$,placement(O)), inverse(value($T1$,placement(O))),shape(O)).

Lemma 23:
throughout(TS, TE, simpleBox$^{\#}(\uparrow O) \wedge^{\#}$ maxBox$^{\#}(\uparrow O, Q)$) $\Rightarrow$
continuousVolume(Q, TS, TE)

Proof: Let T be any time between TS and TE. Let $E > 0$. Using corollary 14, choose $D1 > 0$ such that
volumeOf(expand(value(T,boundary$^{\#}\uparrow O \cup$ boundary$^{\#}(Q)$), $D1$) $< E$.
Since O moves continuously, choose D such that, for any time $T1$ between TS and TE and between $T - D$ and $T + D$, maxShift1($T1, T, O$) $< D1/4$. Using corollary 21 and lemma 22,
volumeOf(value($T1, Q$) $\ominus$ value(T, Q)) $\leq$
volumeOf(expand(value(T,boundary$^{\#}(\uparrow O) \cup^{\#}$ boundary$^{\#}(Q)$), $D1$) $< E$.

Lemma 24:
object(O) $\wedge$ throughout(TS, TE,maxBox$^{\#}(\uparrow O, Q)$) $\wedge$ continuousVolume(Q, TS, TE) $\Rightarrow$
continuous(top$^{\#}(Q), TS, TE$).

Proof: There are two cases:

Case 1: Every point P in Q such that height(P)=top(Q) is in the boundary of O. In that case, O is a closed box and Q is always an entire thickly connected component of the complement of O; that is, Q is a pseudo-object of constant shape moving with O. Since the shape of Q is constant and its placement tracks the placement of O and is continuous, top(Q) is continuous.

Case 2: There exists a point P in the top surface of Q such that the ball of radius $D > 0$ does not intersect O. Over a small enough time interval, O does not come inside that ball. A discontinuous change in top(Q) would cause the corresponding slice of that ball, of finite volume, to come in or out of Q, leading to a volume discontinuity of Q. (This is loosely worded, but can easily be made tight.)

∎

We could weaken the condition "maxBox(O, Q)" in the preceding lemma to be just "openBox(O, Q)", but the analysis of case 1 becomes a little trickier, and we do not need it.

Corollary 25:
object(O) $\wedge$
throughout(TS, TE, simpleBox$^{\#}$($\uparrow O$) $\wedge^{\#}$ maxBox$^{\#}$($\uparrow O, Q$)) $\Rightarrow$
continuous(top$^{\#}$(Q),TS, TE)

Proof: Immediate from lemmas 23 and 24.

Lemma 26:
simpleBox(RB) $\wedge$ openBox($RB, RI1$) $\wedge$ openBox($RB, RI2$) $\Rightarrow$
$RI1 \subset RI2 \vee RI2 \subset RI1$.

Proof of the contrapositive. Suppose that neither $RI1$ nor $RI2$ is a subset of the other. Let $P1$ be a point in the interior of $RI1$ and let $D1$ be the distance from $P1$ to the boundary of $RI1$; thus $RI1$ contains a sphere of radius $D1$ centered at $P1$. Define $P2$ and $D2$ correspondingly for $RI2$. By lemma 7, rccDS($RI1, RI2$). Let $RA1$ be the closure of the union of all regions $RK1$ such that $RI1 \subset RK1$, openBox($RB, RK1$) and rccDS($RK1, RI2$), and let $RA2$ be the closure of the union of all regions $RK2$ such that $RI2 \subset RK2$, openBox($RB, RK2$) and rccDS($RK1, RI2$). It is easily verified that openBox($RB, RA1$), openBox($RB, RA2$), and rccDS($RA1, RA2$). Moreover suppose that $RQ1$ is any proper superset of $RA1$ such that openBox(RB, RQ). Then rccO($RQ1, RI2$) by construction of $RA1$. Hence by lemma 7, $RI2 \subset RQ1$ (since RQ is clearly not a subset of $RI2$); so RQ contains P and thus contains a point that is at least $D1$ from any point in $RA1$. Thus $RA1$ is a localMaxBox for RB. Likewise $RB1$ is a localMaxBox for RB. But since RB has two localMaxBoxes, it does not satisfy simpleBox(RB).

Lemma 27:
openBox(RB, RI) $\wedge$ regDif(convexHull(RB),RB, RD), regionBelow(RD, top(RI),RC) $\Rightarrow$
thicklyConnectedComponent(RI, RC).

Proof: By lemma 16, RI is a subset of convexHull(RB). Since RI does not overlap RB, RI is a subset of RD. By assumption RI is thickly connected. Suppose that RO is a thickly connected set such that $RI \subset RO \subset RC$. If RO is a proper superset of RI, then some part of the boundary of RI must lie in the interior of RO; but this is impossible, since the interior of RO is entirely below top(RI) and entirely disjoint from RB. Hence $RO = RI$, so RI is a thickly connected component of RC.

Definition 6:
regInt($R1, R2, R3$: region).

regInt($R1, R2, R3$) $\equiv$
$\forall_{R:\text{region}}\ R \subset R3 \Leftrightarrow R \subset R1 \wedge R \subset R2$.

Lemma 28:
continuousVolume(QP, TS, TE) $\wedge$ continuousVolume(QQ, TS, TE) $\wedge$
throughout(TS, TE, regInt$^{\#}$(QP, QQ, Q)) $\Rightarrow$
continuousVolume(Q, TS, TE).

Proof: Note that $[RA \cap RB] \ominus [RC \cap RD] \subset [RA \ominus RC] \cup [RB \ominus RD]$
and therefore volumeOf($[RA \cap RB] \ominus [RC \cap RD]$) $\leq$ volumeOf($RA \ominus RC$) + volumeOf($RB \ominus RD$).
Let $T1$ and $T2$ be two times between TS and TE. Then taking RA=value($T1, QP$), RB=value($T1, QQ$), RC=value($T2, QP$), RD=value($T2, QQ$) gives
volumeOf(value($T1, Q$) $\ominus$ value($T2, Q$)) $<$
volumeOf(value($T1, QP$) $\ominus$ value($T2, QP$)) + volumeOf(value($T1, QQ$) $\ominus$ value($T2, QQ$)).

Since QP and QQ are volume-continuous, the summands on the right-hand side of the inequality can be made arbitrarily small by requiring that $T1$ and $T2$ lie close enough; hence the term on the left-hand side, which is the definition of Q being volume continuous. ∎

Corollary 29:
continuousVolume(QP, TS, TE) $\wedge$ continuousVolume(QQ, TS, TE) $\wedge$
throughout(TS, TE, regDif$^{\#}$(QP, QQ, QR)) $\Rightarrow$
continuousVolume(QR, TS, TE)

Proof: Immediate from lemma 28 using the fact that the regularized difference of QP and QQ is the regularized intersection of QP with the complement of QQ.

Lemma 30:
continuousVolume(Q, TS, TE) $\Rightarrow$ continuous(volumeOf$^{\#}$(Q)$, TS, TE$)

Proof: Immediate.

Definition 7:
intersectVolume(RA, RB: region) $\rightarrow$ volume

intersectVolume(RA, RB)=V $\Leftrightarrow$
[rccDS(RA, RB) $\wedge$ $V = 0$] $\vee$ [$\exists_{RI}$ regInt(RA, RB, RI) $\wedge$ V=volumeOf(RI)]

Corollary 31:
continuousVolume($Q1, TS, TE$) $\wedge$ continuousVolume($Q2, TS, TE$) $\Rightarrow$
continuous(intersectVolume($Q1, Q2$),TS, TE).

Proof: The proof of Lemma 28 extends immediately to the case where either or both intersections involved are the null set. The result then follows from lemma 30.

Definition 8:
allLiquidIn(R:region, L:liquidChunk) $\rightarrow$ fluent[Bool]
volumeOfLiquidIn(R:region) $\rightarrow$ fluent[volume]

allLiquidIn(R, L) = regInt$^{\#}$(liquidSpace,R, L).

value(T,volumeOfLiquidIn(R)) = V $\Leftrightarrow$
[holds(T,allLiquidIn(R, L)) $\wedge$ liqVolume(L)=V] $\vee$ [emptyLiquid(T, R) $\wedge$ V=0.]

Corollary 32:
continuousVolume(Q, TS, TE) $\Rightarrow$ continuous(volumeOfLiquidIn$^{\#}$(Q)$, TS, TE$)

Proof: Immediate by applying corollary 31 to the intersection of Q with liquidSpace.

Definition 9:
netInflow(L:liquidChunk, Q:fluent[region], TS, TE: time)
netOutflow(L:liquidChunk, Q:fluent[region], TS, TE: time)
netInflowVolume(Q:fluent[region], TS, TE: time) $\rightarrow$ volume.
netOutflowVolume(Q:fluent[region], TS, TE: time) $\rightarrow$ volume.

netInflow(L, Q, TS, TE) $\equiv$
flowsIn(L, Q, TS, TE) $\wedge$ [$\forall_{L1}$ flowsIn($L1, Q, TS, TE$) $\Rightarrow$ subChunk($L1, L$)]

netOutflow(L, Q, TS, TE) $\equiv$
flowsOut(L, Q, TS, TE) $\wedge$ [$\forall_{L1}$ flowsOut($L1, Q, TS, TE$) $\Rightarrow$ subChunk($L1, L$)]

netInflowVolume(Q, TS, TE) = V $\Leftrightarrow$
[netInflow(L, Q, TS, TE) $\wedge$ volumeOf(L)=V] $\vee$
[noInflow(Q, TS, TE) $\wedge$ $V = 0$].

netOutflowVolume(Q, TS, TE) = $V \Leftrightarrow$
[netOutflow(L, Q, TS, TE) $\wedge$ volumeOf(L)=V] $\vee$
[noOutflow(Q, TS, TE) $\wedge$ $V = 0$].

Corollary 33:
Let TS and TE be times and let Q be a region-valued fluent such that continuousVolume(Q, TS, TE). Define the functions of time $f(T)$ = netInflowVolume(Q, TS, T) and $g(T)$ = netOutflowVolume(Q, TS, T). Then f and g are continuous.

Proof: Let $L1, L2$ be such that holds(st(H, T),allLiquidIn($Q, L1$)) and holds(start(H),allLiquidIn($Q, L2$)). By definition netInflow(Q, TS, T)= $L1 - L2$ and netOutflow(Q, TS, T)= $L2 - L1$. (The minus signs here are set difference.) The result is then immediate from Corollary 32. (The case where either or both of $L1, L2$ are empty are trivial extensions.)

Lemma 34:
slowObjectsInContact(Q, TS, TE) $\wedge$ continuousVolume(Q, TS, TE) $\wedge$
allLiquidIn(TS, Q, L) $\wedge$
throughout(TS, TE, cuppedRegion$^{\#}$(Q) $\wedge^{\#}$ liqVolume(L) $=^{\#}$ volume$^{\#}$(Q) $\wedge^{\#}$ noDrivenLiqIn(Q))
$\Rightarrow$
netOutflow(Q, TS, TE) = netInflow(Q, TS, TE).

Proof: by contradiction. Suppose netOutflow(Q, TS, TE) $<$ netInflow(Q, TS, TE), Then by lemma AK
value(TE,volumeOfLiquidIn(Q)) =
value(TS,volumeOfLiquidIn(Q)) + netInflowVolume(Q, TS, TE) $-$ netOutflowVolume(Q, TS, TE)
$>$
value(TS,volumeOfLiquidIn(Q)) = liqVolume(L) = value(TE,volume$^{\#}$(Q)),
which is impossible.

Suppose netOutflowVolume(Q, TS, TE) $>$ netInflowVolume(Q, TS, TE). For T between TS and TE let $h(T)$ = netOutflowVolume(Q, TS, T) $-$ netInflowVolume(Q, TS, T).
Let $\epsilon = h(TE) > 0$. By corollary 29, h is continuous; hence there exists a $T1$ such that $h(T1) = \epsilon/2$, and for all T between $T1$ and TE $h(T) > \epsilon/2$. Then for all such T,
value(T,volumeOfLiquidIn(Q)) = value(TS,volumeOfLiquidIn(Q))+ $h(T) <$
value(T,volumeOf(Q)).
Thus, if we bind TS of CUP.2 to T here and TE of CUP.2 to TE here, then the conditions of CUP.2 are satisfied. The conclusion of CUP.2 asserts that there is no outflow from T to TE which contradicts the fact the the volume of liquid in Q decreases from T to TE. ∎.

Corollary 35:
slowObjectsInContact(Q, TS, TE) $\wedge$ continuousVolume(Q, TS, TE) $\wedge$
allLiquidIn(TS, Q, L) $\wedge$
throughout(TS, TE, cuppedRegion$^{\#}$(Q) $\wedge^{\#}$ liqVolume(L) $\leq$ volume$^{\#}$(Q) $\wedge^{\#}$ noDrivenLiqIn(Q))
$\Rightarrow$
netOutflowVolume(Q, TS, TE) $\leq$ netInflowVolume(Q, TS, TE).

Proof: Immediate from CUP.2 and Lemma 34.

Lemma 36:
slowObjectsInContact(Q, TS, TE) $\wedge$ continuousVolume(Q, TS, TE) $\wedge$
$VMIN \leq$ value(TS,volumeOfLiquidIn(Q)) $\wedge$
throughout(TS, TE, cuppedRegion$^{\#}$(Q) $\wedge^{\#}$ $VMIN \leq^{\#}$ volume$^{\#}$(Q) $\wedge^{\#}$ noDrivenLiqIn(Q))
$\Rightarrow$
$VMIN \leq$ value(TE,volumeOfLiquidIn(Q)).

Proof: Similar to the proof of lemma 34.

Suppose that value(TE,volumeOfLiquidIn(Q)) $< VMIN$.
Let $\epsilon = VMIN -$value(TE,volumeOfLiquidIn(Q)).
By continuity there exists a time $T1$ such that
$VMIN -$value($T1$,volumeOfLiquidIn(Q)) $= \epsilon/2$ and such that
$VMIN -$value($T1$,volumeOfLiquidIn(Q))$> \epsilon/2$ for all T between $T1$ and TE. Axiom CUP.2 then applies over the subhistory between $T1$ and TE, so there is no outflow from Q in that period; but that is inconsistent with the fact that the volume of liquid in Q decreases from $T1$ to TE.

Corollary 37:
slowObjectsInContact(Q, TS, TE) $\wedge$ continuousVolume(Q, TS, TE) $\wedge$
simpleOverflows(L, Q, TS, TE) $\Rightarrow$
throughout(TS, TE,fullOfLiquid(Q)).

Proof: Let T be any time between TS and TE. Let $VMIN$ be the volume of liquid in Q at time T. The result is then immediate from lemma 36 and definition SPILLD.4.

Lemma 38:
throughout(TS, TE,bounded(Q)) $\wedge$ continuousVolume(Q, TS, TE) $\wedge$ continuous(QZ, TS, TE) $\wedge$
throughout(TS, TE,regionBelow$^{\#}$(Q, QZ, QB)) $\Rightarrow$
continuousVolume(QB, TS, TE).

Proof: Let $T1, T2$ be two times between TS and TE. RZ be the vertical column bounded below and above by value($T1, QZ$) and value($T2, QZ$) and whose horizontal cross-section is the union of the xy-projections of value($T1, Q$) and value($T2, Q$). Then it is easily seen that value($T1, QB$) $\ominus$ value($T2, QB$) $\subset$ [value($T1, Q$) $\ominus$ value($T2, Q$)] $\cup$ RZ]. Since Q and QZ are continuous, the volumes of the terms on the right can be made arbitrarily small by requiring $T1$ and $T2$ to be sufficiently close. Thus, the same is true of the volume of value($T1, QB$) $\ominus$ value($T2, QB$), so QB is volume-continuous.

Lemma 39:

source($BSPOUT$)=OB $\wedge$ $DB > 0$ $\wedge$
throughout(TS, TE, spout1$^{\#}$($\uparrow OB, QI1, QI2, \uparrow BSPOUT, QOPEN, DB$) $\wedge^{\#}$ simpleBox$^{\#}$($\uparrow OB$) $\wedge$
regDif($QI1 \cup QI2, \uparrow BSPOUT, QS$)) $\Rightarrow$
continuousHausdorff(QS, TS, TE) $\wedge$ continuousVolume(QS, TS, TE).

Proof: The boundaries of QS are formed by OB, $BSPOUT$, and the top of $QI2$, which is always DB above top($QI1$), and is thus a continuous function of time. Continuity in the Hausdorff metric is immediate. Continuity in the volume metric follows directly from corollary 15 and lemma 38.

Definition 10: flatBottom(R:region).

flatBottom(R) $\equiv$
$\forall_P$ bottomPoint(P, R) $\Rightarrow$ height(P)=bottom(R).

Lemma 40:
thicklyConnected(R) $\wedge$ flatBottom(R) $\Rightarrow$ connected(bottomSurface(R)).

Proof by contradiction. Suppose that the conditions hold but bottomSurface(R) is not connected. Let $P1$ and $P2$ be points in two different connected components of bottomSurface(R). Since R is thickly connected, there is a path PS from $P1$ to $P2$ through R. Let $PS1$ be the projection of PS onto the horizontal plane at height($P1$). Since $PS1$ goes from $P1$ to $P2$ through bottomSurface(R), there must be a point PA in $PS1$ that is not a bottom point of R. Let PB be a point in PS that is directly above PA. There must be a bottom point PC of R directly below PB. Since $PC \neq PA$ and they are on the same vertical line, height(PC) $\neq$ height(PA)=height($P1$), which contradicts the assumption that connected(bottomSurface(R)).

Lemma 41:
flatBottom(R) $\wedge$ openBox(RB, RI) $\wedge$ rccO(RI, R) $\wedge$ rccDS(RB, R) $\Rightarrow$
bottomSurface(R) $\subset RI$

Proof: Let P be a point in interior(RI) $\cup$ interior(R). Since openBox(RB, RI) there is a point PB on boundary RB directly below P. Since flatBottom(R) there is a point PC on bottomSurface(R) directly below P. Since R and RB are disjoint, height(PB) $\leq$ height(PC)

Suppose that there is a point $P1$ $\in$bottomSurface(R) which is not in RI. By lemma 40 there is a path form $P1$ to PC through bottomSurface(R). Since openBox(RB, RI), this path must meet boundary(RB) at a point PD. Since the path is not at top(RI), there is open set in RB above PD; this must overlap interior(R), which contradicts the assumptions.

Problem Specific Results

Lemma 42:
$\forall_T$ T $\geq$t0 $\Rightarrow$
$\exists_{RB}$ holds(T, regionBelow$^{\#}$($\uparrow$bInsidePitcher, bottom$^{\#}$($\uparrow$bTopPitcher),RB)).

Proof: Immediate from PS.6 and lemma 3.

Lemma 43:
$\forall_T$ T $\geq$t0 $\Rightarrow$
$\exists_R$ holds(T,cuppedRegion(R)) $\wedge$ holds($T, R \subset^{\#}$ $\uparrow$bInsidePitcher)

Proof: Let RB be as in Lemma 42. From corollary 5, PS.2, PS.3, PS.20, RB is a disconnected open box. If we choose R to be a thickly connected component of RB then by lemma 10 R is a cupped region.

Lemma 44:
$\exists^1_Q$ everAfter(t0,rccO$^{\#}$(Q, $\uparrow$bInsidePitcher) $\wedge^{\#}$ maxCuppedRegion$^{\#}$(Q)).

Proof: Immediate from lemmas 43 and corollary 9, with axiom T.2.

Definition 11:
Let qIn be the region-valued fluent satisfying lemma 44.

Lemma 45:
$\forall_{TE}$ t0 $< TE \Rightarrow$ continuousVolume(qIn,t0,TE).

Proof: Immediate from lemma 23, PD.3, PD.4, PS.3.

Lemma 46:
everAfter(t0,noDrivenLiqIn(qIn)).

Proof: By SPILLD.5-7, a driven liquid $L1$ can only exist in upExpand of some liquid $L2$ in a localMaxCup that is overflowing. Since qIn is cupped by oPitcher, if $L1$ is in qIn then $L2$ must also be in qIn. Since there are no object other than OB that border any part of qIn (PS.22), the localMaxCup for $L2$ must be OB itself; but this is impossible since qIn contains it and OB is a simpleBox with only one localMaxBox.

Lemma 47:
t0 $< TE \Rightarrow$ throughout(t0,TE, volumeOf$^{\#}$(qIn) $\geq^{\#}$liqVolume(l0)) $\Rightarrow$
throughout(t0,TE,$\uparrow$l0 $\subset$ qIn).

Proof: By PS.20, the only object in contact with qIn is oPitcher.

By PS.5, CUPD.2, slowObjectsInContact(qIn,t0,TE). By lemma 45, continuousVolume(qIn,t0,TE).
By construction, throughout(t0,TE,cuppedRegion(qIn)).
By hypothesis throughout(t0,TE, volumeOf$^{\#}$(qIn) $\geq^{\#}$liqVolume(l0)).
By lemma 46, throughout(t0,TE,noDrivenLiqIn(qIn)).
Hence by corollary 35, netOutflowVolume(qIn,t0,TE) $\leq$ netInflowVolume(qIn,t0,TE).
However, since qIn is isolated from all liquids but l0 (PS.20), there is no inflow into qIn (FLOW.1, FLOW.3) so the next inflow volume is 0; hence the net outflow volume is 0. Since there is no outflow, l0 remains in qIn throughout t0, TE. (FLOW.2, FLOW.4).

Corollary 48:
throughout(t0,t1, $\uparrow$l0 $\subset$ qIn).

Proof: Immediate from lemma 47, PS.8, PD.7.

Lemma 49:
$\exists_L$ subChunk(L,l0) $\wedge$ everAfter(t0, L $\subset$bInsidePitcher)

Proof: By lemma 42, ever after t0 there is a region of bInsidePitcher below bottom(bTopPitcher). By lemma 4, this is a disconnected open box. By lemma 26, it must contain only one thickly connected component; thus it is a connected open box. Let q1 be the fluent whose value at a time is this region. By lemmas 38 and 30, volumeOf(q1) is a continuous function of time. Since volumeOf(q1) is always positive, it attains a positive minimum vMin over the closed time intervals [t0,t2]. Since the pitcher is motionless after t2, q1 and volumeOf(q1) are constant after t2, and thus volumeOf(q1) is at least vMin ever after t2. Thus, the conditions for lemma 36 are met, and there is always at least a volume vMin of liquid inside bInsidePitcher.

Lemma 50:
everAfter(t0, volumeOfLiquidIn(bInsidePail) $<$ volume$^{\#}$(bInsidePail))

Proof: By PS.21, ever after t0 the liquid in the pail is a subchunk of l0. By PS.13
volumeOf(l0) $<$ volumeOf(bInsidePail).

Definition 12: Using PS.13, let zp1 be the height such that the volume of the part of bInsidePail below zp1 is equal to liqVolume(l0). By PS.13, zp1 $<$ top(bInsidePail)$-$maxOutflow. Let re0 be the part of pouringRegion above top(bInsidePail). It is easily seen that flatBottom(re0).

Definition 13:
horizExpand(PS:pointSet, D:distance) $\rightarrow$ pointSet.

P $\in$horizExpand$(PS, D) \Leftrightarrow$
$\exists_{PC \in PS}$ dist$(P, PC) \leq D \wedge$ height(P) = height(PC)

Let rccDC$(R1, R2)$ be the RCC relation "$R1$ is disconnected from $R2$".

Lemma 51:
t0 $<$ T $\wedge$ liquidChunk(L) $\wedge$ holds(T,openBox(oPail,L)) $\Rightarrow$
holds(T, top(L) $<$ zp1) $\wedge$ holds(T, rccDC(L,re0)).

Proof: By PS.14 L $\subset$bInsidePail in T. By lemma 50 L $\subset$l0. The result is immediate from definition 12.

Lemma 52:
$\neg\exists_{T1,L}$ simpleOverflows(L,bInsidePail,t0,$T1$).

Proof: Immediate from lemma 50 and SPILLD.5.

Lemma 53:
t1 $\leq$ T $\wedge$ liquidChunk(L) $\wedge$ holds(T,cuppedRegion$^{\#}$($\uparrow L$) $\wedge^{\#}$ rccO(L,re0)) $\Rightarrow$

holds(T,L $\subset$qIn).

Proof: By lemma 41, there are two cases to consider: either L contains bottomSurface(re0) or solidSpace overlaps re0.

If L contains bottomSurface(re0) then L overlaps with bInsidePail. By corollary 7, either L is a subset of bInsidePail or vice versa. By lemma 51, if L is a subset of bInsidePail, then L does not overlap re0. If bInsidePail is a subset of L then liqVolume(L) > volume(bInsidePail) > liqVolume(l0) so L contains liquid other than l0; but this is impossible by the isolation condition PS.21.

By the construction of pouringRegion PS.19 and the isolation condition PS.21, the only object that can overlap re0 is oPitcher. By definition 11, qIn is the unique maximal cupped region formed by oPitcher. By the isolation condition PS.20, oPitcher does not form any cupped region in combination with any other objects. ▮

Lemma 54:
$\forall_T$ t1$\leq T \Rightarrow$
$\exists_{RI2,ROPEN}$ holds(T, spout1$^{\#}$(oPitcher,qIn,$RI2$,bSpout,$ROPEN$,maxOutflow))

Proof: Immediate from PS.9, PD.5. It is easily shown that the value of the quantified variable $RI1$ in PD.5 is uniquely determined and must be equal to qIn.

Definition 14:

Using lemma 54, let qAbove, qOpen be fluents whose value at every time T after t1 satisfies
holds(T, spout1$^{\#}$($\uparrow$oPitcher,qIn,qAbove,$\uparrow$bSpout,qOpen,maxOutflow))

Let qSource = qIn $\cup^{\#}$ qAbove $-^{\#}$ $\uparrow$bSpout.

Definition 15:

Let qExpand be the fluent equal to the union of all regions R such that drivenReg(R) (at times when some region is driven).

$\forall_T$ [t1$\leq T$ $\wedge$
[$\exists_R$ holds(T,drivenReg(R))] $\Rightarrow$
$\forall_P$ [holds($T, P \in^{\#}$qExpand) $\Leftrightarrow$
[$\exists_R$ $P \in R$ $\wedge$ holds(T,drivenReg(R))]].

qNearPitcher= qIn $\cup^{\#}$ qAbove $\cup^{\#}$ qExpand.

Lemma 55:
t0$\leq T \Rightarrow$ holds(T, qExpand $\subset^{\#}$ qAbove $\cup^{\#}$ expand$^{\#}$(bSpout,maxOutflow)).

Proof: Let P be a point in qExpand. Let R satisfy the conditions of definition 15. By SPILLD.6, R is a subset of some thickly connected component $R1$ of upExpand(topSurface(qIn), maxOutflow, solidFreeSpace). By definitions PD.4, PD.3, the boundaries of qAbove are the top surface of qIn, the boundary of bPitcher, the horizontal plane at height top(qIn)+maxOutflow, and qOpen. $R1$ does not penetrate into bPitcher, because it is disjoint from solidSpace; it does not penetrate into qIn, because it is entirely above top(qIn); and it does not go above top(qAbove) because any point above top(qAbove) is more than maxOutflow from topSurface(qIn).

The boundary of qIn consists of the boundary of oPitcher and the top surface of qIn. As stated, $R1$ does not overlap oPitcher or qIn; therefore, it meets the top surface of qIn from above. But the entire region immediately above topSurface(qIn) is either oPitcher or qAbove.

Suppose that P is outside qAbove. By SPILLD.6, there is a line PL of length at most maxOutflow from P to a point PB in topSurface(qIn) that goes through $R1$. It is easily shown that for $\epsilon > 0$ there exist points $P1$ and $PB1$ within ϵ of $P1$ and $PB1$ respectively such that $PB1$ is in the interior of qAbove and such that the line from $P1$ to $PB1$ stays in the interior of $R1$. Since this line goes from inside to outside qAbove, it crosses the boundary of qAbove; since the crossing point PC is in the interior of $R1$, it must be in qOpen. But then the distance from PC to $P1$ is at most maxOutflow, so the distance from PC to P is at most maxOutflow+ϵ. Since ϵ can be made arbitrarily small, and since qOpen $\subset$ bSpout, dist(P,bSpout) $\leq$ dist(P,qOpen) $\leq$ maxOutflow. ∎

Lemma 56:
t1$\leq T \Rightarrow$
holds(T,(nonFlowingSpace $\cap^{\#}$ re0) $\subset^{\#}$ ($\uparrow$oPitcher $\cup^{\#}$ qNearPitcher))

Proof: By PS.17, PS.21 the only solid object that enters re0 is oPitcher. By construction, the only liquid cupped by oPitcher is in qIn.

By SPILLD.6, SPILLD.7 any driven liquid must be within upExpand of some overflowing cupped liquid. Any driven liquid associated with the overflow of oPitcher is a subset of qExpand. Any weakly cupped liquid bounded by such a driven liquid together with oPitcher is a subset of qAbove.

Since oPitcher is isolated, there cannot be any cupped region involving oPitcher in combination with some other object.

By lemma 52, bInsidePail does not overflow, and by 53, there is never a filled cupped region that overlaps bInsidePail.

Suppose that there is a cupped region RC created by objects other than bPitcher. By the isolation condition PS.23, all such objects are at least maxOutflow from pouringRegion. Let P be a point and let PL be the shortest line from P to re0. If PL is horizontal or moves downward from P to re0, then it must go through one of the solid objects that bounds RC; hence dist(P,re0) $>$ maxOutflow. If PL goes upward, then PL must intersect a bottom point of re0; but these are all in bInsidePail. In either case, there is no way for a driven chunk of liquid that stays within maxOutflow of RC to overlap the inside of re0. The same is trivially true of weakly cupped liquids associated with an overflow of some other object.

Corollary 57:
t1$\leq T \Rightarrow$
holds(T,top$^{\#}$(nonFlowingSpace $\cap^{\#}$ re0 $-^{\#}$ $\uparrow$oPitcher) $\leq^{\#}$ top$^{\#}$(qAbove))

Proof: Immediate from lemma 56 plus the fact that
top(qExpand) $\leq$ top(qIn)+maxOutflow = top(qAbove).

Lemma 58:
t1$\leq T \Rightarrow$
holds(T, R $\subset$re0 $\wedge^{\#}$ rccDC$^{\#}$(R,qNearPitcher) $\Rightarrow^{\#}$ canFlowDown(R)).

Proof: By lemma 56 the only non flowing space in $RE0$ is oPitcher $\cup$ qExpand $\cup$ qIn. By PS.7, PD.9, the only flow stopping points of oPitcher are in qIn.

Lemma 59:
t1$\leq T \Rightarrow$
holds(T,top$^{\#}$($\uparrow$l0) $\leq$ top$^{\#}$(qAbove))

Proof by contradiction: Suppose this is false. Then there is a time TE after t1 and a liquid chunk $L1$ which is entirely above top(qAbove) at TE. Using KIN.5, KIND.1, let $L2$ be a subchunk of $L1$ that is continuous Hausdoff from t1 to TE. For any time T between t1 and TE, let $f(T)$=value(T,top($L2$)$-$top(qAbove)). Since l0 $\subset$ qIn at t1, we have f(t1)$<$ 0 and $f(TE) > 0$). Since f is continuous, there exists a time TM in $H1$ such that $f(TM) = f(TE)/2$ and such that for all T between TM and TE, $f(T) > f(TM)$. Let $L3$ be a subchunk of $L2$ whose bottom is greater than top(qAbove) in TM. By lemma 57, $L3$ is disconnected from nonFlowingSpace in TM; hence there is a finite time interval over which $L3$ can flow down (DOWND.8, DOWND.9); hence $L3$ does flow down (DOWN.2); however, this contradicts the choice of TM. ∎.

Lemma 60:

$$\begin{array}{l} \forall_{L,TS,TE}\ \text{subchunk}(L,\text{l0}) \wedge \text{t1} \leq TS \leq TE \wedge \text{flowsOut}(L,\text{qSource},TS,TE) \wedge DA > 0 \Rightarrow \\ \quad \exists_{TM,L2}\ TS \leq TM \leq TE \wedge \text{subchunk}(L2,L) \wedge \\ \quad\quad \text{holds}(TM, \uparrow L2 \subset^{\#} \text{expand}^{\#}(\uparrow\text{bSpout},DA)) \wedge \\ \quad\quad \forall_T\ TM \leq T \leq TE \Rightarrow \text{holds}(T,\text{rccDC}^{\#}(\uparrow L2,\text{qSource})). \end{array}$$

Proof: By KIN.5, KIND.1 there exists a subchunk $L2$ of $L1$ such that throughout(TS,TE,thicklyConnected($L2$)), continuousHausdorff($L2,H$), and throughout(TS,TE,diameter($L2$) $\leq DA$).

Let TM be the greatest upper bound of all times when rccC($L2$,qSource); that is, rccC($L2$,qSource) at times prior to and arbitrarily close to $T1$ and rccDC($L2$,qSource) from TM to TE.

By lemma 39 continuousVolume(qSource,TS,TE) and continuousHausdorff(qSource,TS,TE). By corollary 31, volumeOf($L2 \cap$qSource) and distance($L2$,qSource) are continuous functions of time. Since volumeOf($L2 \cap$qSource)=0 arbitrarily soon after TM and dist($L2$,qSource)=0 arbitrarily soon before TM, it follows that in TM, volumeOf($L2 \cap$qSource)=0 and dist($L2 \cap$qSource)=0; hence $L2$ is externally connected to qSource at TM. But the boundary of qSource consists of oPitcher, a top surface at height top(qAbove), and the current value of $QOPEN$, which is a surface inside bSpout. Since $L2$ cannnot overlap with oPitcher or with the region above top(qExpand) (Lemma 59), it must meet qSource in $QOPEN$. Since $QOPEN \subset$bSpout and diamater($L2$) $< DA$ it follows that in TM, $L2 \subset$expand(bSpout,DA). ∎

Definition 16:
ql0Place: fluent[region].

ql0Place = qSource $\cup^{\#}$ re0 $\cup^{\#}$ $\uparrow$bInsidePail.

Lemma 61:

$$\begin{array}{l} \text{t1} \leq T \wedge \text{holds}(T,\uparrow\text{l0} \subset^{\#} \text{ql0Place}) \Rightarrow \\ \exists_D\ D > 0 \wedge \\ \quad \forall_P\ \text{holds}(T,P \in^{\#} \uparrow\text{l0} \cap^{\#} (\text{nonFlowingSpace} \cup^{\#} \text{flowDisruptedSpace}) \wedge^{\#} \\ \quad\quad\quad P \not\in^{\#} \uparrow\text{oPitcher} \cup^{\#} \text{qSource}) \Rightarrow \\ \quad\quad \text{horizExpand}(P,D) \subset \text{re0}. \end{array}$$

Proof: Let D=dist(boundary(pouringRegion), expand(bSpout,2·maxOutflow). By PS.18 $D > 0$. Assume that P satisfies the conditions of the implication in S. There are two cases: Either $P \in$nonFlowingSpace or $P \in$flowDisruptedSpace.

By lemma 56, if $P \in$nonFlowingSpace $\cap$ re0, then P is in oPitcher $\cup$ qNearPitcher. By assumption P is not in oPitcher $\cup$ qSource. By lemma 55, P is in expand(bSpout,maxOutflow). The result then follows from PS.18.

By DOWND.5, if $P \in$flowDisruptedSpace $\cap$ re0, then P is in a thickly connected region R filled with liquid inside upExpand($P1$,maxOutflow,solidFreeSpace) for some weak top point $P1$ of nonFlowingSpace. By PS.21, the liquid filling R is part of l0, so by lemma 59, R does not go higher than top(qAbove).

Since $P1$ is in l0, by the assumption l0 $\subset$ ql0Place, $P1$ is either in qSource, in re0, or in bInsidePail. By lemma 56, if $P1$ is in re0, then $P1$ is either in oPitcher, in qAbove, or in qExpand. There are thus five possibilities, which we consider in turn.

1. $P1$ ∈qSource. By the identical argument as in lemma 55, P is in qSource ∪ expand(bSpout,maxOutflow) (because R can't go out through the other boundaries of qSource). Since P is not in qSource by assumption, P in expand(bSpout,maxOutflow), so expand(P,maxOutflow) ⊂ re0 by PS.20.

2. $P1$ ∈oPitcher. By PS.21, PD.13, either $P1$ is in bSpout, horizExpand($P1$,maxOutflow) is in re0, or P is in qIn. If $P1$ is in bSpout then horizExpand($P1$,maxOutflow) is in re0 by PS.16, PS.18. If $P1$ is in qIn, then $P1$ ∈qSource, which is case 1.

3. $P1$ ∈qAbove. By definition of qSource, $P1$ is either in qSource, covered in case 1, or in bSpout, covered in PS.18.

4. $P1$ ∈qExpand. By Lemma 55, $P1$ is either in qAbove, covered in case 3, or in expand(bSpout,maxOutflow), covered in PS.18.

5. $P1$ is in bInsidePail. Impossible by lemma 51.

▮

Lemma 62:
$\forall_{TE}$ t1$<TE \wedge$ throughoutxE(t1,TE,ql0InPlace) ⇒ hold(TE,ql0InPlace).

Proof: Let QV=volumeOf(l0 ∩ (qSource ∪ re0 ∪ bInsidePail). Since l0 (KIN.4), qSource (lemma 39), re0, and bInsidePail are all continuousVolume, QV is a continuous function of time (lemmas 28 and 30). Since QV is equal to liqVolume(l0) from t1 up until TE, it is still equal to liqVolume(l0) at TE. ▮.

Lemma 63:

t1$\leq TS \wedge DX > 0 \Rightarrow$
$\exists_{TE}\ \forall_{L:\text{liquidChunk},TA,TB}\ TS \leq TA \leq TE \wedge TS \leq TB \leq TE \wedge$
holds(TA,↑$L \subset^{\#}$ re0 $\cap^{\#}$flowUndisruptedSpace) ⇒
hausdorff(value(TA, ↑L),value(TB, ↑L)) $< DX$.

Proof: Immediate from DOWN.4. Choose D of DOWN.4 to be $D1/2$ here, and observe that hausdorff(value(SA, ↑L)),value(SB, ↑L)) ≤
hausdorff(value(SA, ↑L)),value(start(H),↑L)) + hausdorff(value(SB, ↑L)),value(start(H),↑L)) since the Hausdorff distance is a metric.

Lemma 64:
t1$\leq TS \wedge$ holds(TS,ql0InPlace) ⇒
$\exists_{TQ}\ TS < TQ \wedge$ throughout(TS, TQ,ql0InPlace).

Proof: Let $D1$ satisfy lemma 61. Let DX=min($D1$,maxOutflow, bottom(re0)−(zp1+maxOutflow)/3. (By lemma 51, zp1 is an upper bound on the height of cupped liquid in the pail.) Let TE satisfy lemma 63 for DX and TS.

We begin with three general observations:

Observation 1: Let $T2$ be between TS and TE . Suppose that L is in re0 in $T2$ and at least $3 \cdot DX$ from boundary(re0) and that L is disjoint from qSource throughout $[T2, TE]$. Then L is inside re0 ∪ bInsidePail throughout $[T2, TE]$. Proof: Suppose that L goes outside re0 ∪ bInsidePail some time between $T2$ and TE. By KIND.1, KIN.5 there is a subchunk $L2$ of L such that $L2$ is continuous Hausdorff from $T2$ to TE, and the diameter of $L2$ is less than DX. Let $T3$ be the first time after $T2$ where $L2$ meets the complement of re0. By continuity, $L2$ is in re0 in $S3$. Since $L2$ is disjoint from qSource in $T3$, and has diameter less than DX, by lemma 61 and construction of DX, $L2$ is in flowUndisruptedSpace in $T3$. But then the Hausdorff distance between the position of $L2$ in $T3$ and its position in $T2$ is at least DX, contradicting the definition of TE.

Observation 2: If L is a subset of qSource at some time $T2$ between TS and TE, then L is in qSource $\cup$ re0 $\cup$ bInsidePail throughout $[TS, TE]$. Proof by contradiction: Suppose that $L1$ is a subchunk of L that is outside qSource $\cup$ re0 $\cup$ bInsidePail at time $T2$. By lemma 60 there exists a subchunk $L2$ of $L1$ and a time TM between TS and $T2$ such that $L2$ is in expand(bSpout,maxOutflow) at TM and $L2$ is disjoint from qSource between TM and $T2$. But then by Observation 1, $L2$ remains in re0 $\cup$ bInsidePail throughout $[TM, T2]$, which is a contradiction.

Observation 3: Suppose that L is in re0 in TS and at least $3 \cdot DX$ from boundary(re0). Then L is inside re0 $\cup$ bInsidePail from TS to TE. Proof by contradiction: Suppose that subchunk $L1$ of L is outside re0 $\cup$ bInsidePail between TS and TE. There are two cases:

- Case 3.A: Some of $L1$ goes inside qSource between TS and TE. Let $L2$ be a subchunk of $L2$ that is inside qSource at some time between TS and TE. Then $L2$ violates observation 2.
- Case 3.B: None of $L1$ goes inside qSource during between TS and TE. Then $L1$ violates obervation 1.

We now divide l0 into the following parts by location at TS (these are exhaustive but not mutually exclusive).

LA is the part of l0 in bInsidePail.
LB is the part of l0 in flowUndisruptedSpace $\cap$ re0 below top(qIn).
LC is the part of l0 in qSource.
LD is the part of l0 in expand(bSpout,maxFlow).
LE is the part of l0 in (flowDisruptedSpace $\cap$ re0)$-$qSource.
LF is the part of l0 in flowUndisruptedSpace $\cap$ re0 between top(qIn) and top(qAbove).

Using lemmas 55 and 56 and the assumption that holds(TS,l0 $\subset$ ql0Place) it is immediate $LA \cup LB \cup LC \cup LD \cup LE \cup LF =$ l0.

We consider these 6 subchunks of l0 one by one:

- LA is the part of l0 in bInsidePail. By CUP.1, LA remains in bInsidePail.
- LB is the part of l0 in flowUndisruptedSpace $\cap$ re0 between top(bInsidePail) and top(qIn). Since LB is in flowUndisruptedSpace, the conditions of DOWN.3 are satisfied; hence there exists a TB and LX satisfying the conclusions of DOWN.3. Since LX is thickly connected and is below top(qIn) it does not overlap with qSource; to reach qSource it would have to go through the box OB. By PS.22 and PS.23, l0Place does not come into contact with any liquid other than l0; hence LX is a subchunk of l0 and is inside l0Place; thus it is in re0. By DOWN.3 LX flows straight down during $[TS, TB]$ Since LX is in re0 in S, the region directly below LX is in re0 $\cup$ bInsidePail. Thus, LX is re0 $\cup$ bInsidePail throughout $[TS, TB]$.
- LC is the part of l0 in qSource. By observation 2, LC remains in qSource $\cup$ re0 $\cup$ bInsidePail throughout $[TS, TE]$.
- LD is the part of l0 in expand(bSpout,maxFlow). By PS.20 this satisfies observation 3.
- LE is the part of l0 in (flowDisruptedSpace $\cap$ re0)$-$qSource. By lemma 61, this satisfies observation 3.

- LF is the part of l0 in flowUndisruptedSpace $\cap$ re0 between top(qIn) and top(qAbove). Since LF is in flowUndisruptedSpace, the conditions of DOWN.3 are satisfied; hence there exists an TF and LX satisfying the conclusions of DOWN.3. By PS.22 and PS.23, l0Place does not come into contact with any liquid other than l0; hence LX is a subchunk of l0 and is inside l0Place; thus it is in re0 $\cup$ qSource. By DOWN.3 LX flows straight down during $[TS, TF]$. Let $TF1 = \min(TF, TE)$.

  Suppose that there is a thickly connected subchunk $L2$ of LX that is outside ql0Place at time $T2$ between TS and $TF1$. By DOWND.12 there exists a continuous fluent $Q2$ of constant xy projection that coincides with $L2$ at $T2$ and throughout $[TS, T2]$ is thickly connected and inside L. Since $L2$ is outside re0 and $Q2$ moves vertically, $Q2$ is outside re0 throughout $[TS, T2]$. Therefore in S, $Q2$ is in qSource. Using KIND.2-4, let $Q3$ be any subchunk of $Q2$ and let $Q4$ be a subchunk of $Q3$ of diameter less than maxOutflow. Since $Q4$ is inside qSource at the start and outside qSource at the end, by continuity it must be partially inside qSource in the middle. Since $Q4$ is thickly connected, it must cross the boundary of qSource. It can't go above the top of qSource, because l0 does not go above qSource. It can't go through oPitcher. Therefore it must go through bSpout (not impossible, if bSpout moves horizontally, while $Q4$ moves downward). But in that case $Q4$ must be in re0 while it crosses bSpout; but this is a contradiction.

Therefore, if we choose $T1 = \min(TB, TF1)$, the lemma is satisfied. ∎

Corollary 65: foreverAfter(t0, $\uparrow$l0 $\subset^{\#}$ql0Place)

Proof: Immediate from corollary 48 and lemmas 62, 64, and 1.

Theorem 1:
$\exists_{L1,L2:}$liquidChunk
eventuallyForever($\uparrow$l0 $=^{\#}$ $\uparrow L1 \cup^{\#} \uparrow L2 \wedge^{\#}$ liqInContainer($L1$,oPitcher) $\wedge^{\#}$ liqInContainer($L2$,oPail)).

Proof: By corollary 65, all of l0 is in ql0Place throughout j0. By lemma O, some of l0 is always inside oPitcher. By PS.12, PS.13 the capacity of oPitcher after t2 is less than the volume of l0; hence, not all l0 can be in oPitcher. By DOWN.5, re0 must eventually be empty sometime after t2. Hence, the part of l0 not in oPitcher must be in oPail.