Elementarily Equivalent Structures for Topological Languages over Regions in Euclidean Space

Ernest Davis*
Dept. of Computer Science
New York University
davise@cs.nyu.edu

June 1, 2012

Abstract

We prove that the class of rational polyhedra and the class of topologically regular regions definable in an o-minimal structure over the reals are each elementarily equivalent to the class of polyhedra for topological languages.

Keywords: Elementary equivalence, first-order equivalence, topological language.

1 Introduction

The study of qualitative spatial reasoning using topological relations over spatial regions has flourished since the seminal papers of Egenhofer and Franzosa [3] and of Randell, Cui, and Cohn [8, 9]. (See [2] for a recent survey.) One frustrating aspect of this research programme, however, is that the important logical characteristics of the theories involved often depend very sensitively on rather fine details of the language or the model. For example Kontchakov et al. [4, 5] study a variety of existential languages, with different dimensionalities of space, domains of regions, and different collections of predicates, and they demonstrate many differences between these in terms of expressivity and of computational complexity. Since there is often no very principled way of deciding which particular language is most reasonable, one ends up with a large number of equally plausible theories, each with its own characteristics.

In this paper we present some results in the opposite direction, discussing some distinctions that do *not* make a difference. We show that a number of collections of spatial regions are *elementarily equivalent* to the space of polyhedra, for topological languages over regions. That is, suppose you have a collection of topological relations over spatial regions, such as "Regions **P** and **Q** are externally connected," or "Region **Z** is the union of **X** and **Y**", and you have a first-order language $\mathcal{L}$ whose predicates refer to these relations. Then any sentence in $\mathcal{L}$ that is true over the domain of polyhedra in Euclidean space is also true over any of the other domains of regions we will discuss here. Specifically, we show that the class of rational polyhedra and the class of topologically regular regions definable in an o-minimal structure over the reals are each elementarily equivalent to the class of polyhedra in $\mathbb{R}^k$.

*Many thanks to Ian Pratt-Hartmann for helpful discussions, and to the reviewers for valuable suggestions. This research was supported in part by NSF grant #IIS-0534809.

It should be emphasized that these results apply to a language of *any* collection of topological relations — i.e. relations that are invariant under homeomorphisms of the entire space to itself — even including, for what it is worth, non-computable relations such as "the number of connected components of region $\mathbf{P}$ is the index of a non-halting Turing machine."

Pratt-Hartmann has proved a strong result of this kind in [7], corollary 2.174, p. 89:

Theorem 1 *All splittable, finitely decomposable mereotopologies over $\mathbb{S}^2$ with curve-selection have the same $\mathcal{L}_\Sigma$-theory for any topological signature Σ.*

In other words, if $\mathcal{C}$ and $\mathcal{D}$ are mereotopologies satisfying the specified conditions and Σ is a collection of topological relations, then the structures $\langle \mathcal{C}, \Sigma \rangle$ and $\langle \mathcal{D}, \Sigma \rangle$ are elementarily equivalent. For the definitions of the terms "mereotopology", "splittable", "finitely decomposable", and "curve selection" see the cited paper. The key points here are (a) that these are topological features of a collection of regions; (b) that the theorem is only proven in two-dimensional space. The results in the current paper apply to Euclidean space of arbitrary finite dimension, but they place conditions on the collection of regions that are much more restrictive and that are formulated in algebraic and structural terms rather than in geometric terms.

Section 2 presents a general meta-logical theorem giving sufficient conditions that two structures are elementarily equivalent. Section 3 presents the proof that the collection of rational polyhedra is elementarily equivalent to the collection of polyhedra, relative to a topological language. Section 4 presents the proof that any o-minimal collection of regions over the reals is elementarily equivalent to the collection of polyhedra.

2 A general meta-logical theorem

In this section, we prove a general meta-logical theorem, giving sufficient conditions that two structures are elementarily equivalent. First, let us briefly discuss structures and elementary equivalence.

Definition 1 *An* L-structure *is a triple $\langle \mathcal{D}, \sigma, \mathcal{I} \rangle$ where $\mathcal{D}$ is a domain; $\sigma = \langle \sigma_1, \ldots, \sigma_m \rangle$ is a signature of m formal symbols; and $\mathcal{I}$ is an interpretation mapping each σ_i to a relation over $\mathcal{D}$.*

Definition 2 *A* structure *is a tuple $\langle \mathcal{D}, \mathcal{P}_1, \ldots, \mathcal{P}_m \rangle$ where each $\mathcal{P}_i$ is a relation over $\mathcal{D}$. The L-structure $\langle \mathcal{D}, \langle \sigma_1, \ldots, \sigma_m \rangle, \mathcal{I} \rangle$ corresponds to the structure $\langle \mathcal{D}, \mathcal{I}(\sigma_1), \ldots, \mathcal{I}(\sigma_m) \rangle$*

Definition 3 *Let $\langle \mathcal{D}, \sigma, \mathcal{I} \rangle$ and $\langle \mathcal{E}, \sigma, \mathcal{J} \rangle$ be two L-structures with the same signature σ. These are* elementarily equivalent *if, for every sentence ϕ in the first-order language over σ, $\mathcal{I} \models \phi$ if and only if $\mathcal{J} \models \phi$.*

Two structures are elementarily equivalent if the corresponding L-structures with the same signature σ are elementarily equivalent.

Throughout the remainder of this section, let $\mathcal{Z}$ be a set. Let Γ be a set of bijections from $\mathcal{Z}$ to itself that forms a group; that is, Γ is closed under composition and inverse.

We will use boldface capitals such as $\mathbf{R}$ to denote elements of $\mathcal{Z}$. We will use the composition operator $\alpha \circ \phi$ in analysts' style rather than algebraicists'; that is, $(\alpha \circ \phi)(\mathbf{R}) = \alpha(\phi(\mathbf{R}))$. If $\mathcal{P}$ is a relation over $\mathcal{Z}$ and $\mathcal{C} \subset \mathcal{Z}$ then $\mathcal{P}|_\mathcal{C}$ will denote the restriction of $\mathcal{P}$ to $\mathcal{C}$.

Definition 4 *A relation $\mathcal{P}(\mathbf{R}_1, \ldots, \mathbf{R}_n)$ over elements of $\mathcal{Z}$ is an* invariant *of Γ if, for all $\phi \in \Gamma$, $\mathcal{P}(\mathbf{R}_1, \ldots, \mathbf{R}_n)$ if and only if $\mathcal{P}(\phi(\mathbf{R}_1), \ldots, \phi(\mathbf{R}_n))$.*

Definition 5 *Let $\mathcal{C}$ and $\mathcal{D}$ be subsets of $\mathcal{Z}$. $\mathcal{C}$ is* finitely embeddable *in $\mathcal{D}$ with respect to Γ if it satisfies the following condition: For any $m \geq 0$, let $\{\mathbf{P}_1, \ldots, \mathbf{P}_m\}$ be a set of m elements in $\mathcal{C}$. Then there exists a bijection $\alpha \in \Gamma$ such that $\alpha(\mathbf{P}_i) \in \mathcal{D}$ for $i = 1, \ldots, m$.*

For readability, we will often omit the reference to Γ in using "finitely embeddable" and similar terms.

Definition 6 *Let $\mathcal{C}$ and $\mathcal{D}$ be subsets of $\mathcal{Z}$. $\mathcal{C}$ is* extensible *in $\mathcal{D}$ with respect to Γ if it satisfies the following condition. For any $m \geq 0$, let $\{\mathbf{P}_1, \ldots, \mathbf{P}_m\}$ be a set of m elements in $\mathcal{C}$, and let $\alpha \in \Gamma$ be a bijection over $\mathcal{Z}$ such that $\alpha(\mathbf{P}_i) \in \mathcal{D}$ for $i = 1, \ldots, m$. Then for any element $\mathbf{P}_{m+1} \in \mathcal{C}$ there exists a bijection $\alpha' \in \Gamma$ such that $\alpha'(\mathbf{P}_i) = \alpha(\mathbf{P}_i)$ for $i = 1, \ldots, m$ and $\alpha(\mathbf{P}_{m+1}) \in \mathcal{D}$.*

For the case $m = 0$, this condition asserts that for any $\mathbf{P} \in \mathcal{C}$ there exists $\alpha \in \Gamma$ such that $\alpha(\mathbf{P}) \in \mathcal{D}$.

Note that we cannot simply choose $\alpha' = \alpha$, because $\alpha(\mathbf{P}_{m+1})$ is not necessarily in $\mathcal{C}$.

The following simple example illustrates the distinction between embeddability and extensibility. Let $\mathcal{Z} = \mathbb{R}$ the set of real numbers. Let Γ be the collection of bijections that are monotonic under ordering; that is, for $\alpha \in \Gamma$ and for $x, y \in \mathbb{R}$, if $x < y$ then $\alpha(x) < \alpha(y)$. Let $\mathcal{C} = \mathbb{R}$, and let $\mathcal{D} = \mathbb{Z}$, the set of integers. Then $\mathcal{C}$ is finitely embeddable in $\mathcal{D}$; given any set of real numbers $\{\mathbf{P}_1, \ldots, \mathbf{P}_m\}$, sort them, and then map them to the corresponding integers in sequence. However $\mathcal{C}$ is not extensible in $\mathcal{D}$; if we have chosen $\mathbf{P}_1 = 0, \mathbf{P}_2 = 2$, and $\alpha(x) = x/2$, then $\alpha(\mathbf{P}_1)$ and $\alpha(\mathbf{P}_2)$ are in $\mathcal{D}$, but there is no way to extend α so that $\alpha(1) \in \mathcal{D}$. This same example illustrates that $\mathbb{Z}$ is not extensible in itself with respect to Γ.

If we use the same $\mathcal{Z}$, Γ and $\mathcal{C}$ but let $\mathcal{D} = \mathbb{Q}$, the set of rational numbers, then $\mathcal{C}$ is extensible within $\mathbb{Q}$. If α is a function mapping the set of real numbers $S = \{\mathbf{P}_1, \ldots, \mathbf{P}_m\}$ to a set of rationals, and we are given the next element $\mathbf{P}_{m+1}$, then we can extend α by finding the $\mathbf{P}_i$ and $\mathbf{P}_j$ which are the elements of S immediately above and below $\mathbf{P}_{m+1}$, and then choose $\alpha'(\mathbf{P}_{m+1})$ to be a rational between $\alpha(\mathbf{P}_i)$ and $\alpha(\mathbf{P}_j)$. (Applying Theorem 6 below, one can go on to show that the structures $\langle \mathbb{R}, < \rangle$ and $\langle \mathbb{Q}, < \rangle$ are elementarily equivalent.)

Lemma 2 *If $\mathcal{C}$ is extensible in $\mathcal{D}$ with respect to Γ, then $\mathcal{C}$ is finitely embeddable in $\mathcal{D}$.*

Proof: The fact that a subset of $\mathcal{C}$ with m elements is embeddable in $\mathcal{D}$ is trivial by induction on m.

Lemma 3 *If $\mathcal{C}$ is finitely embeddable in $\mathcal{D}$ with respect to Γ, and $\mathcal{D}$ is extensible in $\mathcal{E}$, then $\mathcal{C}$ is extensible in $\mathcal{E}$.*

Proof: Let $\mathbf{P}_1, \ldots, \mathbf{P}_m$ be m elements in $\mathcal{C}$; let $\alpha \in \Gamma$ be a bijection over $\mathcal{Z}$ such that $\alpha(\mathbf{P}_i) \in \mathcal{E}$ for $i = 1, \ldots, m$; and let $\mathbf{P}_{m+1} \in \mathcal{C}$. Since $\mathcal{C}$ is finitely embeddable in $\mathcal{D}$, choose $\theta \in \Gamma$ such that $\theta(\mathbf{P}_i) \in \mathcal{D}$ for $i = 1, \ldots, m+1$. Let $\mathbf{Q}_i = \theta(\mathbf{P}_i)$ for $i = 1, \ldots, m+1$. Let $\phi = \alpha \circ \theta^{-1}$. Then, for $i = 1, \ldots, m$, $\phi(\mathbf{Q}_i) = \alpha(\mathbf{P}_i) \in \mathcal{E}$. Since $\mathcal{D}$ is extensible in $\mathcal{E}$, there exists $\phi' \in \Gamma$ such that $\phi'(\mathbf{Q}_i) = \phi(\mathbf{Q}_i)$ for $i = 1, \ldots, m$ and $\phi'(\mathbf{Q}_{m+1}) \in \mathcal{E}$. Now let $\alpha' = \phi' \circ \theta$. Then for $i = 1, \ldots, m$, $\alpha'(\mathbf{P}_i) = \phi'(\mathbf{Q}_i) = \phi(\mathbf{Q}_i) = \alpha(\mathbf{P}_i)$; and $\alpha'(\mathbf{P}_{m+1}) = \phi'(\mathbf{Q}_{m+1}) \in \mathcal{E}$. ∎

Definition 7 *Two sets $\mathcal{C} \subset \mathcal{Z}$ and $\mathcal{D} \subset \mathcal{Z}$ are* mutually extensible *with respect to Γ if each is extensible in the other. A set $\mathcal{C}$ is* self-extensible *if it is extensible in itself.*

Corollary 4 *If $\mathcal{C}$ is finitely embeddable in $\mathcal{D}$ and $\mathcal{D}$ is self-extensible, then $\mathcal{C}$ is extensible in $\mathcal{D}$.*

Proof: Immediate from Lemma 3 with $\mathcal{E} = \mathcal{D}$.

Lemma 5 *Let $\mathcal{C}$ and $\mathcal{D}$ be mutually extensible subsets of $\mathcal{Z}$ with respect to Γ. Let $\mathcal{P}_1, \ldots, \mathcal{P}_m$ be relations over $\mathcal{Z}$ that are invariants of Γ. Let $\sigma = \langle \sigma_1, \ldots, \sigma_m \rangle$ be a signature with m symbols. Let $\mathcal{I}$ and $\mathcal{J}$ be the interpretations of σ such that $\mathcal{I}(\alpha_i) = \mathcal{P}_i|_{\mathcal{C}}$ and $\mathcal{J}(\alpha_i) = \mathcal{P}_i|_{\mathcal{D}}$. Let ϕ be a prenex first-order formula over σ. Let $\mu_1, \ldots, \mu_n$ be the free variables in ϕ. Let $\mathcal{U}$ be a valuation from $\mu_1, \ldots, \mu_n$ to $\mathcal{C}$. Let $\alpha \in \Gamma$ be a bijection such that $\alpha(\mathcal{U}(\mu_i)) \in \mathcal{D}$. Let $\mathcal{V}$ be the valuation from $\mu_1, \ldots, \mu_m$ to $\mathcal{D}$, $\mathcal{V}(\mu_i) = \alpha(\mathcal{U}(\mu_i))$. Then $\mathcal{C}, \mathcal{I}, \mathcal{U} \models \phi$ if and only if $\mathcal{D}, \mathcal{I}, \mathcal{V} \models \phi$.*

Proof by induction on the number of quantifiers in ϕ.

Base case : ϕ is a quantifier-free formula with variables $\mu_1, \ldots, \mu_n$. Let $\mathcal{U}$ be a valuation of $\mu_1, \ldots, \mu_n$ onto $\mathcal{C}$ such that $\mathcal{C}, \mathcal{I}, \mathcal{U} \models \phi$. Since $\mathcal{C}$ is embeddable in $\mathcal{D}$, there is a bijection $\alpha \in \Gamma$ such that $\alpha(\mathcal{U}(\mu_i)) \in \mathcal{D}$ for $i = 1, \ldots, n$. Let $\mathcal{V}(\mu_i) = \alpha(\mathcal{U}(\mu_i))$. Since $\mathcal{P}_i$ is invariant under Γ for $i = 1, \ldots, m$, it follows that each atomic formula occurring in ϕ has the same value under $\mathcal{J}, \mathcal{V}$ as under $\mathcal{I}, \mathcal{U}$. Hence each subformula of ϕ, including ϕ itself has the same value under $\mathcal{J}, \mathcal{V}$ as under $\mathcal{I}, \mathcal{U}$.

The proof of the reverse implication is symmetric.

Recursive case: Suppose that the statement is true for all formulas with no more than q quantifiers. Let ϕ be a formula with $q + 1$ quantifiers; thus ϕ has either the form $\exists \mu \psi$ or $\forall \mu \psi$ where ψ is a formula with m quantifiers. Let $\mu_1, \ldots, \mu_n$ be the free variables in ϕ; then the free variables in ψ are $\mu_1, \ldots, \mu_n, \mu$.

Let $\mathcal{U}$, α and $\mathcal{V}$ be as in the statement of the lemma.

I. Suppose that ϕ has the form $\exists \mu \psi$, and suppose that this is true under $\mathcal{I}, \mathcal{U}$. Then there exists $\mathbf{R} \in \mathcal{C}$ such that ψ is true under $\mathcal{I}, \mathcal{U}'$ where $\mathcal{U}' = \mathcal{U} \cup \{\mu \rightarrow \mathbf{R}\}$. Since $\mathcal{C}$ is extensible in $\mathcal{D}$, there exists a bijection $\alpha' \in \Gamma$ such that $\alpha'(\mathcal{U}(\mu_i)) = \mathcal{V}(\mu_i)$ for $i = 1, \ldots, n$. and such that $\alpha'(\mathbf{R}) \in \mathcal{D}$. Let $\mathcal{V}' = \mathcal{V} \cup \{\mu \rightarrow \alpha'(\mathbf{R})\}$. Then $\psi, \mathcal{U}', \alpha'$, and $\mathcal{V}'$ satisfy the induction hypothesis, so $\mathcal{J}, \mathcal{V}' \models \psi$. Therefore $\mathcal{J}, \mathcal{V} \models \phi$.

II. Suppose that ϕ has the form $\exists \mu \psi$, and suppose that this is true under $\mathcal{J}, \mathcal{V}$. Then there exists $\mathbf{Q} \in \mathcal{D}$ such that ψ is true under $\mathcal{J}, \mathcal{V}'$ where $\mathcal{V}' = \mathcal{V} \cup \{\mu \rightarrow \mathbf{Q}\}$. Since $\mathcal{D}$ is extensible in $\mathcal{C}$, using the bijection $\theta = \alpha^{-1}$, there exists a bijection $\theta' \in \Gamma$ such that $\theta'(\mathcal{V}(\mu_i)) = \mathcal{U}(\mu_i)$ for $i = 1, \ldots, n$. and such that $\theta'(\mathbf{Q}) \in \mathcal{C}$. Let $\mathcal{U}' = \mathcal{U} \cup \{\mu \rightarrow \theta(\mathbf{Q})\}$. Let $\alpha' = \theta'^{-1}$. Then $\psi, \mathcal{U}', \alpha'$, and $\mathcal{V}'$ satisfy the induction hypothesis, so $\mathcal{I}, \mathcal{U}' \models \psi$. Therefore $\mathcal{I}, \mathcal{U} \models \phi$.

III. Suppose that ϕ has the form $\forall \mu \psi$, and suppose that this is true under $\mathcal{I}, \mathcal{U}$. Then $\exists \mu \neg \psi$ is false under $\mathcal{I}, \mathcal{U}$, so by the contrapositive of (II) $\exists \mu \neg \psi$ is false under $\mathcal{J}, \mathcal{V}$, so $\forall \mu \psi$ is true under $\mathcal{J}, \mathcal{V}$.

IV. Suppose that ϕ has the form $\forall \mu \psi$, and suppose that this is true under $\mathcal{J}, \mathcal{V}$. Then by the contrapositive of (I), $\forall \mu \psi$ is true under $\mathcal{I}, \mathcal{U}$.

∎

Theorem 6 *Let $\mathcal{C}$ and $\mathcal{D}$ be mutually extensible subsets of $\mathcal{Z}$ with respect to Γ. Let $\mathcal{P}_1, \ldots, \mathcal{P}_m$ be relations over $\mathcal{Z}$ that are invariants of Γ. Then the structures $\langle \mathcal{C}, \mathcal{P}_1|_{\mathcal{C}}, \ldots, \mathcal{P}_m|_{\mathcal{C}} \rangle$ and $\langle \mathcal{D}, \mathcal{P}_1|_{\mathcal{D}}, \ldots, \mathcal{P}_m|_{\mathcal{D}} \rangle$ are elementarily equivalent.*

Proof: Let σ be a signature of m symbols and let ϕ be a sentence over σ. By Lemma 5, since ϕ has no free variables, it holds in the L-structure $\langle \mathcal{D}, \sigma, \mathcal{J} \rangle$ if and only if it holds in the L-structure $\langle \mathcal{C}, \sigma, \mathcal{I} \rangle$. ∎

It will be convenient to abbreviate the structure $\langle \mathcal{C}, \mathcal{P}_1|_{\mathcal{C}}, \ldots, \mathcal{P}_m|_{\mathcal{C}} \rangle$ as $\langle \mathcal{C}, \mathcal{P}_1, \ldots, \mathcal{P}_m \rangle$; this is unambiguous, as the relations in a structure are necessarily limited to its domain.

Lemma 7 below gives a set of sufficient conditions for mutual extensibility.

Definition 8 *Let $\mathcal{D} \subset \mathcal{Z}$. Let Γ and Ψ be groups of bijections from $\mathcal{Z}$ to itself. We say that Γ is* rectifiable *to Ψ over $\mathcal{D}$ if the following condition holds: for all $\alpha \in \Gamma$, and all $\mathbf{D}_1, \ldots, \mathbf{D}_m \in \mathcal{D}$, if $\alpha(\mathbf{D}_i) \in \mathcal{D}$ for $i = 1, \ldots, m$, then there exists $\phi \in \Psi$ such that $\phi(\mathbf{D}_i) = \alpha(\mathbf{D}_i)$ for $i = 1, \ldots, m$.*

Lemma 7 *Let $\mathcal{D} \subset \mathcal{C} \subset \mathcal{Z}$. Let Γ be a group of bijections from $\mathcal{Z}$ to itself and let Ψ be a subgroup of Γ. If the following conditions hold:*

a. $\mathcal{C}$ is closed under Γ.

b. $\mathcal{D}$ is closed under Ψ.

c. $\mathcal{C}$ is embeddable in $\mathcal{D}$ under Γ.

d. Γ is rectifiable to Ψ over $\mathcal{D}$.

Then $\mathcal{C}$ and $\mathcal{D}$ are mutually extensible with respect to Γ.

Proof: We first show that $\mathcal{D}$ is self-extensible under Γ. Let $\alpha \in \Gamma$, and $\mathbf{D}_1, \ldots, \mathbf{D}_m \in \mathcal{D}$ such that $\alpha(\mathbf{D}_i) \in \mathcal{D}$ for $i = 1, \ldots, m$. Let $\mathbf{D}_{m+1} \in \mathcal{D}$. Then since Γ is rectifiable to Ψ over $\mathcal{D}$, there exists $\alpha' \in \Psi$ such that $\alpha'(\mathbf{D}_i) = \alpha(\mathbf{D}_i)$ for $i = 1, \ldots, m$. Since $\mathcal{D}$ is closed under Ψ, $\alpha'(\mathbf{D}_{m+1}) \in \mathcal{D}$.

Therefore, by corollary 4, $\mathcal{C}$ is extensible in $\mathcal{D}$.

The fact that $\mathcal{D}$ is extensible in $\mathcal{C}$ under Γ is immediate from the facts that $\mathcal{D} \subset \mathcal{C}$ and $\mathcal{C}$ is closed under Γ. Thus, if $\alpha \in \Gamma$, and $\mathbf{D}_{m+1} \in \mathcal{D}$, then $\alpha(\mathbf{D}_{m+1}) \in \mathcal{C}$. ▮

3 Rational polyhedra

In this section we prove that the domain of rational polyhedra in $\mathbb{R}^k$ (i.e. polyhedra with rational coordinates) is elementarily equivalent to the domain of general polyhedra for a topological language over regions.

We begin by defining conventions of notation and standard terminology. The index k will be the dimension of the Euclidean space, throughout. We use boldface lower-case letters like $\mathbf{x}$ for geometric points in $\mathbb{R}^k$. We use boldface upper-case letters like $\mathbf{R}$ for sets of points, called *regions*; this includes finite sets of points, extended regions, faces of regions, affine spaces, and so on. We use vector notation $\vec{v}$ for vectors. We use calligraphic letters like $\mathcal{C}$ for sets of regions and for relations over regions. We use lower-case Greek letters like α for homeomorphisms and other functions. We use upper-case Greek letters like Ψ for sets of functions. $\mathbb{F}$ will be a subfield of the reals. For other entities, we use italicized letters.

The distance between points $\mathbf{x}$ and $\mathbf{y}$ is denoted $d(\mathbf{x}, \mathbf{y})$. If $\mathbf{R}$ is a region then the topological boundary of $\mathbf{R}$, denoted $\partial\mathbf{R}$, is defined as the set of points in closure($\mathbf{R}$) but not in interior($\mathbf{R}$).

Definition 9 *A subset $\mathbf{R}$ of $\mathbb{R}^k$ is* topologically closed regular *(generally abbreviated to "closed regular") if $\mathbf{R} = \text{closure}(\text{interior}(\mathbf{R}))$.*

Definition 10 *A relation $\mathcal{P}(\mathbf{R}_1, \ldots, \mathbf{R}_n)$ over regions in $\mathbb{R}^k$ is* topological *if it is invariant under the space of homeomorphisms of $\mathbb{R}^k$ to itself.*

We use the standard theory of simplices, complexes, abstract complexes, and piecewise-linear (PL) mappings [10, 6].

Definition 11 *For $q \leq k$, a set of points $\{\mathbf{p}_0, \ldots, \mathbf{p}_q\}$ is* affine-independent *if the vectors $\mathbf{p}_1 - \mathbf{p}_0, \ldots, \mathbf{p}_q - \mathbf{p}_0$ are linearly independent. Let $\mathbf{P} = \langle \mathbf{p}_0, \ldots, \mathbf{p}_q \rangle$ be a $q+1$-tuple of affine-independent points. For any point $\mathbf{x}$, if there exist coefficients $t_0, \ldots, t_q$ such that $\sum_{i=0}^{q} t_q = 1$ and $\sum_{i=0}^{q} t_i \mathbf{p}_i = \mathbf{x}$, then $\langle t_0, \ldots, t_q \rangle$ are the* barycentric coordinates *of $\mathbf{x}$ with respect to $\mathbf{P}$. (If these exist, they are unique.) The set of points that have barycentric coordinates with respect to $\mathbf{P}$ is the* affine space spanned by $\mathbf{P}$. *The* open simplex *spanned by $\mathbf{P}$, denoted "$S(\mathbf{P})$" is the set of points $\mathbf{p}$ such that all the barycentric coordinates of $\mathbf{p}$ are in (0,1). The corresponding* closed simplex, *denoted $\bar{S}(\mathbf{P}) = \text{closure}(S(\mathbf{P}))$. The* dimension *of both the affine space and the open simplex is q. If $\mathbf{W} \subset \mathbf{P}$, then $S(\mathbf{W})$ is a* face *of $S(\mathbf{P})$ and $\bar{S}(\mathbf{W})$ is a face of $\bar{S}(\mathbf{P})$.*

The elements $\langle \mathbf{p}_0, \ldots, \mathbf{p}_m \rangle$ are the vertices *of $S(\mathbf{P})$ and $\bar{S}(\mathbf{P})$. A single vertex is considered both an open and a closed simplex of dimension 0.*

A point with coordinates in $\mathbb{F}$ will be called an "$\mathbb{F}$-point"; likewise "$\mathbb{F}$-simplex" and so on.

Definition 12 *An* open $\mathbb{F}$-half-space *is the set of points defined by a linear inequality $a_0 + \Sigma_{i=1}^{k} a_i x_i > 0$ where $a_i \in \mathbb{F}$ for $i = 0, \ldots, k$. A* basic $\mathbb{F}$-polytope *is the intersection of finitely many open $\mathbb{F}$-half-spaces. A* closed $\mathbb{F}$-polytope *is the closure of the union of finitely many basic $\mathbb{F}$-polytopes. An* $\mathbb{F}$-polyhedron *is a compact $\mathbb{F}$-polytope.*

We denote the collection of $\mathbb{F}$-polyhedra as $Poly[\mathbb{F}]$. The set of all polyhedra, $Poly[\mathbb{R}]$ will be abbreviated as $Poly$.

Definition 13 *A homeomorphism α from $\mathbb{R}^k$ to itself is a* piecewise-linear (PL) mapping *if for some integer m there exist a sequence of m polytopes $\mathbf{P}_1, \ldots, \mathbf{P}_m$; a sequence of m matrices $M_1, \ldots, M_m$; and a sequence of m vectors $\vec{c}_1, \ldots, \vec{c}_m$ such that:*

- $\cup_{i=1}^{m} \mathbf{P}_i = \mathbb{R}^k$.
- *for $i = 1, \ldots, m$, if $\mathbf{p} \in \mathbf{P}_i$ then $\alpha(\mathbf{p}) = M_i \cdot \mathbf{p} + \vec{c}_i$.*

α is a bounded PL mapping *if $\alpha(\mathbf{p})$ is the identity over each unbounded $\mathbf{P}_i$.*

The polytopes $\mathbf{P}_1, \ldots, \mathbf{P}_m$ are the cells *of α.*

An $\mathbb{F}$-PL mapping *is a PL mapping such that all the cells are $\mathbb{F}$-polytopes, and all the elements of M_i and $\vec{c}_i$ are in $\mathbb{F}$.*

Definition 13 departs slightly from the definition in [10] in that here the collection of cells is required to be finite, whereas in [10] it is only required to be locally finite.

We will denote the class of all bounded $\mathbb{F}$-PL mappings as $\Pi[\mathbb{F}]$.

Definition 14 *A* closed complex *is a set $\mathcal{C}$ of closed simplices such that*

a. If $\mathbf{P}$ is in $\mathcal{C}$ then any face of $\mathbf{P}$ is in $\mathcal{C}$.

b. If $\mathbf{P}, \mathbf{Q} \in \mathcal{C}$, then $\mathbf{P} \cap \mathbf{Q}$ is either empty or a face of both $\mathbf{P}$ and $\mathbf{Q}$.

An open complex *is the set of open simplices corresponding the simplices of a closed complex.*

Definition 15 *Let Z be a finite set. An* abstract simplex *over Z is a subset of Z. An* abstract complex *C over Z is a collection of abstract simplices over Z such that, if $A \in C$ and $B \subset A$ then $B \in C$. A* realization *of C is a function α from Z to $\mathbb{R}^k$. A realization is* proper *if $\alpha(C)$ is a complex satisfying Definition 14.*

Lemma 8 *Let α be a realization of abstract complex C. α is proper if the following condition is met: For every pair of abstract simplices $F, G \in C$, if $F \cap G = \emptyset$ then $\bar{S}(\alpha(F)) \cap \bar{S}(\alpha(G)) = \emptyset$.*

Proof: Immediate from Lemma 2.1, p. 8 of [6]. ▮

Since a realization α of Z is an assignment of a point in $\mathbb{R}^k$ to each element of Z, it can be considered a point in $\mathbb{R}^{k|Z|}$.

Definition 16 *Let Z be a set of abstract vertices. We define the following metric over the realizations of Z (the $\mathcal{L}^\infty$ metric): For any two realizations ϕ and α, $d^\infty(\phi, \alpha) = \max_{z \in Z} d(\phi(z), \alpha(z))$.*

The associated metric topology is, of course, the standard topology on $\mathbb{R}^{k|Z|}$.

Lemma 9 *Let C be an abstract complex over Z. The set of proper realizations of C is an open set within $\mathbb{R}^{k|Z|}$.*

Proof: Define the function $f_C(\alpha)$ from $\mathbb{R}^{k|Z|}$ to $\mathbb{R}$ as the minimum distance from $\bar{S}(\alpha(U))$ to $\bar{S}(\alpha(V))$ over all pairs of abstract simplices $U, V \in C$ such that $U \cap V = \emptyset$. Clearly this is a continuous function. By Lemma 8, the set of proper realizations is equal to the set of α where $f_C(\alpha) > 0$, and hence an open set. ▮

Definition 17 *Let $\mathcal{P}$ be a set of closed polyhedra. A* triangulation *of $\mathcal{P}$ is a complex $\mathcal{C}$ such that, for any polyhedron $\mathbf{P} \in \mathcal{P}$ and for any face $\mathbf{Q}$ of $\mathbf{P}$, $\mathbf{Q}$ is the union of simplices in $\mathcal{C}$.*

Lemma 10 *Let $\{\mathbf{P}_1, \ldots, \mathbf{P}_m\}$ be a finite set of $\mathbb{F}$-polyhedra. Let $\mathbf{B}$ be an $\mathbb{F}$-polyhedron such that $\mathbf{P}_i$ is disjoint from $\partial\mathbf{B}$ for $i = 1, \ldots, m$. Let $\mathcal{P} = \{\mathbf{B}, \mathbf{P}_1, \ldots, \mathbf{P}_m\}$. Then there exists an $\mathbb{F}$-triangulation $\mathcal{T}$ of $\mathcal{P}$ such that, if $\mathbf{t}$ is a vertex of $\mathcal{T}$ and $\mathbf{t} \in \partial\mathbf{B}$ then $\mathbf{t}$ is a vertex of $\mathbf{B}$.*

Proof: By Definition 11, each polyhedron $\mathbf{P}_i$ is the finite union of simplices. Let $\mathcal{V}$ be the set of all intersections between component simplices of polyhedra in $\mathcal{P}$. Then $\mathcal{V}$ is a collection of cells, as defined in [10] p. 13. Since $\mathbb{F}$ is a field, all the vertices of cells in $\mathcal{V}$ are $\mathbb{F}$-vertices. By [10], p. 16, proposition 2.9, $\mathcal{V}$ has a triangulation whose vertices are exactly the vertices of $\mathcal{V}$. Since the only vertices of $\mathcal{V}$ in $\partial\mathbf{B}$ are vertices of $\mathbf{B}$ itself, the conclusion of the lemma follows. ▮.

Lemma 11 *Let $\mathcal{P} = \{\mathbf{B}, \mathbf{P}_1, \ldots, \mathbf{P}_m\}$ be a set of $\mathbb{F}$-polyhedra such that $\mathbf{P}_i \subset$ interior($\mathbf{B}$), for $i = 1, \ldots, m$. Let $\mathcal{T}$ be an $\mathbb{F}$-triangulation of $\mathcal{P}$. Let C be an abstract complex and let α be a realization such that $\alpha(C) = \mathcal{T}$. Let β be a proper $\mathbb{F}$-realization of C such that, for every abstract vertex $z \in Z$, if $\alpha(z) \in \partial\mathbf{B}$, then $\beta(z) = \alpha(z)$. Then there exists a PL $\mathbb{F}$-homeomorphism ψ from $\mathbb{R}^k$ to itself such that for every z, $\psi(\alpha(z)) = \beta(z)$ and such that ψ is the identity outside $\mathbf{B}$.*

Proof: We use barycentric coordinates to construct ψ. For any point $\mathbf{x} \in \mathbf{B}$, let $\mathbf{X}$ be the open simplex in $\mathcal{T}$ containing $\mathbf{x}$. Let $\mathbf{x}_1, \ldots, \mathbf{x}_q$ be the vertices of $\mathbf{X}$, and let $\langle t_1, \ldots, t_q \rangle$ be the barycentric coordinates of $\mathbf{x}$. Define $\psi(\mathbf{x})$ such that, for $\mathbf{x} \in \mathbf{B}$, $\psi(\mathbf{x}) = \sum_{i=1}^{q} t_i \beta(\alpha^{-1}(\mathbf{x}_i))$; for $\mathbf{x} \notin \mathbf{B}$, $\psi(\mathbf{x}) = \mathbf{x}$.

Since β and α map the vertices of each simplex to an affine independent set, ψ is a bijection within the simplex. Since β and α are proper realizations, the open simplices of $\mathcal{T}$ are disjoint, and the open simplices of $\psi(\mathcal{T})$ are disjoint, so ψ is overall a bijection. As a point $\mathbf{x}$ in $\mathbf{X}$ approaches a face $\mathbf{F}$ of $\mathbf{X}$, the barycentric coordinates corresponding to vertices of $\mathbf{X}$ outside $\mathbf{F}$ go to 0, so ψ is continuous in moving from simplices to their faces and back. The same argument shows that ψ^{-1} is continuous. Thus ψ is a homeomorphism from $\mathbf{B}$ to $\psi(\mathbf{B})$.

It is obvious that ψ is piecewise linear. Since every vertex of $\mathcal{T}$ is an $\mathbb{F}$-vertex and is mapped onto an $\mathbb{F}$-vertex, ψ is an $\mathbb{F}$-mapping. Clearly ψ is the identity on $\partial\mathbf{B}$, so its continuation as the identity outside $\mathbf{B}$ is continuous. ▮

Lemma 12 *For any subfield $\mathbb{F}$ of $\mathbb{R}$, Poly is embeddable in Poly$(\mathbb{F})$ under $\Pi[\mathbb{R}]$.*

Proof: Let $\mathbf{D}_1, \ldots, \mathbf{D}_m$ be polyhedra in *Poly*. Let $\mathbf{B}$ be an $\mathbb{F}$-polyhedron such that $\mathbf{D}_i \subset \text{interior}(\mathbf{B})$ for $i = 1, \ldots, m$. Let $\mathcal{D} = \{\mathbf{B}, \mathbf{D}_1, \ldots, \mathbf{D}_m\}$. Since none of the $\mathbf{D}_i$ intersect $\partial\mathbf{B}$, we can use Lemma 10 with $\mathbb{F} = \mathbb{R}$ to construct a triangulation $\mathcal{T}$ of $\mathcal{D}$ such that the only vertices of $\mathcal{T}$ in $\partial\mathbf{B}$ are the vertices of $\mathbf{B}$ itself.

Let C be an abstract complex and α be a realization such that $\alpha(C) = \mathcal{T}$. Let N be the number of vertices in C. By Lemma 9 there exists a neighborhood U of α in realization-space such that all realizations in U are proper. Since $\mathbb{F}^{Nk}$ is dense within $\mathbb{R}^{Nk}$, there exists $\beta \in U$ such that $\beta(z)$ is an $\mathbb{F}$-point for all z, and further $\beta(z) = \alpha(z)$ for all vertices z of $\mathbf{B}$.

By Lemma 11 there exists a bounded homeomorphism ψ from $\mathbb{R}^k$ to itself such that $\psi(\alpha(z)) = \beta(z)$ for all vertices z of $\mathbf{B}$. Since each of the polyhedra $\mathbf{D}_i$ is the union of the simplices in T, each of the polyhedra $\psi(\mathbf{D}_i)$ is the union of simplices in $\phi(C)$, and hence is in *Poly*$[\mathbb{F}]$. ▮

Lemma 13 *Let $\mathbf{P}$ be a rational polyhedron. Let $\mathcal{T}$ be a triangulation of $\mathbf{P}$, which corresponds to realization α of abstract complex C. Let $\epsilon > 0$. Then there exists a rational realization β of C such that $\beta(C)$ is a triangulation of $\mathbf{P}$ and $d^\infty(\beta, \phi) < \epsilon$.*

Proof: This is Theorem 1 of [1], with straightforward translation of terminology. ▮

Lemma 14 *Let $\mathcal{P} = \{\mathbf{P}_1, \ldots, \mathbf{P}_m\}$ be a set of $\mathbb{F}$-polyhedra. Let $\mathcal{T}$ be a triangulation of $\mathcal{P}$, which corresponds to realization α of abstract complex C. Let $\epsilon > 0$. Then there exists a $\mathbb{F}$-realization β of C such that $\beta(C)$ is a triangulation of $\mathbf{P}$, $d^\infty(\beta, \phi) < \epsilon$, and $\beta(\mathbf{v}) = \alpha(\mathbf{v})$ for every vertex $\mathbf{v}$ of each polygon $\mathbf{P}_i$ in $\mathcal{P}$.*

Proof: The proof is a trivial extension of the proof given by Beynon [1] of Lemma 13 above (his Theorem 1). In particular, it may be noted that $\mathbb{Q}$ in Beynon's proof can be changed to $\mathbb{F}$ without requiring any other change; that changing one polyhedron to multiple polyhedra likewise requires only minor rephrasing; and that the construction in Beynon's proof already leaves any rational point unchanged. ▮

Lemma 15 *For any subfield $\mathbb{F}$ of $\mathbb{R}$, $\Pi[\mathbb{R}]$ is rectifiable to $\Pi[\mathbb{F}]$ over Poly$[\mathbb{F}]$.*

Proof: Let $\mathcal{D} = \mathbf{D}_1, \ldots, \mathbf{D}_m$ be in *Poly*$[\mathbb{F}]$. Let ϕ be a bounded PL mapping such that $\phi(\mathbf{D}_i) \in$ *Poly*$[\mathbb{F}]$ for $i = 1, \ldots, m$. We need to show that there exists an $\mathbb{F}$-PL mapping ψ such that $\psi(\mathbf{D}_i) = \phi(\mathbf{D}_i)$ for $i = 1, \ldots, m$.

Let $\mathcal{C}$ be the bounded cells of ϕ. Let $\mathbf{B}$ be an $\mathbb{F}$-polyhedron such that, for each $\mathbf{R}$ in $\mathcal{C} \cup \mathcal{D}$, $\mathbf{R} \subset \text{interior}(\mathbf{B})$. Note that since $\partial\mathbf{B}$ is outside all the bounded cells of ϕ, and since ϕ is bounded,

ϕ is the identity on $\partial\mathbf{B}$. Using Lemma 10, let $\mathcal{W}$ be a triangulation of $\mathcal{C} \cup \mathcal{D} \cup \{\mathbf{B}\}$ such that all the vertices of $\mathcal{W}$ in $\partial\mathbf{B}$ are vertices of $\mathbf{B}$. Then $\phi(\mathcal{W})$ is a triangulation of $\{\mathbf{B}, \mathbf{D}_1, \ldots, \mathbf{D}_m\}$, though not in general an $\mathbb{F}$-triangulation, since the cells of $\mathcal{C}$ are not in general $\mathbb{F}$-polytopes.

Let C be an abstract complex and let α be a realization such that $\mathcal{W} = \alpha(C)$; obviously α is a proper realization of C. Let Z be the set of abstract vertices in C. Let $V \subset Z$ be the set of abstract vertices corresponding to vertices of $\{\mathbf{B}\} \cup \mathcal{D}$. Using Lemma 14, let β be an $\mathbb{F}$-realization of Z such $\beta(C)$ is a triangulation of $\{\mathbf{B}, \mathbf{D}_1, \ldots, \mathbf{D}_m\}$, and such that, for every abstract vertex $v \in V$, $\beta(v) = \alpha(v)$.

We next carry out the analogous construction on the range side of the mapping ϕ. Let $\mathcal{T} = \phi(\mathcal{W})$. Since ϕ is a PL homeomorphism, it follows that $\mathcal{T}$ is a triangulation of $\phi(\mathcal{C}) \cup \phi(\mathcal{D}) \cup \{\mathbf{B}\}$; and that all the vertices of $\mathcal{T}$ in $\partial\mathbf{B}$ are vertices of $\mathbf{B}$. (Note again that $\phi(\mathbf{B}) = \mathbf{B}$.) Let γ be the realization of Z such that $\gamma(C) = \mathcal{T}$. Again using Lemma 14, let δ be an $\mathbb{F}$-realization of Z such that $\delta(C)$ is a triangulation of $\phi(\mathcal{D}) \cup \{\mathbf{B}\}$ such that, for every abstract vertex v corresponding to a vertex of $\phi(\mathcal{D}) \cup \{\mathbf{B}\}$, $\delta(v) = \gamma(v)$.

Thus $\beta(C)$ is an $\mathbb{F}$-triangulation of $\{\mathbf{D}_1, \ldots, \mathbf{D}_m, \mathbf{B}\}$, and $\delta(C)$ is an $\mathbb{F}$-triangulation of $\{\phi(\mathbf{D}_1), \ldots, \phi(\mathbf{D}_m), \mathbf{B}\}$. Using Lemma 11 construct a PL $\mathbb{F}$-homeomorphism ψ such that $\psi(\beta(z)) = \delta(z)$ for all $z \in Z$ and such that ψ is the identity outside $\mathbf{B}$.

What remains to be shown is that for $\mathbf{D} \in \mathcal{D}$, $\psi(\mathbf{D}) = \phi(\mathbf{D})$, that all of our original polyhedra have the same image under ψ as under ϕ. Proof by induction over the dimensionality of the faces of $\mathbf{D}$ (*not* the faces of the triangulation). If $\mathbf{x}$ is a vertex of $\mathbf{D}$ then $\mathbf{x}$ is an $\mathbb{F}$-vertex so $\psi(\mathbf{x}) = \phi(\mathbf{x})$ by construction. If $\mathbf{F}$ is a q-dimensional face, then by induction, each of its boundary faces is the same under ψ as under ϕ; hence, so is $\mathbf{F}$. ∎

Theorem 16 *Let $\mathbb{F}$ be a subfield of $\mathbb{R}$. Let $Poly[\mathbb{F}]$ be the collection of $\mathbb{F}$-polyhedra in $\mathbb{R}^k$. Let $\mathcal{P}_1, \ldots, \mathcal{P}_n$ be topological relations over $\mathbb{R}^k$. Then the structures $\langle Poly, \mathcal{P}_1, \ldots, \mathcal{P}_n\rangle$ and $\langle Poly[\mathbb{F}], \mathcal{P}_1, \ldots, \mathcal{P}_n\rangle$ are elementarily equivalent.*

Proof: It is immediate that *Poly* is closed under $\Pi[\mathbb{R}]$ and that $Poly[\mathbb{F}]$ is closed under $\Pi[\mathbb{F}]$. By Lemma 12 *Poly* is embeddable in $Poly[\mathbb{F}]$ under $\Pi[\mathbb{R}]$ and by Lemma 15 $\Pi[\mathbb{R}]$ is rectifiable to $\Pi[\mathbb{F}]$ over $Poly[\mathbb{F}]$. Hence by Lemma 7, *Poly* and $Poly[\mathbb{F}]$ are mutually extensible. The result then follows from Theorem 6. (Note that in applying Theorem 6, the universe $\mathcal{Z}$ is the set of regions in $\mathbb{R}^k$, and thus, from the standpoint of Theorem 6, $\Pi[\mathbb{R}]$ and $\Pi[\mathbb{F}]$ are viewed as sets of bijections over the space of regions, rather than over the space of points. The same applies in the proof of Theorem 27 below.) ∎

We now extend Theorem 16 to polytopes. Let $UPoly[\mathbb{F}]$ (U for "unbounded") be the collection of polytopes, bounded and unbounded, and let $\Lambda[\mathbb{F}]$ be the collection of $\mathbb{F}$-PL mappings, bounded and unbounded.

To transfer the above results on bounded polyhedra to the space of unbounded polytopes, we will use piecewise projective transformations.

Definition 18 *An* $\mathbb{F}$-projective mapping *is a function $\alpha(\mathbf{x}) = (M \cdot \mathbf{x} + \vec{c})/(\vec{a} \cdot \mathbf{x} + b)$ where M is an $\mathbb{F}$-matrix, $\vec{c}$ and $\vec{a}$ are $\mathbb{F}$-vectors and $b \in \mathbb{F}$.*

Let $\mathbf{U}$ and $\mathbf{V}$ be $\mathbb{F}$-polytopes. Let $\mathbf{C}_1, \ldots, \mathbf{C}_m$ be a set of $\mathbb{F}$-polytopes such that $\cup_{i=1}^m \mathbf{C}_i = \mathbf{U}$. Let ϕ be a homeomorphism from $\mathbf{U}$ to $\mathbf{V}$, and let $\phi_i, i = 1, \ldots, m$ be $\mathbb{F}$-projective mappings, such that, for $\mathbf{x} \in \mathbf{C}_i$, $\phi(\mathbf{x}) = \phi_i(\mathbf{x})$. Then ϕ is an $\mathbb{F}$-piecewise projective mapping.

Lemma 17 *Let $\mathbf{P}$ be an unbounded closed polytope, and let α be a projective transformation such that $\alpha(\mathbf{P})$ is also an unbounded closed polytope. Then α is a linear transformation.*

Proof of the contrapositive. Suppose that α is not a linear transformation; then α^{-1} is likewise not a linear transformation. Since α^{-1} is a projective transformation which is not a linear transformation, it has the form $\alpha^{-1}(\mathbf{x}) = (M\mathbf{x} + \vec{c})/(\vec{a} \cdot \mathbf{x} + b)$, where $\vec{a} \neq \vec{0}$. Thus α^{-1} maps the hyperplane $\mathbf{H} = \{\mathbf{x} \mid \vec{a} \cdot \mathbf{x} + b = 0\}$ to the hyperplane at infinity, so α maps the hyperplane at infinity to $\mathbf{H}$. Considered as a function over the projective space, α is a continuous function. Since $\mathbf{P}$ borders the hyperplane at infinity, $\alpha(\mathbf{P})$ borders but does not include $\mathbf{H}$, so $\alpha(\mathbf{P})$ is not an unbounded closed polytope. ∎

In the proofs below, we will use extensively one particular piecewise projective mapping, which we will denote $\theta(\mathbf{p})$. It is defined as follows. For the remainder of this paper, let $\mathbf{B}^o$ be the open box $(-1,1)^k$ and let $\bar{\mathbf{B}}$ be the closed box $[-1,1]^k$. Let $\mathbf{p} = \langle p_1, \ldots, p_k \rangle$ be a point in $\mathbf{B}^\circ$ and let $u = \max_{i=1}^k |v_i|$. Define $\theta(\mathbf{p}) = \mathbf{p}/(1-u)$. Note that:

- θ is a homeomorphism from $\mathbf{B}^\circ$ to $\mathbb{R}^k$.
- θ is a piecewise projective transformation. It has $2k$ cells, $\mathbf{C}_{i+}$ and $\mathbf{C}_{i-}$ for $i = 1, \ldots, k$, defined as follows:
  $\mathbf{C}_{i+} = \{\mathbf{p} \mid 0 \leq p_i < 1, p_i \geq p_j, p_i \geq -p_j, \text{ for all } j \neq i\}$
  $\mathbf{C}_{i-} = \{\mathbf{p} \mid -1 < p_i \leq 0, p_i \leq p_j, p_i \leq -p_j, \text{ for all } j \neq i\}$.
- Likewise, the cells of θ^{-1} are the regions
  $\mathbf{D}_{i+} = \{\mathbf{p} \mid p_i \geq 0, p_i \geq p_j, p_i \geq -p_j, \text{ for all } j \neq i\}$ and
  $\mathbf{D}_{i-} = \{\mathbf{p} \mid -p_i \leq 0, p_i \leq p_j, p_i \leq -p_j, \text{ for all } j \neq i\}$.

For example in $\mathbb{R}^2$ the four cells of θ are the left, right, up, and down quadrants of $\mathbf{B}^\circ$, bounded by the diagonals $x = \pm y$, and the cells of the θ^{-1} are the quadrants of the plane.

Since θ is piecewise projective, it maps any open polytope in $\mathbf{B}^\circ$ to an open polytope in $\mathbb{R}^k$. If $\mathbf{R}$ is an open polyhedron such that closure$(\mathbf{R}) \subset \mathbf{B}^\circ$, then $\theta(\mathbf{R})$ is a bounded polyhedron, and conversely. If $\mathbf{R}$ is an open polyhedron such that $\mathbf{R} \subset \mathbf{B}^\circ$ but closure$(\mathbf{R}) \not\subset \mathbf{B}^\circ$ then $\theta(\mathbf{R})$ is an unbounded open polytope, and conversely.

The following lemma is analogous to Lemma 11.

Lemma 18 *Let $\mathbf{P}_1, \ldots, \mathbf{P}_m$ be polyhedra such that for $i = 1, \ldots, m$, $\mathbf{P}_i \subset \bar{\mathbf{B}}$, and let $\mathcal{P} = \{\bar{\mathbf{B}}, \mathbf{P}_1, \ldots, \mathbf{P}_m\}$. Let $\mathcal{T}$ be a triangulation of $\mathcal{P}$. Let C be an abstract complex and let α be a realization such that $\alpha(C) = \mathcal{T}$. Let β be a proper realization of C such that, for every abstract vertex $z \in Z$, if $\alpha(z) \in \partial\bar{\mathbf{B}}$, then $\beta(z)$ is on the same face of $\partial\bar{\mathbf{B}}$ as $\alpha(z)$. Then there exists a PL homeomorphism ψ from $\bar{\mathbf{B}}$ to itself such that for every z, $\psi(\alpha(z)) = \beta(z)$ and such that, if $\mathbf{F}$ is a face of $\bar{\mathbf{B}}$, then $\psi(\mathbf{F}) = \mathbf{F}$. ($\psi$ need not be defined outside $\bar{\mathbf{B}}$.)*

Proof: As in the proof of Lemma 11, for any point $\mathbf{x} \in \mathbf{B}$, let $\mathbf{x}_1, \ldots, \mathbf{x}_q$ be the vertices of the open simplex in $\mathcal{T}$ containing $\mathbf{x}$, and let $\langle t_1, \ldots, t_q \rangle$ be the barycentric coordinates of $\mathbf{x}$. Define $\psi(\mathbf{x})$ such that, for $\mathbf{x} \in \mathbf{B}$, $\psi(\mathbf{x}) = \sum_{i=1}^q t_i \beta(\alpha^{-1}(\mathbf{x}_i))$; The proof that ψ is a homeomorphism proceeds as in Lemma 11. If $\mathbf{x}$ is in a face $\mathbf{F}$ of $\bar{\mathbf{B}}$, let C be the abstract simplex $C = \alpha^{-1}(\{\mathbf{x}_1, \ldots, \mathbf{x}_q\})$. By assumption $\beta(C) \subset \mathbf{F}$, so $\phi(\mathbf{x}) \in \mathbf{F}$. ∎

Lemma 19 *Let χ be an $\mathbb{F}$-piecewise projective transformation that is a homeomorphism from $\mathbb{R}^k$ to $\mathbb{R}^k$. Let $\{\mathbf{C}_1, \ldots, \mathbf{C}_m\}$ be the cells of χ. Then there exists an $\mathbb{F}$-PL homeomorphism ϕ from $\mathbb{R}^k$ to itself such that, for $i = 1, \ldots, m$, $\phi(\mathbf{C}_i) = \chi(\mathbf{C}_i)$.*

Proof: Let χ_i be the restriction of χ to $\mathbf{C}_i$. Since χ is a homeomorphism of $\mathbb{R}^k$ to itself, for any unbounded cell $\mathbf{C}_i$, $\alpha(\mathbf{C}_i)$ must be unbounded, so by Lemma 17 χ_i is a linear transformation.

We define ϕ as follows. Let $\mathcal{T}$ be a triangulation of the bounded polytopes in $\{\mathbf{C}_1, \ldots, \mathbf{C}_m\}$.

- If $\mathbf{x}$ is in one of the simplices in $\mathcal{T}$, let $\langle \mathbf{s}_1, \ldots, \mathbf{s}_q \rangle$ be the open simplex of $\mathcal{T}$ containing $\mathbf{x}$; let $\langle t_1, \ldots, t_q \rangle$ be the barycentric coordinates of $\mathbf{x}$; and let $\phi(\mathbf{x}) = \sum_{i=1}^{q} t_i \cdot \chi(\mathbf{s}_i)$.
- Otherwise, let $\phi(\mathbf{x}) = \chi(\mathbf{x})$.

The proof that ϕ is continuous across bounded simplices is the same argument as in Lemma 11. The fact it is consistently defined going from the bounded to the unbounded cells follows from the fact that ψ_2 is linear on the unbounded cells, and hence must agree with the barycentric mapping on the bounded faces of the unbounded cells.

It is immediate by construction that ϕ is an $\mathbb{F}$-mapping.

∎

Lemma 20 *$UPoly[\mathbb{R}]$ is embeddable in $UPoly[\mathbb{F}]$ under $\Lambda[\mathbb{R}]$.*

Proof: Let $\langle \mathbf{P}_1, \ldots, \mathbf{P}_m \rangle$ be a tuple of polytopes in $UPoly[\mathbb{R}]$. For $i = 1, \ldots, m$ let $\mathbf{Q}_i = \text{closure}(\theta^{-1}(\mathbf{P}_i))$. Thus $\langle \mathbf{Q}_1, \ldots, \mathbf{Q}_m \rangle$ is a tuple of polyhedra in $\bar{\mathbf{B}}$. Let $\mathcal{Q} = \{\bar{\mathbf{B}}, \mathbf{Q}_1, \ldots, \mathbf{Q}_m\}$. Let $\mathcal{T}$ be a triangulation of $\mathcal{Q}$. Let C be an abstract simplex and α a realization of C such that $\alpha(C) = \mathcal{T}$. By Lemma 9 there exists a proper realization β such that $\beta(z)$ is an $\mathbb{F}$-point for all $z \in C$ and such that, if $\alpha(z)$ is in a face $\mathbf{F}$ of $\bar{\mathbf{B}}$ then $\beta(z) \in \mathbf{F}$. By Lemma 18 there exists an PL homeomorphism ψ from $\mathbf{B}$ to itself such that for every z, $\psi(\alpha(z)) = \beta(z)$ and such that, if $\mathbf{F}$ is a face of $\bar{\mathbf{B}}$, then $\phi(\mathbf{F}) = \mathbf{F}$.

Now let $\chi = \theta \circ \psi \circ \theta^{-1}$. Clearly this is a homeomorphism from $\mathbb{R}^k$ to itself. Since it is the composition of piecewise projective mappings, χ is itself a piecewise projective mapping. Since χ satisfies the conditions of Lemma 19, there exists an $\mathbb{F}$-PL homeomorphism ϕ such that $\phi(\mathbf{P}_i) = \chi(\mathbf{P}_i)$. ∎

Lemma 21 *$\Lambda[\mathbb{R}]$ is rectifiable to $\Lambda[\mathbb{F}]$ over $UPoly[\mathbb{F}]$.*

Proof: Let $\mathbf{P}_1, \ldots, \mathbf{P}_m$ be $\mathbb{F}$-polytopes and let α be a PL mapping such that $\alpha(\mathbf{P}_i)$ is an $\mathbb{F}$-polytope for $i = 1, \ldots, m$. Note that $\alpha(\mathbb{R}^k) = \mathbb{R}^k$. For $i = 1, \ldots, m$, let $\mathbf{Q}_i = \theta^{-1}(\mathbf{P}_i)$ and let $\mathbf{W}_i = \theta^{-1}(\alpha(\mathbf{P}_i))$; these are all $\mathbb{F}$-polyhedra. Let $\psi = \theta^{-1} \circ \alpha \circ \theta$. Thus ψ is a piecewise projective mapping from $\bar{\mathbf{B}}$ to itself such that $\psi(\mathbf{Q}_i) = \mathbf{W}_i$. Moreover, since ψ is both a homeomorphism and a projective mapping, it preserves betweenness relations on the points in $\bar{\mathbf{B}}$. That is, if $\mathbf{c}_1, \ldots, \mathbf{c}_q$ are in $\bar{\mathbf{B}}$ and point $\mathbf{x}$ is in the open simplex with vertices $\mathbf{c}_1, \ldots, \mathbf{c}_q$, then $\psi(\mathbf{x})$ is in the open simplex with vertices $\psi(\mathbf{c}_1), \ldots, \psi(\mathbf{c}_q)$.

Therefore, let $\mathcal{Q} = \{\bar{\mathbf{B}}, \mathbf{Q}_1, \ldots, \mathbf{Q}_m\}$. Let $\mathcal{W}$ be the union of $\mathcal{Q}$ with the cells of ψ. Let $\mathcal{T}$ be a triangulation of $\mathcal{W}$. Define the PL-mapping ψ_2 from $\bar{\mathbf{B}}$ to itself as follows: For any point $\mathbf{x}$ let $\mathbf{c}_1, \ldots, \mathbf{c}_q$ be the open simplex in $\mathcal{T}$ containing $\mathbf{x}$, let $t_1, \ldots, t_q$ be the barycentric coordinates of $\mathbf{x}$, and let $\psi_2(\mathbf{x}) = \Sigma_{i=1}^{q} t_i \psi(\mathbf{c}_i)$. Then ψ_2 is a PL homeomorphism from $\bar{\mathbf{B}}$ to itself. Moreover, for any simplex $\mathbf{S}$ in $\mathcal{T}$, $\psi_2(\mathbf{S}) = \psi(\mathbf{S})$; hence $\psi_2(\mathbf{Q}_i) = \psi(\mathbf{Q}_i)$.

By Lemma 15, ψ_2 is rectifiable to an $\mathbb{F}$-PL mapping; that is, there exists an $\mathbb{F}$-PL mapping ϕ_2 such that $\phi_2(\mathbf{Q}) = \psi_2(\mathbf{Q})$ for $\mathbf{Q} \in \mathcal{Q}$.

Now, let $\chi = \theta \circ \phi_2 \cup \theta^{-1}$. This is a piecewise $\mathbb{F}$-projective mapping from $\mathbb{R}^k$ to itself. For any $\mathbf{P}_i$ we have

$$\chi(\mathbf{P}_i) = \theta(\phi_2(\theta^{-1}(\mathbf{P}_i))) = \theta(\phi_2(\mathbf{Q}_i)) = \theta(\psi(\mathbf{Q}_i)) = \theta(\theta^{-1}(\alpha(\theta(\theta^{-1}(\mathbf{P}_i))))) = \alpha(\mathbf{P}_i)$$

Since χ satisfies the conditions of Lemma 19, there exists an $\mathbb{F}$-PL mapping ϕ such that $\phi(\mathbf{P}_i) = \chi(\mathbf{P}_i) = \alpha(\mathbf{P}_i)$. ∎

Theorem 22 *Let $\mathcal{P}_1, \ldots, \mathcal{P}_n$ be topological relations. Then the structures $\langle UPoly[\mathbb{R}], \mathcal{P}_1, \ldots, \mathcal{P}_n\rangle$ and $\langle UPoly[\mathbb{F}], \mathcal{P}_1, \ldots, \mathcal{P}_n\rangle$ are elementarily equivalent.*

Proof: Identical to the proof of Theorem 16, replacing $\Pi[\mathbb{R}]$ and $\Pi[\mathbb{F}]$ with $\Lambda[\mathbb{R}]$ and $\Lambda[\mathbb{F}]$, and replacing Lemmas 12 and 15 by Lemmas 20 and 21. ∎

4 O-minimal domains

Using the powerful theory of o-minimal domains [12] we can show that all o-minimal domains are elementarily equivalent with respect to topological languages over regions in $\mathbb{R}^k$.

We begin with the set-theoretic definition of an o-minimal domain (there is also an equivalent model-theoretic definition):

Definition 19 *For $m = 1, 2, \ldots$ let $\mathcal{O}_m$ be a collection of subsets of $\mathbb{R}^m$. The sequence $\mathcal{O} = \langle \mathcal{O}_1, \mathcal{O}_2, \ldots\rangle$ is an* o-minimal domain over $\mathbb{R}$ *if the following conditions are satisfied:*

- *$\mathcal{O}_1$ is the set of all finite unions of points and intervals in $\mathbb{R}$.*
- *The graph of the order relation $\{\langle x, y\rangle \mid x < y\}$ is an element of $\mathcal{O}_2$.*
- *The graph of the addition function $\{\langle x, y, z\rangle \mid z = x + y\}$ and the graph of the multiplication function $\{\langle x, y, z\rangle \mid z = xy\}$ are elements of $\mathcal{O}_3$.*
- *The set $\{\langle x_1, \ldots, x_m\rangle \mid x_1 = x_m\}$ is an element of $\mathcal{O}_m$*
- *$\mathcal{O}_m$ is closed under pairwise union, intersection, and complementation.*
- *If $\mathbf{A} \in \mathcal{O}_m$ then $\mathbf{A} \times \mathbb{R}$ and $\mathbb{R} \times \mathbf{A}$ are in $\mathcal{O}_{m+1}$.*
- *If $A \in \mathcal{O}_{m+1}$ then the projection of A,onto the first m coordinates,*
  $$\pi(A) = \{\langle x_1, \ldots, x_m\rangle \mid \exists_y \langle x_1, \ldots, x_m, y\rangle \in A\}$$
  is in $\mathcal{O}_m$.

Examples of o-minimal structures include the class of semi-algebraic regions and the class of sub-analytic regions.

Definition 20 *Let $\mathcal{O}$ be an o-minimal collection over $\mathbb{R}$. A function α from $\mathbb{R}^m$ to $\mathbb{R}^n$ is* definable *with respect to $\mathcal{O}$ if the graph $\{\langle x, \alpha(x)\rangle \mid x \in \mathbb{R}^m\}$ is an element of $\mathcal{O}_{m+n}$.*

For the remainder of this section, let $\mathcal{O}$ be an o-minimal domain over the reals. Let Δ be the class of homeomorphisms from $\mathbb{R}^k$ to itself that are definable relative to $\mathcal{O}$. Let $\Lambda[\mathbb{R}]$ be the class of piecewise-linear homeomorphisms over $\mathbb{R}^k$. Let $UDef$ be the class of closed regular regions in $\mathcal{O}_k$, and let Def be the class of bounded regions in $UDef$.

Define the open box $\mathbf{B}^\circ = (-1, 1)^k$, the closed box $\bar{\mathbf{B}} = [-1, 1]^k$, and the piecewise projective mapping $\theta : \mathbf{B}^\circ \rightarrow \mathbb{R}^k$ as in section 3.

Lemma 23 *For any finite set of definable regions $\mathcal{D} \subset UDef$ there exists a definable homeomorphism ψ over their union such, for every $\mathbf{D} \in \mathcal{D}$, $\psi(\mathbf{D})$ is the union of simplices in $\mathbb{R}^k$.*

Proof: This is the triangulation theorem for o-minimal structures. See [12] p. 130. ∎

Lemma 24 *Let $\langle \mathbf{X}_1, \ldots, \mathbf{X}_m \rangle$ and $\langle \mathbf{Y}_1, \ldots, \mathbf{Y}_m \rangle$ be sequences of polyhedra. If there is a definable homeomorphism α such that $\mathbf{Y}_i = \alpha(\mathbf{X}_i)$ for $I = 1, \ldots, m$, then there exists a PL homeomorphism ϕ such that $\mathbf{Y}_i = \phi(\mathbf{X}_i)$ for $I = 1, \ldots, m$.*

That is, the space of definable homeomorphisms is rectifiable to the space of piecewise linear homeomorphisms over the domain of polyhedra.

Proof: See [11], p. 5, Theorem 2.1. ▮

Lemma 25 *Let $\mathcal{D}$ be a finite subset of $UDef$. Then there exists a definable homeomorphism ϕ from $\mathbb{R}^k$ to $\mathbb{R}^k$ such that $\phi(\mathcal{D}) \subset UPoly$. That is, $UDef$ is embeddable in $UPoly$ over Δ.*

Likewise, Def is embeddable in $Poly$ over Δ.

Proof: Let $\mathcal{E} = \{\text{interior}(\mathbf{D}) \,|\, \mathbf{D} \in \mathcal{D}\} \cup \{\mathbb{R}^k\}$ By Lemma 23 there exists a homeomorphism ψ_1 such that $\psi_1(\mathbf{E})$ is the union of simplices for all $\mathbf{E} \in \mathcal{E}$. Note that $\psi_1(\mathbf{E}) \subset \psi(\mathbb{R}^k)$ for each $\mathbf{E} \in \mathcal{E}$. Also, $\psi_1(\mathbb{R}^k)$ is a bounded open polyhedron homeomorphic to $\mathbb{R}^k$ and therefore to the open box $\mathbf{B}^\circ$. By Lemma 24 there exists a PL homeomorphism ψ_2 from $\phi_1(\mathbb{R}^k)$ to $\mathbf{B}^\circ$. Since ψ_2 is piecewise linear, for each $\mathbf{E} \in \mathcal{E}$, $\psi_2(\psi_1(\mathbf{E}))$ is an open polyhedron inside $\mathbf{B}^\circ$. Now apply the piecewise projective transformation θ and let ϕ be the composition $\phi = \theta \circ \psi_2 \circ \psi_1$. Then for each $\mathbf{E}_i \in \mathcal{E}$, $\phi(\mathbf{E})$ is an open polytope, so $\phi(\mathbf{D}_i) = \text{closure}(\phi(E_i))$ is a closed polytope.

If all of the $\mathbf{D}$ are bounded, then $\phi(\mathbf{D})$ is bounded and thus in $Poly$. ▮.

Lemma 26 *The space of definable homeomorphisms Δ is rectifiable to the space of piecewise linear homeomorphisms $\Lambda[\mathbb{R}]$ over $UPoly$.*

Proof: Let $\mathcal{D} = \mathbf{D}_1, \ldots, \mathbf{D}_m$ be a set of polytopes, and let α be a homeomorphism in Δ such that $\alpha(\mathbf{D}_i)$ is a polytope for $i = 1, \ldots, m$. For $i = 1, \ldots, m$ let $\mathbf{E}_i = \text{interior}(\mathbf{D}_i)$ and let $\mathbf{P}_i = \theta^{-1}(\mathbf{E}_i)$; thus $\mathbf{P}_i \subset \mathbf{B}^\circ$. Consider the composition $\phi = \theta^{-1} \circ \alpha \circ \theta$, which is a definable homeomorphism from $\mathbf{B}^\circ$ to itself. Note that for any $\mathbf{P}_i$, $\phi(\mathbf{P}_i) = \theta^{-1}(\alpha(\mathbf{E}_i))$ is an open polyhedron in $\mathbf{B}^\circ$; thus ϕ is a definable homeomorphism mapping polyhedra in $\mathbf{B}^\circ$ to polyhedra in $\mathbf{B}^\circ$. By Lemma 24, there exists a PL-mapping ψ such that $\psi(\mathbf{P}_i) = \phi(\mathbf{P}_i)$ for $i = 1, \ldots, m$. Now let χ be the composition $\theta \circ \psi \circ \theta^{-1}$. For any $\mathbf{E}_i$,

$$\chi(\mathbf{E}_i) = \theta(\psi(\mathbf{P}_i)) = \theta(\phi(\mathbf{P}_i)) = \theta(\theta^{-1}(\alpha(\mathbf{E}_i))) = \alpha(\mathbf{E}_i)$$

Then χ satisfies the conditions of Lemma 19, so there exists a PL-mapping ϕ such that $\phi(\mathbf{E}_i) = \alpha(\mathbf{E}_i)$, and therefore $\phi(\mathbf{D}_i) = \alpha(\mathbf{D}_i)$. ▮.

Theorem 27 *Let $\mathcal{O}$ be an o-minimal structure over the reals. Let $UDef$ be the class of regular regions in $\mathbb{R}^k$ definable in $\mathcal{O}$. Let $\mathcal{P}_1, \ldots, \mathcal{P}_m$ be topological predicates over regions in $\mathbb{R}^k$. Then the structures $\langle UDef, \mathcal{P}_1, \ldots, \mathcal{P}_m \rangle$ and $\langle UPoly, \mathcal{P}_1, \ldots, \mathcal{P}_m \rangle$ are elementarily equivalent.*

Let Def be the collection of bounded regions in $UDef$. Then $\langle Def, \mathcal{P}_1, \ldots, \mathcal{P}_m \rangle$ and $\langle Poly, \mathcal{P}_1, \ldots, \mathcal{P}_m \rangle$ are elementarily equivalent.

Proof: It is immediate that $UDef$ is closed under Δ and that $UPoly$ is closed under $\Lambda[\mathbb{R}]$. Together with the results of Lemma 23 and Lemma 25, the conditions of Lemma 7 are satisfied, so $UDef$ and $UPoly$ are mutually extendible relative to Δ. The result then follows Theorem 6.

The same argument applies to Def and $Poly$. ▮

5 Conclusion

We have demonstrated that the domain of rational polyhedra and the domain of topologically regular, definable regions relative to an o-minimal structure over the reals are each elementarily equivalent to the domain of polyhedra.

The most obvious problem left open is to find a theorem comparable to Theorem 1, giving *geometric* criteria rather than algebraic or structural criteria sufficient to guarantee elementary equivalence to polyhedra that will apply in dimensions higher than 2. One would think it should be possible to find some reasonable set of geometric conditions that guarantee a simplicial structure that will support the kind of construction used in the proof of Theorem 16, but I have not been able to work it through.

References

[1] W.M. Beynon. On rational subdivisions of polyhedra with rational vertices. *Canadian Journal of Mathematics,* **19**(2), 238-242, 1977.

[2] A.G. Cohn and J. Renz. Qualitative spatial representation and reasoning. In *The Handbook of Knowledge Representation,* F. van Harmelen, V. Lifschitz, and B. Porter (eds.), Elsevier, Oxford, 2008.

[3] M. Egenhofer and R. Franzosa. Point-set topological spatial relations. *International Journal of Geographical Information Systems,* **5**(2), 161-174, 1991.

[4] R. Kontchakov, I. Pratt-Hartmann, and M. Zakharyaschev. Interpreting topological logics over Euclidean space. In *Twelfth International Conference on Principles of Knowledge Representation and Reasoning (KR-10).* 2010.

[5] R. Kontchakov, I. Pratt-Hartmann, F. Wolter. and M. Zakharyaschev. Spatial logics with connectedness predicates. *Logical Methods in Computer Science,* 2010.

[6] J. Munkres. *Elements of Algebraic Topology.* Menlo Park, CA: Addison Wesley, 1984.

[7] I. Pratt-Hartmann. First-order mereotopology. In *Handbook of Spatial Logics,* M. Aiello, I. Pratt-Hartmann, and J. van Benthem (eds.), Springer-Verlag, 2007.

[8] D.A. Randell, Z. Cui, and A.G. Cohn. Modelling topological and metrical properties of physical properties, *First International Conference on Principles of Knowledge Representation and Reasoning (KR-89).* 55-66, 1989.

[9] D.A. Randell, Z. Cui, and A.G. Cohn. A spatial logic based on regions and connection. *Third International Conference on Principles of Knowledge Representation and Reasoning (KR-92)* 1992. 165-176.

[10] C.P. Rourke, and B.J. Sanderson. *Introduction to Piecewise-Linear Topology.* Springer-Verlag, 1982.

[11] M. Shiota. PL and differential topology in o-minimal structure. http://arxiv.org/abs/1002.1508v3, 2010.

[12] L. van den Dries. *Tame Topology and O-minimal Structures.* Cambridge University Press, 1998.