Propositional Calculus

Atomic sentence: \( p, q, r \ldots \)

Boolean operators:

\[ \neg p \quad \text{not } p. \]
\[ p \land q \quad p \text{ and } q. \]
\[ p \lor q \quad p \text{ or } q. \]
\[ p \rightarrow q \quad \text{if } p \text{ then } q \]
\[ p \leftrightarrow q \quad p \text{ if and only if } q. \]

Sentence: Either an atomic sentence or a Boolean operator applied to sentences.

Examples:

\[ p, \quad p \lor q, \quad \neg p \leftrightarrow (q \lor p). \]

A literal is either an atomic sentence or the negation of an atomic sentence.

Examples: \( p, q, \neg p, \neg q. \)

A sentence is in conjunctive normal form (CNF) if it is the disjunction of literals. A set of sentences is in CNF if each sentence is in CNF.

Example: The following set of sentences is in CNF.

\[ p, \quad \neg p \lor q \lor r, \quad q \lor \neg r. \]

Converting a sentence to CNF:

1. Replace every occurrence of \( \leftrightarrow \) by \((\alpha \Rightarrow \beta) \land (\beta \Rightarrow \alpha). \)
   When this is complete, the sentence will have no occurrence of \( \leftrightarrow. \)

2. Replace every occurrence of \( \Rightarrow \) by \( \neg \alpha \lor \beta. \) When this is complete, the only Boolean operators will be \( \lor, \neg, \text{ and } \land. \)

3. Replace every occurrence of \( \neg (\alpha \lor \beta) \) by \( \neg \alpha \land \neg \beta; \) every occurrence of \( \neg (\alpha \land \beta) \) by \( \neg \alpha \lor \neg \beta; \) and every occurrence of \( \neg \neg \alpha \) by \( \alpha. \) Repeat as long as applicable. When this is done, all negations will be next to an atomic sentence.

4. Replace every occurrence of \( (\alpha \land \beta) \lor \gamma \) by \((\alpha \lor \gamma) \land (\beta \lor \gamma), \) and every occurrence of \( \alpha \lor (\beta \land \gamma) \) by \((\alpha \lor \beta) \land (\alpha \lor \gamma). \) Repeat as long as applicable. When this is done, all conjunctions will be at top level.

5. Break up the top-level conjunctions into separate sentences. That is, replace \( \alpha \land \beta \) by the two sentences \( \alpha \) and \( \beta. \) When this is done, the set will be in CNF.

Example:
Start: \((p \Rightarrow q) \Leftrightarrow r\).
After step 1: \(((p \Rightarrow q) \Rightarrow r) \land (r \Rightarrow (p \Rightarrow q))\).
After step 2: \((-p \lor q) \lor r) \land (-r \lor (-p \lor q))\).
Step 3(a): \(((\neg p \land \neg q) \lor r) \land (-r \lor (-p \lor q))\).
After step 3: \(((p \land q) \lor r) \land (-r \lor (-p \lor q))\).
After step 4: \(((p \lor r) \land (-q \lor r)) \land (-r \lor (-p \lor q))\).
After step 5: \{p \lor r\}
\begin{align*}
\neg q \lor r. \\
\neg r \lor -p \lor q. 
\end{align*}