Independent Evidence

The independence assumption can also be used in evidence combination. Suppose that we have two events E and F, both of which are evidence for another event X. If we can assume that E and F are independent given X, and that E and F are independent given $\neg X$ then it is easy to compute $\mathrm{P}(X|E \wedge F)$ from $\mathrm{P}(X|E)$ and $\mathrm{P}(X|F)$ as follows:

Let E and F be independent, both with respect to X and with respect to $\neg X$; thus $\mathrm{P}(E \wedge F|X) = \mathrm{P}(E|X)\ \mathrm{P}(F|X)$ and $\mathrm{P}(E \wedge F|\neg X) = \mathrm{P}(E|\neg X)\ \mathrm{P}(F|\neg X)$. By Bayes' formula

1.
$$\mathrm{P}(X|E \wedge F) = \frac{\mathrm{P}(E \wedge F|X)\mathrm{P}(X)}{\mathrm{P}(E \wedge F)}$$

Similarly,

2.
$$\mathrm{P}(\neg X|E \wedge F) = \frac{\mathrm{P}(E \wedge F|\neg X)\mathrm{P}(\neg X)}{\mathrm{P}(E \wedge F)}$$

Dividing formula (1) by (2) yields

3.
$$\frac{\mathrm{P}(X|E \wedge F)}{\mathrm{P}(\neg X|E \wedge F)} = \frac{\mathrm{P}(E \wedge F|X)\mathrm{P}(X)}{\mathrm{P}(E \wedge F|\neg X)\mathrm{P}(\neg X)}$$

Using our independence assumptions, we can rewrite this.

4.
$$\frac{\mathrm{P}(X|E \wedge F)}{\mathrm{P}(\neg X|E \wedge F)} = \frac{\mathrm{P}(E|X)\mathrm{P}(F|X)\mathrm{P}(X)}{\mathrm{P}(E|\neg X)\mathrm{P}(F|\neg X)\mathrm{P}(\neg X)}$$

Let us define the odds on A as the ratio $\mathrm{P}(A)/\mathrm{P}(\neg A)$. Thus, if $\mathrm{P}(A) = 1/4$, then $\mathrm{P}(\neg A) = 1 - \mathrm{P}(A) = 3/4$ so $\mathrm{Odds}(A) = (1/4)/(3/4) = 1/3$. Analogously, we define $\mathrm{Odds}(A|B)$ as the ratio $\mathrm{P}(A|B)/\mathrm{P}(\neg A|B)$. Using Bayes' rule

5.
$$\mathrm{Odds}(A|B) = \frac{\mathrm{P}(A|B)}{\mathrm{P}(\neg A|B)} = \frac{\mathrm{P}(B|A)\mathrm{P}(A)/\mathrm{P}(B)}{\mathrm{P}(B|\neg A)\mathrm{P}(\neg A)/\mathrm{P}(B)} = \frac{\mathrm{P}(B|A)}{\mathrm{P}(B|\neg A)} \cdot \mathrm{Odds}(A)$$

so 6.
$$\frac{\mathrm{P}(B|A)}{\mathrm{P}(B|\neg A)} = \frac{\mathrm{Odds}(A|B)}{\mathrm{Odds}(A)}$$

Let us now define the odds updating function $\mathrm{OU}(A|B)$ as the ratio $\mathrm{Odds}(A|B)/\mathrm{Odds}(A)$, the change that evidence B makes in the odds of A. From (6), we have

7.
$$\frac{\mathrm{P}(B|A)}{\mathrm{P}(B|\neg A)} = \mathrm{OU}(A|B)$$

We can therefore rewrite formula (4) as

8.
$$\mathrm{OU}(X|E \wedge F) = \mathrm{OU}(X|E) \cdot \mathrm{OU}(X|F)$$

For example, suppose that $\mathrm{P}(X) = 1/3$, $\mathrm{P}(X|E) = 2/3$, and $\mathrm{P}(X|F) = 3/4$. Then $\mathrm{Odds}(X) = 1/2$, $\mathrm{Odds}(X|E) = 2$, $\mathrm{Odds}(X|F) = 3$; so $\mathrm{OU}(X|E) = 4$ and $\mathrm{OU}(X|F) = 6$. Therefore $\mathrm{OU}(X|E \wedge F) = 24$, so $\mathrm{Odds}(X|E \wedge F) = \mathrm{OU}(X|E \wedge F) \cdot \mathrm{Odds}(X) = 12$, and

$$\mathrm{P}(X|E \wedge F) = \frac{\mathrm{Odds}(X|E \wedge F)}{1 + \mathrm{Odds}(X|E \wedge F)} = 12/13$$

From *Representations of Commonsense Knowledge,* Ernest Davis, Morgan Kaufmann, 1990, pp. 128-130.