Subdivision Approximation for Curves and Integral Analysis

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Joint work with
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ABSTRACT

Geometric operations on curves and surfaces can be based on algebraic techniques (e.g., cylindrical algebraic decomposition, resultants) or on numerical/geometric techniques (e.g., subdivision methods, marching cubes). The latter techniques have adaptive complexity but are usually incomplete. To achieve completeness, hybrid techniques which combine numerical with algebraic techniques are usually used. In this talk, we focus on purely numerical techniques.

Vegter-Plantinga gave the first numerical subdivision algorithm that is guaranteed to compute isotopic approximations for implicit curves and surfaces that are non-singular. The computational model is non-algebraic, using only evaluation functions and the interval evaluation of functions and their derivatives. We show how to achieve isotopic approximation of implicit curves with isolated singularities within the Vegter-Plantinga model.

The complexity analysis of adaptive algorithms is an major challenge. We shall consider the 1-D version of their algorithm: this amounts to real root isolation. We introduce general framework and a novel integral formula for the complexity of EVAL, a version of their root isolation algorithm. We also show that for the benchmark problem where the input polynomial f is a integer polynomial of degree d with L-bit coefficient size, EVAL has $O(d^2L)$ complexity.

Our analysis technique might be called "continuous amortization argument", and exploits an evaluation-form of the Mahler-Davenport type bounds.

PART I.
Introduction

Computational Curves and Surfaces

- TWO APPROACHES
  - ∗ Algebraic Approach
  - ∗ Numerical/Geometric Approach

- PROS and CONS
  - ∗ ALGEBRAIC: robust and complete BUT inefficent, hard-to-implement, non-local
  - ∗ GEOMETRIC: fast, simple-to-implement, local BUT incomplete and non-robust
  - ∗ HYBRID METHOD: e.g., subdivision WITH algebraic primitives

- SUBDIVISION METHODs in Geometric Approaches
  - ∗ STRONG FORM: Snyder, Mourrain
  - ∗ WEAK FORM: Vegter-Plantinga (see below)

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General Objectives (CHALLENGES)

- OF COURSE...
  - ∗ Correct, Robust, Efficient

- (A) ADAPTIVE METHODS
  - ∗ Fast on typical or non-degenerate inputs
  - ∗ THIS TALK: subdivision method

- (B) PURELY NUMERICAL METHODS
  - ∗ No manipulation of algebraic numbers, resultants, root isolation
  - ∗ ADVANTAGES: simpler, applies to non-algebraic geometry
  - ∗ THIS TALK: purely numerical subdivision

- (C) COMPLETE METHODS
  - ∗ AVOID Assumptions on Inputs
  - ∗ E.g., non-singularity, Morseness, general position
  - ∗ THIS TALK: isolated singularities

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  - How can we detect tangential intersection?
  - Adaptive Application of Geometric Separation Bounds
- Near-Optimal Analysis of Descartes' Method [ISSAC'06]
  - Amortized Analysis – what does it mean?
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- PART II: Problem and Review
  - ∗ The Vegter-Plantinga computational model
- PART III: One-Dimensional Case
  - ∗ I.e., root isolation in presence of multiple roots
- PART IV: Integral Complexity Analysis
  - ∗ Continuous amortization arguments
  - ∗ Integral Analysis for non-singular 1-D
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PART II.
PROBLEM AND REVIEW

Problem Statement, Computational Model

- Meshing of Implicit Curve
  - ∗ GIVEN: $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ and $\varepsilon > 0$
  - ∗ COMPUTE: a polygonal approximation $\widetilde{S}$ for the surface $S : f = 0$.
  - ∗ WHERE: $\widetilde{S}$ is isotopic to S, and $d(\widetilde{S}, S) \leq \varepsilon$.

- The Vegter-Plantinga Computational Model
  - ∗ (1) [Like Marching Cube] Evaluate sign of $f(x)$ for $x \in \mathbb{F}^n \subseteq \mathbb{R}^n$
  - ∗ (2) [Interval Arithmetic] Evaluate interval versions of f and its derivatives

- NOTE: Model is applicable to non-algebraic f
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Interval Arithmetic

- Let $[a, b] = I \subseteq \mathbb{R}$ be a (closed) interval
  - $*$ Width: $w(I) = b - a$.
  - $*$ Midpoint: $m(I) = (a + b)/2$.

- For $S \subseteq \mathbb{R}$, let $\square S$ denote the set of all intervals contained in S
  - $*$ n-Boxes: $B \in \square\mathbb{R}^n$
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Review of Vegter-Plantinga Algorithm

- Assume $S : f = 0$ is a non-singular curve
  - ∗ Begin with any square box B_0
  - ∗ We want to approximate $S \cap B_0$

- TWO TESTS
  - ∗ $C_0(B)$: $\qquad 0 \notin \square f(B)$
  - ∗ $C_1(B)$: $\qquad 0 \notin (\square f_x(B))^2 + (\square f_y(B)^2)$

- STEP 1: MAIN LOOP

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        Initialize $Q$ with $\{B_0\}$
        while $Q$ is non-empty
              Remove $B$ from $Q$
1.            if $C_0(B)$, discard $B$
2.            elif $C_1(B)$, put $B$ into $Q^*$
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  - I.e., adjacent boxes in Q^* differ by at most one level

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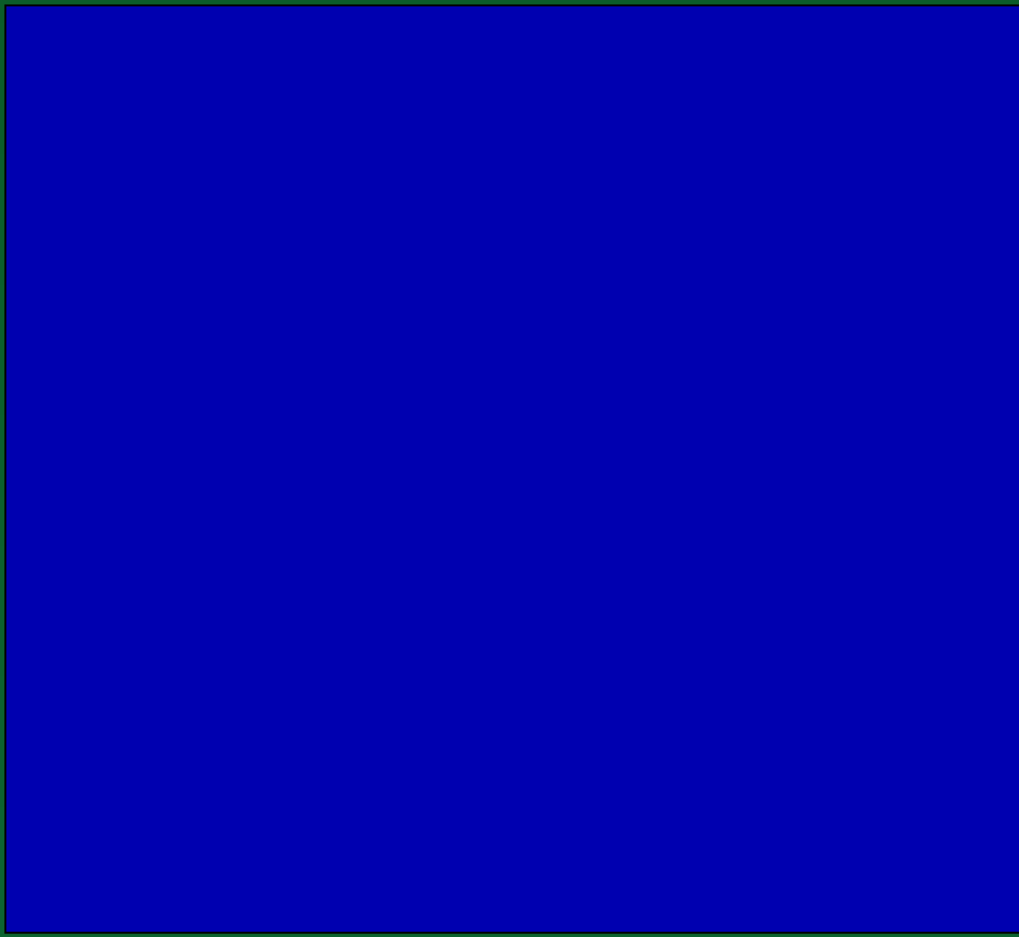

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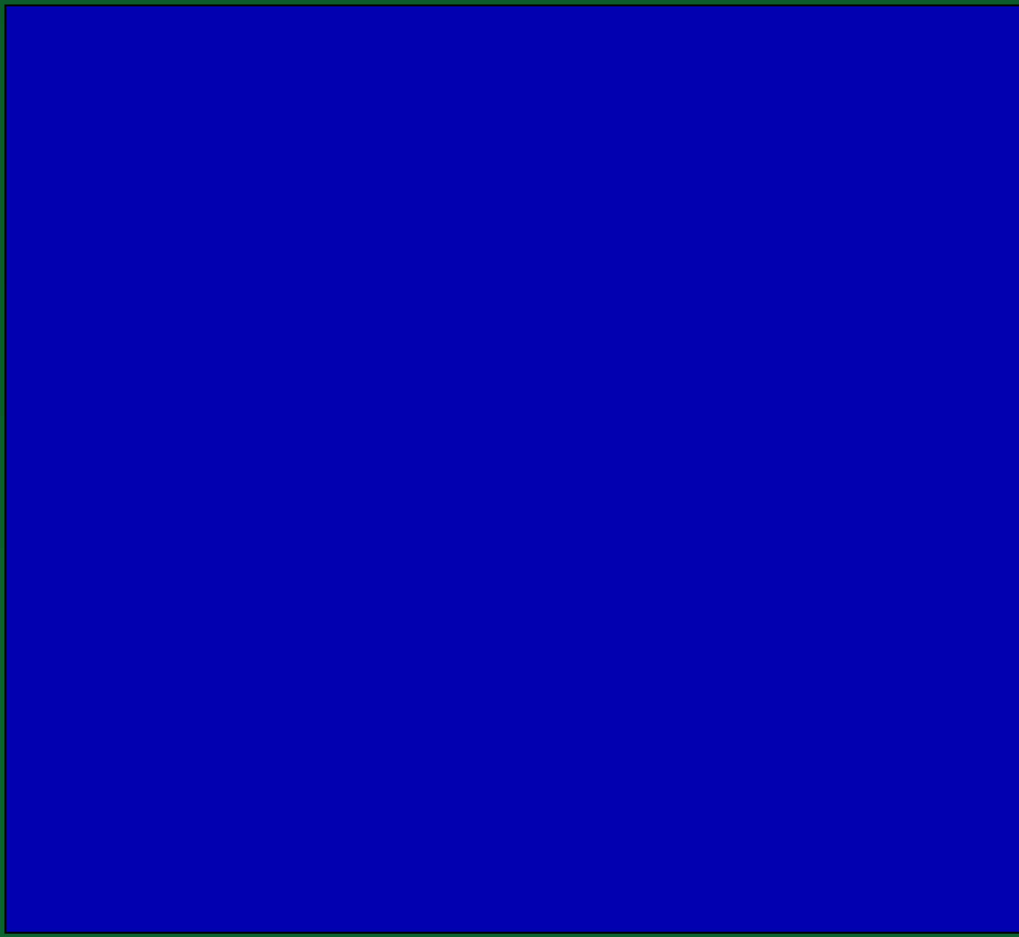

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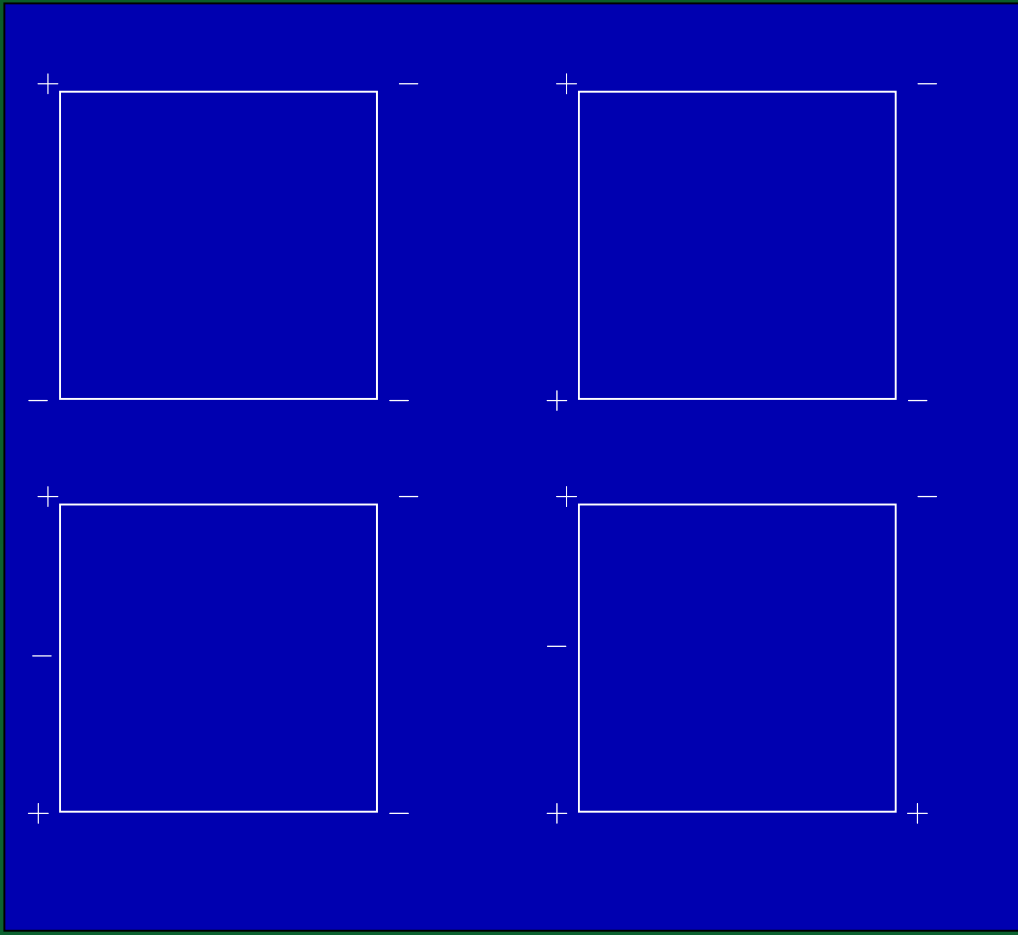

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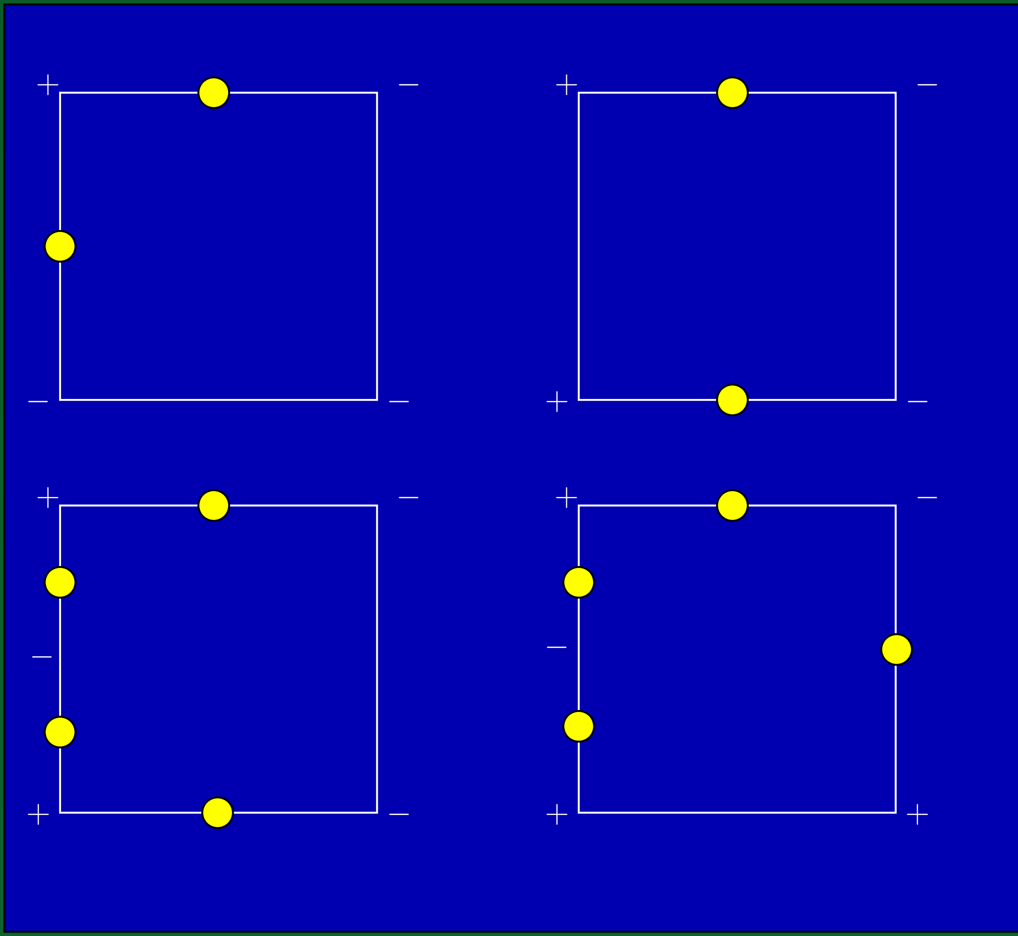

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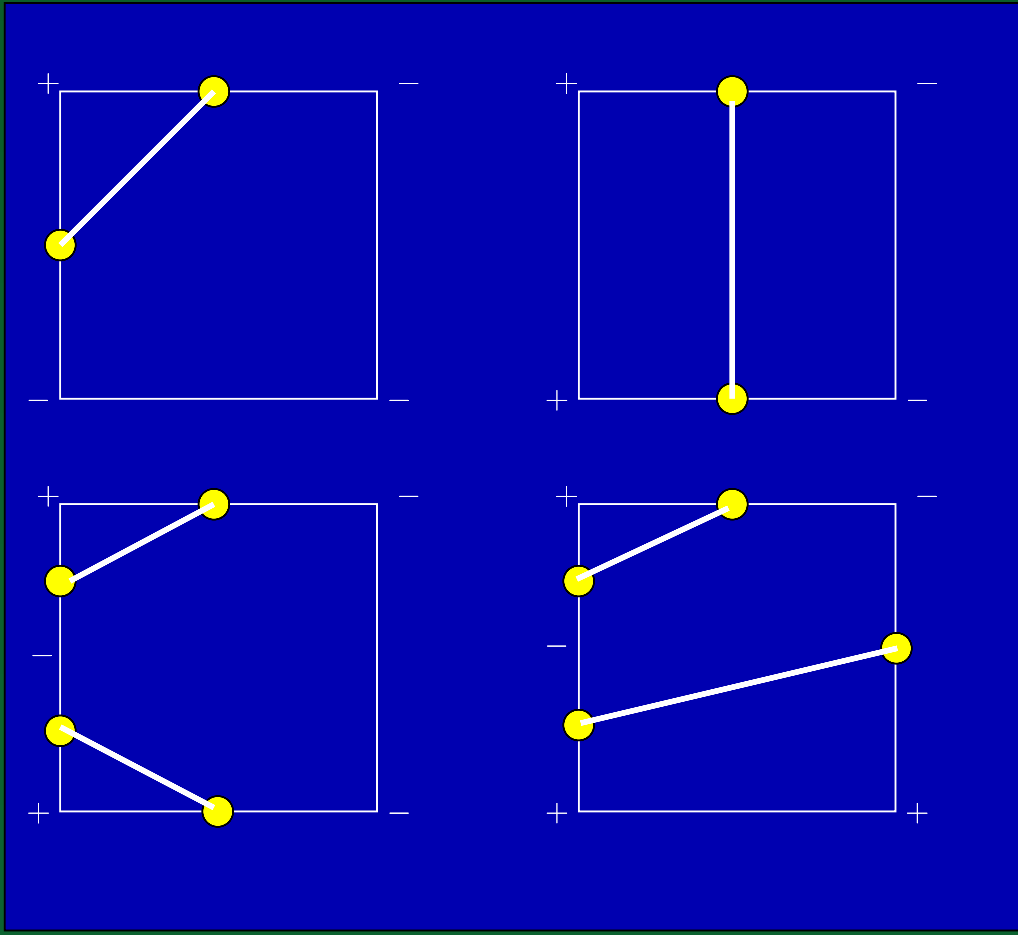

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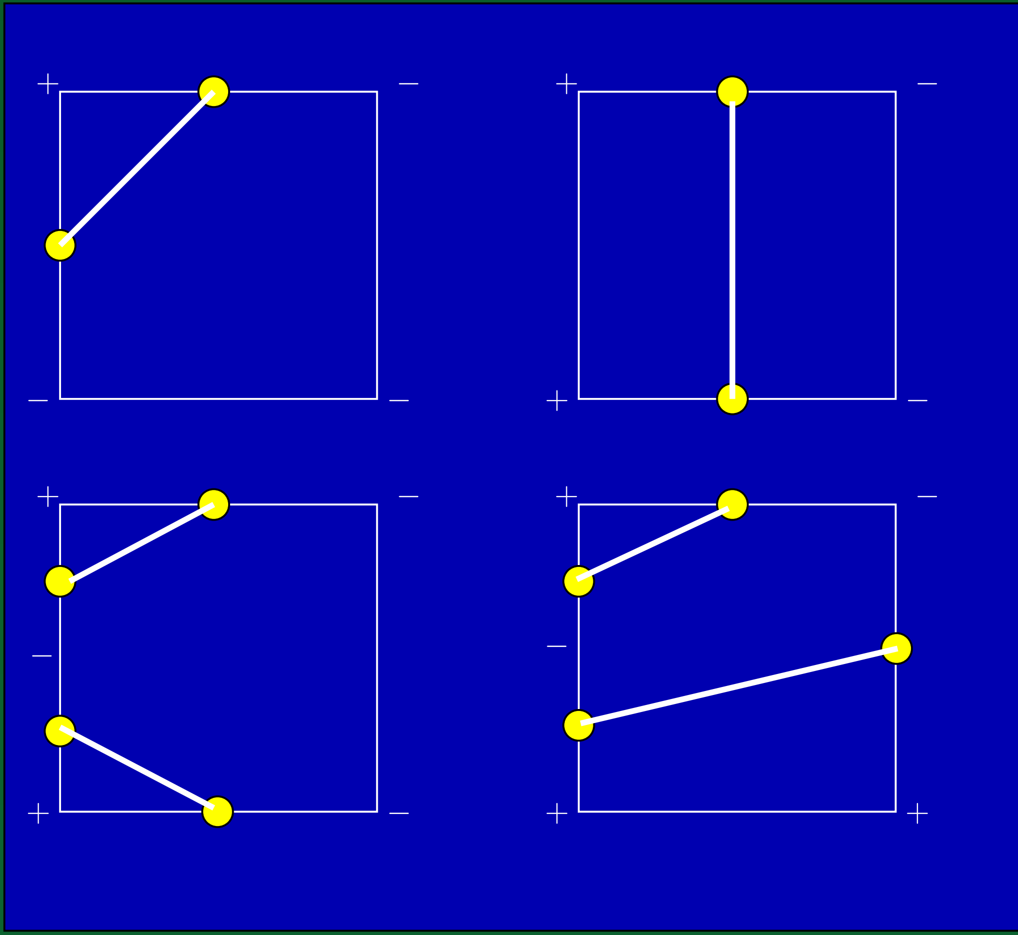

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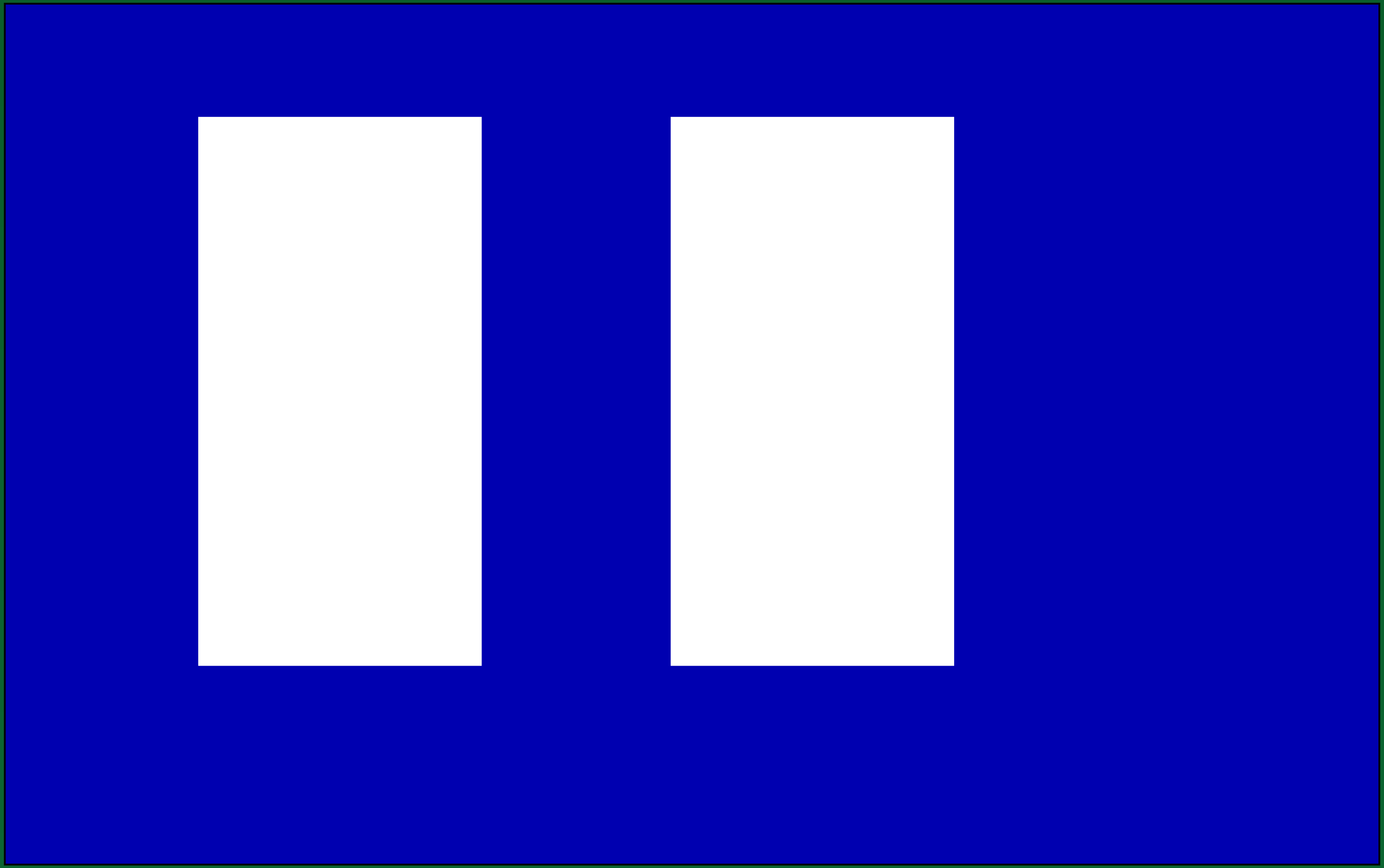

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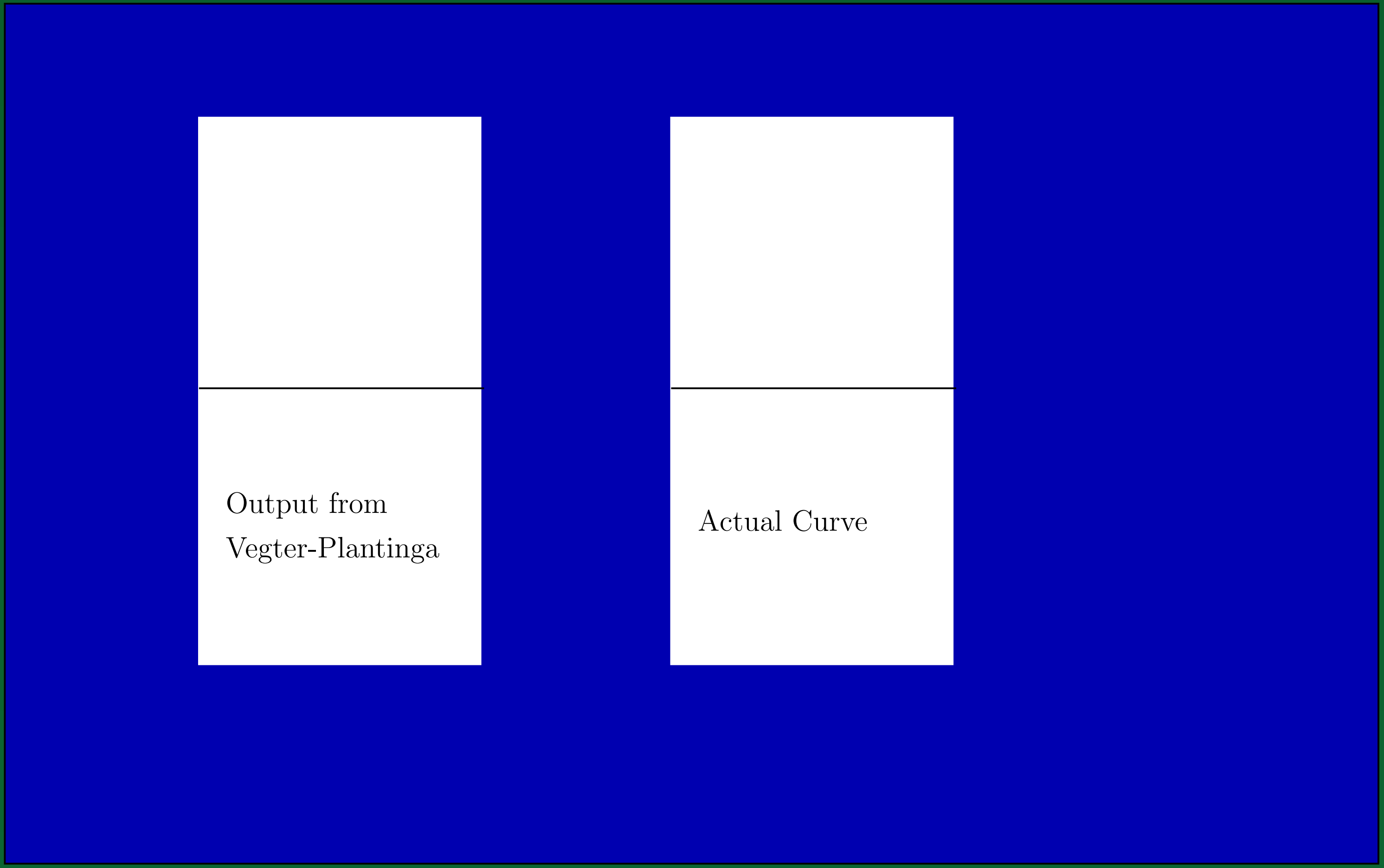

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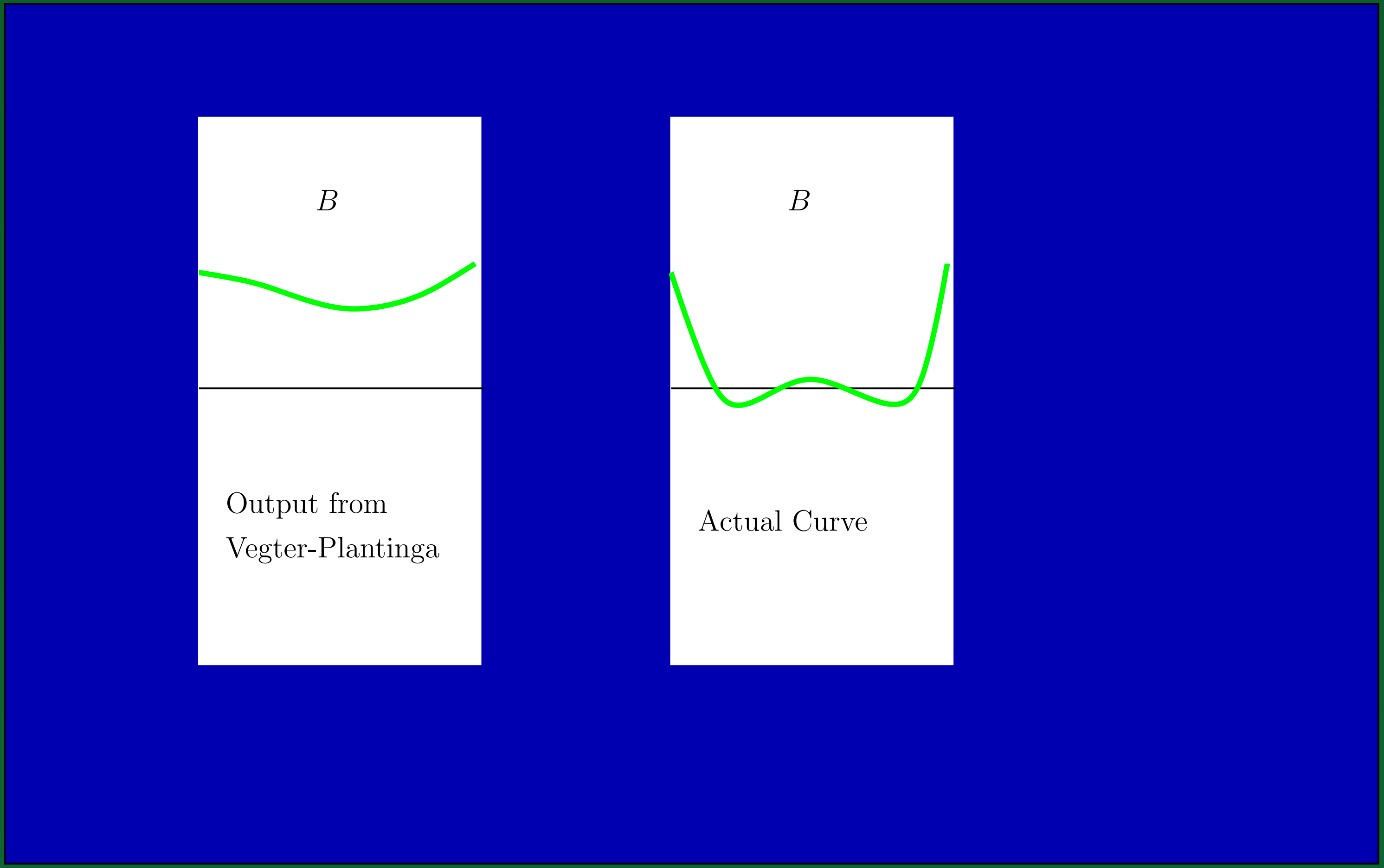

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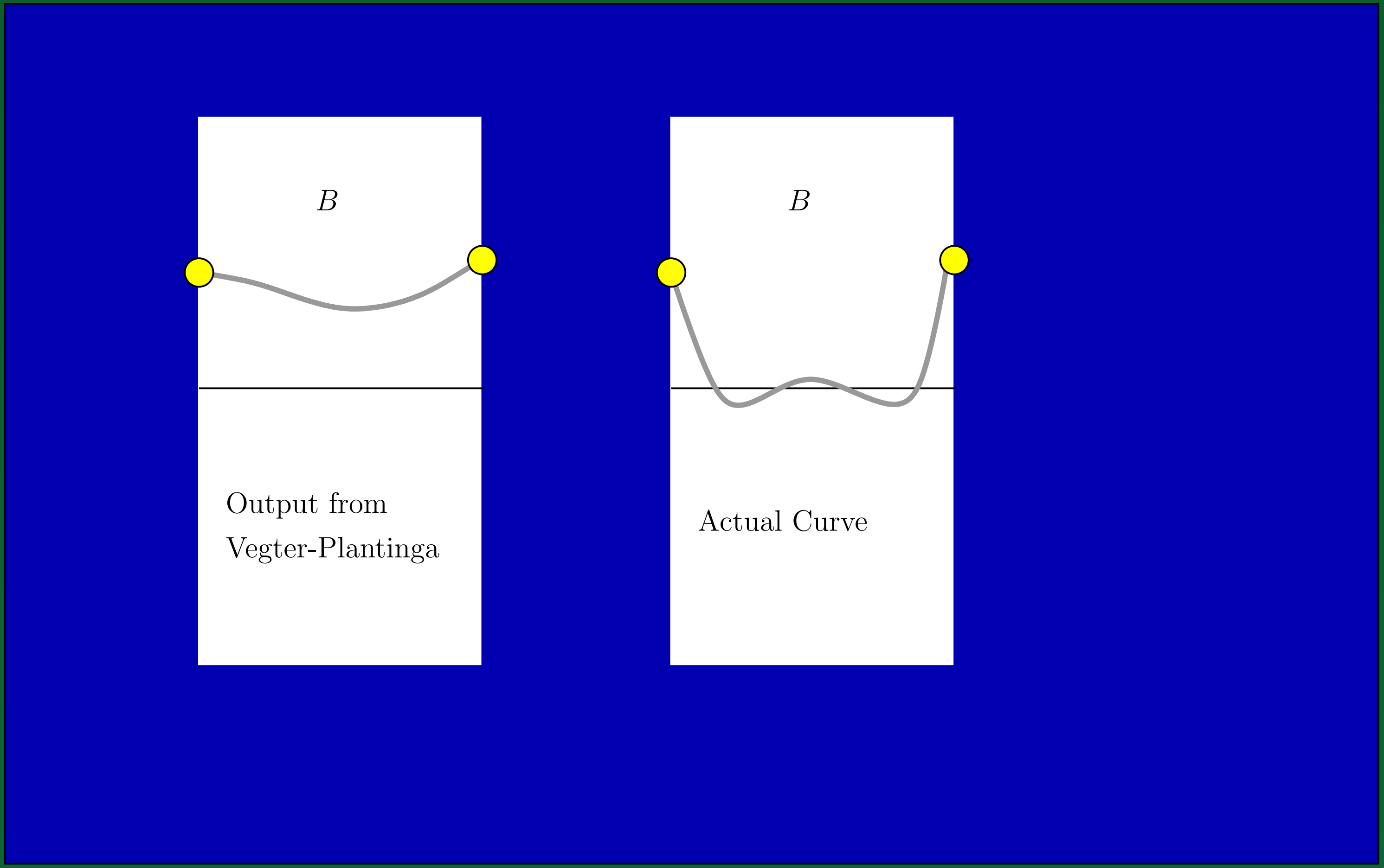

- THEOREM (Vegter-Plantinga)
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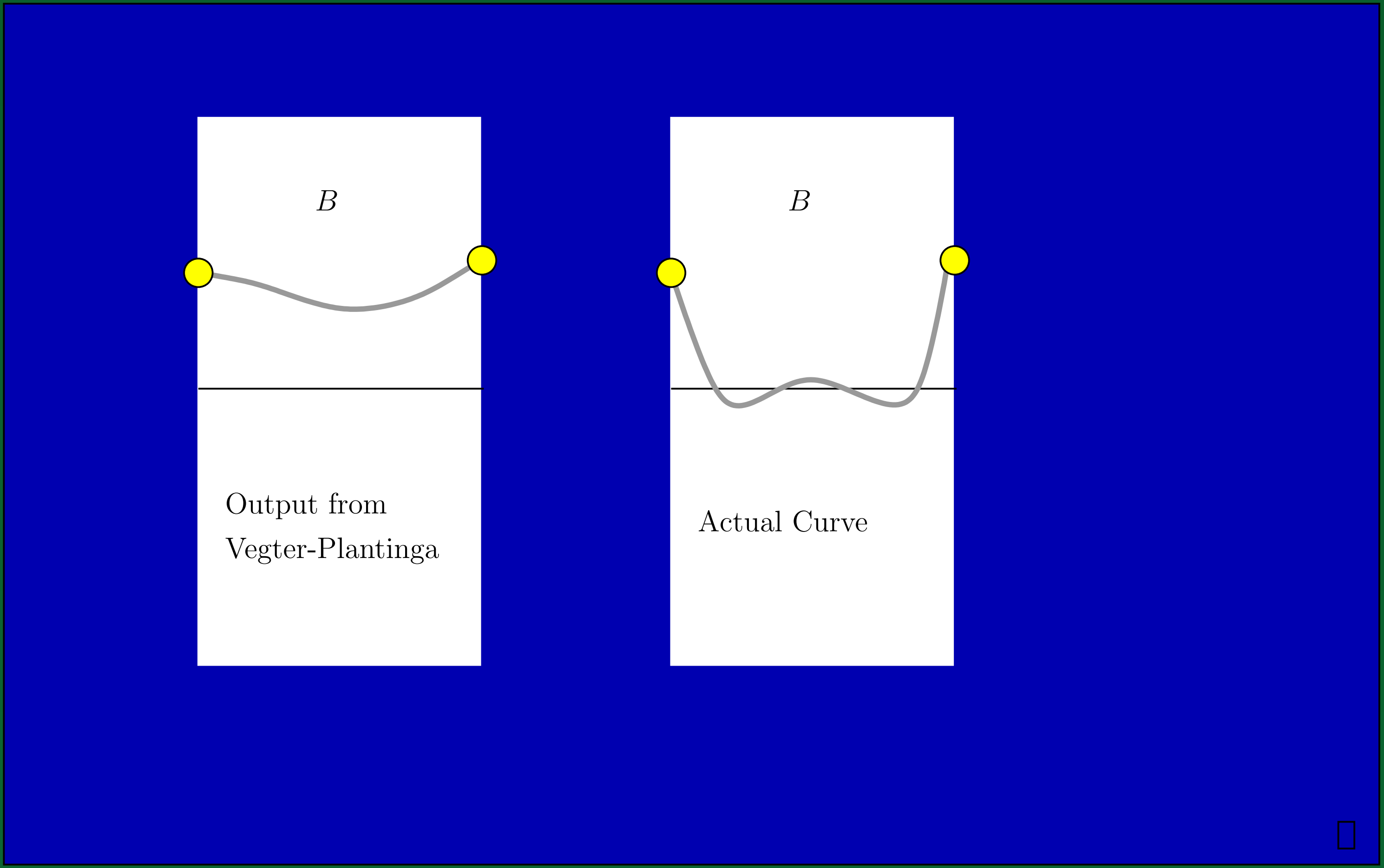

- We will need EXTENSION 1: If S is not contained in B_0
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- Completeness Issue:
  - ∗ How to confirm singularities in the in the Vegter-Plantinga Model?
- Meshing curves is a 2-D problem
  - ∗ The 1-D analogue is Root Isolation!
- STRATEGY: First understand the 1-D case

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PART III.
THE SINGULAR CASE IN 1-D

The Algorithm EVAL

- EVAL: an adaptation of Vegter-Plantinga to 1-D
  - ∗ STEP 1: MAIN LOOP

```
        Initialize $Q$ with $\{B_0\}$
        while $Q$ is non-empty
              Remove $B$ from $Q$
1.            if $C_0(B)$, discard $B$
2.            elif $C_1(B)$, put $B$ [if necessary] into $Q^*$
3.            else split $B$ into 4 [2 not 4] squares and place in $Q$
```

- STEP 2: Process Q^*
  - ∗ NOTE: $C_1(B)$ says "$0 \notin \square f'(B)$"
  - ∗ Output $B = [a, b]$ iff $f(a)f(b) \leq 0$.

- Bolzano Theorem: if $f(a)f(b) < 0$, then $f(c) = 0$ for some $a < c < b$
  - ∗ Other Evaluation Methods: based on Newton operator

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- We need various bounds:
  - K_I: Lipschitz constant for f over I
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- ALSO: let $\Delta(f) \equiv \Delta(f,f')$, $\mathrm{EV}(f) \equiv \mathrm{EV}(f,f')$

- NonAdaptic MultipleRoot Isolation
  - Subdivide as long as interval is larger than

$$BOUND := \min\left\{\Delta(f),\ \Delta(f'),\ \Delta(f,f'),\ \frac{\mathrm{EV}(f)}{3K},\ \frac{\mathrm{EV}(f',f)}{2K'}\right\}$$

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```
ROOTISOL Input: $F : \mathbb{R} \longrightarrow \mathbb{R}$, and interval $I_0$
Output: A list $F$ of isolating intervals
      Initialize $Q$ to $I_0$.
      while $Q \neq \emptyset$
            $I = [a, b] \leftarrow Q.remove()$.
            if $|I| > BOUND$
                  if the midpoint of $I$, $m = (a + b)$, is a root of $F$
                        if midpoint of $I$ is also a root of $F'$, put $[m, m]$ into $L$ (singular).
                        else put $[m, m]$ into $L$ (nonsingular)
                  Split $I$ in two equal halves, and put them in $Q$.
            else
                  if $(F'(a)F'(b) \leq 0)$
1.                      if $|f(a)| \leq \mathrm{EV}(F, F')/3$, put $I$ into $L$ (singular).
2.                      else Discard $I$.
                  else % Thus, $F'(a)F'(b) > 0$
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4.                            else put $I$ into $L$ (nonsingular).
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PART IV.
COMPLEXITY ANALYSIS OF EVAL

Perspective on EVAL

- 3 Subdivision Methods for Isolating real zeros

$$STURM > DESCARTES > BOLZANO$$

- PRIMITIVES (decreasing complexity):
  - ∗ Sturm Query: Exact number of roots in I
  - ∗ Descartes/Bernstein Query: Rule of Sign
  - ∗ Bolzano Queries: $C_0(I)$ and $C_1(I)$

- WHAT IS THE SIZE OF THE SEARCH TREE?
  - ∗ Sturm Tree: $O(dL)$ [Davenport'86]
  - ∗ Descartes Tree: $O(dL)$ [Eigenwillig-Sharma-Y'06]
  - ∗ Bolzano Tree: $O(dL)$ for optimal □ f [Burr-Sharma-Y'07]

- Empirically, Descartes is faster than Sturm
  - ∗ Attributable to the cheaper primitive!
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Lipschitz Constants and Box Functions

- Analysis of EVAL for non-singular function $f : \mathbb{R} \to \mathbb{R}$

- Lipschitz Constant of f for interval X:
  - $K_X = K_X(f) := \max_{a \in X} \sum_{i=1}^{n} \frac{|f^{(i)}(a)|}{i!} w(X)^{i-1}$.
  - ALSO: let K'_X for $K_X(f')$

- Centered Form box function [Ratschek-Rokne]
  - $\square f(X) := \sum_{i=0}^{n} \frac{|f^{(i)}(m(X))|}{i!} \left(\frac{w(X)}{2}[-1,1] \right)^i$.

- PROPERTY: Let $I \subseteq X$
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General Framework of Stopping Functions

- BASIC LEMMA
  - ∗ Let $a \in J$ and $0 \in \Box f(J)$
  - ∗ Then $w(J) \geq |f(a)|/K_J$

- Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous.
  - ∗ An interval J is large (relative to g) if for all $a \in J$, $w(J) \geq g(a)$.
  - ∗ Call J big (relative to g) if for all $a \in J$, $w(J) \geq \frac{1}{2}g(a)$.
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- A subdivision P of I is a partition
  - ∗ obtained from $P = \{I\}$ by performing repeated bisections of intervals $J \in P$
  - ∗ P is a big subdivision if each $J \in P$ is big
  - ∗ Let $\#(P)$ be the size of P.

- LEMMA
  - ∗ Let P be a big subdivision of I relative to g. Then $\#(P) \leq 2 \int_I \frac{da}{g(a)}$.

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  - ∗ Let P be a big subdivision of I relative to g. Then $\#(P) \leq 2\int_I \frac{da}{g(a)}$.

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  - ∗ The Subdivision P of I at the end of STEP 1 of EVAL has size $\max\{1, \int_I \frac{2da}{g(a)}\}$.

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- Global Lipschitz Constants
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Bounding the Ideal Stopping Function

- BENCHMARK PROBLEM: let $f(x)$ an integer polynomial of degree d and the logarithmic height is $L = \lg \|f\|$
  - $*$ What is the complexity of isolating all the roots of f?
  - $*$ We know that $O(dL)$ is optimal (assuming $L \geq \log d$)

- Consider ideal case: $R := \int_a^b \frac{da}{g_*(a)} := \int_a^b \min\{\left|\frac{|f'(a)|}{f(a)}\right|, \left|\frac{|f''(a)|}{f'(a)}\right|\}da.$
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- Amortization Arguments for Evaluation: let $f, h \in \mathbb{Z}[x]$ have degrees m, n (resp.)
  - $*$ Let $\texttt{lead}(h) = b$ and $\beta_1, \ldots, \beta_k$ be complex zeros of h.
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PART V.
Isotopic Approximation of Singular Curves

Ensuring Isolated Singularities

- Let $f \in \mathbb{Z}[X_1, \ldots, X_n]$.
  - ∗ If $n = 1$, then f is square-free implies $Zero(f)$ has no singularites
  - ∗ WANTED: generalization for $n > 1$

- THEOREM: Let $f_1, \ldots, f_m \in \mathbb{Z}[X_1, \ldots, X_n]$ be non-constant.
  - ∗ IF $GCD(f_1, \ldots, f_m) = 1$
  - ∗ THEN $Zero(f_1, \ldots, f_m) \subseteq \mathbb{C}^n$ has dimension at most $n - 2$.
  - ∗ FURTHER, the dimension $n - 2$ can be achieved

- COROLLARY: Let $f \in \mathbb{Z}[X_1, \ldots, X_n]$ be squarefree.
  - ∗ The singularities of the surface $f = 0$ has dimension at most $n - 2$.

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- PREVIOUS METHODS:
  - ∗ Requires resultants (E.g., [Wolpert-Seidel]
  - ∗ Requires root isolation on box boundary (E.g., using topological degree, [Mourrain]

- STEP 1: CONSTRUCT $F = f^2 + f_x^2 + f_y^2$
  - ∗ So: $F(p) = 0$ iff p is a singular point of $f = 0$

- STEP 2: Perturb F
  - ∗ Let $\varepsilon_1 = \frac{1}{2}\mathrm{EV}(F)$
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  - ∗ The curve of $\{F = \varepsilon_1\}$ can be solved by Vegter-Plantinga!

- STEP 3: Consider singular regions of $\{F < \varepsilon_1\}$
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How to Determine the Singularity Degree

- GOAL: Determine the Degree of an isolated singularity
  - i.e., its vertex degree when the curve S is viewed as a graph

- Let δ be smaller than:
  - (1) The separation bound between singular points [SoCG'06]
  - (2) $\|p-q\|$ where p is singular point, $q \in S$ and $\nabla f(q) \| (p-q)$

- PROPERTY: if p is singular point
  - If $q \in S$ and $\|p-q\| < \delta$, then one of the paths from q will converge monontonically to p

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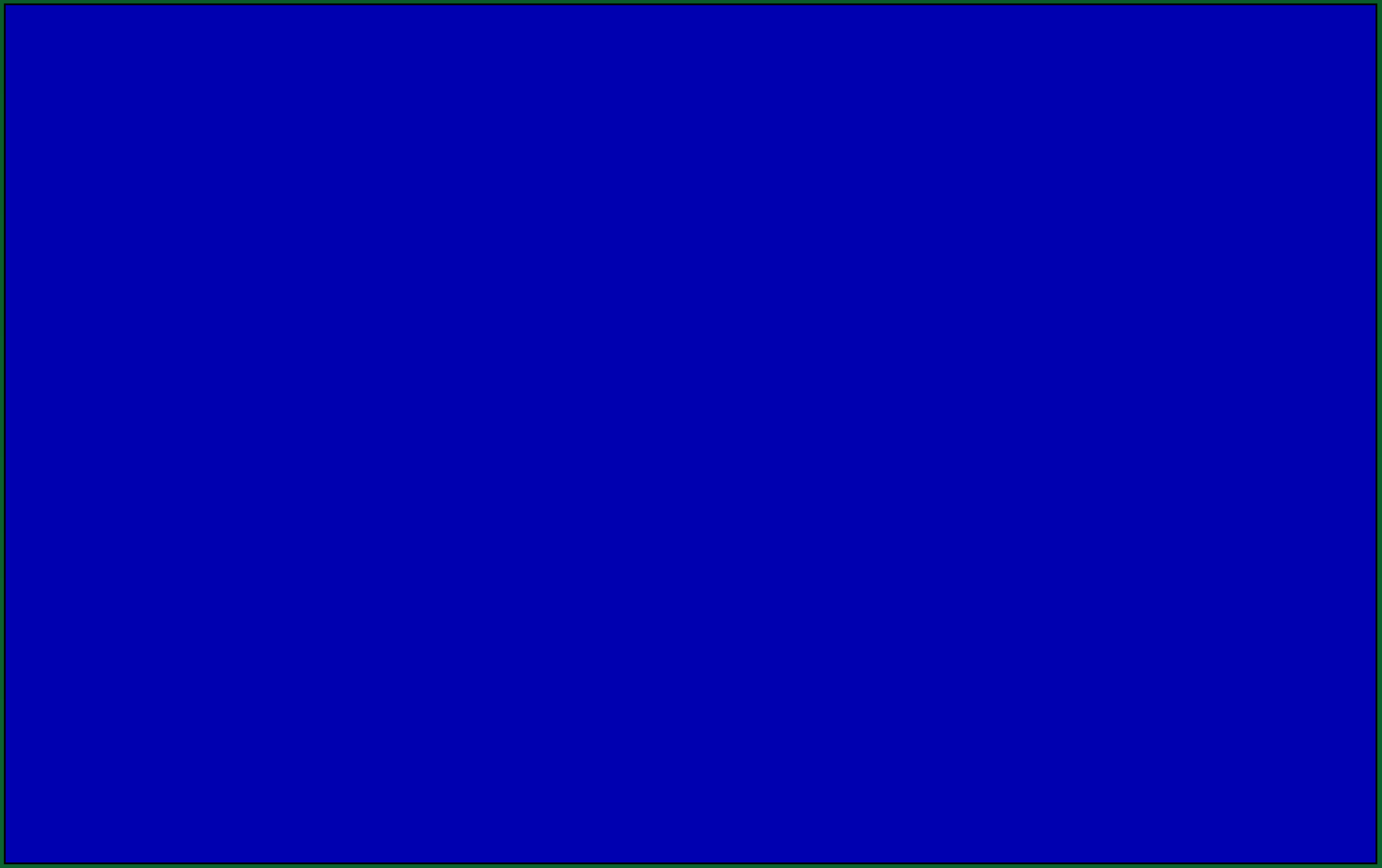
  - $*$ $w(B) < \delta$ and B' contains a singular point p



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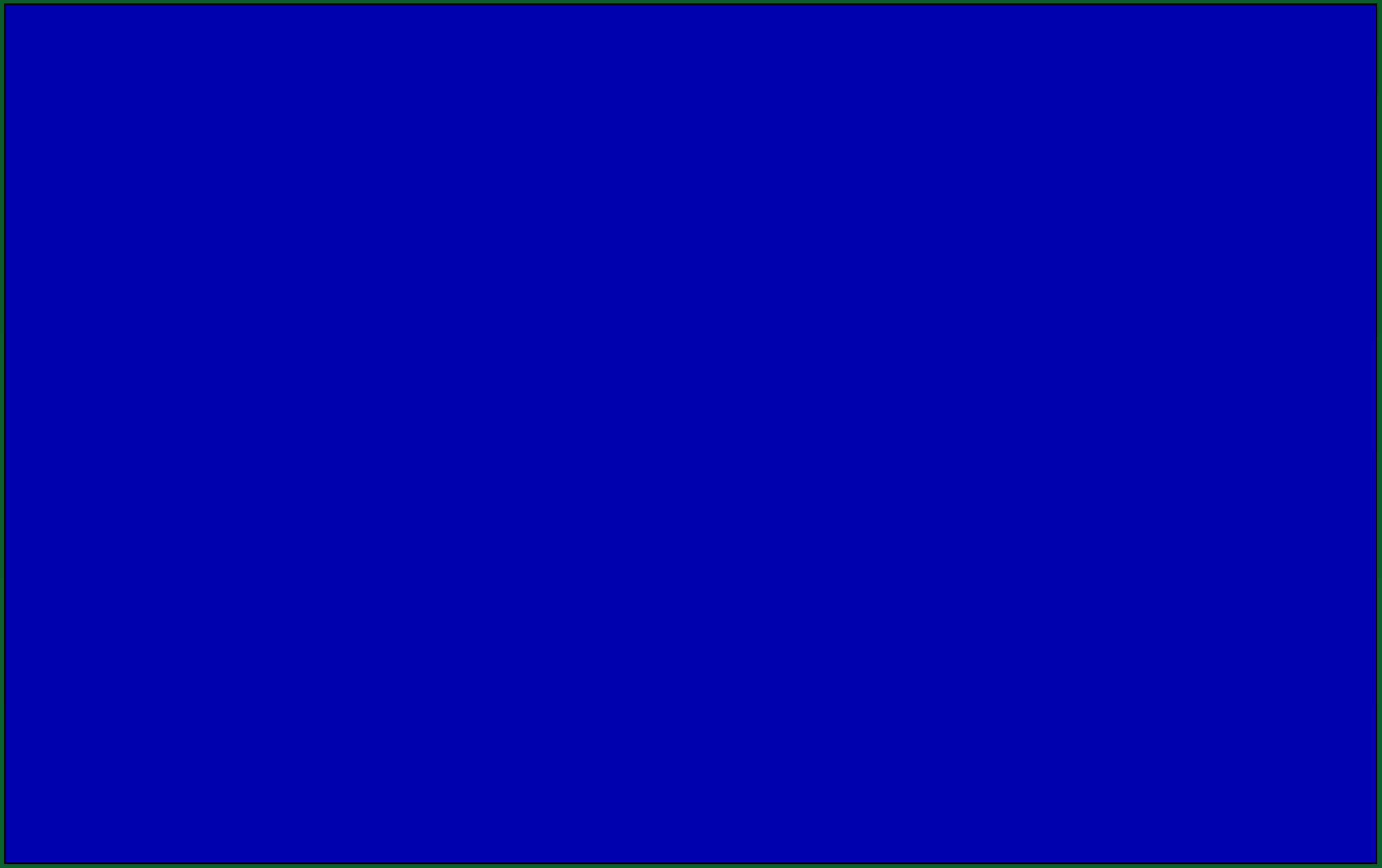

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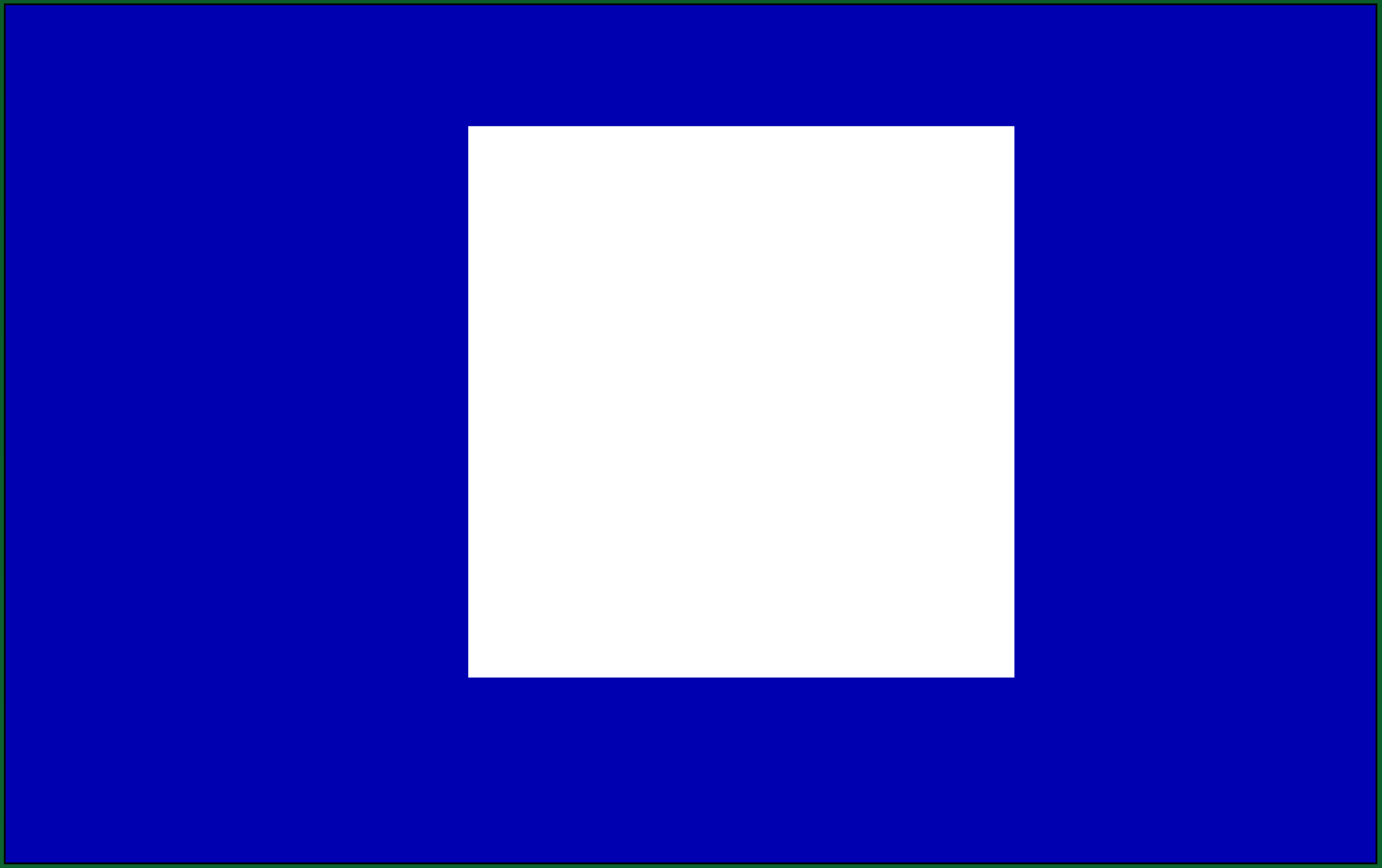

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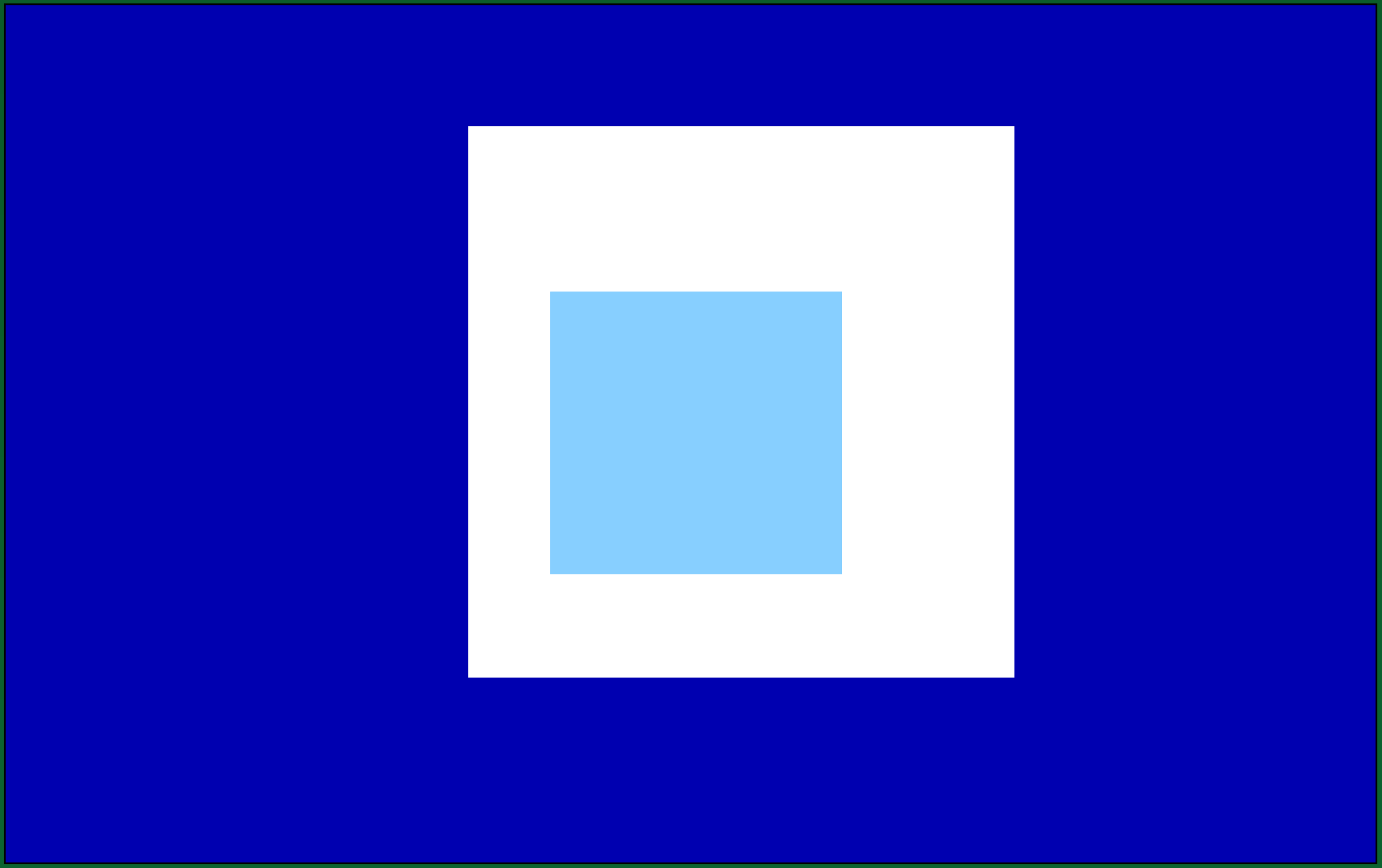

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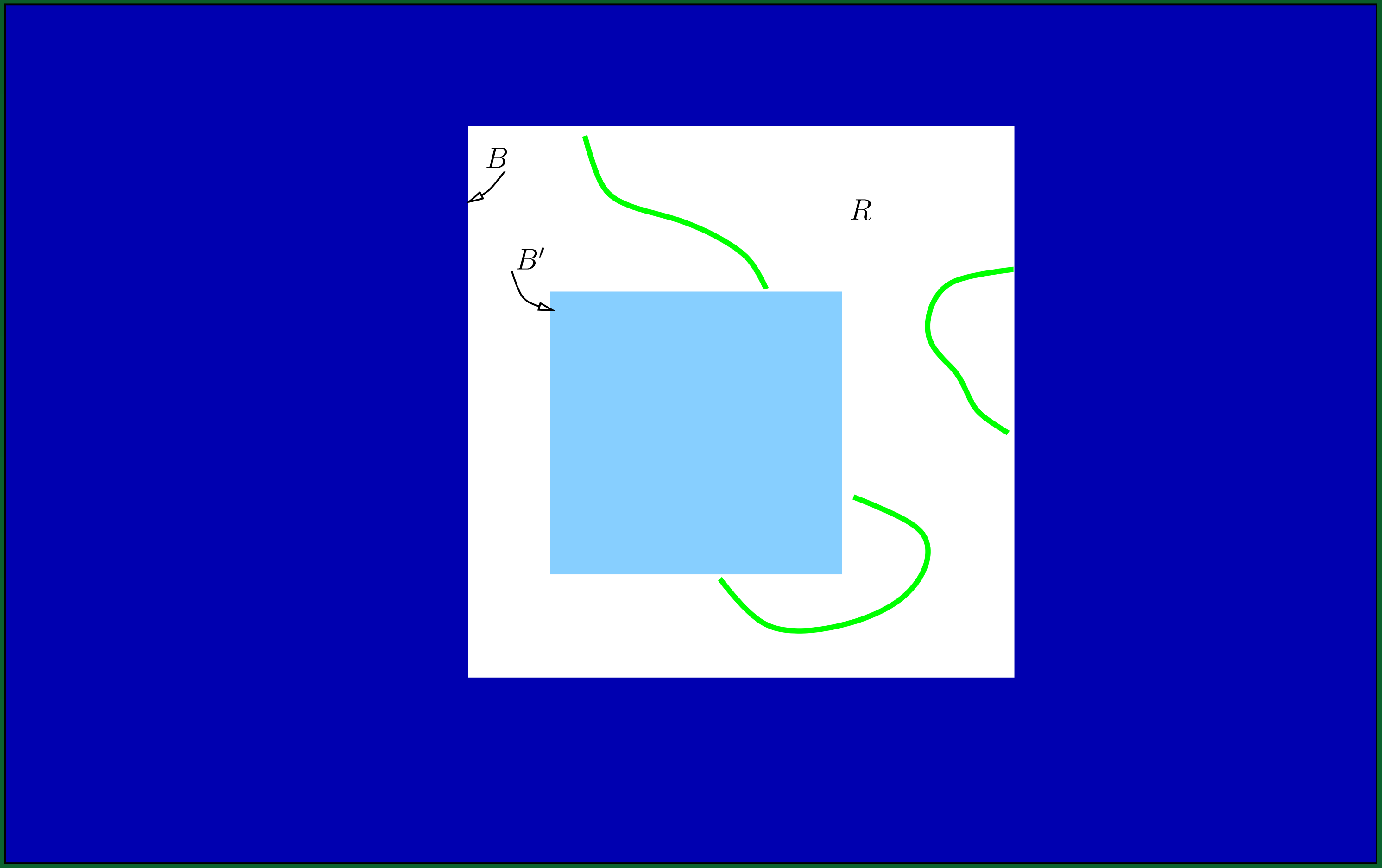

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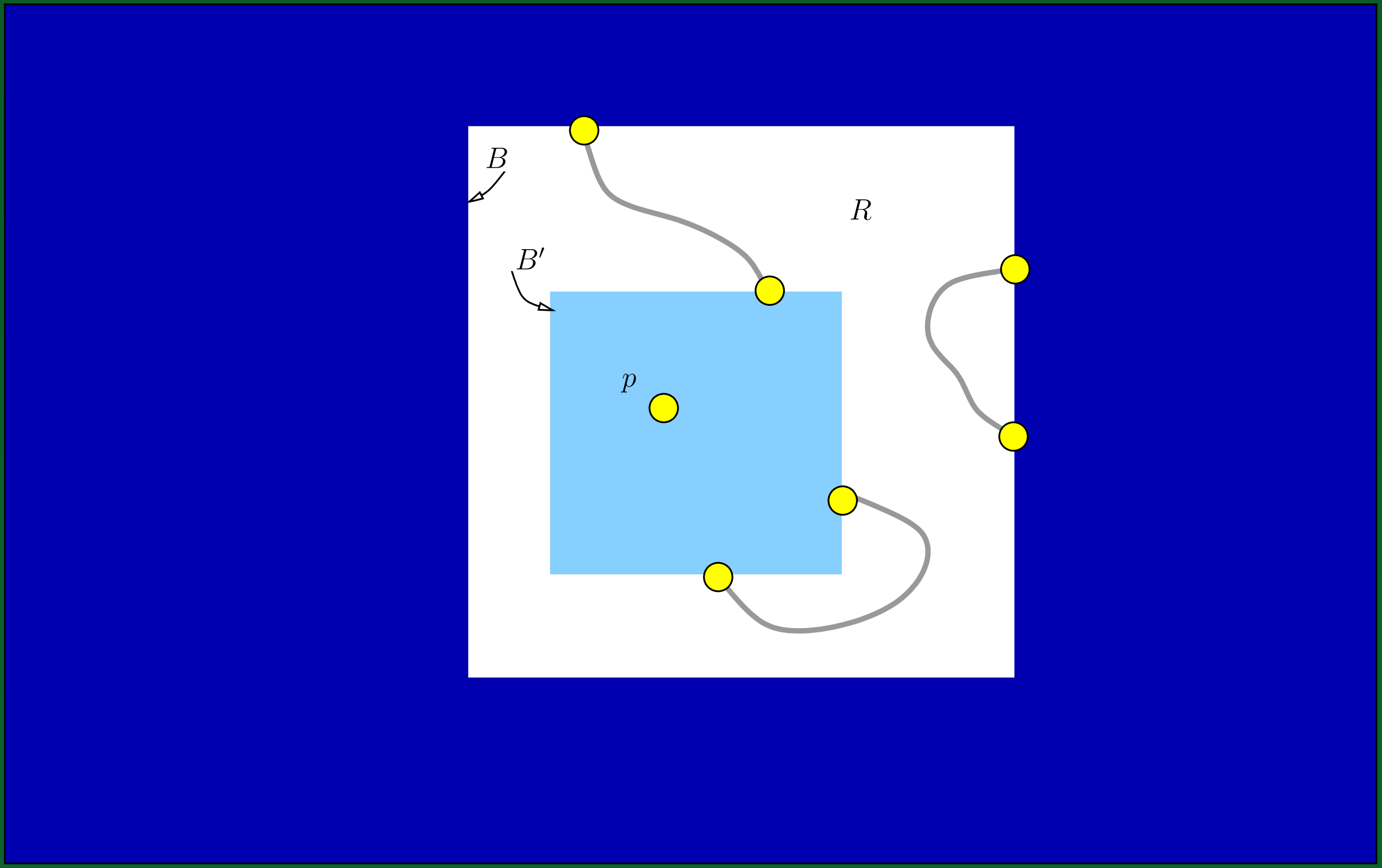

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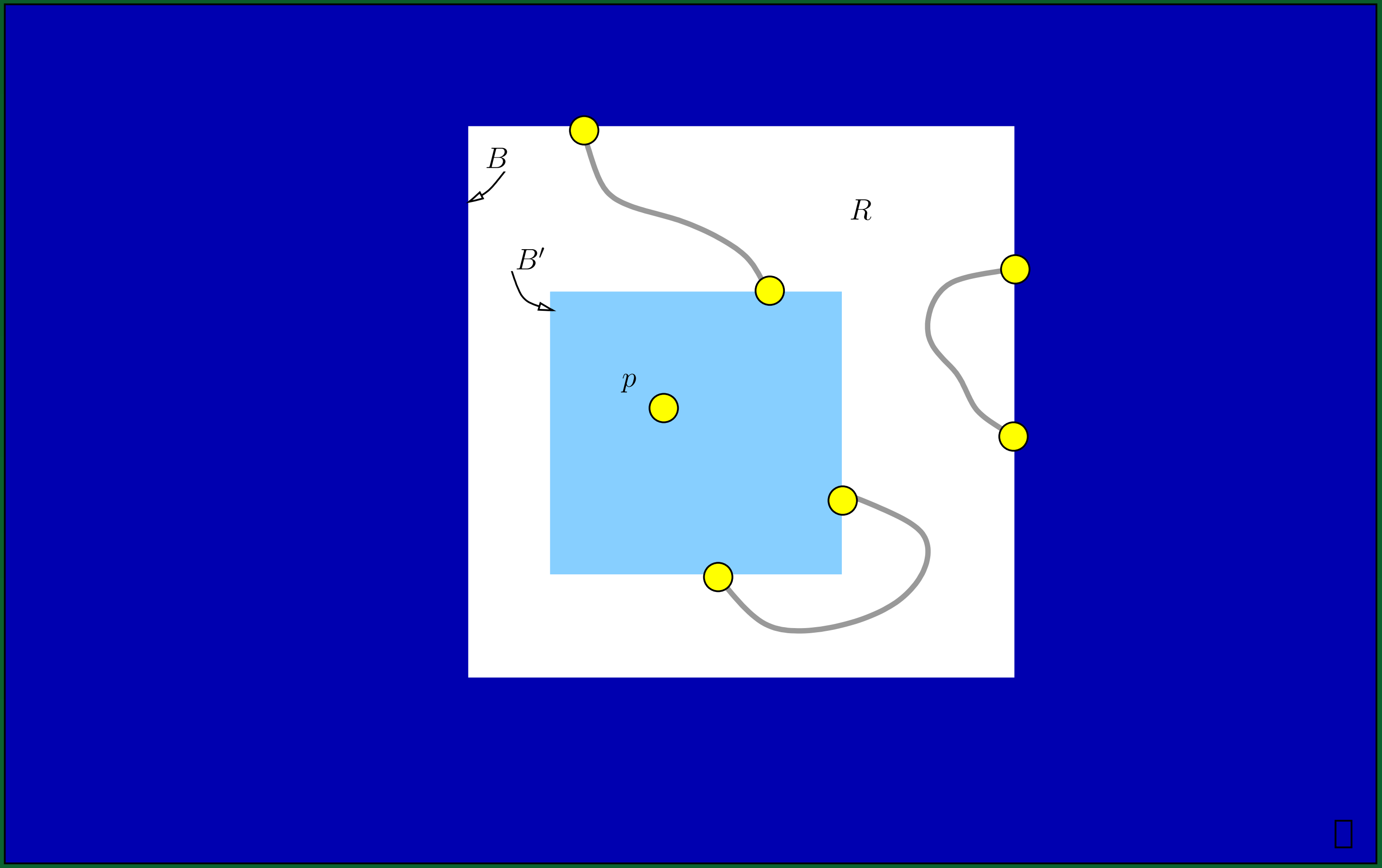

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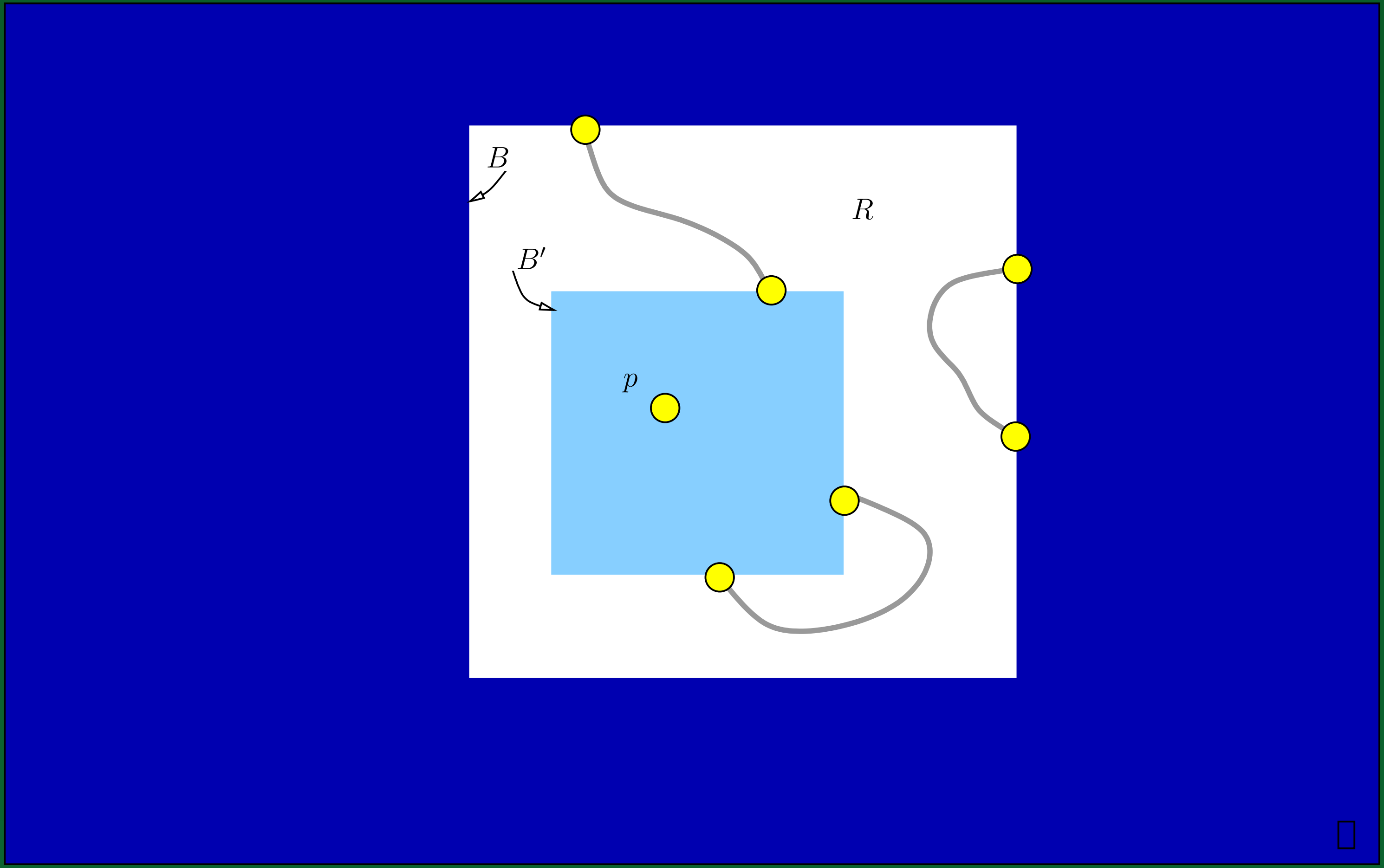

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  - ∗ Simpler algorithms
  - ∗ Exploits local geometry
  - ∗ Generality – not restricted to algebraic case
  - ∗ Algorithm uses, but is decoupled from, root bounds

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  - ∗ (1) Achieve completeness via purely numerical means
  - ∗ (2) Complexity Analysis

- This talk
  - ∗ (1) First complete numerical solution for curve approximation
  - ∗ (2) Integral analysis of EVAL
  - ∗ (3) EVAL has complexity $O(d^2L)$ for benchmark problem

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END of TALK

Thanks for Listening!

FURTHER INFORMATION

- "Complete Subdivision Algorithms I: Intersection of Bezier Curves"
  - ∗ SoCG'06, C.Yap
- "Complete Subdivision Algorithms II: Isotopic Meshing of Singular Algebraic Curves"
  - ∗ M.Burr, S.W.Choi, B.Galehouse and C.Yap
- "Evaluation-based Root Isolation"
  - ∗ M.Burr, V.Sharma, C.Yap
- "Integral Analysis of Evaluation-based Root Isolation"
  - ∗ M.Burr, F.Krahmer, C.Yap

Extra Slides