PAPER

A Practical Method of Numerical Calculation of the Mapping Degree

Sunao MURASHIGE[†a)], ***Member***

SUMMARY This paper proposes a simple and efficient method to numerically obtain the mapping degree $\deg(f, 0, B)$ of a C^1 map $f : \mathbb{R}^n \to \mathbb{R}^n$ at a regular value 0 relative to a bounded open subset $B \subset \mathbb{R}^n$. For practical application, this method adopts Aberth's algorithm which does not require computation of derivatives and determinants, and reduces the computational cost with two additional procedures, namely preconditioning using the coordinate transformation and pruning using Krawczyk's method. Numerical examples show that the proposed method gives the mapping degree with $2n + 1$ operations using interval arithmetic.

key words: *mapping degree, intersection number, numerical calculation, interval arithmetic*

1. Introduction

Topological indices such as the mapping degree help us systematically understand complicated problems in engineering and science. The mapping degree is a crucial index for some typical properties of a map, for example, existence and number of equilibrium points, bifurcation, complexity of dynamical system, and so on [1]–[4]. In general, it is hard to analytically determine the mapping degree. Then some numerical methods have been proposed [5]–[9], but it is not straightforward to obtain it for a general system of large dimension with practical computational cost. This work considers a simple and efficient method of numerical calculation of the mapping degree for a finite dimensional system of equations $f(x) = 0$.

The mapping degree $\deg(f, 0, B)$ can be defined as follows:

Definition 1. *Let $f : \overline{B} \to \mathbb{R}^n$ be a C^1 map, $B \subset \mathbb{R}^n$ a bounded open subset, $\overline{B}$ the closure of B, $0 \in \mathbb{R}^n$ a regular value of f, namely the Jacobi matrix $\partial f/\partial x$ at $x = f^{-1}(0) \in B$ being non-singular, and $0 \notin f(\partial B)$ where ∂B denotes the boundary of B, respectively. Then the mapping degree* $\deg(f, 0, B)$ *is defined by*

$$\deg(f, 0, B) = \sum_{x^* \in f^{-1}(0)} \operatorname{sgn}\left(\det\left(\frac{\partial f}{\partial x}\right)\Big|_{x=x^*}\right), \tag{1}$$

where $f^{-1}(0) = \{x \in B \ : \ f(x) = 0\}$. When $0 \in \mathbb{R}^n$ be a critical value of f, the degree can be defined with small change of f ([3], p.523).

This definition directly leads to Kronecker's existence principle, namely

$$\deg(f, 0, B) \neq 0 \ \Rightarrow \ \exists x^* \in B \ : \ f(x^*) = 0. \tag{2}$$

Another important property of the mapping degree is the homotopy invariance. This can be stated that, if a map f is continuously changed such that $f(\partial B) \not\ni 0$, then $\deg(f, 0, B)$ remains the same.

Most of previous works on numerical calculation of the mapping degree [6]–[8] are based on Stenger's method [5] using

$$\deg(f, 0, B) = \sum_{j \in \Lambda} \deg(\hat{f}^1, 0, \partial B_j), \tag{3}$$

where $f = (f_1, f_2, \cdots, f_n)$, $\hat{f}^1 = (f_2, f_3, \cdots, f_n)$, $\partial B = \sum_{j=1}^{N} \partial B_j$ with $\partial B_i \cap \partial B_j = \varnothing \ (i \neq j)$, and Λ denotes a set of indices j of ∂B_j on which $f_1 > 0$, respectively. Equation (3) shows that the mapping degree depends on the value on the boundary $f(\partial B)$. In addition, Kearfott et al. [8] applied this idea to numerical verification of a finite dimensional singular system of equations using (2).

The mapping degree can be interpreted as the oriented intersection number, namely

$$\deg(f, 0, B) = I(f(B), \{0\}), \tag{4}$$

where $I(f(B), \{0\})$ denotes the oriented intersection number defined by sum of the orientation number $+1$ or -1 at the intersection points of $f(B)$ and $\{0\}$ [1], [2], [4]. Aberth [9] proposed a numerical method of the mapping degree, based on (4) and using an equivalent relation to (3) and interval arithmetic.

The aim of this work is to develop a practical numerical method of the mapping degree which can be applied to a general system of large dimension n. For that, although Kearfott et al. [8] demonstrated effectiveness of preconditioning for their method, it may be worth trying to improve Aberth's method [9]. It is because the algorithm does not require computation of derivatives and determinants. This work tries to reduce its computational cost with some additional procedures.

In this paper, Sect. 2 summarizes Aberth's method, Sect. 3 proposes two additional processes using the coordinate transformation and Krawczyk's method, Sect. 4 shows numerical examples, and Sect. 5 concludes this work.

Manuscript received September 6, 2005.
Manuscript revised February 6, 2006.
Final manuscript received February 24, 2006.

[†]The author is with the Department of Complexity Science and Engineering, Graduate School of Frontier Sciences, The University of Tokyo, Kashiwa-shi, 277-8561, Japan.
a) E-mail: s-murashige@nds.k.u-tokyo.ac.jp

DOI: 10.1093/ietfec/e89–a.6.1813

2. Aberth's Method

This section summarizes the algorithm, proposed by Aberth [9], for numerical calculation of the mapping degree $\deg(f, 0, B)$ for the C^1 map $f : \overline{B}(\subset \mathbb{R}^n) \to \mathbb{R}^n$. Let $\overline{B}$ be given by a product of intervals as follows:

$$\overline{B} = [\underline{x}_1, \overline{x}_1] \times [\underline{x}_2, \overline{x}_2] \times \cdots \times [\underline{x}_n, \overline{x}_n]. \tag{5}$$

The key of Aberth's method is to successively apply the following formula to the boundary elements of B :

Lemma 1. ([4] p.84) *The oriented intersection number $I(f(B), \{0\})$ can be written in the form*

$$I(f(B), \{0\}) = I(f(\partial B), H_1), \tag{6}$$

where $H_1 = \{(h_1, 0, \cdots, 0) \in \mathbb{R}^n \ : \ h_1 > 0\}$.

The boundary ∂B of the domain B can be divided into the boundary elements $\{\partial B_j^{n-1}\}$ such that $\partial B = \sum_{j=1}^{N_1} \partial B_j^{n-1}$ with $\partial B_k^{n-1} \cap \partial B_\ell^{n-1} = \varnothing$ $(k \neq \ell)$, the dimension of ∂B_j^{n-1} is $n-1$, and the sign of the first component f_1 of f is constant in each interior of ∂B_j^{n-1}. From additivity of the mapping degree, $I(f(\partial B), H_1)$ in (6) can be expressed as

$$I(f(\partial B), H_1) = \sum_{j=1}^{N_1} I(f(\partial B_j^{n-1}), H_1). \tag{7}$$

It follows from (4) that the intersection number $I(f(\partial B_j^{n-1}), H_1)$ can be interpreted as

$$I(f(\partial B_j^{n-1}), H_1) = \begin{cases} \deg(\hat{f}^1, 0, \partial B_j^{n-1}) & \text{for } f_1 > 0 \text{ on } \partial B_j^{n-1} \\ 0 & \text{for } f_1 \leq 0 \text{ on } \partial B_j^{n-1}, \end{cases} \tag{8}$$

where $\hat{f}^1 = (f_2, f_3, \cdots, f_n)$. Therefore (6) is equivalent to (3). Furthermore, by taking the boundary of ∂B_j^{n-1} and using (6), we can get

$$I(f(\partial B_j^{n-1}), H_1) = \sum_{j_2=1}^{N_j} I(f(\partial B_{j_2}^{n-2}), H_2), \tag{9}$$

where $\sum_{j_2}^{N_j} \partial B_{j_2}^{n-2}$ is the boundary of ∂B_j^{n-1} and $H_2 = \{(h_1, h_2, 0, \cdots, 0) \in \mathbb{R}^n \ : \ h_1, h_2 > 0\}$, respectively.

From these considerations, we have

$$\begin{aligned} \deg(f, 0, B) &= I(f(B), \{0\}) \\ &= \sum_{j_1=1}^{N_1} I(f(\partial B_{j_1}^{n-1}), H_1) \\ &= \sum_{j_1=1}^{N_1} \sum_{j_2=1}^{N_{j_1}} I(f(\partial B_{j_2}^{n-2}), H_2) \\ &\quad \vdots \\ &= \sum_{j_1=1}^{N_1} \cdots \sum_{j_n=1}^{N_{j_{n-1}}} I(f(\partial B_{j_n}^{0}), H_n), \end{aligned} \tag{10}$$

where $\sum_{j_k}^{N_{j_{k-1}}} \partial B_{j_k}^{n-k}$ is the boundary of $\partial B_{j_{k-1}}^{n-k+1}$ and $H_k = \{(h_1, \cdots, h_k, 0, \cdots, 0) \in \mathbb{R}^n \ : \ h_1, \cdots, h_k > 0\}$, respectively. Since the dimension of the boundary element $\partial B_{j_k}^{n-k}$ is $n-k$, (10) indicates that the mapping degree can be determined by sum of the sign of f at finite number of vertices of boundary elements ∂B_j^{n-1} which are sufficiently fine.

Aberth's method is based on (10) and searches the boundary elements ∂B_j^{n-k} contributing to the oriented intersection number $I(f(\partial B_j^{n-k}), H_k)$ which includes a point $x \in \partial B_j^{n-k}$ such that

$$f_\ell(x) \begin{cases} > 0 & (\ell = 1, 2, \cdots, k), \\ = 0 & (\ell = k+1, k+2, \cdots, n). \end{cases} \tag{11}$$

This condition (11) can be estimated using interval arithmetic, because each boundary element is expressed as a product of intervals. The orientation $\sigma(\partial B_j^{n-k})$ of each boundary element ∂B_j^{n-k} is determined using the same idea as that of a convex cell [2] and $\sigma(B)$ is set to $+1$. The algorithm of Aberth's method is summarized as follows:

Step 1. : Set $k := 1$. Fix the domain B and the orientation of each boundary element ∂B_j^{n-k}.

Step 2. : Express each boundary element ∂B_j^{n-k} as a product of intervals, and compute $f_\ell(\partial B_j^{n-k})$ for $\ell = k, k+1, \cdots, n$ using interval arithmetic.

Step 3. : Examine the condition (11).

If $f_k > 0$ and $f_\ell \ni 0$ for $\forall \ell = k+1, \cdots, n$, then keep the corresponding boundary element and its orientation number in the list.

If $f_k < 0$ or $f_\ell \not\ni 0$ for $\exists \ell = k+1, \cdots, n$, then omit the corresponding boundary element from the list.

If $f_k \ni 0$ and $f_\ell \ni 0$ for $\forall \ell = k+1, \cdots, n$, then divide the corresponding boundary element and return to *Step 2*.

Step 4. : After *Step 2* and *Step 3* for $\forall j$, set $k := k+1$ and return to *Step 2* for estimate of $I(f(\partial B_j^{n-k}), H_k)$. When $k = n$, the sum of orientation numbers in the list is equal to $I(f(B), \{0\})$, namely $\deg(f, 0, B)$.

On the assumption of $0 \notin f(\partial B)$, we can complete *Steps 2* and *3*, and reach *Step 4* with a finite division of elements [9].

The set of boundary elements $\{\partial B_j^{n-k}\}$ is tree structured as shown in Fig. 1. The above algorithm searches the bound-

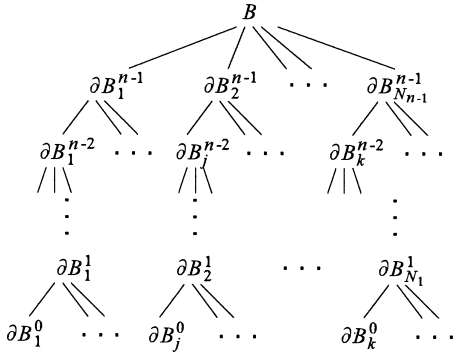


Fig. 1 The set of boundary elements $\{\partial B_j^{n-k}\}$.

ary elements which may satisfy the condition (11). When there exist the boundary elements contributing computation of the mapping degree, examination of (11) is performed for all branches of the corresponding element. This requires a considerable amount of work for a general system of large dimension n. Next section tries some additional procedures to reduce this computational cost.

3. Preconditioning and Pruning

3.1 Preconditioning Using the Coordinate Transformation

As indicated in Sect. 1, the mapping degree is invariant under homotopic change of a map f. It may be natural to consider that suitable change of a map leads to efficient computation of the mapping degree. For example, when an equilibrium of a two-dimensional map f is a saddle point as shown in Fig. 2(a), the corresponding mapping degree is the same as that for a normal form in Fig. 2(b). Numerical examples in Sect. 4 will show that the algorithm in Sect. 2 produces the mapping degree with much less computational cost for the normal form in Fig. 2(b) than for the original form in Fig. 2(a). This deformation of the vector field can be done using the coordinate transformation.

Consider an equilibrium $\tilde{x}$ of f, namely a solution of $f(x) = 0$. Around $x = \tilde{x}$, $f(x)$ can be expressed as

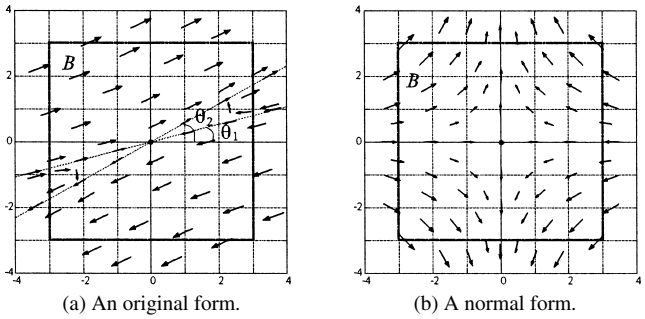
$$\begin{aligned} f(x) &= g(\xi) = f(\tilde{x}) + A\,\xi + \cdots \\ &= A\,\xi + q(\xi), \end{aligned} \tag{12}$$

where $A = (\partial f/\partial x)_{x=\tilde{x}}$ and $\xi = x - \tilde{x}$, respectively. For the sake of simplicity, assume that A is diagonalizable, and that all eigenvalues λ_j of A are real and mutually different. Then the $n \times n$ matrix A can be written in the form

$$A = \sum_{j=1}^{n} \lambda_j P_j, \tag{13}$$

where the projection matrices $P_j : \mathbb{R}^n \to E_j$ $(j = 1, 2, \cdots, n)$ can be obtained using Lagrange's interpolating polynomial $P_j = \sum_{i=1, i\neq j}^{n} (A - \lambda_i I)/\sum_{i=1, i\neq j}^{n} (\lambda_j - \lambda_i)$, and E_j denotes the eigenspace for the eigenvalue λ_j of A, respectively, and

$$P_j^2 = P_j \ (\forall j), \quad P_i P_j = 0 \ (i \neq j). \tag{14}$$



(a) An original form. (b) A normal form.

Fig. 2 The vector field around a saddle point.

Equations (12), (13) and (14) yield

$$P_j g(\xi) = \lambda_j P_j \xi + P_j q(\xi) \quad (j = 1, 2, \cdots, n). \tag{15}$$

Here it should be noted that $P_j g \in E_j$ and $P_j \xi \in E_j$, and that g and ξ can be expressed as

$$g = \sum_{j=1}^{n} g_j v_j, \quad \xi = \sum_{j=1}^{n} \xi_j v_j \quad (v_j \in E_j), \tag{16}$$

where v_j denotes the normalized eigenvectors for λ_j. Thus $P_j g$ and $P_j \xi$ corresponds to g_j and ξ_j, respectively. Equation (15) indicates that, when $\lambda_j \xi_j = \lambda_j P_j \xi$ is dominant in the right hand side, the sign of $g_j = P_j g$ is constant and the same as that of the eigenvalue λ_j. This is desirable for efficient examination of the condition (11). This idea of preconditioning is essentially the same as that using Gaussian elimination in [8]. The present work applies this to Aberth's method.

In order to realize this preconditioning using the coordinate transformation, we need to get the components g_j and ξ_j. Let f_j and $x_j - \tilde{x}_j$ $(j = 1, 2, \cdots, n)$ be components of f and $\xi = x - \tilde{x}$ in the original coordinate system, respectively. Then g_j and ξ_j can be obtained using

$$(g_j)^\top = Q^{-1}(f_j)^\top, \quad (\xi_j)^\top = Q^{-1}(x_j - \tilde{x}_j)^\top, \tag{17}$$

where $(g_j) = (g_1, g_2, \cdots, g_n)$, $(f_j) = (f_1, f_2, \cdots, f_n)$, $(\xi_j) = (\xi_1, \xi_2, \cdots, \xi_n)$, $(x_j - \tilde{x}_j) = (x_1 - \tilde{x}_1, x_2 - \tilde{x}_2, \cdots, x_n - \tilde{x}_n)$, $^\top$ denotes the transpose, and $Q = (v_1 v_2 \cdots v_n)$ the $n \times n$ matrix, respectively. After this preconditioning, the algorithm in Sect. 2 can be directly applied for $g = (g_1, g_2, \cdots, g_n)$ with estimation of the condition (11) on the boundary elements of $\overline{B} = [\underline{\xi}_1, \overline{\xi}_1] \times [\underline{\xi}_2, \overline{\xi}_2] \times \cdots \times [\underline{\xi}_n, \overline{\xi}_n]$.

In numerical examples of Sect. 4, the $n \times n$ matrix $Q = (v_1 v_2 \cdots v_n)$ and its inverse Q^{-1}, where v_j is the eigenvector of the Jacobi matrix $A = (\partial f/\partial x)_{x=\tilde{x}}$, are numerically obtained using LAPACK.

Equation (15) shows that this preconditioning may not work well for a nonlinear system of equation and a large size of domain B, because the nonlinear term q in (15) can be dominant at points far from $\tilde{x}$. This point will be considered in Sect. 4 using numerical examples.

3.2 Pruning Using Krawczyk's Method

In the algorithm in Sect. 2, if an oriented intersection number $I(f(\partial B_j^{n-k}), H_k)$ can be determined for some $k = \check{k}$ $(1 \leq \check{k} \leq n-1)$, then search for all boundaries of the corresponding boundary element is not necessary. This is "pruning" for the tree structured set of boundary elements $\{\partial B_j^{n-k}\}$ in Fig. 1, and may lead to reduction of the computational cost. This section shows that this pruning can be performed using Krawczyk's method.

As (8) shows, the oriented intersection number $I(f(\partial B_j^{n-k}), H_k)$ is related to solutions of

$$\hat{f}(x) = (f_{k+1}, f_{k+2}, \cdots, f_n)(x) = 0 \quad \text{on } \partial B_j^{n-k}, \tag{18}$$

where $f_1, f_2, \cdots, f_k > 0$ on ∂B_j^{n-k}. If the solution $\hat{x}$ of (18) uniquely exists on ∂B_j^{n-k}, $I(f(\partial B_j^{n-k}), H_k)$ is given by the orientation $\sigma(\partial B_j^{n-k})$ and $\mathrm{sgn}(\det(\partial \hat{f}/\partial u))$ at $x = \hat{x}$ where $u = (u_1, u_2, \cdots, u_{n-k})$ denotes the variables for ∂B_j^{n-k}. Krawczyk's method can be applied to examine unique existence of a solution of (18). When the boundary ∂B_j^{n-k} is expressed as a product of intervals X, Krawczyk's form $G(X)$ for $\hat{f}$ is written in the form

$$G(X) = g(\tilde{x}) + C_X(X - \tilde{x}), \tag{19}$$

where $g(x) = x - J_{\tilde{x}}{}^{-1}\hat{f}(x)$, $J_y = (\partial \hat{f}/\partial u)|_{x=y}$, $C_X = I - J_{\tilde{x}}{}^{-1}J_X$, and $\tilde{x}$ denotes an approximate solution of $\hat{f}(x) = 0$, respectively (for example, see [10]). Then Krawczyk's form (19) can be used for pruning as follows:

Lemma 2. *For Krawczyk's form $G(X)$ in (19), in the case of*

$$G(X) \subset X \ \text{ and } \ \|C_X\|_\infty < 1, \tag{20}$$

the contraction principle shows unique existence of a solution $\hat{x}$ of $\hat{f}(x) = 0$ on ∂B_j^{n-k}. Then the oriented intersection number $I(f(\partial B_j^{n-k}), H_k)$ is given by

$$I(f(\partial B_j^{n-k}), H_k) = \sigma(\partial B_j^{n-k}) \cdot \mathrm{sgn}\left(\det\left(\frac{\partial \hat{f}}{\partial u}\right)\Big|_{x=\hat{x}}\right). \tag{21}$$

On the other hand, in the case of

$$G(X) \cap X = \varnothing, \tag{22}$$

there is no solution $\hat{f}(x) = 0$ on ∂B_j^{n-k}. Then $I(f(\partial B_j^{n-k}), H_k) = 0$.

When (20) or (22) is satisfied, search for all boundaries of the corresponding boundary element is not necessary. Otherwise, namely when both (20) and (22) are not satisfied, the original algorithm in Sect. 2 is used again. In numerical examples of next section, when (20) is satisfied, the solution $\hat{x}$ of $\hat{f}(x) = 0$ is approximately obtained using Newton's method.

4. Numerical Examples

This section shows computed results of the mapping degree for three examples using the proposed method in Sects. 2 and 3. The first example is a two-dimensional linear system corresponding to Fig. 2(a), the second example is a nonlinear system of large dimension which is derived from a water wave problem, and the third example is two-dimensional systems of which the mapping degree is greater than one.

4.1 Example 1

The first example is a two-dimensional linear system given by

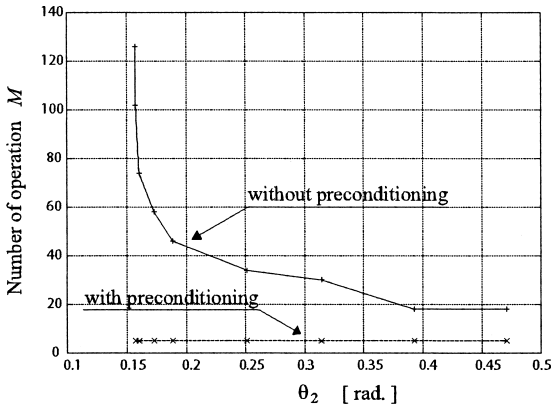


Fig. 3 Comparison of the number of operation for example 1 (23) without and with preconditioning in Sect. 3.1. M: the number of examination of the conditions (11), (20), and (22).

$$\begin{cases} f_1(x) &= a_{11}x_1 + a_{12}x_2 \\ f_2(x) &= a_{21}x_1 + a_{22}x_2 \end{cases}, \tag{23}$$

where

$$\begin{aligned} A &= (a_{ij}) \\ &= \frac{1}{\sin(\theta_2 - \theta_1)} \\ &\quad \times \begin{pmatrix} -\sin(\theta_1 + \theta_2) & 2\cos\theta_1 \cos\theta_2 \\ -2\sin\theta_1 \sin\theta_2 & \sin(\theta_1 + \theta_2) \end{pmatrix}. \end{aligned} \tag{24}$$

The vector field of (23) is roughly depicted in Fig. 2(a) where $\theta_1 = \pi/12$ and $\theta_2 = \pi/6$, and the equilibrium $x = 0$ of $f = (f_1, f_2)$ is the saddle point. The eigenvalues of the Jacobi matrix $A = (\partial f/\partial x)$ at $x = 0$ are $\lambda_1 = -1$ and $\lambda_2 = 1$, and the corresponding eigenvectors are $v_1 = (\cos\theta_1, \sin\theta_1)$ and $v_2 = (\cos\theta_2, \sin\theta_2)$, respectively. When the domain B includes the origin $x = 0$, the degree $\deg(f, 0, B)$ is equal to -1. The numerical methods in Sects. 2 and 3 were used for different values of θ_2. In all cases, the correct degree $(= -1)$ was obtained. Figure 3 compares the number of operation M, namely the number of examination of the conditions (11), (20), and (22), without and with preconditioning using the coordinate transformation in Sect. 3.1. Here θ_1 is fixed to $\pi/20$, θ_2 is varied in $\pi/20 < \theta_2 \le \pi/2$, and the width w of $B = [-w, w] \times [-w, w]$ is set to 0.1, respectively. Figure 4 shows the corresponding computation time T in sec. and reflects the number of operation in Fig. 3. In these computations, preconditioning helped us obtain the correct degree with the minimum number of operation $M = 5$, namely $2n = 4$ for (11) and 1 for (20). From these results, it is found that preconditioning using the coordinate transformation in Sect. 3.1 is very effective for the case of strongly distorted vector fields, namely the directions of the eigenvectors v_1 and v_2 being very close.

4.2 Example 2

The second example is a nonlinear map $f : \mathbb{R}^n \to \mathbb{R}^n$ given by

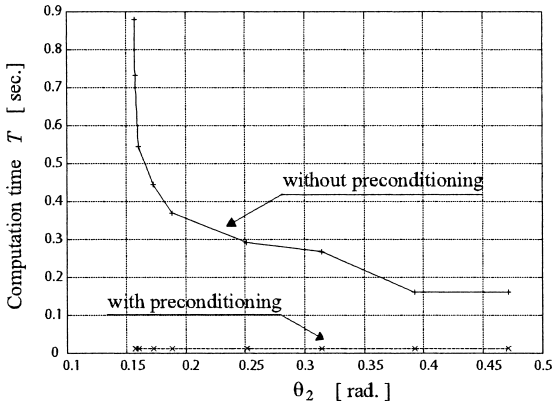


Fig. 4 Comparison of the computation time T [sec.] for example 1 (23) without and with preconditioning in Sect. 3.1.

$$\begin{cases} f_1(x) &= x_1 + x_2x_2 + 2x_3x_3 + \cdots \\ &\quad +(n-1)x_nx_n + c^2 \\ f_2(x) &= x_2 + x_1x_2 + 2x_2x_3 + \cdots \\ &\quad +(n-1)x_{n-1}x_n \\ &\vdots \\ f_n(x) &= x_n + x_{n-1}x_2 + 2x_{n-2}x_3 + \cdots \\ &\quad +(n-1)x_1x_n, \end{cases} \tag{25}$$

where $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$. This map approximates a mathematical model of water waves progressing into one direction with a constant speed c in water of infinite depth, and x_j $(j = 1, 2, \cdots, n)$ corresponds to the Fourier coefficient of surface shape of water waves [11]. For systematic comparison of computed results, it is convenient to introduce the parameter Q defined by [11]

$$Q = 1 + \frac{1}{2}a_0 + \sum_{j=1}^{n} a_j. \tag{26}$$

Physically, this parameter Q corresponds to $1 - \frac{1}{2}{q_0}^2$ for $n \to \infty$, where q_0 denotes the velocity of fluid at the crest of waves.

Figures 5 and 6 compare the number of operation M and the computation time T, respectively, for different dimension n $(3 \le n \le 100)$. Definition of M is the same as that in example 1. Here the parameter Q and the width w of $B = \prod_{j=1}^{n}[\tilde{x}_j - w, \tilde{x}_j + w]$ are fixed to $Q = 0.8$ and $w = 10^{-10}$, respectively. The approximate solution $\tilde{x} = (\tilde{x}_j)$ of $f(x) = 0$ is obtained using Newton's method. All of eigenvalues of the Jacobi matrix $A = (\partial f/\partial x)$ at $x = \tilde{x}$ are real. In these computations, preconditioning and pruning helped us obtain the correct degree with the minimum number of operation $M = 2n + 1$, namely $2n$ for (11) and 1 for (20). These results show that the proposed methods in Sect. 3 work very well even for a system of large dimension n.

Figure 7 shows that the number of operation M depends on the width w of the domain $B = \prod_{j=1}^{n}[\tilde{x}_j - w, \tilde{x}_j + w]$. It is because the map f in (25) is nonlinear and the second term q of the right hand side of (15) becomes dominant at the points far from $\tilde{x}$. These results suggest that it is necessary

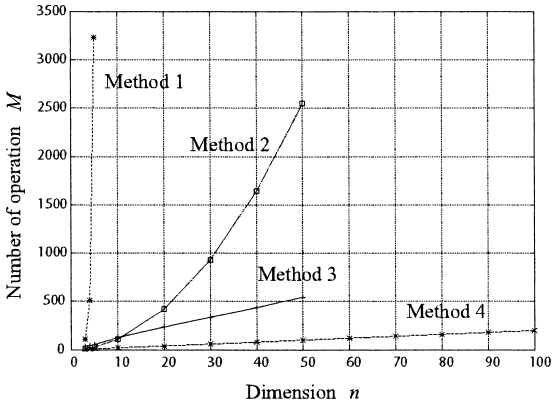


Fig. 5 Comparison of the number of operation for example 2 (25) with change of dimension n. M : the number of examination of the conditions (11), (20), and (22). Method 1: without preconditioning and without pruning. Method 2: with preconditioning and without pruning. Method 3: without preconditioning and with pruning. Method 4: with preconditioning and with pruning.

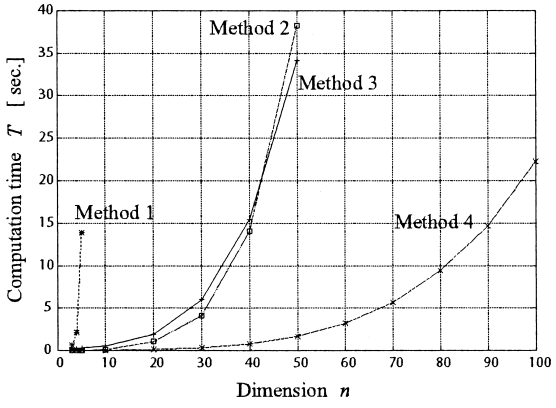


Fig. 6 Comparison of the computation time T [sec.] for example 2 (25) with change of dimension n. See caption in Fig. 5.

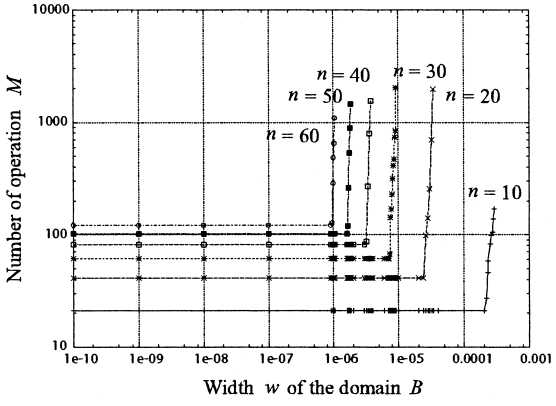


Fig. 7 Effects of the width w of the domain $B = \prod_{j=1}^{n}[\tilde{x}_j - w, \tilde{x}_j + w]$ on the number of operation M for example 2 (25). Both axes have a logarithmic scale. n: dimension.

to determine suitable size of the domain B for a nonlinear system of large dimension. Although choice of the suitable size depends on complexity of f, the order of the leading

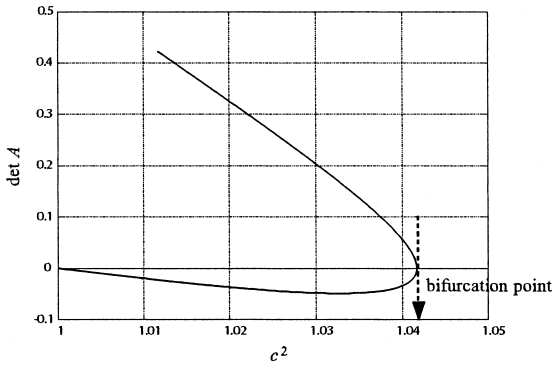


Fig. 8 The relation between det$A$ and the parameter c^2 for example 2 (25). $A = (\partial f/\partial x)|_{x=\tilde{x}}$. $n = 3$.

term of q in (15), namely $(\partial^2 f/\partial x_i \partial x_j)|_{x=\tilde{x}}\xi_i\xi_j$, would be a useful index to judge it.

It is known that the nonlinear map f in (25) produces bifurcation phenomena with change of the parameter Q or c [11]. Figure 8 shows the relation between the parameter c^2 and the determinant $\det(\partial f/\partial x)|_{x=\tilde{x}} = \det A$ for $n = 3$. Here $\tilde{x} = (\tilde{x}_j)$ is an approximate solution of (25) and the center of the domain $B = \prod_{j=1}^{n}[\tilde{x}_j - w, \tilde{x}_j + w]$. The corresponding mapping degree $\deg(f, 0, B)$ is -1 for $\det A < 0$ and $+1$ for $\det A > 0$. This jump of the mapping degree gives the bifurcation condition. Near the bifurcation point at which $\det A = 0$, the domain B can include two solutions of (25) at which the determinants $\det A$ have different signs, and then the mapping degree $\deg(f, 0, B) = 0$. These typical properties of a nonlinear system can be estimated using the proposed method with practical computational cost.

4.3 Example 3

The third example is a two-dimensional map $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ given by the real and imaginary parts of

$$(x_1 + \mathrm{i}\, x_2)^m = f_1(x_1, x_2) + \mathrm{i}\, f_2(x_1, x_2), \tag{27}$$

where i denotes the imaginary unit. The solution of $f(x) = 0$ is located at the origin $x = 0$ and the corresponding mapping degree $\deg(f, 0, B)$ for $B \ni 0$ is equal to m. Since this is the case of critical value, namely the Jacobi matrix $A = (\partial f/\partial x)|_{x=0}$ being singular, preconditioning in Sect. 3.1 cannot be directly applied.

Figures 9 and 10 show the number of operation M and the computation time T, using the algorithm in Sect. 2 and pruning in Sect. 3.2. These results show that the computational cost considerably increases with the magnitude m of the mapping degree due to complexity of the vector field and overestimation of interval arithmetic.

A system with critical value can be transformed to that with regular value using homotopic change. This may lead to reduction of the computational cost shown in Figs. 9 and 10. This problem remains as future works.

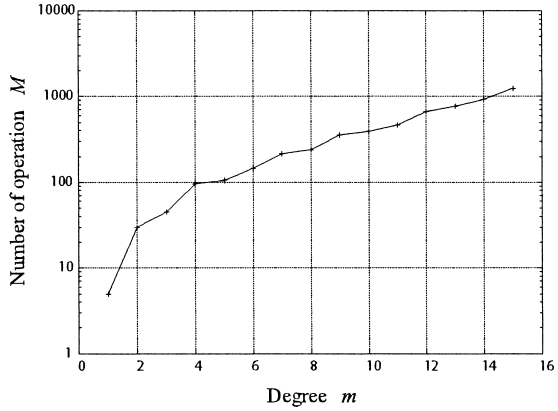


Fig. 9 Comparison of the number of operation for example 3 (27). M: the number of examination of the conditions (11), (20), and (22). m: the mapping degree. The axis of ordinate has a logarithmic scale.

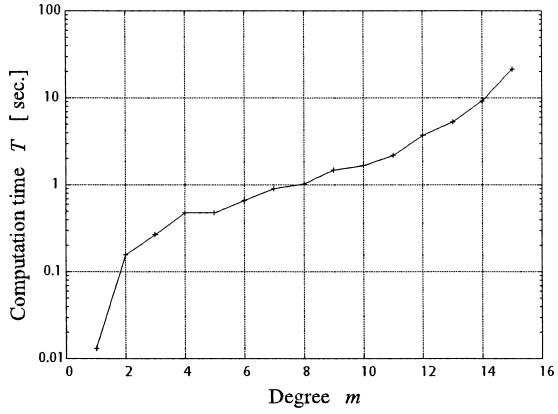


Fig. 10 Comparison of the computation time T [sec.] for example 3 (27). m : the mapping degree. The axis of ordinate has a logarithmic scale.

5. Discussions

As shown in Figs. 5 and 6, Krawczyk's method efficiently reduces the computational cost for some systems of large dimension. This type of Newton-like method has been used to obtain the mapping degree [8]. Here it should be noted that this method does not always work, in particular when the Jacobi matrix $J_{\tilde{x}}$ in (19) is singular or nearly singular.

For example, in the case of $n = 3$ of Example 2 in Sect. 4.2, (25) is expressed as

$$\begin{cases} f_1(x) = x_1 + x_2x_2 + 2x_3x_3 + c^2 \\ f_2(x) = x_2 + x_1x_2 + 2x_2x_3 \\ f_3(x) = x_3 + x_2x_2 + 2x_1x_3. \end{cases} \tag{28}$$

For this map, when the parameter c^2 is set to 1.03 and one of boundary elements ∂B^2 is located at $x_1 = -1$, the following conditions are satisfied at $(x_2, x_3) = (0, 0)$:

$$f_1 > 0, \quad f_2 = f_3 = 0, \tag{29}$$

and

$$\det J_x = \begin{vmatrix} \dfrac{\partial f_2}{\partial x_2} & \dfrac{\partial f_2}{\partial x_3} \\ \dfrac{\partial f_3}{\partial x_2} & \dfrac{\partial f_3}{\partial x_3} \end{vmatrix} = 0, \tag{30}$$

where J_x corresponds to the Jacobi matrix $\partial \hat{f}/\partial u$ used in (19). Thus, if this point is chosen as $\tilde{x}$ in Krawczyk's form (19), the method in Sect. 3.2 fails even with further subdivision of the boundary element.

For this singular case, the proposed method does not fail and proceeds to examine lower dimensional boundary elements. Thus this method efficiently works for regular cases, similarly to conventional algorithms using Newton-like methods, and also does work even for singular cases using Aberth's algorithm. This flexibility is the advantageous point of this method.

6. Conclusions

This work has considered a simple and efficient method to numerically obtain the mapping degree $\deg(f, 0, B)$ for a C^1 map $f : \overline{B} \to \mathbb{R}^n$ at a regular value 0 relative to a bounded open subset $B \subset \mathbb{R}^n$. This method adopts Aberth's method as a basic algorithm, which utilizes dependence of the mapping degree on the boundary ∂B, because computation of derivatives and determinants is not necessary. In order to reduce the computational cost, this work proposes two additional procedures, namely preconditioning using the coordinate transformation and pruning using Krawczyk's method. Numerical results show that these methods work very well even for a nonlinear system of large dimensions.

This work has restricted the conditions of the Jacobi matrix $\partial f/\partial x$ at a solution of $f(x) = 0$. For further practical application, more general cases including critical value will be considered in future works.

Acknowledgments

The author thanks anonymous reviewers for their valuable comments.

References

[1] J. Cronin, Fixed Points and Topological Degree in Nonlinear Analysis, Amer. Math. Soc., Providence, RI, 1964.
[2] W. Guillemin and A. Pollack, Differential Topology, Prentice-Hall, 1974.
[3] E. Zeidler, Nonlinear Functional Analysis and Its Applications I, Springer, 1986.
[4] E.H. Rothe, Introduction to Various Aspects of Degree Theory in Banach Spaces, American Mathematical Society, 1986.
[5] F. Stenger, "Computing the topological degree of a mapping in $\mathbb{R}^n$," Numeri. Math., vol.25, pp.23–38, 1975.
[6] M. Stynes, "A simplification of Stenger's topological degree formula," Numeri. Math., vol.33, pp.147–156, 1979.
[7] R.B. Kearfott, "An efficient degree-computation method for a generalized method of bisection," Numeri. Math., vol.32, pp.109–127, 1979.
[8] R.B. Kearfott, J. Dian, and A. Neumaier, "Existence verification for singular zeros of complex nonlinear systems," SIAM J. Numeri. Anal., vol.38, no.2, pp.360–379, 2000.
[9] O. Aberth, "Computation of topological degree using interval arithmetic, and applications," Mathematics of Computations, vol.62, no.205, pp.171–178, 1994.
[10] A. Neumaier, Introduction to Numerical Analysis, Cambridge University Press, 2001.
[11] M.S. Longuet-Higgins, "Bifurcation in gravity waves," J. Fluid Mech., vol.151, pp.457–475, 1985.

Sunao Murashige received the B.E., M.E., and Dr. Eng. degrees from the University of Tokyo, Japan, in 1986, 1988, and 1991, respectively. His research interests are numerical calculation and dynamical system.