Tutorial: Exact Numerical Computation in Algebra and Geometry

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- Many problems in Computational Science & Engineering (CS&E) are defined on the continuum. Standard algorithms for these problems are numerical and approximate. Their computational techniques include iteration, subdivision, and approximation. Such techniques are rarely seen in exact or algebraic algorithms. In this tutorial, we discuss a mode of computation called Exact Numerical Computation (ENC) that achieves exactness through numerical approximation. Through ENC, we naturally incorporate iteration, subdivision and approximation into our design of algorithms for computer algebra and computational geometry. Such algorithms are both novel and practical. This tutorial on ENC is divided into three equal parts:
  (a) ENC and Zero Problems
  (b) Explicitization and Subdivision Algorithms
  (c) Complexity Analysis of Adaptivity

Overview of Tutorial

- Background is algebraic and geometric computation
- Motivation: much of computing world (CS&E) is continuous
- But Theoretical Computer Science has gone completely discrete
- The discrete view alone is inadequate for CS&E.
- What role for exact computation in the continuum?
- Geometric insights holds the key
- Exact Numerical Computation (ENC) is a proposed synthesis
- Lecture in 3 parts
  - (a) ENC and Zero Problems
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PART 1

Exact Numeric Computation and the Zero Problem

"The history of the zero recognition problem is somewhat confused by the fact that many people do not recognize it as a problem at all."

— DANIEL RICHARDSON (1996)

"Algebra is generous, she often gives more than is asked of her."

— JEAN LE ROND D'ALEMBERT (1717-83)

Coming Up Next

Introduction to Geometric Computation

- PUZZLE 1:
  Is Geometry discrete or continuous?
- PUZZLE 2:
  How come numbers do not arise in Computational Geometry?

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What is Computational Geometry?

Geometric Objects

- Prototype: Points, Lines, Circles (Euclidean Geometry)
- Arrangement of hyperplanes and hypersurfaces
- Zero sets and their Singularities
- Semi-algebraic sets
- Non-algebraic sets
- Geometric complexes

Geometric Problems

- Constructing geometric objects
- Searching in geometric complexes or structures

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Dual Descriptions of Geometry

Where do Geometric Objects Live?

- As Points in Parametric Space $\mathcal{P}$
  - E.g., for lines given by $L(a,b,c) := aX + bY + c = 0$, the space is $\mathcal{P} := \{(a,b,c) : a^2 + b^2 > 0\} \subseteq \mathbb{R}^3$.
- As Loci in Ambient Space $\mathcal{A}$
  - E.g., Locus of the $Line(1,-2,0)$ is the set $\{(x,y) \in \mathbb{R}^2 : x - 2y = 0\} \subseteq \mathcal{A} = \mathbb{R}^2$.
- More involved example: Cell Complexes (in the sense of algebraic topology)

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Computation: Geometric vs. Algebraic

Where is the Computation?

- **Algebraic Computation:** in parameter space $\mathcal{P}$
  - E.g., Gröbner bases
  - Polynomial manipulation, Expensive (double exponential time)
- **Geometric Computation:** in ambient space $\mathcal{A}$
  - E.g., Finding Zeros of Polynomials in $\mathbb{R}^n$
  - Numerical, Combinatorial, Adaptive (single exponential time)

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(Contd.) Computation: Geometric vs. Algebraic

Answer to PUZZLE 1: “BOTH”

- Geometry is discrete (in $\mathcal{P}$) (algebraic computation)
- Geometry is continuous (in $\mathcal{A}$) (analytic computation)

Actions in the Ambient Space

- Geometric Relationships on different Object types arise in $\mathcal{A}$
  - E.g., Point is ON/LEFT/RIGHT of a Line
- Analytic properties of Objects comes from their loci
  - E.g., Proximity, Approximations, Isotopy, etc

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- Geometry is continuous (analytic view)
- Up Next : What do Computational Geometers think?

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Coming Up Next

Five Examples of Geometric Computation

- (I) Convex Hulls
- (II) Euclidean Shortest Path
- (III) Disc Avoiding Shortest Path
- (IV) Mesh Generation
- (V) Discrete Morse Theory

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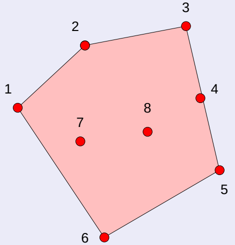

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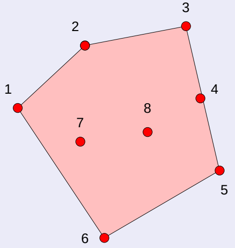

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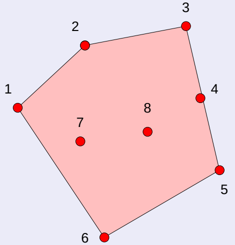

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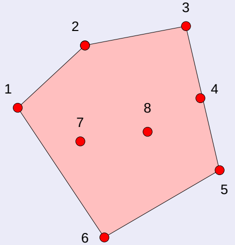

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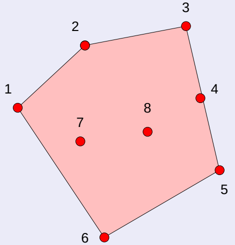

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(II) Euclidean Shortest Path (ESP)

Shortest Path amidst Polygonal Obstacles

- Shortest path from p to q avoiding A, B, C

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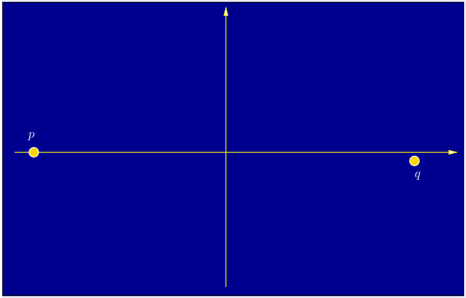

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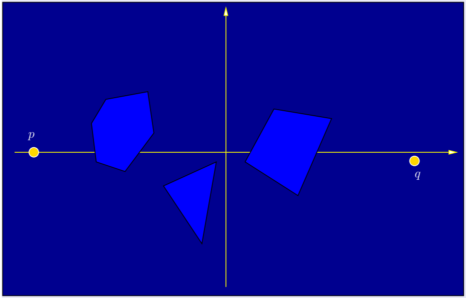

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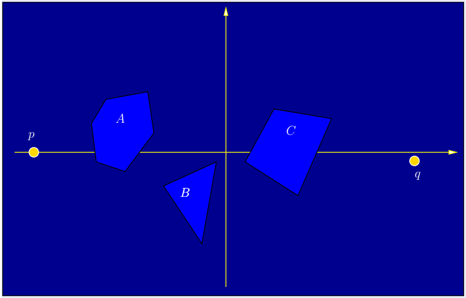

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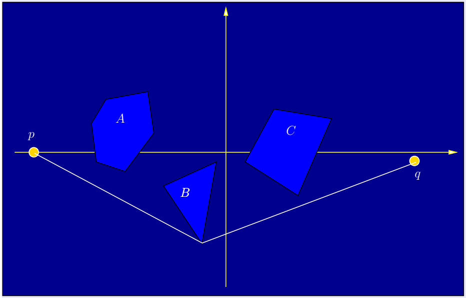

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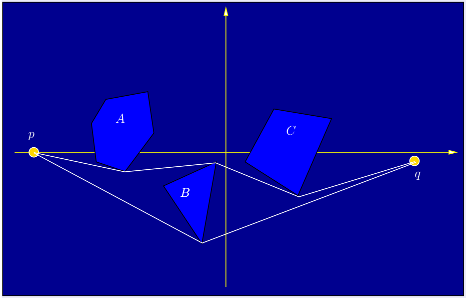

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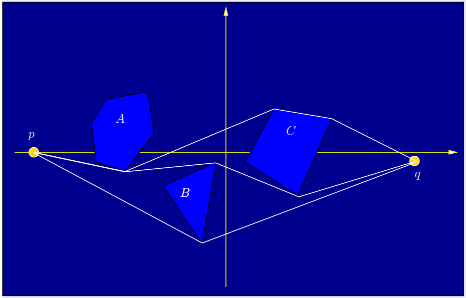

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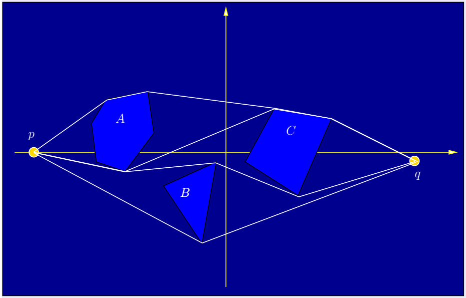

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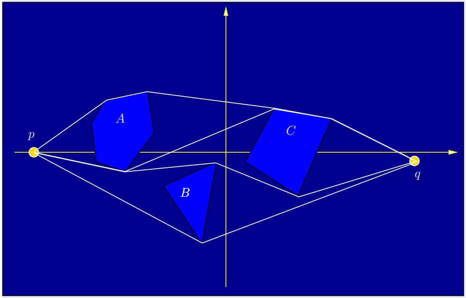

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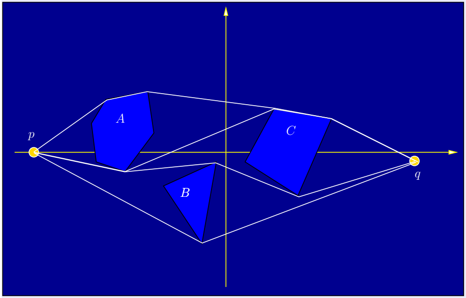

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- Combinatorial complexity: $O(n^2 \log n)$ (negligible)
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- Not known to be polynomial-time!
- Algebraic Approach: Repeated Squaring Method (Nontrivial for Inequalites!)
  - $\Omega(2^m)$ (slow, unless you are lucky! (Illustrate))
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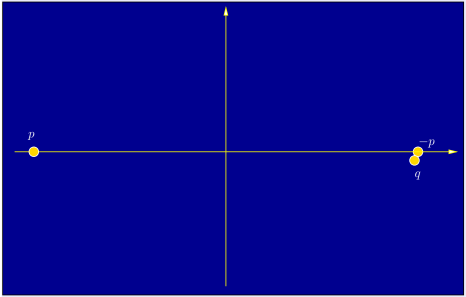

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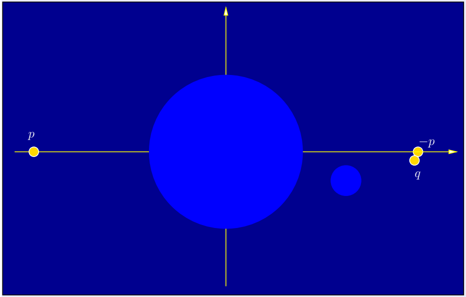

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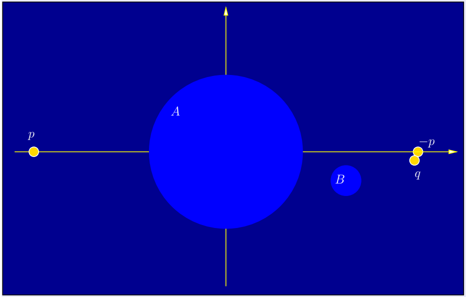

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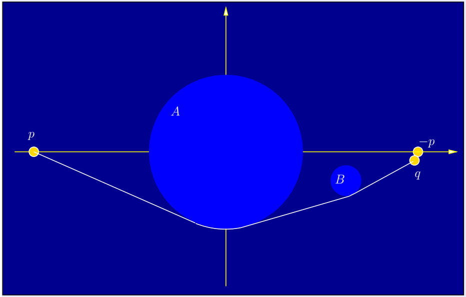

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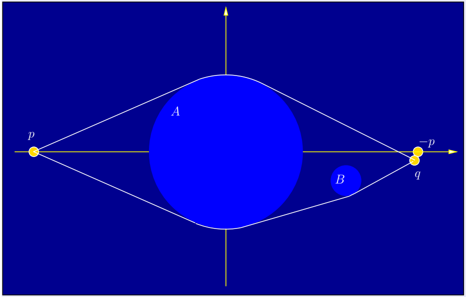

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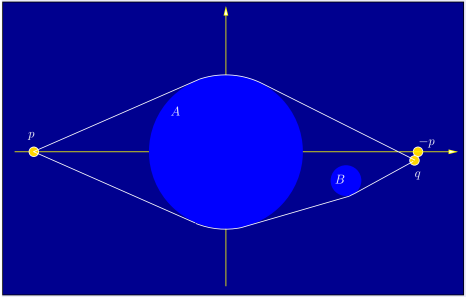

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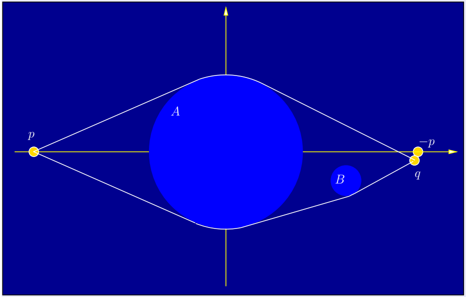

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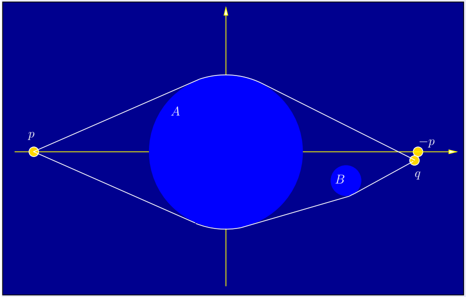

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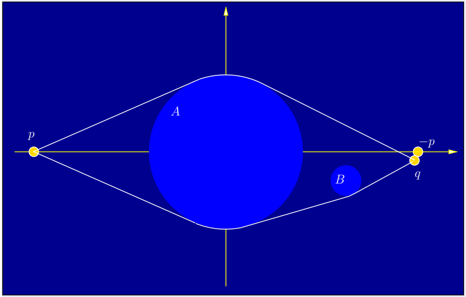

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Disc Obstacles, contd.

Reduction to Dijkstra's Algorithm (Again?)

- Combinatorial complexity: $O(n^2 \log n)$ (negligible, exercise)
- Path length $= \gamma + \alpha$
  where γ is algebraic, but α is transcendental
- Not even clear that we can compute this!
  - Why? Numerical Halting Problem
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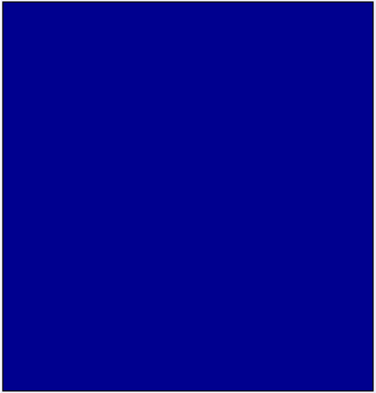
Let $A = [C, p, q, n]$ and $A' = [C', p', q', n']$ encode two arc lengths.



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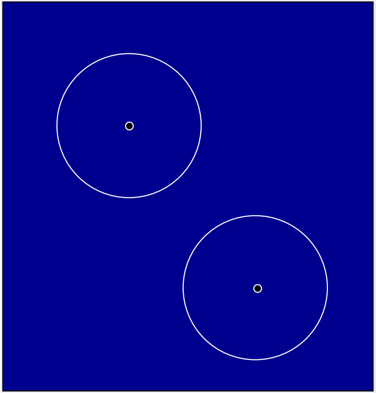

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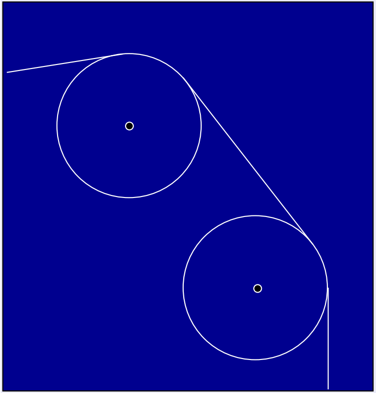

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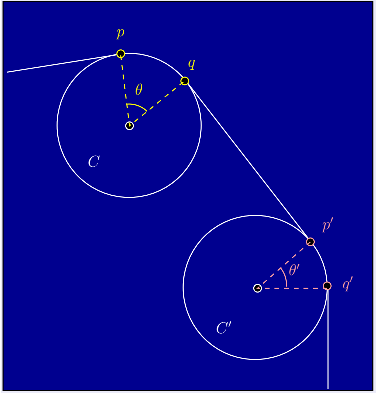

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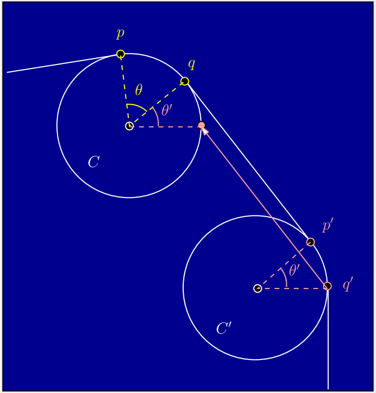

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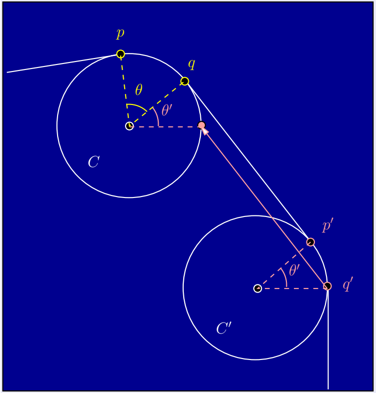

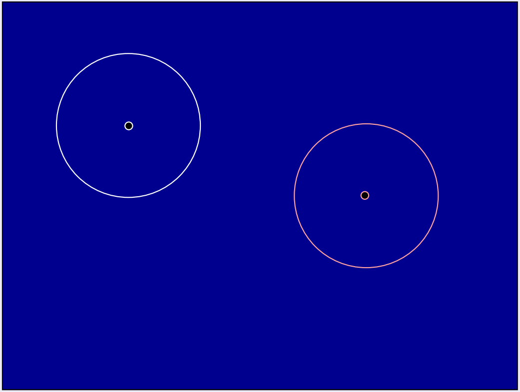

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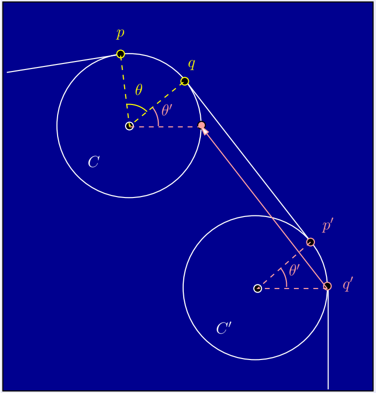

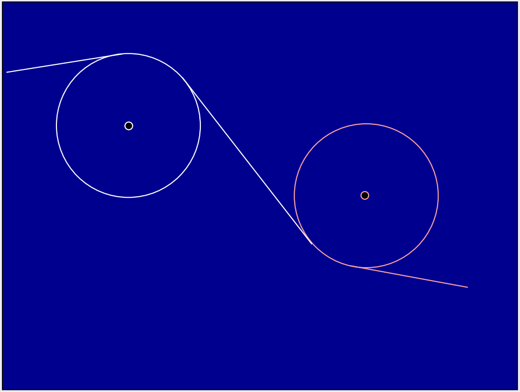

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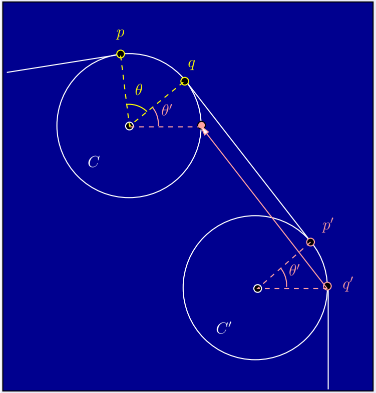

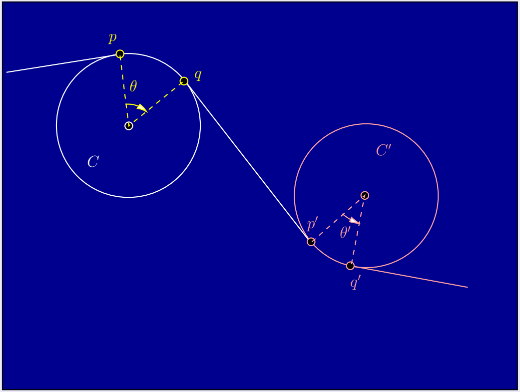

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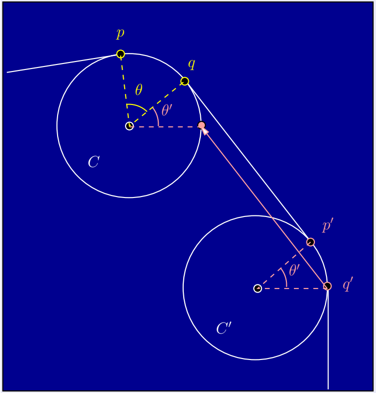

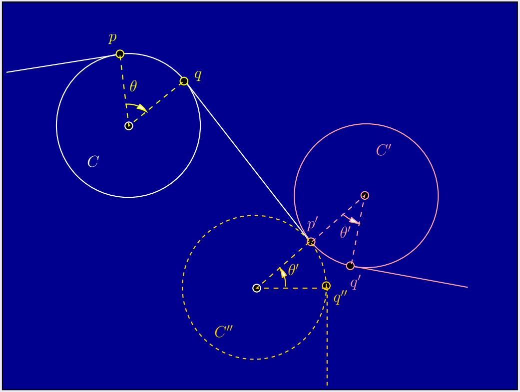

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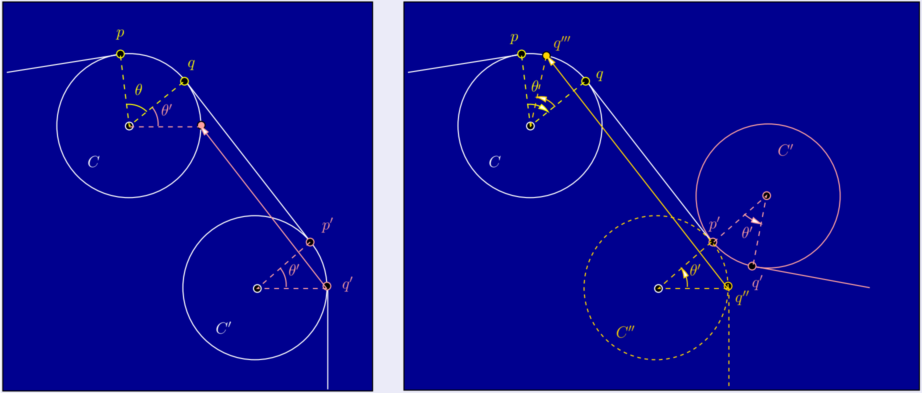

Computability of Shortest Paths

[Chang/Choi/Kwon/Park/Y. (2005)]

Is it really Transcendental?

LEMMA: $\cos\theta_i$ is algebraic.
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Theorem (Unit Disc)

Shortest Path for unit disc obstacles is computable.

Rational Case

Much harder – use Chebyshev functions of first kind.
Main issue: how to transfer arc lengths to circles with different radii.

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Shortest Paths for rational discs is in single-exponential time.

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Computability of Shortest Paths (contd.)

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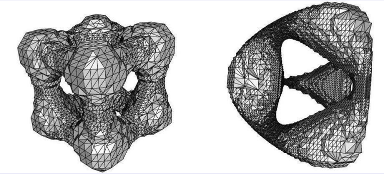

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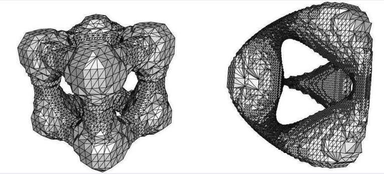

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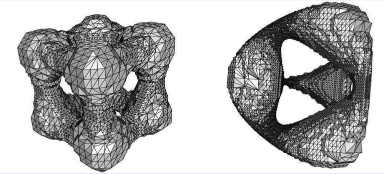

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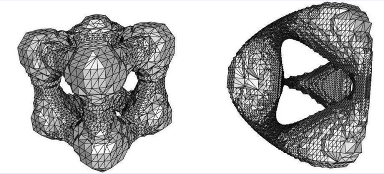

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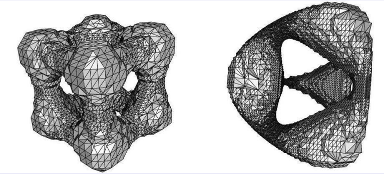

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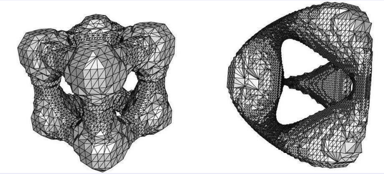

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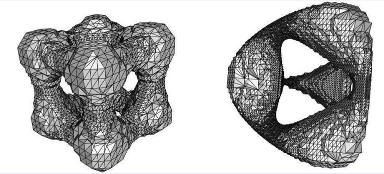

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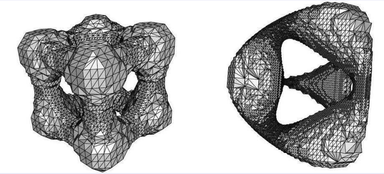

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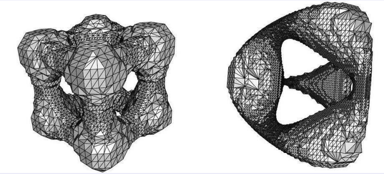

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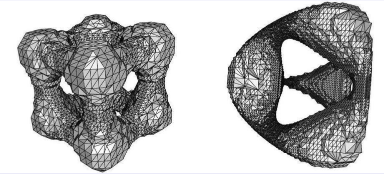

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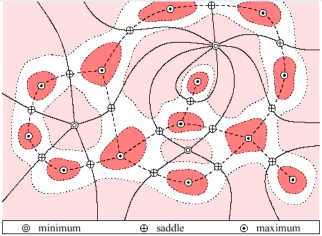

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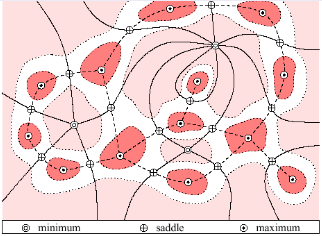

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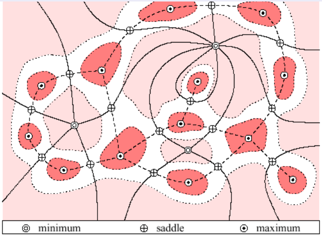

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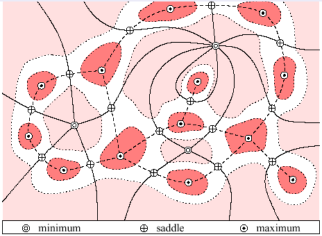

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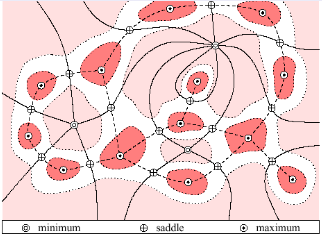

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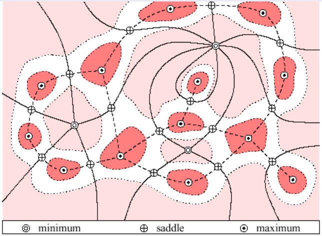

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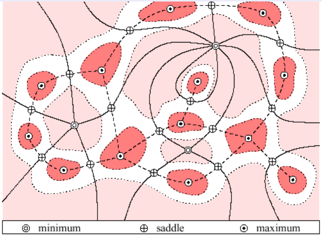

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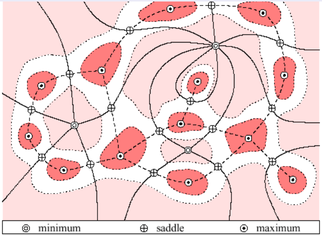

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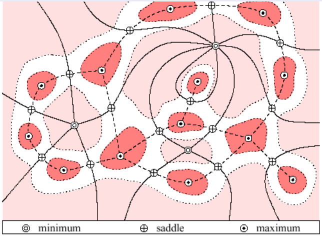

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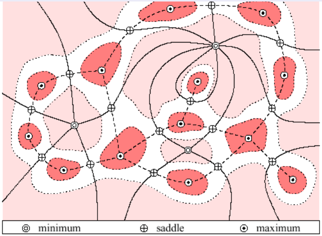

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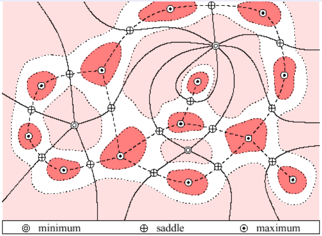

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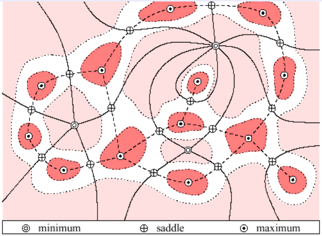

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  I=classic, II=hard, III=very hard, IV=current, V=open
- Up Next : Let us examine their underlying computational models...

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Coming Up Next

1. Introduction: What is Geometric Computation?
2. Five Examples of Geometric Computation
3. Exact Numeric Computation – A Synthesis
4. Exact Geometric Computation
5. Constructive Zero Bounds

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  - Problems too hard in exact framework (e.g., 3D Ising Model)
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Geometry is always about zeros

- Problem (I): Is a Point on a Hyperplane?
- Problems (II),(III): Are two path lengths are equal?
- Problems (IV),(V): Continuous-to-discrete transformations, defined by zero sets
- These zero decisions are captured by geometric predicates
- View developed by CG'ers in robust geometric computation

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Four Computational Models for Geometry

How to compute in a Continuum ($\mathbb{R}^n$)?

- (EX) Algebraic Computational Model
  (e.g., Real RAM, Blum-Shub-Smale model, Disc Shortest Path)
  - PROBLEM: Zero is trivial
- (EX') Abstract Operational Models
  (e.g., CG, Traub, Orientation, Ray shooting, Giftwrap)
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Duality in Numbers

- Physics Analogy:

	Discrete	Continuous
Light	particle	wave
$\mathbb{R}$	field	metric space
Numbers	algebraic	analytic
α	$=\sqrt{15-\sqrt{224}}$	≈ 0.0223

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- Geometry is decided by Zeros
- Zero is a special number
- Numbers have a dual nature: need dual representation
- Up Next : A General Solution

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Coming Up Next

The Universal Solution (EGC)

Key Principle of Exact Geometric Computation (EGC)

- Algorithm = Sequence of Steps
- Steps = Construction $\boxed{x := y + 2;}$ or Tests $\boxed{\text{if } x = 0 \text{ goto L}}$
- Geometric relations determined by Tests (Zero or Sign)
- THUS: if Tests are error free, the Geometry is exact
- Numerical robustness follows! Take-home message

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Implementing the Universal Solution (Core Library)

Any programmer can access this capability

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#define Core_Level 3
#include "CORE.h"
.... Standard C++ Program ....
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Numerical Accuracy API

- Level 1: Machine Accuracy (int, long, float, double)
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What is Achieved?

Features

- Removed numerical non-robustness from geometry (!)
- Algorithm-independent solution to non-robustness
- Standard (Euclidean) geometry (why important?)
- Exactness in geometry (can use approximate numbers !)
- Implemented in LEDA , CGAL , Core Library

Other Implications

- A new approach to do algebraic number computation
- Euclidean Shortest Path need signs of expressions like $\sum_{i=1}^{100} a_i \sqrt{b_i}$.

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- Standard (Euclidean) geometry (why important?)
- Exactness in geometry (can use approximate numbers !)
- Implemented in LEDA , CGAL , Core Library

Other Implications

- A new approach to do algebraic number computation
- Euclidean Shortest Path need signs of expressions like $\sum_{i=1}^{100} a_i \sqrt{b_i}$.

  Standard algebraic approach is doomed

Coming Up Next

Adaptive Zero Determination

Core of Core Library

- Must use numerical method based on Zero Bounds
  - Must NOT use algebraic methods!
- Let $\Omega = \{+, -, \times, \ldots\} \cup \mathbb{Z}$ be a class of operators
  - $ZERO(\Omega)$ is the corresponding zero problem
- Zero Bound for Ω is a function $B : Expr(\Omega) \to \mathbb{R}_{\geq 0}$ such that $e \in Expr(\Omega)$ is non-zero implies

$$|e| > B(e)$$

- How to use zero bounds? Combine with approximation.
- Zero Bound is the bottleneck only in case of zero.

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Some Constructive Bounds

- Degree-Measure Bounds [Mignotte (1982)], [Sekigawa (1997)]
- Degree-Height, Degree-Length [Yap-Dubé (1994)]
- BFMS Bound [Burnikel et al (1989)]
- Eigenvalue Bounds [Scheinerman (2000)]
- Conjugate Bounds [Li-Yap (2001)]
- BFMSS Bound [Burnikel et al (2001)]
    - One of the best bounds
- k-ary Method [Pion-Yap (2002)]
    - Idea: division is bad. k-ary numbers are good

An Example

- Consider the $e = \sqrt{x} + \sqrt{y} - \sqrt{x + y + 2\sqrt{xy}}$.
- Assume $x = a/b$ and $y = c/d$ where a, b, c, d are L-bit integers.
- Then Li-Yap Bound is $28L + 60$ bits, BFMSS is $96L + 30$ and Degree-Measure is $80L + 56$.
- Timing in seconds (Core 1.6):

L	50	100	500	5000
BFMS	0.637	9.12	101.9	202.9
Measure	0.063	0.07	1.93	15.26
BFMSS	0.073	0.61	1.95	15.41
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- There is a “Universal Solution” for synthesizing the Algebraic and the Geometric viewpoints
- Slogan: $\boxed{\text{Algebraic computation without Algebra}}$
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- PUZZLE 3: What was the answer to PUZZLE 2?

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- Nature of Geometric Computation:
  - Discrete as well as Continuous
  - Algebraic as well as Analytic
- It is possible to provide a fairly general solution (ENC) that combines the dual nature of numbers
- Up Next : Directly design ENC algorithms

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PART 2

Explicitization and Subdivision

"It can be of no practical use to know that π is irrational, but if we can know, it surely would be intolerable not to know."

— E.C. Titchmarsh

Coming Up Next

Towards Exact Numerical Computation (ENC)

Beyond the Universal Solution

Design algorithms directly incorporating the principles of EGC

- What do we need? What are its features?
  - It must be numerical in nature
  - It must be arbitrary precision
  - It must respect zero
  - It must be adaptive
    - actively control precision
    - exploit filters

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Computational Ring Approach

Computational Ring $(\mathbb{D}, 0, 1, +, -, \times, \div 2)$

- $\mathbb{D}$ is countable, dense subset of $\mathbb{R}$
- $\mathbb{D}$ is a ring extension of $\mathbb{Z}$
- Efficient representation $\rho : \{0,1\}^* \dashrightarrow \mathbb{D}$ for implementing ring operations, and exact comparison.

Examples of $\mathbb{D}$

- BigFloats or dyadic numbers:

$$\mathbb{F} := \{m2^n : m, n \in \mathbb{Z}\} = \mathbb{Z}[\frac{1}{2}]$$

- Rationals: $\mathbb{Q}$ (avoid, if possible)
- Real Algebraic Numbers: $\mathbb{A}$ (AVOID!)

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What else is needed in ENC Algorithms?

Intervals

- $\square\mathbb{D}$: set of dyadic intervals
- $\square\mathbb{D}^n$: set of n-boxes

Box Functions

Box function

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From Implicit to Explicit Representation

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- Discrete Morse-Smale complex [Problem (V)]
- Arrangement of hypersurfaces
- Voronoi diagram of a collection of objects
- Cell complex approximation of algebraic variety
- Representation of Flow fields

From Parameter Space to Ambient Space

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- Why this class? Interface between Continuous and Discrete!
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- Interplay of Topological and Geometric requirements
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Our Target: Explicitization Problems

From Implicit to Explicit Representation

- Mesh generation [Problem (IV)]
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1. Algebraic Approach

- Projection Based (Refinements of CAD)

  E.g., [Mourrain and Tecourt (2005); Cheng, Gao, and Li (2005)]
- Algebraic Subdivision Schemes

  E.g., [Wolpert and Seidel (2005)]
- Properties Exact; complete (usually); slow (in general); hard to implement

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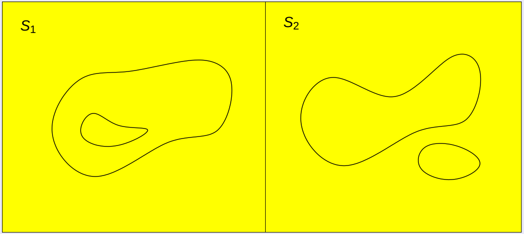
Two Criteria of Meshing

I. Topological Correctness

The approximation $\widetilde{S}$ is isotopic to the S.



- S_1 and S_2 are homeomorphic, but not isotopic
- Ambient space property!

(contd.) Two Criteria of Meshing

II. Geometrical Accuracy (ε-closeness)

For any given $\varepsilon > 0$, the Hausdorff distance $d(S, \widetilde{S})$ should not exceed ε.

- Set $\varepsilon = \infty$ to focus on isotopy.

Mini Summary

- Want ENC algorithms for Explicitization Problems
- Focus on (purely) Numerical Subdivision methods
- Algorithms for Meshing Curves (and Surfaces)
- What will be New?
  Numerical methods that are exact and can handle singularities

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Coming Up Next

Subdivision Algorithms

- Viewed as generalized binary search, organized as a quadtree.
- Here is a typical output:

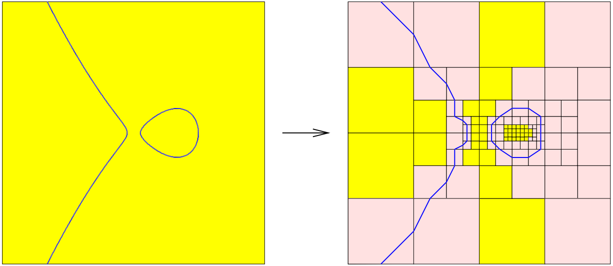

Figure: Approximation of the curve $f(X, Y) = Y^2 - X^2 + X^3 + 0.02 = 0$

The Generic Subdivision Algorithm

- INPUT: Curve $S = f^{-1}(0)$, box $B_0 \subseteq \mathbb{R}^2$, and $\varepsilon > 0$
- OUTPUT: Graph $G = (V, E)$, representing an isotopic ε-approximation of $S \cap B_0$.

1. Let $Q_{in} \leftarrow \{B_0\}$ be a queue of boxes
2. SUBDIVISION PHASE: $Q_{out} \leftarrow SUBDIVIDE(Q_{in})$
3. REFINEMENT PHASE: $Q_{ref} \leftarrow REFINE(Q_{out})$
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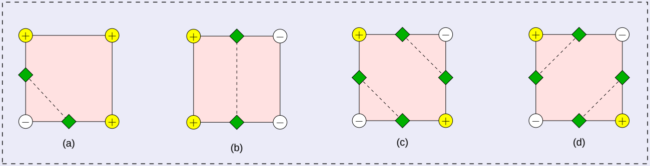
E.g., Marching Cube

Subdivision Phase

Subdivide until size of each box $\leq \varepsilon$.

Construction Phase

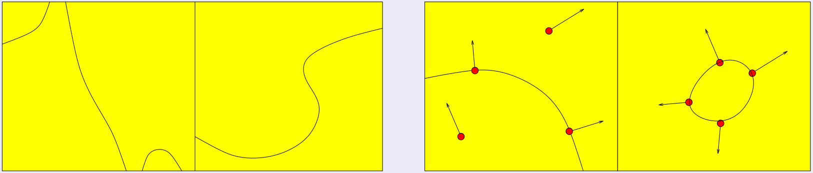
(1) Evaluate sign of f at grid points, (2) insert vertices, and (3) connect them in each box:



Cannot guarantee the topological correctness

Parametrizability and Normal Variation

Parametrizable in X-direction



(a) (b) (c) (d)

- (a) Parametrizable in X-direction
- (b) Non-parametrizable in X- or Y-direction
- (c) Small normal variation
- (d) Big normal variation

Box Predicates

Three Conditions (Predicates)

	C0	$0 \notin \square f(B)$	Exclusion
•	Cxy	$0 \notin \square f_x(B)$ or $0 \notin \square f_y(B)$	Parametrizability
	C1	$0 \notin \square f_x(B)^2 + \square f_y(B)^2$	Small Normal Variation

Implementation: e.g., $f(x,y) = x^2 - 2xy + 3y$

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Snyder's Algorithm

Subdivision Phase

For each box B:

- $C_0(B) \Rightarrow$ discard
- $\neg C_{xy}(B) \Rightarrow$ subdivide B

Construction Phase

- Determine intersections on boundary
- Connect the intersections
- (Non-trivial, unbounded complexity)

Boundary Analysis is not good (may not even terminate).

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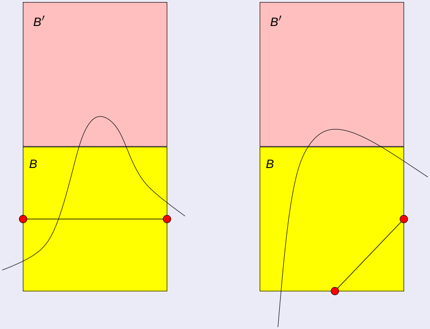
Idea of Plantinga and Vegter

Introduce a strong predicate $C1$ predicate

- Allow local NON-isotopy
  Local incursion and excursions



- Locally, graph is not isotopic

- Simple box geometry
  (simpler than Snyder, less simple than Marching Cube)

Plantinga and Vegter's Algorithm

Exploit the global isotopy

- Subdivision Phase: For each box B:
  - $C_0(B) \Rightarrow$ discard
  - $\neg C_1(B) \Rightarrow$ subdivide B
- Refinement Phase: Balance!

(contd.) Plantinga and Vegter's Algorithm

Global, not local, isotopy

- **Construction Phase:**

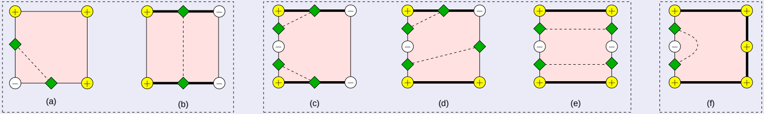


Figure: Extended Rules

- Local isotopy is NOT good !

Coming Up Next

Idea of Cxy Algorithm

Replace C1 by Cxy

- $C_1(B)$ implies $C_{xy}(B)$
- This would produce fewer boxes.

Exploit local non-isotopy

- Local isotopy is an artifact!
- This also avoid boundary analysis.

Obstructions to Cxy Idea

Replace C1 by Cxy

- Just run PV Algorithm but using C_{xy} instead:
- What can go wrong?

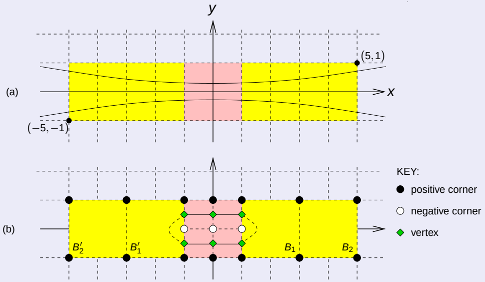


Figure: Elongated hyperbola

Cxy Algorithm

- Subdivision and Refinement Phases: As before
- Construction Phase:

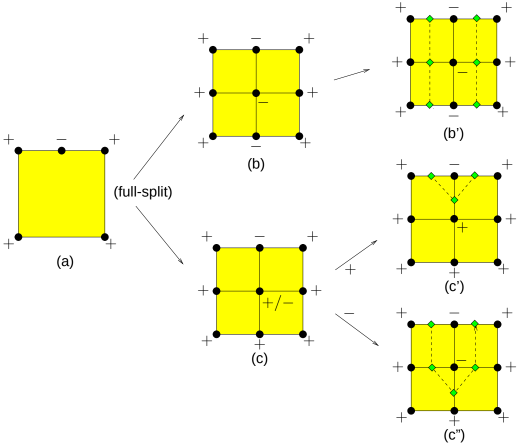


Figure: Resolution of Ambiguity

Mini Summary

- What has Cxy Algorithm done?
  - Exploit Parametrizability (like Snyder)
  - Rejected local isotopy (like PV)
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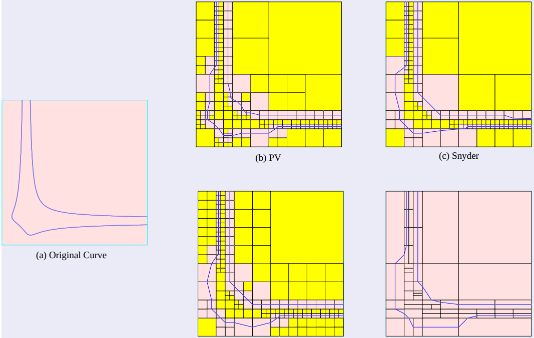
Coming Up Next

Idea of Rectangular Cxy Algorithm

Exploit Anisotropy



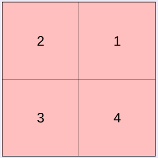
(a) Original Curve (b) PV (c) Snyder (d) Balanced Cxy (e) Rectangular Cxy

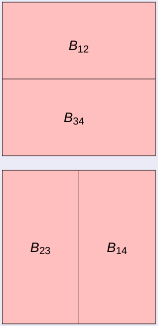
- "Heel Curve" $X^2Y^2 - X + Y - 1 = 0$ in box $B = [(-2,-10),(10,2)]$
- Comparing PV, Snyder, Cxy, Rect Cxy

Partial Splits for Rectangles

Splits

Full-splits:
$B \longrightarrow (B_1, B_2, B_3, B_4)$





- Horizontal Half-split:

  $B \longrightarrow (B_{12}, B_{34})$

- Vertical Half-split:

  $B \longrightarrow (B_{14}, B_{23})$

Rectangular Cxy Algorithm

What is needed

- Aspect Ratio Bound: $r > 1$ arbitrary but fixed.
- Splitting Procedure: do full-split if none of these hold

L_0 :	$C_0(B), C_{xy}(B)$	Terminate
L_{out} :	$C_0(B_{12}), C_0(B_{34}), C_0(B_{14}), C_0(B_{23})$	Half-split
L_{in} :	$C_{xy}(B_{12}), C_{xy}(B_{34}), C_{xy}(B_{14}), C_{xy}(B_{23})$	Half-split

- Axis-dependent balancing: each node has a X-depth and Y-depth.

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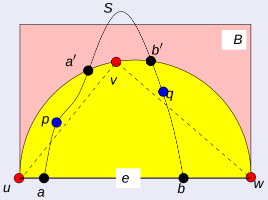
Ensuring Geometric Accuracy

Buffer Property of C1 predicate

- Aspect Ratio ≤ 2:



Half-circle argument

- Generalize $C_1(B)$ to $C_1^*(B)$. for any box B

Comparisons

- Compare Rect Cxy to PV (note: Snyder has degeneracy).
  - Curve $X(XY-1)=0$, box $B_s := [(-s,-s),(s,s)]$, Aspect ratio bound $r=5$: (JSO=Java stack overflow)

#Boxes/Time(ms)	$s=15$	$s=60$	$s=100$
PV	5686/157	JSO	JSO
Cxy	2878/125	45790/2750	JSO
Rect	258/32	3847/766	11196/7781

- Increasing r can increase the performance of Rect Cxy.
  - $r=80, s=100 \Rightarrow Boxes/Time(ms) = 751/78$

Comparisons (2)

- Compare to Snyder's Algorithm.
  - Curve $X(XY-1)=0$, box $B_n := [(-14\times 10^n, -14\times 10^n), (15\times 10^n, 15\times 10^n)]$. Maximum aspect ratio $r = 257$.

#Boxes/Time(ms)	$n=-1$	$n=0$	$n=1$
Snyder	10/15	1306/125	JSO
Cxy	13/0	1510/62	JSO
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Summary of Experimental Results

- Cxy combines the advantages of Snyder & PV Algorithms.
- Can be significantly faster than PV & Snyder's algorithm.
- Rectangular Cxy Algorithm can be significantly faster than Balanced Cxy algorithm.

Coming Up Next

Boundary (Summary)

- An Obvious Way and a Better Way
  - Exact Way : Recursively solve the problem on ∂B_0
  - Better Way : Exploit isotopy
- Price for Better Way: Weaker Correctness Statement

$$\text{For some } B_0 \subseteq B_0^+ \subseteq B_0 \oplus B(\varepsilon),$$
$$G \text{ is isotopic to } S \cap B_0^+ .$$

- APPLICATIONS:
  - Singularity (below)
  - Input region B_0 to have "any" geometry (even holes), provided it contains no singularities.

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Coming Up Next

Singularity : Algebraic Preliminary

- Square-free part of $f(X_1, \ldots, X_n) \in \mathbb{Z}[X_1, \ldots, X_n]$:
  $$\frac{f}{\mathtt{GCD}(f, \partial_1 f, \ldots, \partial_n f)} = \frac{f}{\mathtt{GCD}(f, \nabla(f))}$$
- For $n = 1$: square-free implies no singularities
- Generally:
  Singular set $sing(f) := \mathtt{Zero}(f, \nabla(f))$ has co-dimension ≥ 2.
- For Curves:
  we now assume $f(X, Y) \in \mathbb{Z}[X, Y]$ has isolated singularities.

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Some Zero Bounds

Evaluation Bound Lemma

If $f(X,Y)$ has degree d and height L then

$$\boxed{-\log EV(f) = O(d^2(L + d\log d))}$$

where $EV(f) := \min\{|f(\alpha)| : \nabla(\alpha) = 0, f(\alpha) \neq 0\}$

Singularity Separation Bound [Y. (2006)]

Any two singularities of $f = 0$ are separated by

$$\boxed{\delta_3 \geq (16^{d+2} 256^L 81^{2d} d^5)^{-d}}$$

Closest Approach Bound

The "locally closest" approach of a curve $f = 0$ to its own singularities is

$$\boxed{\delta_4 \geq (6^2 e^7)^{-30D} (4^4 \cdot 5 \cdot 2^L)^{-5D^4}}$$

where $D = \max\{2, \deg f\}$

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Isolating Singularities

Mountain Pass Theorem

Consider $F := f^2 + f_X^2 + f_Y^2$.

Any 2 singularities in B_0 are connected by paths $\gamma : [0,1] \to \mathbb{R}^2$ satisfying

$$\min \gamma(F([0,1])) \geq \varepsilon_0$$

where

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Can provide a subdivision algorithm using F, ε_0 to isolate regions containing singularities.

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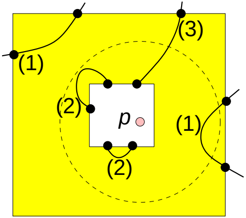
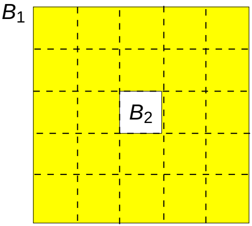
Degree of Singularites

- Degree of singularity $:=$ number of half-branches
- Use two concentric boxes $B_2 \subseteq B_1$:
  inner box has singularity, outer radius less than δ_3, δ_4



(a)



(b)

(a) Singularity p with 3 types of components

(b) Concentric boxes (B_1, B_2)

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- We have seen how to combine Snyder and PV, and make several practical improvements
- Future Work: Extend *3D* (and beyond?)
- Improve efficiency of refinement
- Improve efficiency of singularity
  - Nonsingular case is fast
  - Singular case is (currently) not fast

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- ENC Algorithms for them are novel
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PART 3

Complexity Analysis of Adaptivity

"A rapacious monster lurks within every computer,
and it dines exclusively on accurate digits."

— B.D. McCullough (2000)

Coming Up Next

Towards Analysis of Adaptive Algorithms

- Major Challenge in Theoretical Computer Science
  - Analysis of discrete algorithms is highly developed
  - What about continuous, adaptive algorithms?
- Previous such analysis requires probabilistic assumptions.
  - Basically in Linear Programming: [Smale, Borgwardt, Teng-Spielman]
- We focus on the recursion tree size
  - Return to 1-D !
- Adaptive algorithms may have some deep paths, but overall size is only polynomial in depth.
  - Previous (trivial) result – size is exponential in depth [Kearfott (1987)]

Towards Analysis of Adaptive Algorithms

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  - Analysis of discrete algorithms is highly developed
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- Suppose you want to isolate real roots of $f(x)$ in $I = [a, b]$
- Midpoint $m(I) := (a+b)/2$, Width $w(I) := b - a$
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2. WHILE $(Q \neq \emptyset)$ ◁ *Subdivision Phase*
3. $\quad I \leftarrow Q.remove()$
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Benchmark Problem in Root Isolation

- **Problem:** isolate ALL (real) roots of square-free $f(X) \in \mathbb{Z}[X]$ of degree $\leq d$ and height $< 2^L$.
- Highly classical problem:
  - Bit complexity is $\widetilde{O}(d^3 L)$ [Schöhage 1982].
    - Improvement: [illegible] arithmetic complexity [Pan]
  - Sturm tree size is $O(d(L + \log d))$ [Davenport, 1985]
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- **MAIN RESULT:** Bolzano tree size is $O(d^2(L + \log d))$
  - Sketch in this lecture. See [Burr-Krahmer-Y-Sagraloff, 2008-9]

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Main Complexity Goal – Benchmark Problem

Benchmark Problem in Root Isolation

- Problem: isolate ALL (real) roots of square-free $f(X) \in \mathbb{Z}[X]$ of degree $\leq d$ and height $< 2^L$.
- Highly classical problem:
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Warm Up Technique: Algebraic Amortization

Idea of Amortization [Davenport (1985), Du/Sharma/Y. (2005)]

- Let $A(X) \in \mathbb{Z}[X]$ have degree n and L-bit coefficients.
- Root separation bound: $-\log|\alpha - \beta| = O(n(L + \log n))$
- Amortized bound: $\boxed{-\prod_{(\alpha,\beta)\in E} |\beta - \alpha| = O(n(L + \log n))}$
- What are restrictions on set E?

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The Davenport–Mahler Bound

Theorem ([Davenport (1985), Johnson (1991/98), Du/Sharma/Y. (2005)])

Consider a polynomial $A(X) \in \mathbb{C}[X]$ *of degree n. Let* $G = (V, E)$ *be a digraph whose node set* V *consists of the roots* $\vartheta_1, \ldots, \vartheta_n$ *of* $A(X)$*. If*

(i) $(\alpha, \beta) \in E \implies |\alpha| \le |\beta|$,

(ii) $\beta \in V \implies \mathrm{indeg}(\beta) \le 1$, *and*

(iii) G *is acyclic,*

then

$$\prod_{(\alpha,\beta)\in E} |\beta - \alpha| \ge \frac{\sqrt{|\mathrm{discr}(A)|}}{\mathrm{M}(A)^{n-1}} \cdot 2^{-O(n \log n)},$$

where

$$\mathrm{discr}(A) := a_n^{2n-2} \prod_{i>j} (\vartheta_i - \vartheta_j)^2 \quad \textit{and} \quad \mathrm{M}(A) := |a_n| \prod_i \max\{1, |\vartheta_i|\}.$$

Mini Summary

- Adaptive analysis is important but virgin territory
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Coming Up Next

What is the Descartes Method?

Same framework as EVAL or Sturm

- To isolate roots of square-free $A(X)$ in interval I
- Routine $DescartesTest(A(X), I)$ gives an upper estimate on the number of real roots in I.
- If $DescartesTest(A(X), I) \in \{0, 1\}$ then estimate is exact.
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Analysis of Descartes Method

Two-circle Theorem

[Ostrowski (1950), Krandick/Mehlhorn (2006)]

If DescartesTest$(A(X),[c,d]) \geq 2$, then the two-circles figure in $\mathbb{C}$ around interval $[c,d]$ contains two roots α,β of $A(X)$.

Corollary

Can choose α,β to be complex conjugate or adjacent real roots.
Moreover, $|\beta-\alpha| < \sqrt{3}(d-c)$; i.e., $(d-c) > |\beta-\alpha|/\sqrt{3}$.

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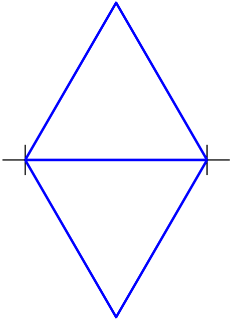

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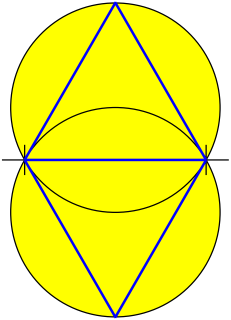

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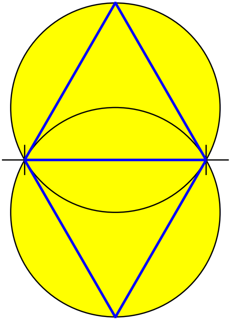

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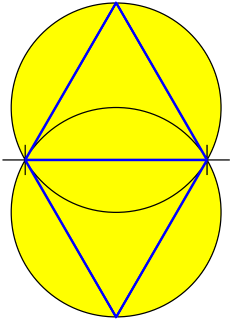

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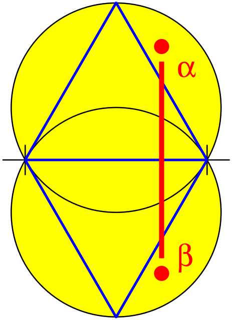

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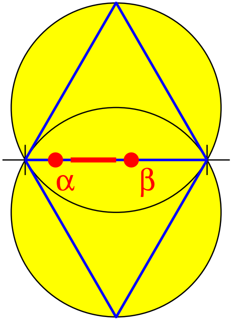

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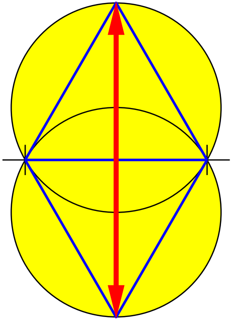

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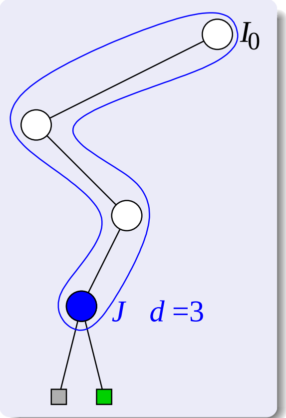

A bound on path length

1. Consider any path in the recursion tree from I_0 to a parent J of two leaves.
2. At depth d, interval width is $2^{-d}|I_0|$. Hence depth of J is $d = \log|I_0|/|J|$.
3. The path consists of $d+1$ internal nodes.
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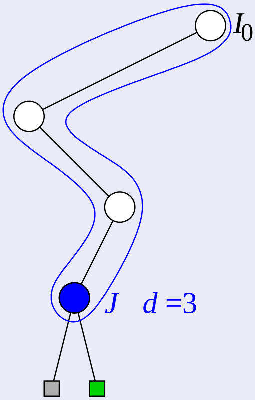

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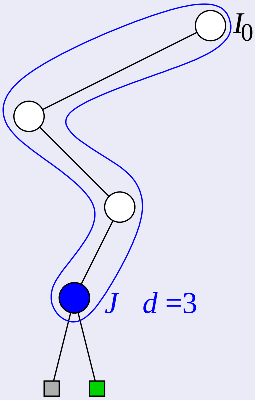

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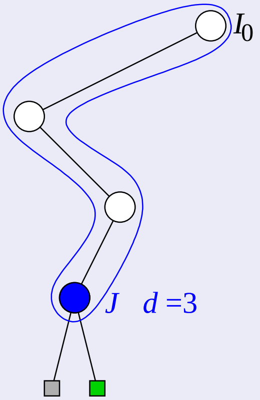

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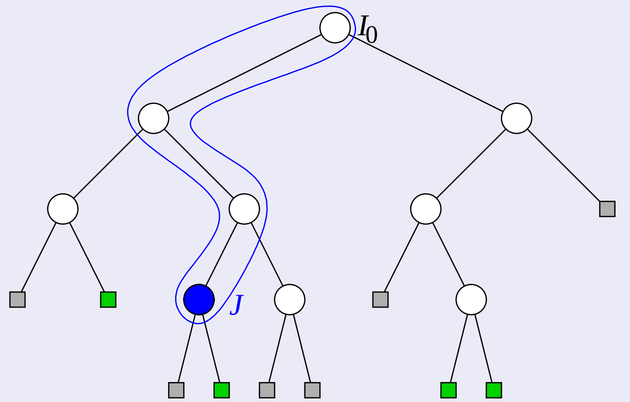

$$\begin{array}{lcr}
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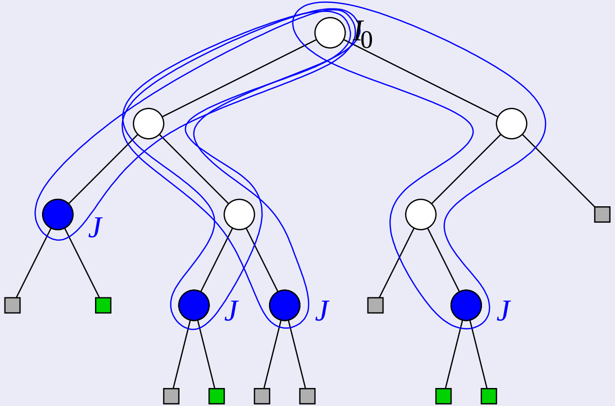

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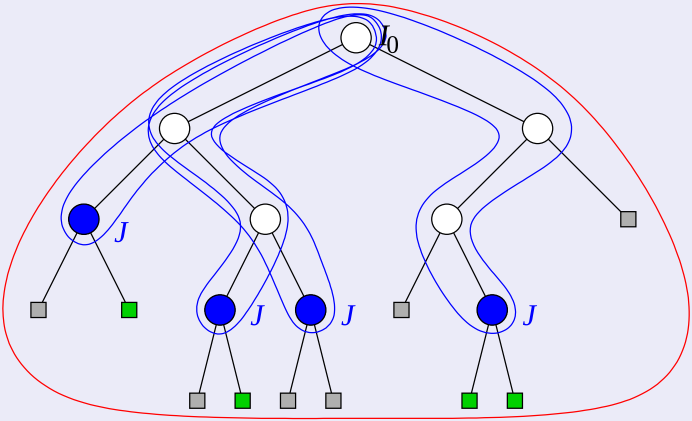

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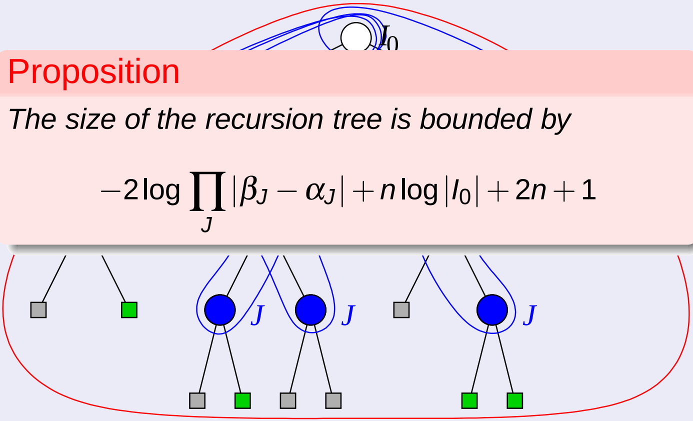

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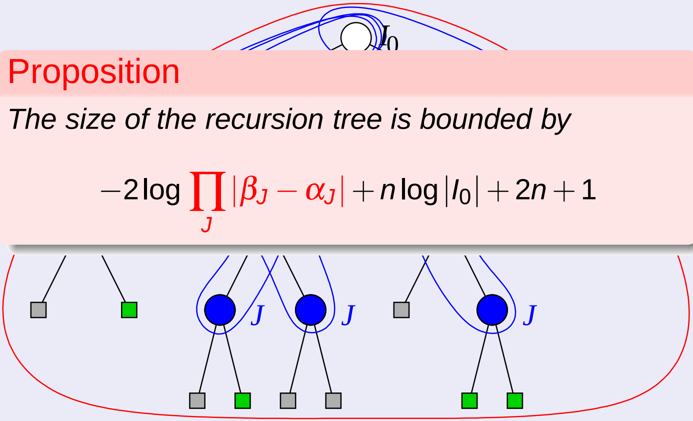

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Turning our Product into an Admissible Graph

We want to rewrite

$$\prod_J |\beta_J - \alpha_J| \quad \text{as} \quad \prod_{(\alpha,\beta)\in E} |\beta - \alpha|.$$

How often $|\beta_J - \alpha_J|$ appears?

- adjacent real: ≤ 1
- complex conjugate ≤ 2

We need two graphs. (Paper: just 1)

Conditions on $G = (V, E)$

(i) $(\alpha, \beta) \in E \implies |\alpha| \leq |\beta|$ ✓

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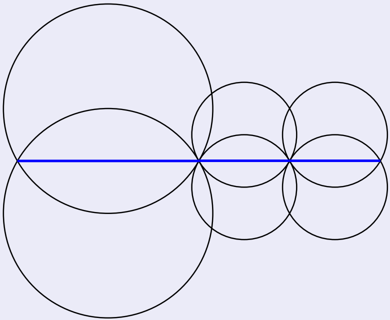

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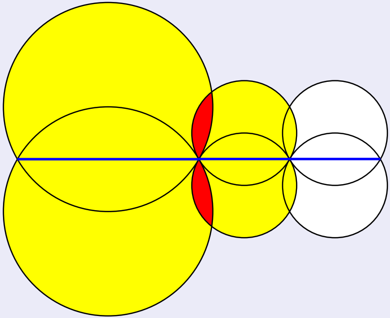

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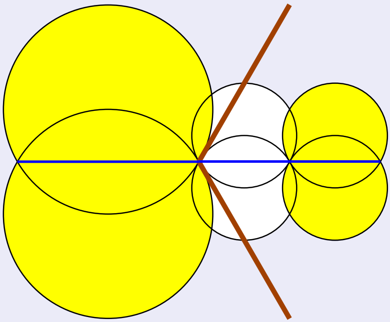

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Turning our Product into an Admissible Graph

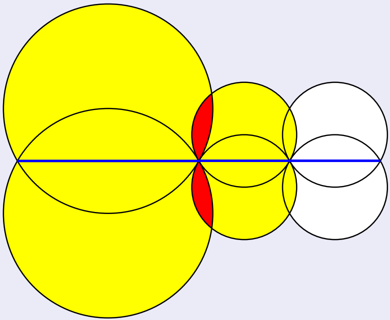
We want to rewrite

$$\prod_J |\beta_J - \alpha_J| \quad \text{as} \prod_{(\alpha,\beta)\in E} |\beta - \alpha|.$$

How often $|\beta_J - \alpha_J|$ appears?

- adjacent real: ≤ 1
- complex conjugate $\leq$ 2

We need two graphs. (Paper: just 1)



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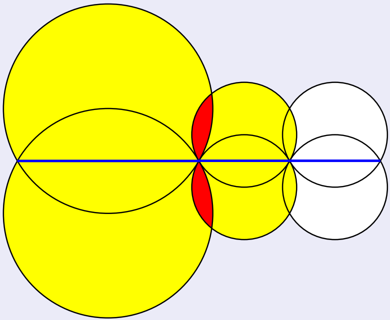

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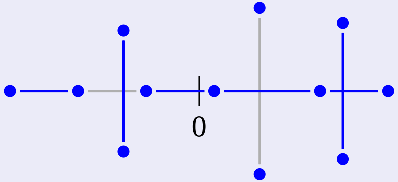

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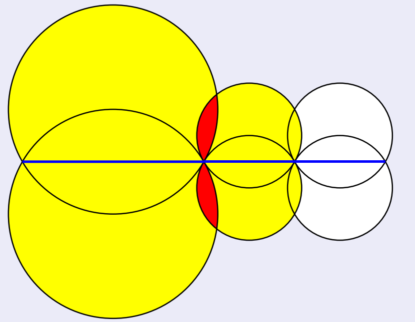

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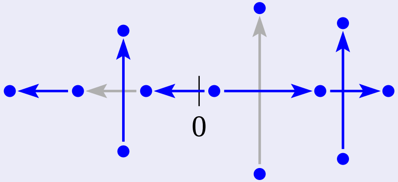

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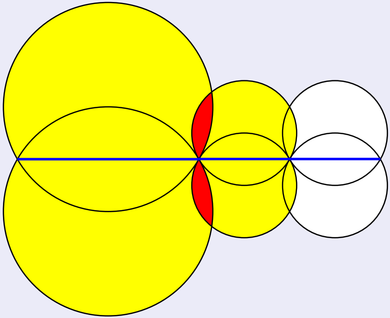

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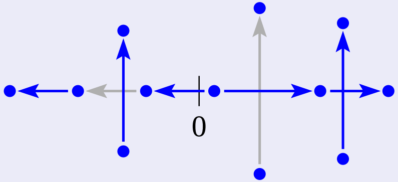

Main Result on Descartes Analysis

Theorem (Eigenwillig/Sharma/Y. (2006))

On the Benchmark Problem, we obtain

$$|\mathcal{T}| = O(n(L + \log n)).$$

For $L \geq \log n$, this is optimal.

Argument of [Krandick/Mehlhorn, 2006]: $|\mathcal{T}| = O(n \log n (L + \log n))$.

Mini Summary

- Almost Tight Bound on Descartes Method based on Algebraic Amortization
- Benchmark complexity of Sturm and Descartes are the same
  - "theory caught up with practice"
- What about EVAL?
  - New ideas needed – one is Amortized Evaluation Bounds

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Coming Up Next

12 Analysis of Adaptive Complexity

13 Analysis of Descartes Method

14 Integral Bounds and Framework of Stopping Functions

Subdivision Phase

Subdivision based on a Predicate $C(I)$

- Initialize a queue $Q \leftarrow \{I_0\}$
  1. WHILE $(Q \neq \emptyset)$
  2. $I \leftarrow Q.remove()$
  3. IF ($C(I)$ holds), output I
  4. ELSE
  5. Split I and insert children into Q

Goal – Bound the size of recursion tree $T(I_0)$

- NOTE: $C(I) = C_0(I) \vee C_1(I)$ in EVAL
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Framework of Stopping Functions

Stopping Function for $C(I)$ is $F : \mathbb{R} \to \mathbb{R}_{\geq 0}$

For all interval I:

$$\text{If } (\exists b \in I)[w(I) < F(b)],$$

$$\text{then } C(I) \text{ holds.}$$

How to use F? The Penultimate Property

- Similar to Descartes proof
- If $J \in P(I_0)$, its parent ("penultimate leaf") has width $2w(J)$.
- Conclude from definition of stopping function:

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Theorem (Integral Bound [Burr/Krahmer/Y.])

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Proof.

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An Integral Bound

Theorem (Integral Bound [Burr/Krahmer/Y.])

$$\#P(I_0) \le \max\left\{1, \int_{I_0} \frac{2dx}{F(x)}\right\}$$

Proof.

1. If $\#P(I_0) = 1$, result is true.
2. Else pick any $J \in P(I_0)$: it has the penultimate property.
3. Choosing $c^* \in J$ such that $F(c^*)$ is maximum

Pf (contd)

$$\begin{aligned}
\int_J \frac{2dx}{F(x)} &\ge \int_J \frac{2dx}{F(c^*)} \\
&\ge \frac{2}{F(c^*)} \int_J dx \\
&= \frac{2w(J)}{F(c^*)} \\
&\ge 1 \; [PenultimateProp.]
\end{aligned}$$

$$\begin{aligned}
\int_{I_0} \frac{2dx}{F(x)} &= \sum_{J \in P(I_0)} \int_J \frac{2dx}{F(c^*)} \\
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\end{aligned}$$

Remarks on Integral Bound

- Too hard to directly bound the integral implied by $C_0(I) \vee C_1(I)$.
  - So we devise stopping functions $F(x)$ that can be analyzed.
- Technique of bounding $\int_I \phi(x)dx$ is Continuous Amortization where $\phi(x)$ is charge function.
  - In discrete "amortization arguments", we bound $\sum_{i=1}^{n} \phi(i)$ where $\phi(i)$ is "charge" for the ith operation.
- Ruppert (1995) introduced a similar integral for triangulation.
  - Unlike us, he does not evaluate his integral.

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An Amortized Evaluation Bound

The Idea

- Want lower bounds on $|f(\alpha)|$
- Multivariate version used in [Cheng/Gao/Y. ISSAC'2007]
- Amortization: give lower bounds on $\prod_{i\in J}|f(\alpha_i)|$.

Theorem

Let $F, H \in \mathbb{Z}[X]$ be relatively prime such that $F = \phi\widetilde{\phi}$, $H = \eta\widetilde{\eta}$ where $\phi, \widetilde{\phi}, \eta, \widetilde{\eta} \in \mathbb{C}[X]$ have degrees $m, \widetilde{m}, n, \widetilde{n}$, respectively. If $\beta_1, \ldots, \beta_n$ are all the zeros of $\eta(X)$, then

$$\prod_{i=1}^{n}|\phi(\beta_i)| \geq \frac{1}{\mathrm{lc}(\eta)^m\left((m+1)\|\phi\|\right)^{\widetilde{n}} M(\widetilde{\eta})^m \left((\widetilde{m}+1)\|\widetilde{\phi}\|\right)^{n+\widetilde{n}} M(H)^{\widetilde{m}}}.$$

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Complex Roots: Lesson from Meshing Curves

How to isolate complex roots?

- Previous subdivision methods:
  - Pan-Weyl Algorithm (Turan Test)
  - Root isolation on boundary of boxes (topological degree)
- Hints from Curve Meshing (Snyder/PV/Cxy) – not good idea

New Result (with Sagraloff)

There is an exact analog CEVAL for complex roots that is simple and easy to implement exactly.
It achieves the same bit complexity bound as in the real case.

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Mini Summary

- The Bolzano approach to Root Isolation is an Exact and Analytic approach to root isolation
- It seems to have complexity that matches Sturm and Descartes
- It is much easier to implement than either

Summary of Lecture 3

- Complexity Analysis of Adaptivity at infancy
- Analysis Techniques we have seen so far:
  - Continuous amortization via integral bounds
  - Amortized root separation bounds
  - Amortized evaluation bounds
  - Cluster analysis
- Major Open Problems
  - How to characterize local complexity?
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- MANY advantages in numerical/analytic approaches to algebraic and geometric problems
  - practical, adaptive, easy to implement
- New ingredient we seek: a priori guarantees and exactness
- Zero problems is the locus of our investigation
- Exact Numerical Computation (ENC) is a suitable computational model
- The explicitization problems are central for ENC
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Thank you!

Website http://cs.nyu.edu/exact/

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GENERAL REFERENCE

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