Sieving a Needle
from
Lovász’s Exponential Haystack

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Lovász Local Lemma

The Framework

Universe $\Omega$.

$i \in \Omega$ make independent choice

$BAD_{\alpha}$ depends on choices $i \in X_{\alpha}$

$\alpha \sim \beta$ if $X_{\alpha} \cap X_{\beta} \neq \emptyset$

Example: Boolean $x_1, \ldots, x_n \leftarrow \{T, F\}$

Clause $C_{\alpha}$, e.g.: $x_{11} \land \overline{x}_{19} \land x_{204}$

$BAD_{\alpha}$: $C_{\alpha}$ false.

Desired Sieve Outcome:

$$\land_{\alpha} \overline{BAD_{\alpha}} \neq \emptyset$$

Example: $\land C_{\alpha}$ satisfiable.
Lovász Local Lemma

The Statement (Symmetric Case)

Assume:

All $\Pr[BAD_\alpha] \leq p$

All $\alpha$: $|\{\beta : \beta \sim \alpha\}| \leq d$

$p \frac{d^d}{(d+1)^{d+1}} \leq 1$ (roughly: $epd < 1$)

Conclusion:

$\wedge_\alpha \overline{BAD_\alpha} \neq \emptyset$

Example: Each $C_\alpha$ of Length 4.

$p = \frac{1}{16}$. Each $C_\alpha$ overlaps $\leq 5$ clauses.

No restriction on number of Clauses!

Satisfiable.
Lovász Local Lemma

Lovász (~1970) Proof

Induction on $|ARB|$:

$$\Pr[BAD_\alpha \land_{ARB} \overline{BAD}_\gamma] \leq x p$$

Renumber $\alpha = 0$, $ARB = \{1, \ldots, n\}$, $0 \sim 1, \ldots d$:

$$\Pr[B_0|B_1 \land \cdots \land B_n] =$$

$$= \frac{\Pr[B_0 \land B_1 \land \cdots \land B_d|B_{d+1} \land \cdots \land B_n]}{Pr[B_1 \land \cdots \land B_d|B_{d+1} \land \cdots \land B_n]} = \frac{NUM}{DEM}$$
\[ NUM \leq \Pr[B_0|\overline{B}_{d+1} \land \cdots \land \overline{B}_n] = \Pr[B_0] \leq p \]

\[ DEN = \prod_{i=1}^{d} \Pr[\overline{B}_i|\overline{B}_{i+1} \land \cdots \land \overline{B}_n] \]

Induction: \( DEN \geq (1 - xp)^d \)

Done if \( p(1 - xp)^{-d} \leq xp, \ 1 \leq x(1 - xp)^d \)

Calculus: Optimal \( x = \frac{1}{p(d+1)}. \)

OK if \( p \leq d^d(d + 1)^{-(d+1)}. \)
Lovász Local Lemma

Moser-Tardos 2009 Algorithm

Each \( i \in \Omega \) makes independent choice

\[
\text{WHILE some } C_{\alpha} \text{ false}
\]

\[
\text{SELECT } * \text{ false } C_{\alpha}
\]

Each \( i \in X_{\alpha} \) reselects

*Use BFS for Efficiency*
\( \text{LOG} = (e_1, \ldots, e_t, \ldots) \), which \( C \)'s called

E.g.: \((\alpha, \gamma, \kappa, \alpha, \beta, \delta, \alpha, \kappa)\)

"Time" \( T = \text{length of LOG} \)

\( T_{\alpha} = \text{number of times } \alpha \text{ called} \)

Key Lemma: \( E[T_{\alpha}] \leq xp \)

As \( E[T] = \sum E[T_{\alpha}] \), Linear Time Algorithm!
**Tree of Relevant History**

$TREE[t]$ has root $e_t$

FOR $i = t - 1$ DOWN TO 1

If $e_i$ overlaps $e_j$ already in $TREE$

(** If not, Ignore **)  

Make $e_i$ child of $e_j$

(*****) If choices, pick node furthest from root
Key Properties

- The $TREE[t]$ are all different
- $e \in TREE[t]$ on same level do not overlap
- If $e_r, e_s \in TREE[t]$, $r < s$, $e_r, e_s$ overlap, then $e_r$ lower than $e_s$
- Let $i \in \Omega$. $i \in f_1, \ldots, f_s \in TREE[t]$.
  Order of $f_j$ in $LOG$ is by depth in $TREE[t]$.

$$E[T_\alpha] = \sum_{TR} \Pr[\exists_t TREE[t] = TR]$$

$TR$ rooted at $\alpha$

$\gamma$ child of $\beta \Rightarrow \gamma, \beta$ overlap.
\[
\Pr[\exists_t TREE[t] = TR] \leq \prod_{\gamma \in TR} \Pr[BAD_\gamma]
\]

Proof: Preprocess Randomness

Each \(i\) chooses \(y_1, y_2, \ldots\)

\(TR\) only \(\alpha\): \(BAD_\alpha\) with first choice

\(TR\) \(\alpha\) with child \(\beta\).

\(BAD_\beta\) with first choice all \(i \in X_\beta\)

\(BAD_\alpha\) with first choice \(i \notin X_\beta\), else second.

General: Choice Number determined by \(TR\).
\[ E[T_\alpha] \leq \sum_{TR} \prod_{\gamma \in TR} \Pr[BAD_\gamma] \]

\[ E[T_\alpha] \leq y := \sum_{TR} p^{|TR|} \]

*TR* subtree of infinite rooted tree.

Each node has \(d\) children (here \(\alpha\) child of \(\alpha\))


\(y/p = \text{expected number of subtrees.}\)

\[ p \leq \frac{(d-1)^{d-1}}{d^d} \Rightarrow y = p(1 + y)^d \leq \frac{1}{d-1} \]

**AlmostProof**

Specifying \(i\) children of root, \(py^i\).

\[ y = p \sum_{i=0}^{d} \binom{d}{i} y^i = p(1 + y)^d \]

Here \(\alpha\) is child of \(\alpha\) so **same** as Lovász!
You don’t have to believe in God
but you should believe in The Book.
– Paul Erdős