Midwest Probability Colloquium

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Sieving a Needle

from

Lovász's Exponential Haystack

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Lovász Local Lemma

The Framework

Universe Ω.

$i \in \Omega$ make independent choice

BAD_α depends on choices $i \in X_\alpha$

$\alpha \sim \beta$ if $X_\alpha \cap X_\beta \neq \emptyset$

Example: Boolean $x_1, \ldots, x_n \leftarrow \{T, F\}$

Clause C_α, e.g.: $x_{11} \wedge \overline{x}_{19} \wedge x_{204}$

BAD_α: C_α false.

Desired Sieve Outcome:

$$\wedge_\alpha \overline{BAD}_\alpha \neq \emptyset$$

Example: $\wedge C_\alpha$ satisfiable.

Lovász Local Lemma

The Statement (Symmetric Case)

Assume:

All $\Pr[BAD_\alpha] \le p$

All α: $|\{\beta : \beta \sim \alpha\}| \le d$

$p\frac{d^d}{(d+1)^{d+1}} \le 1$ (roughly: $epd < 1$)

Conclusion:

$$\wedge_\alpha \overline{BAD}_\alpha \neq \emptyset$$

Example: Each C_α of Length 4.

$p = \frac{1}{16}$. Each C_α overlaps ≤ 5 clauses.

`No restriction on number of Clauses!`

Satisfiable.

Lovász Local Lemma

Lovász ($\sim$ 1970) Proof

Induction on $|ARB|$:

$$\Pr[BAD_\alpha | \wedge_{ARB} \overline{BAD}_\gamma] \le xp$$

Renumber $\alpha = 0$, $ARB = \{1, \ldots, n\}$, $0 \sim 1, \ldots d$:

$$\Pr[B_0 | \overline{B}_1 \wedge \cdots \wedge \overline{B}_n] =$$

$$= \frac{\Pr[B_0 \wedge \overline{B}_1 \wedge \cdots \wedge \overline{B}_d | \overline{B}_{d+1} \wedge \cdots \wedge \overline{B}_n]}{\Pr[\overline{B}_1 \wedge \cdots \wedge \overline{B}_d | \overline{B}_{d+1} \wedge \cdots \wedge \overline{B}_n]} = \frac{NUM}{DEM}$$

$$NUM \leq \Pr[B_0|\overline{B}_{d+1} \wedge \cdots \wedge \overline{B}_n] = \Pr[B_0] \leq p$$

$$DEN = \prod_{i=1}^{d} \Pr[\overline{B}_i|\overline{B}_{i+1} \wedge \cdots \wedge \overline{B}_n]$$

Induction: $DEN \geq (1 - xp)^d$

Done if $p(1 - xp)^{-d} \leq xp$, $1 \leq x(1 - xp)^d$

Calculus: Optimal $x = \frac{1}{p(d+1)}$.

OK if $p \leq d^d(d + 1)^{-(d+1)}$.

Lovász Local Lemma

Moser-Tardos 2009 Algorithm

Each $i \in \Omega$ makes independent choice

WHILE some C_α false

  SELECT * false C_α

  Each $i \in X_\alpha$ reselects

*Use BFS for Efficiency

$LOG = (e_1, \ldots, e_t, \ldots)$, which C's called

E.g.: $(\alpha, \gamma, \kappa, \alpha, \beta, \delta, \alpha, \kappa)$

"Time" $T =$ length of LOG

$T_\alpha =$ number of times α called

Key Lemma: $E[T_\alpha] \leq xp$

As $E[T] = \sum E[T_\alpha]$, Linear Time Algorithm!

Tree of Relevant History

$TREE[t]$ has root e_t

FOR $i = t - 1$ DOWN TO 1

If e_i overlaps e_j already in $TREE$

(** If not, Ignore **)

Make e_i child of e_j

(***) If choices, pick node furthest from root

Key Properties

- The $TREE[t]$ are all different
- $e \in TREE[t]$ on same level do not overlap
- If $e_r, e_s \in TREE[t]$, $r < s$, e_r, e_s overlap,

  Then e_r lower than e_s
- Let $i \in \Omega$. $i \in f_1, \ldots, f_s \in TREE[t]$.

  Order of f_j in LOG is by depth in $TREE[t]$.

$$E[T_\alpha] = \sum_{TR} \Pr[\exists_t TREE[t] = TR]$$

TR rooted at α

γ child of $\beta \Rightarrow \gamma, \beta$ overlap.

$$\text{Pr}[\exists_t TREE[t] = TR] \leq \prod_{\gamma \in TR}^{(rep)} \text{Pr}[BAD_\gamma]$$

Proof: Preprocess Randomness

Each i chooses $y_1, y_2, \ldots$

TR only α: BAD_α with first choice

TR α with child β.

BAD_β with first choice all $i \in X_\beta$

BAD_α with first choice $i \notin X_\beta$, else second.

General: Choice Number determined by TR.

$$E[T_\alpha] \le \sum_{TR} \prod_{\gamma \in TR}^{(rep)} \Pr[BAD_\gamma]$$

$$E[T_\alpha] \le y := \sum_{TR} p^{|TR|}$$

TR subtree of infinite rooted tree.

Each node has d children (here α child of α)

Galton-Watson $BIN[d,p]$ Birth Process.

$y/p =$ expected number of subtrees.

$p \le \frac{(d-1)^{d-1}}{d^d} \Rightarrow y = p(1+y)^d \le \frac{1}{d-1}$

AlmostProof

Specifying i children of root, py^i.

$$y = p \sum_{i=0}^{d} \binom{d}{i} y^i = p(1+y)^d$$

Here α is child of α so **same** as Lovász!

You don’t have to believe in God

but you should believe in The Book.

– Paul Erdős