COCOON 2003

LIAR!
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Joint work with
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Paul versus Carole

$N$ Possibilities

$Q$ Yes/No Paul Queries

$K$ (or fewer) Carole Lies

Try it with $N = 100$, $Q = 10$, $K = 1$

Carole plays Adversary Strategy

$\Rightarrow$ Perfect Information

$\Rightarrow$ Winning Strategy for Paul or Carole

$B_K(Q) =$ maximal $N$ so that Paul Wins

Theorem:

$$B_K(Q) \sim \frac{2^Q}{\binom{Q}{K}}$$
Carole Strategy

Notation
\[
\binom{Q}{\leq K} = \sum_{I=0}^{K} \binom{Q}{I}
\]

Theorem: \( N\binom{Q}{\leq K} > 2^Q \Rightarrow \text{Carole Wins} \)

Proof 1: Preserve Ministrategies

Proof 2: Random Play

Proof 1 \Rightarrow \text{Proof 2: Derandomization}
Vector Format

Position \((3, 14)\) \(((x_0, \ldots, x_K))\)

Paul Move \((1, 9)\) \(((a_0, \ldots, a_K))\)

Yes: \((1, 11)\); No: \((2, 6)\)

Perfect Split: Yes=No

Position \((8, 20)\), Move \((4, 10)\), Yes/No \((4, 14)\)

\[ L : (x, y) \rightarrow (\frac{x}{2}, \frac{x}{2} + \frac{y}{2}) \quad (L : R^{K+1} \rightarrow R^{K+1}) \]

Position after perfect split.

Problem: Integrality

Weight Function \(W_Q(\bar{x}) = L^Q(\bar{x}) \cdot \bar{1}\)

\[ W_Q(x, y) = 2^{-Q}((Q + 1)x + y) \]

\[ (2^{-Q}((\frac{Q}{\leq K})x_0 + \ldots + (Q + 1)x_{K-1} + x_K)) \]
Paul Strategy

Theorem (JS): \((K \text{ fixed, } Q \text{ large})\)

\[ W \leq 1 \text{ and } > cQ^K \text{ “pennies”} \]

\[ \Rightarrow \text{ Paul Win} \]

Keep Weight Equal (Perfect Split if Possible)

\(Q = 10. \text{ Position } (17, 837). \text{ } W = 1\)

Paul \((8, 418 + x) \Rightarrow (8, 427 + x); (9, 427 - x)\)

\[ W_9(1, -2x) = 0 \Rightarrow x = 5 \]

Problem: Nonnegativity

Proof Outline

First \(K\) Moves: Initial Penny Supply

Middle: Pennies Replenished from Nonpennies

End: Endgame Analysis
Halflie: No False Negatives

$N$ Possibilities

$Q$ Queries

$K$ Halflies

$A_K(Q) = \text{maximal } N$, Paul Wins

Theorem (Cicalese/Mundici, COCOON00):

$A_1(Q) \sim \frac{2^{Q+1}}{Q}$

Dumitriu/JS:

$A_K(Q) \sim 2^K B_K(Q) \sim 2^K \frac{2^Q}{\binom{Q}{K}}$
Position $\vec{x} = (x, y) \ ( (x_0, \ldots, x_K))$

Paul Query: $(a, b) \ ( (a_0, \ldots, a_K))$

Yes $(a, b + x - a)$; No $(x - a, y - b)$

Perfect Split $(\frac{x}{2}, \frac{y}{2} - \frac{x}{4})$

Yes/No $L\vec{x} := (\frac{x}{2}, \frac{y}{2} + \frac{x}{4}) \ (L : R^{K+1} \rightarrow R^{K+1})$

Problems: Integrality, Nonnegativity

Weight $W_Q(\vec{x}) = L^Q(\vec{x}) \cdot \vec{1}$

$W_Q(x, y) = 2^{-Q}(x(1 + \frac{Q}{2}) + y)$

$2^{-Q}(x_0p_K(Q) + \ldots + x_{K-1}(1 + \frac{Q}{2}) + x_K)$
Paul Strategy

Start \((N, 0)\), \(N < (1 - \epsilon)2^{Q+1}/Q\)

- Give Ground to \((N, N)\)

\[ T := \lfloor \lg N \rfloor \]

- Roundoff so \(2^T | N\)

- \(T\) perfect splits to \(L^T(N\vec{1})\)

- Endgame: Win in \(R\) from

\[(0, 2^R); (1, 2^R - 1); (2, 2^R - 3); (3, 2^R - 5)\]
A Combinatorial Approach

1-Set: Subset of \( \{Y, N\}^Q \) with

stem \( YNNYNYN \)
child \( YYYNNYN \)
child \( YNYYYYN \)
child \( YNNYYN \)

0-Set: Any Singleton

\( K \)-Set: Depth \( K \) tree with marked “lies.”

parent \( YYYNNYN \)
child \( YYYNYNN \)
grandchild \( YYYNYYY \)

Theorem: Paul Wins from \((x_0, \ldots, x_K)\) in \( Q \)
\( \Leftrightarrow \) Can Pack \( x_i k - i \)-Sets in \( \{Y, N\}^Q \)
Bound Packing of $K$-Sets

- When all have $\geq L Y$, Size $> \left(\frac{L}{\leq K}\right)$

$L \sim \frac{Q}{2}$ Volume Bound $2^Q/(\binom{Q/2}{K})$

$o(2^Q Q^{-K})$ have any $L < (1 - o(1))\frac{Q}{2}$

$A_K(Q) < (1 + o(1))2^Q/(\binom{Q/2}{K})$

Careful Cutoff

Set $L = \frac{Q}{2} + c\sqrt{Q} \sqrt{\ln Q} Y$

$A_K(Q) \leq \frac{2^Q}{\binom{Q/2}{K}}(1 + cQ^{-1/2} \sqrt{\ln Q})$

Yan/JS: Remove $\sqrt{\ln Q}$
Second Order Terms

Theorem (Yan/JS):

\[ A_K(Q) = \frac{2^Q}{\binom{Q}{K}} \left( 1 + \Theta(Q^{-1/2}) \right) \]

Paul Strategy for \( K = 1 \)

\( W = 1 + cQ^{-1/2} \) at start.

Perfect Splits until \( R \sim \frac{Q}{10} \) queries remain

New position \((x, y)\) with

\[ W = 2^{-R} [x(1 + \frac{R}{2})] + y] \sim 1 + c'R^{-1/2} \]

\[ 2^{-R} x(1 + \frac{R}{2}) \sim \epsilon \]

Find \( x \) disjoint “small” 1-Sets
Disjoint Shadows

A $K$-shadow consists of stem $\sigma \in \{Y, N\}^R$ and all $\tau$ derived by changing at most $K$ $Y$ to $N$. $K$-shadows are $K$-sets.

$K = 1$: Stems $(\epsilon_1, \ldots, \epsilon_R)$ with

$$\sum_{\epsilon_i=Y} i \equiv c \mod R + 1$$

have disjoint 1-Shadows

- All stems with $\leq \frac{R}{2} - \sqrt{R} Y$

- Average $c$

More than $x$ disjoint 1-shadows, all small.
\[ x \frac{R}{2} + y \sim 2^R(1 + c'R^{-1/2}) \]
\[ x \frac{R}{2} \sim \epsilon 2^R \]
\[ x\left(\frac{R}{2} - \sqrt{R}\right) + y < 2^R \]

- Find \( x \) small disjoint 1-Shadows

- Add \( y \) 0-Sets as filler.
Arbitrary Channel

T-ary queries

E lie patterns

Example with $T = 3$, $E = 4$

Ternary Answers A/B/C.

Carole may lie B for A, A for B, A or B for C.

Theorem (Dumitriu, JS):

$$A^*_K(Q) \sim \frac{T^K T^Q}{E^K \binom{Q}{K}}$$
Open Question

What is the maximum number $G(R)$ of disjoint $1$-Shadows in $\{Y, N\}^R$?

\[
\frac{2^R}{R + 1} \leq G(R)
\]

\[G(R) \leq 2 \frac{2^R}{R} (1 + o(1))\]

Asymptotic Factor of Two Gap.