Random Graphs Assignment 8 Solutions

1. Show that Carole wins the liar game with $n = 93$, $k = 10$.
   Solution: There are now $93 \cdot 11 = 1023 < 2^{10}$ ministrategies. Paul’s best split is to ask if $x \leq 46$ to which Carole says No. Now there are $46 + 10(47) > 2^9$ ministrategies so Carole wins.

2. Consider the $q$-Chip Liar with initial position $(1, y)$. (That is, there are $q$ rounds. Initially there is one possibility about which one Carole may lie and $y$ for which she cannot.) Find the maximal $y = y(q)$ for which Paul wins. (Hint: The bound given by Theorem 15.2.1 turns out to be precise and you’ll need to prove that by giving a Paul strategy.)
   Solution: As $W = (q + 1) + y$ from Theorem 15.2.1 we must have $y \leq 2^q - q - 1$ for Paul to have a chance. But Paul does win in this case! He asks $(2^q - 1, 0)$. If Yes, the new state is $(2^q - 1, 0)$ – effectively the no lie case – and he wins in the remaining $q - 1$ rounds. If No, he wins by induction (the initial cases $q = 1, 2$ being easy) as the new state can be written $(2^q - 1 - (q - 1) - 1, 1)$.

3. Using the above and that $y(4) = 11$ give an explicit strategy for Paul to win the 1-Liar game with $n = 2^{11}$ and 15 questions. (Idea: Let the values of $x$ be 0 through $2^{11} - 1$ and let the first 11 questions be of the binary bits of $x$.) Show by Theorem 15.2.1 that for $n = 2^{11} + 1$ Carole wins.
   Solution: For $n = 2^{11} + 1$, $n(q + 1) = 2^4n > 2^{15}$ so Carole wins. Now say $n = 2^{11}$. After the first 11 questions from the idea the new state is $(1, 11)$ regardless of Carole’s answers and so Paul wins by the previous problem.

4. Consider two independent Galton-Watson processes, each with $c = 1$, and let $X, Y$ be the numbers of nodes in the respective processes. Set $Z = X + Y$. The following are meant asymptotically in $n$. (Say $n$ odd, though $n$ even is very similar.)
   (a) Find, in $\Theta$-land, $P[Z = n]$. (Idea: $P[Y = y]$ is bounded within constants for $\frac{n}{2} < y \leq n$.)
   Solution: By symmetry
   \[
   P[Z = n] = 2 \sum_{x=1}^{n/2} P[X = x] P[Y = n - x]
   \]
But $\Pr[Y = y] = \Theta(n^{-3/2})$ in the range $\frac{n}{2} < y \leq n - 1$ so that

$$\Pr[Z = n] = \Theta(n^{-3/2}) \sum_{x=1}^{n/2} \Pr[X = x]$$

The sum is asymptotically one so $\Pr[Z = n] = \Theta(n^{-3/2})$.

(b) Show that $\Pr[X = 1|Z = n] = \Omega(1)$. (That is, find a constant lower bound on probability $(X, Y) = (1, n-1)$ given that $X+Y = n$.)

Solution: $\Pr[X = 1, Y = n-1] = e^{-1} \Pr[Y = n-1] = \Theta(n^{-3/2})$ so

$$\Pr[X = 1|Z = n] = \frac{\Pr[X = 1, Y = n-1]}{\Pr[Z = n]} = \frac{\Theta(n^{-3/2})}{\Theta(n^{-3/2})} = \Omega(1)$$

(c) Show that $\Pr[\frac{n}{3} \leq X \leq \frac{2n}{3}|Z = n] = O(n^{-\alpha})$ for an explicit positive $\alpha$.

Solution: For each $x \in [n/3, 2n/3]$, $\Pr[X = x] = \Theta(n^{-3/2})$. Hence in that range all $\Pr[X = x, Y = n - x] = \Theta(n^{-3})$. There are $\Theta(n)$ addends so $\sum_x \Pr[X = x, Y = n - x] = \Theta(n^{-2})$. Hence

$$\sum_{x=n/3}^{2n/3} \frac{\Pr[X = x, Y = n - x]}{\Pr[Z = n]} = \frac{\Theta(n^{-2})}{\Theta(n^{-3/2})} = \Theta(n^{-1/2})$$

Remark: Yet another wierdness about heavy tails!

5. (Just for fun) Play a Half-Liar game with a friend with Carole selecting $1 \leq x \leq 100$, 1 Half-Lie, and Paul having ten questions. Record the play with commentary. Are you still friends?

\footnote{I certainly hope $\exists_y FRIEND(u, y)$.}