Random Graphs Assignment 7

Solutions

1. Let \( H \) have vertices \( \{A, B, C, D, E\} \) and be the complete graph on \( \{A, B, C, D\} \) and the edges \( \{E, A\}, \{E, B\} \). For \( \alpha > 0 \) let \( f(n, \alpha) \) denote the probability that \( G(n, p) \) does not contain a copy of \( H \) when \( p = n^{-\alpha} \). Here we will give \( f(n, \alpha) \) up to a constant in the exponent.

(a) For \( t = 2, 3, 4, 5 \) find the subgraph \( H_t \) of \( H \) on \( t \) vertices with the maximal number of edges and find \( e_t \), the number of edges of \( H_t \).

Solution: \( e_2 = 1 \) (edge), \( e_3 = 3 \) (triangle), \( e_4 = 6 \) \((K_4)\) and \( e_5 = 8 \) \((H\) itself\).

(b) Show that \( H \) is strictly balanced. What is the threshold function for containing a copy of \( H \)? Henceforth, restrict to \( \alpha \) so that \( p = n^{-\alpha} \) is bigger than that threshold.

Solution: We check that \( 1, 3, 3, 6, 4 \) are all less than \( 8 \). So \( n^{-5/8} \) is the threshold function.

(c) Let \( LB_t \) denote the lower bound, from Janson’s Inequality, on the probability that \( G(n, p) \) does not contain a copy of \( H_t \). Set \( LB = \) the maximum of \( LB_t, t = 2, 3, 4, 5 \). Find \( LB \) as a function of \( \alpha \) – there will be three ranges (some graph paper will help!) of \( \alpha \) at which different \( t \) give the maximum.

Solution: With \( p = n^{-\alpha} \) and \( \alpha < 5/8 \):

\[
\Pr[G \text{ has no } K_2] \geq (1 - p)^{\Theta(n^2)} = \exp[-\Theta(n^{2-\alpha})] = LB_2
\]
\[
\Pr[G \text{ has no } K_3] \geq (1 - p^3)^{\Theta(n^3)} = \exp[-\Theta(n^{3-3\alpha})] = LB_3
\]
\[
\Pr[G \text{ has no } K_4] \geq (1 - p^6)^{\Theta(n^4)} = \exp[-\Theta(n^{4-6\alpha})] = LB_4
\]
\[
\Pr[G \text{ has no } H] \geq (1 - p^8)^{\Theta(n^5)} = \exp[-\Theta(n^{5-8\alpha})] = LB_5
\]

To find \( \max LB_t \) we need to maximize \( 2 - \alpha, 3 - 3\alpha, 4 - 6\alpha \) and \( 5 - 8\alpha \). Graphing the four linear functions we see:

For \( 0 < \alpha < \frac{2}{5} \), \( LB \sim LB_2 = \exp[-\Theta(n^{2-\alpha})] \).

For \( \frac{2}{5} < \alpha < \frac{1}{2} \), \( LB \sim LB_4 = \exp[-\Theta(n^{4-6\alpha})] \).

For \( \frac{1}{2} < \alpha < \frac{5}{8} \), \( LB \sim LB_5 = \exp[-\Theta(n^{5-8\alpha})] \).

(At the critical values \( \alpha = \frac{2}{5}, \frac{1}{2} \) both values work.)

(d) Find \( \mu, \Delta \) of the upper bound of the Extended Janson’s Inequality. Show that the \( \Delta \) breaks into a finite number of ranges depending on which addend predominates.
Solution: $\mu = \Theta(n^5 p^8) = \Theta(n^{5-8\alpha})$. $\Delta$ is the sum of three terms, as the vertex overlap can be 2, 3 or 4. They are $\Theta(n^8 p^{15})$ (two $H$s overlapping in an edge), $\Theta(n^7 p^{13})$ (two $H$s overlapping in a triangle), and $\Theta(n^6 p^{10})$ (two $H$s overlapping in a $K_4$). So $\mu^2 / 2\Delta$ would be the smallest of $\Theta(n^{2-r\alpha})$, $\Theta(n^{3-3\alpha})$, $\Theta(n^{4-6\alpha})$.

(e) Combining the lower and upper bounds above get a result of the form:
If $0 < \alpha < \kappa_1$ the $f(n, p) = \exp[-\Theta(n^{\gamma_1+\gamma_2\alpha})]$. If $\kappa_1 < \alpha < \kappa_2$ the $f(n, p) = \exp[-\Theta(n^{\gamma_3+\gamma_4\alpha})]$. If $\kappa_2 < \alpha < \kappa_3$ then $f(n, p) \sim 1$. Here the $\kappa$s and $\gamma$s will be nice rational numbers.

Solution: For $\frac{1}{2} < \alpha < \frac{5}{8}$ we have $\Delta = o(\mu)$ so we would apply Janson’s Inequality (not the Extended Version!) so give an upper bound

$$UB = \exp[\Theta(\mu)] = \exp[-n^{5-8\alpha}]$$

For $\frac{2}{5} < \alpha < \frac{1}{2}$, $\mu^2 / \Delta = \Theta(n^{4-6\alpha})$ and $UB = \exp[-\Theta(n^{4-6\alpha})]$. For $\alpha < \frac{2}{5}$, $\mu^2 / \Delta = \Theta(n^{2-\alpha})$ and $UB = \exp[-\Theta(n^{2-\alpha})]$. That is, the upper and lower bounds match. This actually holds for any fixed graph $H$ and was one of the first applications of Janson’s Inequality.

2. In $G(n, p)$ with $p = c/n$ let $X$ be the number of isolated triangles. Let $\mu = E[X]$. In an earlier assignment you calculated the limiting value of $\mu$.

(a) For $r \geq 1$ give an exact formula for $S^{(r)} = E[\binom{X}{r}]$. (Hint: There is only one “picture” for $r$ isolated triangles!)

Solution: For any $r$, $E[\binom{X}{r}]$ is the expected number of $r$-tuples of isolated triangles, these must perforce be disjoint and so

$$E[\binom{X}{r}] = \frac{1}{r!} \prod_{i=0}^{r-1} \binom{n-3i}{3} p^{3r} (1-p)^{3r(n-3r)+9(r)}$$

(b) For $r$ and $c$ fixed find the limiting (in $n$) value of $S^{(r)}$.

Solution: It is helpful to compare it with $E[X]$. So

$$\prod_{i=0}^{r-1} \binom{n-3i}{3} \sim \binom{n}{3}^r$$
and

\[(1 - p)^{3r(n-3r) + 9r(n)} \sim [(1 - p)^{3n-9}]^r\]

so

\[E\left(\frac{X}{r}\right) \sim \frac{1}{r!}E[X]^r = \frac{\mu^r}{r!}\]

(c) Use Brun’s Sieve to deduce the limiting value of \(\Pr[X = 0]\).

Solution: The limiting value is \(e^{-\mu}\) with \(\mu = c^3 e^{-3c}/6\).

3. The Coupon Collector Problem: Set \(m = n \ln n + cn\) where \(c\) is a constant. (Don’t worry about integrality.) Let \(f\) be a random function from \(\{1, \ldots, m\}\) to \(\{1, \ldots, n\}\). Call \(j \in \{1, \ldots, n\}\) missed if there is no \(i \in \{1, \ldots, m\}\) with \(f(i) = j\). Let \(X\) be the number of missed \(j \in \{1, \ldots, n\}\).

(a) Find \(E[X]\) precisely.

Solution:

\[E[X] = n(1 - \frac{1}{n})^m\]

(b) Find the limiting value of \(E[X]\).

Solution:

\[E[X] = n(1 - \frac{1}{n})^m \sim ne^{-m/n} \sim nn^{-1}e^{-c} \sim e^{-c}\]

More carefully: \( (1 - \frac{1}{n})^m = \exp[m(-\frac{1}{n} + O(n^{-2})] \) and as \(mn^{-2} = o(1)\) the “error” \(O(n^{-2})\) has negligible effect.

(c) For \(r \geq 2\) find \(E[\binom{X}{r}]\) precisely.

Solution:

\[E[\binom{X}{r}] = \binom{n}{r} \left(1 - \frac{r}{n}\right)^m\]

(d) For \(r \geq 2\) find the limiting value of \(E[\binom{X}{r}]\).

Solution:

\[\binom{n}{r} \left(1 - \frac{r}{n}\right)^m \sim \frac{1}{r!} (ne^{-m/n})^r \sim \mu^r/r!\]

(e) Apply Brun’s Sieve to find the limiting value of \(\Pr[X = 0]\).

Solution:

\[\Pr[X = 0] \to e^{-mu} \to e^{-e^{-c}}\]
4. In $G \sim G(n, p)$ let $X$ denote the number of isolated edges – i.e., the number of $v, w$ adjacent to each other and no other vertices.

(a) Find $E[X]$ precisely.
Solution: $\binom{n}{2} p (1 - p)^{2(n-2)}$.

(b) Give an explicit parameterization $p = f_1(n) + cf_2(n)$ so that $E[X] \rightarrow g(c)$ where $g(c)$ will be an explicit continuous function with $\lim_{c \rightarrow -\infty} g(c) = 0$ and $\lim_{c \rightarrow +\infty} g(c) = +\infty$. (When $X$ is the number of isolated vertices the parametrization $p = \frac{\ln n}{n} + \frac{c}{n}$ was given in class. This is similar, though the answers are not the same.)
Solution: Set $p = \frac{\ln n}{2n} + \frac{\ln \ln n}{2n} + \frac{c}{n}$ so that

$$E[X] \sim n^2 p e^{-2pn/2} \sim \frac{n^2 \ln n}{2n} e^{-\ln n - \ln \ln n - c} = e^{-2c/4}.$$

(c) With the above parametrization set $\mu := E[X] \sim g(c)$. Use the Brun’s Sieve method to show that $X$ approaches a Poisson Distribution with mean $\mu$.
Solution: For fixed $r$ there are

$$\frac{1}{r!} \prod_{i=0}^{r-1} \left( n - 2i \right) \sim \binom{n}{2}^r / r!$$

$r$-tuples of vertex disjoint pairs. (Only such $r$ pairs can all be isolated edges.) The probability that such $r$ pairs are all isolated edges is $p^r (1 - p)^{2r(n-2) - 2r(r-1)}$, as there must be $r$ edges and for each edge there are $2(n - 2)$ nonedges except the nonedges have an overcount of $2r(r - 1)$. For $r$ fixed the $(1 - p)^{2r(r-1)} = o(1)$ and so the probability is $\sim [p(1 - p)^{2(n-2)}]^r$ and so the Inclusion-Exclusion term $S^{(r)} \sim \mu^r / r!$ and so $X$ approaches a Poisson distribution.

(d) Put everything together to make a statement analogous to the isolated vertices statement of the form: If $p = \text{blah blah blah}$ then the probability that $G$ has no isolated edges is yadda yadda yadda.
Solution: When

$$p = \frac{\ln n}{2n} + \frac{\ln \ln n}{2n} + \frac{c}{n}$$
the probability that $G$ has no isolated edges is

$$e^{-e^{-2c/4}} + o(1)$$

But how would you find this parametrization? Here is how your instructor thinks about it: Let’s estimate $\binom{n}{2}$ by $n^2/2$ and $(1 - p)$ by $e^{-p}$ and $2(n - 2)$ by $2n$ (after you get the answer you then have to go back and confirm that these approximations were good enough) so you now have

$$A := E[X] \sim n^2 e^{-2np} p$$

That last $p$ is the problem. If you ignore it (the "art") you would set

$$p = \frac{\ln n}{n}$$

(Lets leave the $+\frac{c}{n}$ out for now) But it doesn’t quite work, the $n^2 e^{-2np}$ becomes a constant and you get

$$A \sim K \frac{\ln n}{n}$$

You have to make $p$ smaller so has to bring $A$ up to constant. Challenging! First try

$$p = \frac{\ln n}{2n}$$

Now the $n$ factors will cancel. The new value is still not quite right, we have

$$A \sim K_1 \ln n$$

So now make $p$ bigger by adding $\frac{\ln \ln n}{n}$. So we have

$$p = \frac{\ln n}{2n} + \frac{\ln \ln n}{2n}$$

In adding this $p$ itself is not changed asymptotically but $e^{-2np}$ is decreased by a factor of $\ln n$ which is what we want. Now $A$ is a constant. By adding $\frac{c}{n}$ to $p$ we can adjust the constant.

5. (Just For Fun:) Give the family tree with root your paternal grandmother. (If you’re not comfortable doing this take some other root or just make one up.) True or false: your paternal grandmother has no children that have no children that have no children.

**Solution:** My paternal grandmother’s tree is:
Dora

Henry Rose Alfred Martin

Me Steven

David Danielle

Kiran Thomas

Henry had no children that have no children so it is False that my paternal grandmother had no children that had no children that had no children. My maternal grandmother’s tree is too complicated to write, but her eldest son, my Uncle Bill, had one child Jo-Anne who had three children. So Bill had no children that have no children so it is False that my maternal grandmother had no children that had no children that had no children that had no children.