Random Graphs
Assignment 1. Solutions

1. The Bipartite Ramsey Number $BR(k)$ is the least $n$ so that if $A, B$ are disjoint with $|A| = |B| = n$ and $A \times B$ is two colored there exist $A_1 \subseteq A, B_1 \subseteq B$ with $|A_1| = |B_1| = k$ and $A_1 \times B_1$ monochromatic. Find and prove a theorem which gives a lower bound for $BR(k)$ and explore the asymptotics.

**Solution.** There are $\binom{n}{k}^2$ possible choices for $A_1, B_1$ and each has probability $2^{1-k^2}$ of being monochromatic so the theorem would be:

If $\binom{n}{k}^2 2^{1-k^2} < 1$ then $BR(k) > n$

For the rough asymptotics, using $\binom{n}{k} \leq n^k$, it suffices that $n^{2k} 2^{1-k^2} < 1$ or $n < 2^{k/2}(1 - o(1))$ so $BR(k) > 2^{k/2}(1 + o(1))$. More precisely, as $n$ is exponential in $k$ we have $\binom{n}{k} \sim n^k/k!$ so it suffices that $n^{2k} 2^{1-k^2} < k!^2$. Taking $2k$-th roots we have $(k!)^{2/k} = k!^{1/k} \sim k/e$ so $BR(k) > (k/e)^{2/k}(1 + o(1))$.

2. Let $f(k)$ be the maximal $n$ for which there exists $p$ with $0 \leq p \leq 1$ such that

$n^k p^{k^2 / 2} + n^{2k} (1 - p)^{2k^2} \leq 1$

Let $U(k)$ be the maximal $n$ for which there exists such $p$ with $n^k p^{k^2 / 2} \leq 1$ and $n^{2k} (1 - p)^{2k^2} \leq 1$. Let $L(k)$ be the maximal $n$ for which there exists such $p$ with $n^k p^{k^2 / 2} \leq \frac{1}{2}$ and $n^{2k} (1 - p)^{2k^2} \leq \frac{1}{2}$.

(a) Argue that $L(k) \leq f(k) \leq U(k)$

**Solution.** When both terms are at most one half their sum is tautologically at most one. For the sum to be at most one it is necessary that both terms be at most one.

(b) Find the asymptotics of $U(k)$. (Warning: Do not assume $p = o(1)$ because the optimal $p$ isn’t!) Partial credit for $\lim_k U(k)^{1/k}$.

**Solution:** Taking appropriate roots we need $n \leq p^{-k/2}$ and $n \leq (1-p)^{-k}$ so $U(k) = \max(p, (1-p)^{2})^{-k/2}$. The functions $p, (1-p)^2$ are increasing and decreasing respectively in $p$ in $[0,1]$ so their max is achieved when they are equal, $p = \frac{1}{2}(3 - \sqrt{5})$ and $U(k) = (\frac{1}{2}(3 - \sqrt{5}))^{-k/2}$. But its more clear to write it $U(k) \leq \phi^k$ where $\phi := \frac{1}{2}(1 + \sqrt{5}) = 1.61 \ldots$ is the golden ratio!
(c) Find the asymptotics of $L(k)$, showing that it is the same as that of $U(k)$. (That is, changing 1 to $\frac{1}{2}$ had an asymptotically negligible effect.)

Solution: The key here is to use the results for $U(k)$ and try to turn them around. Here goes: Take $n_0 = \phi^k$ and $p = \frac{1}{2}(3 - \sqrt{5})$ so the upper bound works. Let $n = n_02^{-1/k}$ and keep $p$ the same. Now $n, p$ works for the lower bound, as $n^k$ and $n^{2k}$ have gone down by factors of two and four respectively. Are these the optimal values for the $L(k)$? Well, no, we can’t say that. What we can say is that since $n, p$ works it gives a lower bound and so $L(k) \geq n = n_02^{-1/k}$. As $\lim_k 2^{-1/k} = 1$, $U(k) \sim L(k)$.

(d) Deduce the asymptotics of $f(k)$

Solution: $f(k) \sim \phi^k$

3. Find asymptotic lower bounds on the Ramsey function $R(k, 2k)$. That is, set $g(k)$ to be the maximal $n$ for which there exists $p$ with $0 \leq p \leq 1$ such that

$$\binom{n}{k} p^{(k)} + \binom{n}{2k} (1-p)^{(2k)} < 1$$

Find an asymptotic formula for $g(k)$. (Note: You’ll want to use the ideas of the previous problem. Still, this is not an easy problem. Full marks for $\lim_k g(k)^{1/k} –$ its the same as $\lim_k f(k)^{1/k}$ but you have to prove this. The full asymptotics are if you enjoy a challenge.)

Partial Solution: Let $U^*(k), L^*(k)$ be the bounds when one requires both terms at most one and both terms at most one half respectively. The argument previously used shows these are asymptotic so it suffices to look at $U^*(k)$. Taking appropriate roots

$$U^*(k) = (1 + o(1)) \max_p \left( \frac{k}{e} p^{-(k-1)/2}, \frac{2k}{e} (1-p)^{-(2k-1)/2} \right)$$

so the optimal $p$ is when

$$\frac{k}{e} p^{-(k-1)/2} = \frac{2k}{e} (1-p)^{-(2k-1)/2}$$

Now $p \sim p_0 = \frac{1}{2}(3 - \sqrt{5})$.

One then plugs that $p_0$ back to get $U^*(k)$. This will still be off by a constant.

More precisely: Cancelling the $\frac{k}{e}$ factors we want

$$p^{-(k-1)/2} (1-p)^{(2k-1)/2} = 2$$
At $p = p_0$ the main terms $p_0^{-k/2}(1-p_0)^k$ cancel and the LHS is $p_0^{1/2}(1-p_0)^{-1/2}$. A combination of art, experience and insight – often achieved after several incorrect attempts – brings us (but see notes below) to the parametrization

$$p = p_0 + \frac{x}{k}$$

Now $p$ has changed (from $p_0$) by a $1 + \frac{x}{kp_0}$ factor so that when taken to the $-(k-1)/2$ power it changes up by a $\exp[-x/(2p_0)]$ factor. Simultaneously $1-p_0$ has changed (from $1-p_0$) by a $1 - \frac{x}{k(1-p_0)}$ factor so that when taken to the $(2k-1)/2$ power it changes up by a $\exp[-x/(1-p_0)]$ factor. So we want

$$p_0^{1/2}(1-p_0)^{-1/2} \exp[-x/2p_0] \exp[-x/(1-p_0)] = 2$$

This has a real solution $x$ which gives a better $p$. One then plugs that $p$ back to get $U^*(k)$ with the correct constant.

**Why this parametrization?** Oftentimes when a rough approach gives a $p_0$ and we want a finer analysis we think it will be pretty close to $p_0$ and so it is often good to parametrize $p = p_0 + z$. (Sometimes one sees $p = p_0(1+z)$ instead.) Say we first did that. Changing $p$ by $z$ has changes $p_0$ by a $1 + \frac{z}{p_0}$ factor. This changes $p^{k/2}$ by roughly a $\exp[zk/(2p_0)]$ factor. Simultaneously $1 - p$ changes by a $1 - \frac{z}{1-p_0}$ factor so that $(1-p)^k$ is changing by roughly a $\exp[-zk/(1-p_0)]$ factor. It is looking like (of course, this may turn out to be wrong, we are only indicating the thinking that leads one to a parametrization) the proper scaling would be $zk = \Theta(1)$ so that we are led (maybe!) to the further parametrization $z = x/k$.

4. (The $m = m(n)$ problem is deferred to week 2.)