Random Graphs Assignment 9
Due Tuesday, April 5, 2012

The voyage of discovery lies not in seeking new horizons, but in seeking with new eyes. – Proust

1. For \( \lambda \) an arbitrary real and \( c \) an arbitrary positive real define find an asymptotic (in \( n \)) formula for

\[ \Pr[T^{PO}_{1+\lambda n^{-1/3}} > cn^{2/3}] \]

in the form \( h(\lambda, c)n^{-1/3} \) where \( h(\lambda, c) \) is an explicit function given by a definite integral which you should leave unevaluated. (The sum of \( \Pr[T^{PO} = k] \) becomes an integral under parametrization \( k = xn^{2/3} \). There will be two cases, \( \lambda \leq 0 \) and \( \lambda > 0 \). When \( \lambda > 0 \) you must then add \( \Pr[T^{PO} = \infty] \).

2. Let \( A_1, \ldots, A_n \subseteq \{1, \ldots, m\} \) with \( \sum_{i=1}^{n} 2^{-|A_i|} < 1 \). Paul and Carole alternately select distinct vertices from \( \{1, \ldots, m\} \), Paul having the first move, until all vertices have been selected. Carole wins if she has selected all the vertices of some \( A_i \), Paul wins if Carole does not win. Give a winning strategy for Paul.

3. Let \( p = \frac{1}{n} + \lambda n^{-4/3} \) where \( \lambda \) is a fixed negative real. Let \( k = cn^{2/3} \) with \( c \) an arbitrary positive constant. Let \( X_k \) be the number of \( v \) for which \( |C(v)| \geq k \) and let \( Y_k \) be the number of components with at least \( k \) vertices. Use the general bound

\[ \Pr[|C(v)| \geq k] \leq \Pr[T^{BIN}_{n-1,p} \geq k] \sim \Pr[T^{PO}_{np} \geq k] \]

(don’t worry about the justification for the last part) to give an upper bound on \( E[X_k] \) via the problem 1. Use \( Y_k \leq X_k/k \) to give an asymptotic upper bound on \( E[Y_k] \) as a function \( f(\lambda, c) \). Show that for fixed \( \lambda < 0 \), \( \lim_{c \to -\infty} f(\lambda, c) = 0 \) and that for fixed \( c > 0 \), \( \lim_{\lambda \to -\infty} f(\lambda, c) = 0 \). Find the asymptotics for \( f(\lambda, c) \) for \( \lambda \) fixed and \( c \to 0^+ \). Remark: The actual asymptotic value of \( E[Y_k] \) is smaller.

I doubt sometimes whether a quiet and unagitated life would have suited me – yet I sometimes long for it.

– Byron