Random Graphs Assignment 8  
Due Tuesday, March 29, 2016

Ramanujan would sit working on the pial (porch) of his house on Sarangapani Street, legs pulled into his body, a large slate spread across his lap, madly scribbling,. . . When he figured something out, he sometimes seemed to talk to himself, smile, and shake his head with pleasure.

R. Kanigal, The Man who knew Infinity

Apologies, I use the opposite notation from the one in the book. Here \((a, b)\) means that there are \(a\) chips at position one and \(b\) chips at position zero.

1. Show that Carole wins the liar game with \(n = 93, q = 10\). (Note that this does not follow from the Theorem proven in class. The key will be that Paul doesn’t have a good first move.)

2. Consider the \(q\)-Chip Liar with initial position \((1, y)\). (That is, there are \(q\) rounds. Initially there is one possibility about which one Carole may lie and \(y\) for which she cannot.) Find the maximal \(y = y(q)\) for which Paul wins. (Hint: The bound given by Theorem 15.2.1 turns out to be precise and you’ll need to prove that by giving a Paul strategy.)

3. Using Problem 2 and that \(y(4) = 11\) give an explicit strategy for Paul to with the 1-Liar game with \(n = 2^{11}\) and 15 questions. (Idea: Let the values of \(x\) be 0 through \(2^{11} - 1\) and let the first 11 questions be of the binary bits of \(x\).) Show by Theorem 15.2.1 that for \(n = 2^{11} + 1\) Carole wins.

4. Consider two independent Galton-Watson processes, each with \(c = 1\), and let \(X, Y\) be the numbers of nodes in the respective processes. Set \(Z = X + Y\). The following are meant asymptotically in \(n\). (Idea: Use \(Pr[X = k] = \Theta(k^{-3/2})\).

   (a) Find, in \(\Theta\)-land, \(Pr[Z = n]\). (Idea: \(Pr[Y = y]\) is bounded within constants for \(\frac{n}{2} \leq y \leq n\).)

   (b) Show that \(Pr[X = 1|Z = n] = \Omega(1)\). (That is, find a constant lower bound on probability \((X, Y) = (1, n-1)\) given that \(X+Y = n\).)

   (c) Show that \(Pr[\frac{n}{3} \leq X \leq \frac{2n}{3}|Z = n] = O(n^{-\alpha})\) for an explicit positive \(\alpha\). (Its actually \(\Theta!\))
Remark: Yet another wierdness about heavy tails!

5. (Just for fun) Play a Half-Liar game (§15.6) with a friend\(^1\) with Carole selecting \(1 \leq x \leq 100\), 1 Half-Lie, and Paul having ten questions. Record the play with commentary. Are you still friends?

You don’t have to believe in God but you should believe in The Book.
– Paul Erdős

\(^1\)I certainly hope \(\exists_y FRIEND(u, y)\).