Any new possibility that existence acquires, even the least likely, transforms everything about existence.
– from Slowness by Milan Kundera

Note: Please work on all problems. Hand in either problem one or problem two AND either problem three or problem four, leaving out 4(f).

1. We know that $e = \lim_{n \to \infty} (1 + \frac{1}{n})^n$. Use the Taylor Series for $\ln(1 + \epsilon)$ out to an appropriate term, with error term, for the following:
   (a) Find the asymptotics of $(1 + \frac{1}{n})^n - e$ (in form $an^{-b}$)
   (b) Find the asymptotics of $(1 + \frac{1}{n})^{n+0.5} - e$ (in form $an^{-b}$)
   (c) Which is a better ($n$ large) approximation to $e$.

2. Consider a Galton-Watson process beginning with root Eve in which each node independently has number of children given by a Poisson distribution with mean $c$. Find the probabilities of the following events.
   (Use the nice fact that if a node has $Po(c)$ children and each child has a given property $A$ with independent probability $z$ then the node has $Po(cz)$ children with property $A$.)
   (a) Eve has precisely two children.
   (b) Eve has no children with precisely two children.
   (c) Eve has no children that have no children that have no children.

3. Consider a Galton-Watson process beginning with root Eve in which each node independently has number of children given by a Binomial distribution with parameters $m, p$.
   (a) Find an equation, as in the Poisson case, for $y = \Pr[T = \infty]$.
   (b) Show that this equation has only the solution $y = 0$ when $mp < 1$ and two solutions when $mp > 1$.
   (c) (*) Show that this equation has only the solution $y = 0$ when $mp = 1$ and $m \neq 1$.
   (d) Let $c > 1$. Let $y = y(m, p)$ denote the nonzero solution. Show that $y(m, p) \to y(c)$ when $m \to \infty$ and $mp \to c$. Here $y(c)$ denotes the unique positive $y$ with $e^{-cy} = 1 - y$. 
4. By a **dumbbell** we mean two cycles joined by a path. Let \( r, s \geq 3 \) denote the cycle lengths and \( t \geq 1 \) the number of interior points of the path so that the dumbbell has \( k = r + s + t \geq 7 \) points. The exact formulae below will have an \( (n)k \) term.

(a) Give the exact number of dumbbells with a given \( r, s, t \) on \( n \) vertices. There will be two cases: \( r = s \) and \( r \neq s \). Set \( k = r + s + t \).

(b) Assume \( k \) is even (the odd case is similar) and write \( k = 2w \). Let \( B(n, k) \) denote the number of dumbbells with \( k \) vertices on \( n \) vertices. Find an exact formula for \( B(n, k) \). (When \( r \neq s \), flipping \( r, s \) gives the same dumbbell so we can assume \( r < s \).) Use \( w \) as an auxiliary variable but write the final answer in terms of \( k \).

(c) Find an asymptotic expression (leave the \( (n)k \) term as is) for \( B(n, k) \) with \( k \to \infty \).

(d) Let \( A(n, p) \) denote the expected number of dumbbells in \( G(n, p) \). Give an exact expression, in terms of \( B(n, k) \) and \( (n)k \) and as a sum over \( k \), for \( A(n, p) \). [Note: We allow the dumbbells to have extra edges so there will be no \( 1 - p \) factors.]

(e) Let \( c < 1 \) be fixed and let \( p = c/n \). Assume the asymptotic formula for \( B(n, k) \) found previously is also valid for \( k \) odd. (It is!) Show \( A(n, p) = O(n^{-1}) \). (Bound \( (n)k \leq n^k \).)

(f) (NOT not be submitted!) Let \( p = \frac{\lambda n^{-1/3}}{n} \) where \( \epsilon = \lambda n^{-1/3} \) and \( \lambda (n) \to \infty \). (This is known as the **barely subcritical** range.) Show \( A(n, p) = o(1) \). (Again bound \( (n)k \leq n^k \).) [Note: Connected components of \( G \) which are neither trees nor unicyclic are called **complex**. Complex components either contain dumbbells or two points joined by three paths. It turns out the analysis for two points joined by three paths is quite similar to the dumbbell analysis. For the ranges of \( p \) above one can get that the expected number of such objects is \( o(1) \) and so whp they do not appear – so that complex components do not appear in the subcritical or the barely subcritical range.]

His plan, when he entered the university, was to qualify as a mathematician, then go abroad and devote himself to art. […] While perfecting his poetic skills abroad he will earn a living doing something obscure and respectable.

J.M. Coetzee, *Youth*