Random Graphs Assignment 6
Due Tuesday, March 8, 2016

If you take a number and double it and double it again and then
double it a few more times, the number gets bigger and bigger
and goes higher and higher and only arithmetic can tell you what
the number is when you quit doubling.
from *Arithmetic* by Carl Sandburg

1. Let $H$ have vertices $\{A, B, C, D, E\}$ and be the complete graph on
$\{A, B, C, D\}$ and the edges $\{E, A\}, \{E, B\}$. For $\alpha > 0$ let $f(n, \alpha)$
denote the probability that $G(n, p)$ does not contain a copy of $H$ when
$p = n^{-\alpha}$. Here we will give $f(n, \alpha)$ up to a constant in the exponent.

(a) For $t = 2, 3, 4, 5$ find the subgraph $H_t$ of $H$ on $t$ vertices with the
maximal number of edges and find $e_t$, the number of edges of $H_t$.

(b) Show that $H$ is strictly balanced. What is the threshold function
for containing a copy of $H$? Henceforth, restrict to $\alpha$ so that
$p = n^{-\alpha}$ is bigger than that threshold.

(c) Let $LB_t$ denote the lower bound, from Janson’s Inequality, on the
probability that $G(n, p)$ does not contain a copy of $H_t$. Set $LB$
equal the maximum of $LB_t$, $t = 2, 3, 4, 5$. Find $LB$ as a function
of $\alpha$ – there will be three ranges (some graph paper will help!) of
$\alpha$ at which different $t$ give the maximum.

(d) Find $\mu, \Delta$ of the upper bound of the Extended Janson’s Inequality.
Show that the $\Delta$ breaks into a finite number of ranges de-
pending on which addend predominates.

(e) Combining the lower and upper bounds above get a result of the
form:
If $0 < \alpha < \kappa_1$ the $f(n, p) = \exp[-\Theta(n^{\gamma_1 + \gamma_2 \alpha})]$.
If $\kappa_1 < \alpha < \kappa_2$ the $f(n, p) = \exp[-\Theta(n^{\gamma_3 + \gamma_4 \alpha})]$.
If $\kappa_2 < \alpha < \kappa_3$ the $f(n, p) = \exp[-\Theta(n^{\gamma_5 + \gamma_6 \alpha})]$.
If $\alpha > \kappa_3$ then $f(n, p) \sim 1$. Here the $\kappa$s and $\gamma$s will be nice
e rational numbers.

2. In $G(n, p)$ with $p = c/n$ let $X$ be the number of isolated triangles. Let
$\mu = E[X]$. In an earlier assignment you calculated the limiting value
of $\mu$.

(a) For $r \geq 1$ give an exact formula for $S(r) = E[(X)^r]$. (Hint: There
is only one “picture” for $r$ isolated triangles!)
(b) For $r$ and $c$ fixed find the limiting (in $n$) value of $S^{(r)}$.

(c) Use Brun’s Sieve to deduce the limiting value of $\Pr[X = 0]$.

3. The Coupon Collector Problem: Set $m = n \ln n + cn$ where $c$ is a constant. (Don’t worry about integrality.) Let $f$ be a random function from $\{1, \ldots, m\}$ to $\{1, \ldots, n\}$. Call $j \in \{1, \ldots, n\}$ missed if there is no $i \in \{1, \ldots, m\}$ with $f(i) = j$. Let $X$ be the number of missed $j \in \{1, \ldots, n\}$.

(a) Find $E[X]$ precisely.

(b) Find the limiting value of $E[X]$.

(c) For $r \geq 2$ find $E[(X^r)]$ precisely.

(d) For fixed $r \geq 2$ and $c$ find the limiting value of $E[(X^r)]$.

(e) Apply Brun’s Sieve to find the limiting value of $\Pr[X = 0]$.

4. In $G \sim G(n, p)$ let $X$ denote the number of isolated edges – i.e., the number of $v, w$ adjacent to each other and no other vertices.

(a) Find $E[X]$ precisely.

(b) Give an explicit parameterization $p = f_1(n) + cf_2(n)$ so that $E[X] \to g(c)$ where $g(c)$ will be an explicit continuous function with $\lim_{c \to -\infty} g(c) = 0$ and $\lim_{c \to +\infty} g(c) = +\infty$. (When $X$ is the number of isolated vertices the parameterization $p = \ln n + \frac{n}{n}$ was given in class. This is similar, though the answers are not the same.)

(c) With the above parameterization set $\mu := E[X] \sim g(c)$. Use the Brun’s Sieve method to show that $X$ approaches a Poisson Distribution with mean $\mu$.

(d) Put everything together to make a statement analogous to the isolated vertices statement of the form: If $p = \text{blah blah blah}$ then the probability that $G$ has no isolated edges is yadda yadda yadda.

5. (Just For Fun:) Give the family tree with root your paternal grandmother. (If you’re not comfortable doing this take some other root or just make one up.) True or false: your paternal grandmother has no children that have no children that have no children.

Well, you see, Haresh Chacha, its like this. First you have ten, that’s just ten, that is, ten to the first power. Then you have
a hundred, which is ten times ten, which makes it ten to the second power. Then you have a thousand which is ten to the third power. Then you have ten thousand, which is ten to the fourth power - but this is where the problem begins, don’t you see? We don’t have a special word for that, and we really should. 

...But you know, said Haresh, I think there is a special word for ten thousand. The Chinese tanners of Calcutta once told me that they used the number ten-thousand as a standard unit of counting. What they call it I can’t remember ... Bhaskar was electrified. But Haresh Chacha you must find that number for me, he said. You must find out what they call it. I have to know, he said, his eyes burning with mystical fire and his small frog-like features taking on an astonishing radiance.

– from *A Suitable Boy* by Vikran Seth