Random Graphs Assignment 4
Due Tuesday, February 23, 2016

I like things that look difficult and intractable to solve – they challenge me because they are more interesting to figure out.
Fracne Córdova, Director, National Science Foundation

Special Note: While I hope you will work on all of these problems please submit only problems 1, 2, 5.

1. Note: I’d like you to get upper and lower bounds off by a constant but the marks will be only on the upper bound.

Let \( P, Q, R, S \) be uniformly and independently selected from the unit square. Let \( f(\epsilon) \) be the probability that triangles \( PQR \) and \( QRS \) both have area less than \( \epsilon \). Find the asymptotics of \( f(\epsilon) \) (neglecting constant factors) as \( \epsilon \) approaches zero. [Idea: Integrate over \( r = |QR| \).]
[Warning: For all \( r \) the probability that \( PQR, QRS \) have area less than \( \epsilon \) is at most one.]

2. Let \( X \) be the number of triangles in \( G(n, p) \) with \( p = c/n \). Find both the precise and the asymptotic \([c \text{ fixed}, n \to \infty, \text{ in terms of } c]\) values for the expectation and variance of \( X \).

3. Let \( X \) be the number of isolated triangles in \( G(n, p) \) with \( p = c/n \). Find the asymptotic \([c \text{ fixed}, n \to \infty, \text{ in terms of } c]\) value for the expectation of \( X \). (A triangle \( v, w, u \) is isolated if there are no edge of the form \( vx \) or \( wx \) or \( ux \) with \( x \neq v, w, u \).)

4. For \( 1 \leq i \leq n \) let \( X_i \) be independent random variables with \( \Pr[X_i = 1] = \frac{1}{4}, \Pr[X_i = 0] = 1 - \frac{1}{4} \). Set \( Y_n = \sum_{i=1}^{n} X_i \). Let \( \mu_i, \sigma_i^2 \) equal the mean and variance of \( X_i \) and let \( \mu, \sigma^2 \) denote the mean and variance of \( Y_n \).

(a) Find asymptotic formulae for \( \mu, \sigma^2 \).

(b) Use Chebyschev’s Inequality to show that for any \( \epsilon > 0 \)

\[
\lim_{n \to \infty} \Pr[|Y_n - E[Y_n]| > \epsilon E[Y_n]] = 0
\]

(c) Show that if \( \lambda = o(1) \) then

\[
E[e^{\lambda(X_i - \mu_i)}] = 1 + \frac{\lambda^2}{2} \sigma_i^2 (1 + o(1))
\]
(Hint: For \( t \geq 3 \) bound \( E[(X_i - \mu_i)^t] \leq E[(X_i - \mu_i)^2] \)
as \( |X_i - \mu_i| \leq 1 \) always.\)

(d) Deduce that \( \ln\{E[e^{\lambda(Y_n - \mu)}]\}\sim \frac{1}{2}\lambda^2\sigma^2. \)

(e) Use Chernoff bounds to show that if \( a = o(\sigma) \) then

\[
\Pr[Y_n - E[Y_n] > a\sigma] < e^{-\frac{a^2}{2}(1+o(1))}
\]

5. Let \( X_i, 1 \leq i \leq n, \) be i.i.d. uniform on \{1, \ldots, 6\} (throws of a fair die),
\( Y_i = X_i - \frac{7}{2} \) (to move to zero mean) and \( Y = \sum_{i=1}^{n} Y_i. \) Use Chernoff
Bounds to give \( A = A(n) \) as small (asymptotically) as possible (include
the constant factor!) so that

(a) \( \Pr[Y > A] < n^{-1} \)
(b) \( \Pr[Y > A] < n^{-10} \)
(c) \( \Pr[Y > A] < e^{-\sqrt{n}} \)

6. Let \( P \) be Poisson with mean \( \mu. \) That is

\[
\Pr[P = i] = e^{-\mu}\frac{\mu^i}{i!} \text{ for } i = 0, 1, 2, \ldots
\]

(a) Find the Laplace Transform \( E[e^{\lambda P}] \) in closed form.

(b) Use Calculus to find that \( \lambda > 0 \) which minimizes \( E[e^{\lambda P}]e^{-\lambda x} \) and
give the Chernoff Bound on \( \Pr[P \geq x] \) explicitly.

(c) Assume that \( x \) is an integer. Find an asymptotic formula (here \( \mu \)
is fixed) for \( \Pr[P \geq x] \) (hint: show \( \Pr[P = x + i] \) drops so rapidly
that only the \( i = 0 \) term matters). How far off (as an asymptotic
function of \( x \) with \( \mu \) fixed) is the Chernoff Bound on \( \Pr[P \geq x] \)
from the actual value?

I was interviewed in the Israeli Radio for five minutes and I said
that more than 2000 years ago, Euclid proved that there are
infinitely many primes. Immediately the host interrupted me
and asked “Are there still infinitely many primes?”
Noga Alon