Random Graphs Assignment 12  
Due Tuesday, April 26, 2016

He rarely copied box scores into the Book, but today it seemed the right thing to do. All those zeroes! He decided for zeroes he’d use red ink. Zero: the absence of number, an incredible idea! Only infinity compared to it, and no batter could hit an infinite number of home runs - no, in a way, the pitchers had it better. Perfection was available to them.

– Robert Coover

1. Let $A = (a_{ij})$ be a random $n \times n$ matrix, with each coordinate independently uniform on $[0, 1]$. Set

$$L = L(A) = \min_{\sigma \in S_n} \sum_{i=1}^{n} a_{i,\sigma(i)}$$

Consider the gradation $\emptyset = B_0 \subset B_1 \ldots \subset B_n$ of the $n^2$ positions $(i, j)$ given by adding a row at a time. That is: $B_i = \{(i', j) : i' \leq i\}$.

(a) Show that $L$ is Lipschitz under this gradation.

(b) Apply Azuma’s Inequality to get a concentration result for $L$.

(c) Now assume (this is not so easy to prove!) that with probability $1 - o(1)$ the permutation $\sigma$ giving $X(A)$ has all $a_{i,\sigma(i)} \leq 10\ln n$. Set

$$L^* = L^*(A) = \min_{\sigma \in S_n} \sum_{i=1}^{n} \min\{a_{i,\sigma(i)}, 10\frac{\ln n}{n}\}$$

Apply an appropriately normalized Azuma’s Inequality to get an even better concentration result for $L^*$. As $L = L^*$ with high probability, this implies a concentration result for $L$. **Comment:** We actually know $L = \pi^2/6 + o(1)$ with high probability – this has been the subject of major research by many people.

2. Let $\sigma_u$, $1 \leq u \leq 3$, be permutations of $\{1, \ldots, n\}$. For each $u = 1, 2, 3$ and each $1 \leq i \leq j \leq n$ set

$$S_{iju} = \{\sigma_u(t) : i \leq t \leq j\}$$

Let $\Omega = \{1, \ldots, n\}$ and let $\mathcal{A}$ denote the family of all such $S_{iju}$. Here we give a polylog upper bound for $\text{disc}(\mathcal{A})$. (Remark: It was a famous
conjecture of Jozsef Beck that $\text{disc}(A)$ was bounded by an absolute constant. This was disproved by Alantha Newman and Aleksandar Nikolov. BTW, Nikolov was then a graduate student at Rutgers.) For convenience we shall assume $n$ is a power of two and write $n = 2^s$. For $0 \leq r \leq s$ and $1 \leq w \leq 2^{s-r}$ we define

$$I_{wr} = \{ t : (w-1)2^r < t \leq w2^r \}$$

For each $u = 1, 2, 3$ and such $w, r$ set

$$T_{wru} = \{ \sigma_u(t) : t \in I_{wr} \}$$

and let $\mathcal{B}$ denote the family of all such $T_{wru}$.

(a) Apply the Beck-Fiala Theorem to give an upper bound on $\text{disc}(\mathcal{B})$.

(b) Find a decomposition of an arbitrary $S_{iju} \in \text{disc}(A)$ into a moderate number of sets $T_{wru} \in \text{disc}(\mathcal{B})$. Use that to give an upper bound on $\text{disc}(A)$.

3. Let $G \sim G(n, \frac{1}{2})$. Set $f(k) = \binom{n}{k}2^{-\binom{k}{2}}$. Let $k_0 \sim 2 \lg n$ be such that $f(k_0) > 1 > f(k_0 + 1)$. Set $k = k_0 - 4$. For each set $S$ of $k$ vertices let $I_S$ be the indicator that $S$ is a clique and let $X = \sum I_S$ be the number of $k$-cliques. Set $\mu = E[X]$.

(a) Give an exact expression for the $\Delta$ of Janson’s Inequality. There should be a sum over $i$ where $i = |S \cap T|$.

(b) Let $\Gamma$ denote the $i = 2$ contribution to the $\Delta$ formula. Find the asymptotics of $\Gamma/\mu^2$.

(c) Assume (this is true but technical) that $\Delta \sim \Gamma$ – that is, that the sum is dominated by the $i = 2$ term. Use the Extended Janson Inequality to bound $\Pr[X = 0]$.

Do I contradict myself?
Very well, I contradict myself.
I am large
and contain multitudes.
– Walt Whitman