Deterministic Random Walks on the Integers*

Joshua Cooper† Benjamin Doerr‡ Joel Spencer§

Gábor Tardos¶

Abstract

Jim Propp's P-machine, also known as the 'rotor router model' is a simple deterministic process that simulates a random walk on a graph. Instead of distributing chips to randomly chosen neighbors, it serves the neighbors in a fixed order.

We investigate how well this process simulates a random walk. For the graph being the infinite path, we show that, independent of the starting configuration, at each time and on each vertex, the number of chips on this vertex deviates from the expected number of chips in the random walk model by at most a constant c_1, which is approximately 2.29. For intervals of length L, this improves to a difference of $O(\log L)$, for the L_2 average of a contiguous set of intervals even to $O(\sqrt{\log L})$. All these bounds are tight.

1 The Propp Machine

The following deterministic process was suggested by Jim Propp as an attempt to derandomize random walks on infinite grids $\mathbb{Z}^d$:

*The authors enjoyed the hospitality, generosity and the strong coffee of the Rényi Institute (Budapest) while doing this research. Spencer's research was partially supported by EU Project Finite Structures 003006; Doerr's by EU Research Training Network COMBSTRU; Cooper's by an NSF Postdoctoral Fellowship (USA, NSF Grant DMS-0303272); and Tardos's by the Hungarian National Scientific Research Fund grants OTKA T-046234, AT-048826 and NK-62321.

†ETH-Zürich, Institute of Theoretical Computer Science, jcooper@inf.ethz.ch

‡Max–Planck–Institut für Informatik, Saarbrücken, Germany

§Courant Institute of Mathematical Sciences, New York, U.S.A., spencer@cims.nyu.edu

¶Simon Fraser University, Canada and Rényi Institute, Hungary, tardos@cs.sfu.ca

Rules of the Propp machine: Each vertex $x \in \mathbb{Z}^d$ is associated with a 'rotor' and a cyclic permutation of the $2d$ cardinal directions of $\mathbb{Z}^d$. Each vertex may hold an arbitrary number of 'chips'. In each time step, each vertex sends out all its chips to neighboring vertices in the following manner: The first chip is sent into the direction the rotor is pointing, then the rotor direction is updated to the next direction in the cyclic ordering. The second chip is sent in this direction, the rotor is updated, and so on. As a result, the chips are distributed highly evenly among the neighbors.

This process has attracted considerable attention recently. It turns out that the Propp machine in several respects is a very good simulation of a random walk. Used to simulate internal diffusion limited aggregation (repeatedly, a single chip is inserted at the origin, performs a rotor router walk until it reaches an unoccupied position and occupies it), it was shown by Levine and Peres [LP05] that this derandomization produces results that are extremely close to what a random walk would have produced. See also Kleber's paper [Kle05], which adds interesting experimental results: Having inserted three million chips, the closest unoccupied site is at distance 976.45, the farthest occupied site is at distance 978.06. Hence the occupied sites almost form a perfect circle!

In [CS05, CS04], the authors consider the following question: Start with an arbitrary initial position (that is, chips on vertices and rotor directions), run the Propp machine for some time and compare the number of chips on a vertex with the expected number of chips a random walk run for the same amount of time would have placed on that vertex. Apart from a technicality, which we defer to the end of Section 2, the answer is astonishing: For any grid $\mathbb{Z}^d$, this difference (discrepancy) can be bounded by a constant, independent of the number of chips, the run-time, the initial rotor position and the cyclic permutation of the cardinal directions.

In this paper, we continue this work. We mainly regard the one-dimensional case, but as will be visible from the proofs, our methods can be extended to higher dimensions as well. Besides making the constant precise (approximately 2.29), we show that the differences become even better for larger intervals (both in space and time). We also present a fairly general method to prove lower bounds (the 'arrow forcing theorem'). This shows that all our upper bounds are actually sharp, including the aforementioned constant.

Instead of talking about the expected number of chips the random walk produces on a vertex, we find it more convenient to think of the following 'linear' machine. Here, in each time step each vertex sends out exactly the same (possibly non-integral) number of chips to each neighbor. Hence, for a given starting configuration, after t time-steps the number of chips in the linear model is exactly the expected number of chips in the random walk model.

2 Our Results

We obtain the following results (again, see the end of the section for a slight technical restriction): Fix any starting configuration, that is, the number of chips on each vertex and the position of the rotor on each vertex. Now run both the Propp machine and the linear machine for a fixed number of time-steps. Looking at the resulting chip configurations, we have the following:

- On each vertex, the number of chips in both models deviates by at most a constant $c_1 \approx 2.29$. One may interpret this to mean that the Propp machine simulates a random walk extremely well. In some sense, it is even better than the random walk. Recall that in a random walk a vertex holding n chips only in expectation sends $n/2$ chips to the left and the right. With high probability, the actual numbers deviate from this by $\Omega(n^{1/2})$.
- In each interval of length L, the number of chips that are in this interval in the Propp model deviates from that in the linear model by only $O(\log L)$ (instead of, e.g., $2.29L$).
- If we average this over all length L intervals in some larger interval of $\mathbb{Z}$, things become even better. The average squared discrepancy in the length L intervals also is only $O(\log L)$.

We may as well average over time. In the setting just fixed, denote by $f(x,T)$ the sum of the numbers of chips on vertex x in the last T time steps in the Propp model, and by $E(x,T)$ the corresponding number for the linear model. Then we have the following discrepancy bounds:

- The discrepancy on a single vertex over a time interval of length T is at most $|f(x,T) - E(x,T)| = O(T^{1/2})$. Hence a vertex cannot have too few or too many chips for a long time (it may, however, alternate having too few and too many chips and thus have an average $\Omega(1)$ discrepancy over time).

- We may extend this to discrepancies in intervals in space and time: Let I be some interval in $\mathbb{Z}$ having length L. Then the discrepancy in I over a time interval of length T satisfies

$$\Big|\sum_{x\in I} f(x,T) - \sum_{x\in I} E(x,T)\Big| = \begin{cases} O(LT^{1/2}) & \text{if } L \le 2T^{1/2}, \\ O(T\log(LT^{-1/2})) & \text{otherwise.} \end{cases}$$

  Hence if L is small compared to $T^{1/2}$, we get L times the single vertex discrepancy in a time interval of length T (no significant cancellation in space); if L is of larger order than $T^{1/2}$, we get T times the $O(\log L)$ bound for intervals of length L (no cancellation in time, the discrepancy cannot leave the large interval in short time).

All bounds stated above are sharp, that is, for each bound there is a starting configuration such that after suitable run-time of the machines we find the claimed discrepancy on a suitable vertex, in a suitable interval, etc.

A technicality: There is one limitation, which we only briefly mentioned, but without which our results are not valid. Note that since $\mathbb{Z}^d$ is a bipartite graph, the chips that start on even vertices never mix with those which start on odd positions. It looks as if we would play two games in one. This is not true, however. The even chips and the odd ones may interfere with each other through the rotors. Even worse, we may use the odd chips to reset the arrows and thus mess up the even chips. Note that the odd chips are not visible if we look at an even position after an even run-time. An extension of the arrow-forcing theorem presented below shows that we can indeed use the odd chips to arbitrarily reset the rotors. This is equivalent to running the Propp machine in an adversarial setting, where an adversary may decide each time where the extra odd chips on a position is sent to. It is clear that in this setting, the results above cannot be expected. We therefore assume that *the starting configuration has chips only on even positions* ("even starting configuration") or only on odd positions ("odd starting configuration"). An alternative, in fact equivalent, solution would be to have two rotors on each vertex, one for even and one for odd time steps.

3 The Basic Method

For numbers a and b set $[a..b] = \{z \in \mathbb{Z} \mid a \leq z \leq b\}$ and $[b] = [1..b]$. For integers m and n, we write $m \sim n$ if m and n have the same parity, that is, if $m - n$ is even.

For a fixed starting configuration, we use $f(x,t)$ to denote the number of chips at time t at position x and $\textsc{arr}(x,t)$ to denote the value of the arrow at time t and position x, i.e., $+1$ if it points to the right, and -1 if it points to the left. We have:

$$\begin{aligned}
f(x,t+1) &= f(x-1,t)/2 + f(x+1,t)/2 \\
&\quad +\textsc{arr}(x-1,t)(f(x-1,t) \bmod 2)/2 \\
&\quad -\textsc{arr}(x+1,t)(f(x+1,t) \bmod 2)/2, \\
\textsc{arr}(x,t+1) &= (-1)^{f(x,t)}\textsc{arr}(x,t).
\end{aligned}$$

Note that after an even starting configuration if $x \sim t$ does not hold, then we have $f(x,t) = 0$ and $\textsc{arr}(x,t+1) = \textsc{arr}(x,t)$.

We consider the machine to be started at time $t = 0$. Being a deterministic process, the initial configuration (i.e., the values $f(x,0)$ and $\textsc{arr}(x,0)$, $x \in \mathbb{Z}$) determines the configuration at any time $t > 0$ (i.e., the values $f(x,t)$ and $\textsc{arr}(x,t)$, $x \in \mathbb{Z}$). The totality of all configurations for $t > 0$ we term a *game*. We call a configuration *even* if no chip is at an odd position. Similarly, a position is *odd* if no chip is at an even position. Clearly, an even position is always followed by an odd position and vice versa.

By $E(x,t)$ we denote the expected number of chips on a vertex x after running a random walk for t steps (from the implicitly given starting configuration). As described earlier, this is equal to the number of chips on x after running the linear machine for t time-steps.

In the proofs, we need the following mixed notation. Let $E(x,t_1,t_2)$ be the expected number of chips at location x and time t_2 if a simple random walk were performed beginning from the Propp machine's configuration at time t_1. In other words, this is the number of chips on vertex x after t_1 Propp and $t_2 - t_1$ linear steps.

Let $H(x,t)$ denote the probability that a chip arrives at location x at time $t \geq 0$ in a simple random walk begun from the origin, i.e., $H(x,t) =$

$2^{-t}\binom{t}{(t+x)/2}$, if $t \sim x$, and $H(x,t) = 0$ otherwise. For $t > 0$ let $\textsc{inf}(y,t)$ denote the "influence" of a Propp step of a single chip at distance y with t linear steps remaining (compared to a linear step). More precisely, we compare the two probabilities that a chip on position y reaches 0 if (a) it is first sent to the right (by a single Propp step) and then does a random walk for the remaining $t-1$ time steps, or (b) it just does t random walk steps starting from y. Hence,

$$\textsc{inf}(y,t) := H(y+1,t-1) - H(y,t).$$

A simple calculation yields

$$\textsc{inf}(y,t) = -\tfrac{y}{t} H(y,t). \tag{1}$$

This shows in particular, that $\textsc{inf}(y,t) \leq 0$ for $y \geq 0$ and $\textsc{inf}(y,t) \geq 0$ for $y \leq 0$. We have $\textsc{inf}(0,t) = 0$.

For notational convenience we extend the definitions of $H(x,t)$ and $\textsc{inf}(x,t)$ by letting $H(x,t) = 0$ for $t < 0$ and $\textsc{inf}(x,t) = 0$ for $t \leq 0$.

Note that

$$\textsc{inf}(y,t) = \tfrac{1}{2} H(y+1,t-1) - \tfrac{1}{2} H(y-1,t-1). \tag{2}$$

Therefore, the first Propp step with arrow pointing to the left has an influence of $-\textsc{inf}(y,t)$.

Using this notation, we can conveniently express the (signed) discrepancy $f(x,t) - E(x,t)$ on a vertex x using information about when "odd splits" occurred. It suffices to prove the result for the vertex $x = 0$. Clearly, $E(0,t,t) = f(0,t)$ and $E(0,0,t) = E(0,t)$, so that

$$f(0,t) - E(0,t) = \sum_{s=0}^{t-1} \left(E(0,s+1,t) - E(0,s,t) \right). \tag{3}$$

In comparing $E(0,s+1,t)$ and $E(0,s,t)$, note that whenever there are two chips on some vertex at time s, then these chips can be assumed to behave identically no matter whether the next step is a linear or a Propp step. Denote by $\textsc{odd}_s$ the set of locations which are occupied by an odd number of chips at time s. Then

$$\begin{aligned} & E(0,s+1,t) - E(0,s,t) \\ & \quad = \sum_{y \in \textsc{odd}_s} \left(H(y + \textsc{arr}(y,s), t-s-1) - H(y,t-s) \right) \\ & \quad = \sum_{y \in \textsc{odd}_s} \textsc{arr}(y,s)\, \textsc{inf}(y,t-s). \end{aligned}$$

Therefore, appealing to (3),

$$f(0,t) - E(0,t) = \sum_{s=0}^{t-1} \sum_{y \in \mathrm{ODD}_s} \mathrm{ARR}(y,s)\,\mathrm{INF}(y,t-s).$$

Using $\mathrm{INF}(y,u) = 0$ for $u \leq 0$ we can extend the summation above for all non-negative integers s.

Let $s_i(y)$ be the i^{th} time that y is occupied by an odd number of chips, beginning with $i = 0$. Switching the order of summation and noting that the arrows flip each time there is an odd number of chips on a vertex, we have

$$\begin{aligned} f(0,t) - E(0,t) &= \sum_{y \in \mathbb{Z}} \sum_{i \geq 0} \mathrm{ARR}(y, s_i(y))\,\mathrm{INF}(y, t - s_i(y)) \\ &= \sum_{y \in \mathbb{Z}} \mathrm{ARR}(y,0) \sum_{i \geq 0} (-1)^i\,\mathrm{INF}(y, t - s_i(y)). \end{aligned} \tag{4}$$

This equation will be crucial in the remainder of the paper. It shows that the discrepancy on a vertex only depends on the initial arrow positions and the set of location-time pairs holding an odd number of chips.

In the remainder, we show that we can construct starting configurations with arbitrary initial arrow positions and odd number of chips at arbitrary sets of location-time pairs. This will be the heart of our lower bound proofs in the following sections. Here $\mathbb{N}_0$ denotes the set of non-negative integers.

Theorem 1 (Parity-forcing Theorem)**.** *For any initial position of the arrows and any $\pi : \mathbb{Z} \times \mathbb{N}_0 \to \{0,1\}$, there is an initial even configuration of the chips such that for all $x \in \mathbb{Z}$, $t \in \mathbb{N}_0$ such that $x \sim t$, $f(x,t)$ and $\pi(x,t)$ have identical parity.*

Since rotors change their direction if and only if the vertex has an odd number of chips, the parity-forcing theorem is a consequence of the following arrow-forcing statement.

Theorem 2 (Arrow-forcing Theorem)**.** *Let $\rho(x,t) \in \{-1,+1\}$ be arbitrarily defined for $t \geq 0$ integer and $x \sim t$. Then there exists an even initial configuration that results in a game with $\mathrm{ARR}(x,t) = \rho(x,t)$ for all such x and t. Similarly, if $\rho(x,t)$ is defined for $x \sim t+1$ a suitable odd initial configuration can be found.*

Proof. By symmetry, it is enough to prove the first statement.

Assume the functions f and $\textsc{arr}$ describe the game following an even initial configuration, and for some $T \geq 0$, we have $\textsc{arr}(x,t) = \rho(x,t)$ for all $0 \leq t \leq T+1$ and $x \sim t$. We modify the initial position by defining $f'(x,0) = f(x,0) + \epsilon_x 2^T$ for even x, while we have $f'(x,0) = 0$ for odd x and $\textsc{arr}'(x,0) = \textsc{arr}(x,0)$ for all x. Here, $\epsilon_x \in \{0,1\}$ are to be determined.

Observe that a pile of 2^T chips will split evenly T times so that the arrows at time $t \leq T$ remain the same. Our goal is to choose the values ϵ_x so that $\textsc{arr}'(x,t) = \rho(x,t)$ for $0 \leq t \leq T+2$ and $x \sim t$. As stated above this holds automatically for $t \leq T$ as $\textsc{arr}'(x,t) = \textsc{arr}(x,t) = \rho(x,t)$ in this case. For $t = T+1$ and $x - T - 1$ even we have $\textsc{arr}'(x,T+1) = \textsc{arr}'(x,T) = \textsc{arr}(x,T) = \textsc{arr}(x,T+1) = \rho(x,T+1)$ since we start with an even configuration. To make sure the equality also holds for $t = T+2$ we need to ensure that the parities of the piles $f'(x,T)$ are right. Observe that $\textsc{arr}'(x,T+2) = \textsc{arr}'(x,T)$ if $f'(x,T)$ is even, otherwise $\textsc{arr}'(x,T+2) = -\textsc{arr}'(x,T)$. So for $x - T$ even we must make $f'(x,T)$ even if and only if $\rho(x,T+2) = \rho(x,T)$. At time T the "extra" groups of 2^T chips have spread as in Pascal's Triangle and we have

$$f'(x,T) = f(x,T) + \sum_y \epsilon_y \binom{T}{\frac{T+x-y}{2}}$$

where $x \sim T$ and the sum is over the even values of y with $|y - x| \leq T$. As $f(x,T)$ are already given it suffices to set the parity of the sum arbitrarily. For $T = 0$ the sum is ϵ_x so this is possible. For $T > 0$ we express

$$\sum_y \epsilon_y \binom{T}{\frac{T+x-y}{2}} = \epsilon_{x+T} + h + \epsilon_{x-T}$$

where h depends only on ϵ_y with $x - T < y < x + T$. We now determine the ϵ_y sequentially. We initialize by setting $\epsilon_y = 0$ for $-T < y \leq T$. The values ϵ_y for $y > T$ are set in increasing order. The value of ϵ_y is set so that the sum at $x = y - T$ (and thus $f'(y-T,T)$) will have the correct parity. Similarly, the values ϵ_y for $y \leq -T$ are set in decreasing order. The value of ϵ_y is set so that the sum at $x = y + T$ (and thus the $f'(y+T,T)$) will have the correct parity.

Note that the above procedure changes an even initial configuration that matches the prescription in ρ for times $0 \leq t \leq T+1$ into another even initial configuration that matches the prescription in ρ for times $0 \leq t \leq T+2$. We start by defining $f(x,0) = 0$ for all x (no chips anywhere) and $\textsc{arr}(x,0) = \rho(x,0)$ for even x, while $\textsc{arr}(x,0) = \rho(x,1)$ for odd x. We now have $\textsc{arr}(x,t) = \rho(x,t)$ for $0 \leq t \leq 1$ and $x \sim t$. We can apply the above procedure repeatedly to get an even initial configuration that satisfies the prescription in ρ for an ever increasing (but always finite) time period $0 \leq t < T$. Notice however, that in the procedure we do not change the initial configuration of arrows $\textsc{arr}(x,0)$ at all, and we change the initial number of chips $f(x,0)$ at position x only if $|x| \geq T$. Thus at any given position x the initial number of chips will be constant after the first $|x|$ iterations. This means that the process converges to an (even) initial configuration. It is simple to check that this limit configuration satisfies the statement of the theorem. $\square$

4 Discrepancy on a Single Vertex

Theorem 3. *There exists a constant $c_1 \approx 2.29$, independent of the initial (even) configuration, the time t, or the location x, so that*

$$|f(x,t) - E(x,t)| \leq c_1.$$

The proof needs the following elementary fact. Let $X \subseteq \mathbb{R}$. We call a mapping $f : X \to \mathbb{R}$ *unimodal*, if there is an $m \in X$ such that f is monotonically increasing in $\{x \in X \,|\, x \leq m\}$ and f is monotonically decreasing in $\{x \in X \,|\, x \geq m\}$.

Lemma 4. *Let $f : X \to \mathbb{R}$ be non-negative and unimodal. Let $t_1, \ldots, t_n \in X$ such that $t_1 < \ldots < t_n$. Then*

$$\left| \sum_{i=1}^{n} (-1)^i f(t_i) \right| \leq \max_{x \in X} f(x).$$

Proof of Theorem 3. It suffices to prove the result for $x = 0$. In case t is even we start with an even configuration, if t is odd, then with an odd configuration (otherwise both $f(0,t)$ and $E(0,t)$ would be zero with no discrepancy).

First we show that $\textsc{inf}(y,u)$ with a fixed $y < 0$ is a non-negative unimodal function of u if restricted to the values $u \sim y$. We have already seen that it is non-negative. For the unimodality let $y < 0$ and $u > 2$, $u \sim y$. We have

$$\begin{aligned}\textsc{inf}(y,u) - \textsc{inf}(y,u+2) &= -\frac{y}{u}H(y,u) + \frac{y}{u+2}H(y,u+2) \\ &= \frac{4+3u-y^2}{(u+2-y)(u+2+y)}\textsc{inf}(y,u),\end{aligned}$$

whenever $u \geq y$. Hence the difference is non-negative if $u \geq (y^2-4)/3$ and it is non-positive if $u \leq (y^2-4)/3$. Thus we have unimodality, with $\textsc{inf}(y,u)$ taking its maximum at the smallest value of u exceeding $(y^2-4)/3$ with $u \sim y$. Let $t_{\max}(y) := \lfloor (y^2-4)/3 \rfloor + 2$. It is easy to check that $t_{\max}(y) \sim y$ always holds, so we have that $\textsc{inf}(y,u)$ takes its maximum for fixed $y < 0$ at $u = t_{\max}(y)$. For $y > 0$ the values $\textsc{inf}(y,u)$ are non-positive and by symmetry the minimum is taken at $u = t_{\max}(y)$. For $y = 0$ we have $\textsc{inf}(y,u) = 0$ for all u. We have just proved the following:

Lemma 5. *For $y \in \mathbb{Z}$, the function $|\textsc{inf}(y,t)|$ is maximized over all integers t at $t_{\max}(y) = \lfloor (y^2-4)/3 \rfloor + 2$.*

To bound $|f(0,t) - E(0,t)|$ we use the formula (4) where the inner sums are alternating sums, for which we can apply Lemma 4, as $y \sim t - s_i(y)$ holds by our even or odd starting position assumption. We get

$$\begin{aligned}|f(0,t) - E(0,t)| &\leq \sum_{y\in\mathbb{Z}} \left| \sum_{i\geq 0} (-1)^i \,\textsc{inf}(y, t - s_i(y)) \right| \\ &\leq \sum_{y\in\mathbb{Z}} \max_u |\textsc{inf}(y,u)| \\ &= 2\sum_{y=1}^{\infty} |\textsc{inf}(y, t_{\max}(y))|. \qquad (5)\end{aligned}$$

Here

$$\begin{aligned}|\textsc{inf}(y, t_{\max}(y))| &= \frac{y}{t_{\max}(y)} 2^{-t_{\max}(y)} \binom{t_{\max}(y)}{(t_{\max}(y)+y)/2} \\ &= O(y/(t_{\max}(y))^{3/2}) = O(y^{-2}).\end{aligned}$$

and, therefore, (5) implies that $|f(0,t) - E(0,t)|$ is bounded by

$$c_1 := 2\sum_{y=1}^{\infty} |\textsc{inf}(y, t'_{\max}(y))| \approx 2.29,$$

proving Theorem 3. $\square$

Amazingly, the constant c_1 defined above is best possible. Indeed, let $y > 0$ be arbitrary and even and let $t_0 = t_{\max}(y)$. We apply the Arrow-forcing Theorem to find an even starting position that makes $\textsc{arr}(x,t) = -1$ if $x > 0$ and $t \le t_0 - t_{\max}(x)$ or $x < 0$ and $t > t_0 - t_{\max}(x)$ and makes $\textsc{arr}(x,t) = -1$ otherwise. It is easy to verify that in this case at a position $|x| \le y$, $x \ne 0$ we have an odd number of chips exactly once at time $t_0 - t_{\max}(x)$ and the formula (4) gives

$$f(0,t_0) - E(0,t_0) = 2\sum_{x=1}^{y} |\textsc{inf}(x, t_{\max}(x))|.$$

5 Intervals in Space

In this section, we regard the discrepancy in intervals in $\mathbb{Z}$. For an arbitrary finite subset X of $\mathbb{Z}$ set

$$\begin{aligned} f(X,t) &:= \sum_{x \in X} f(x,t), \\ E(X,t) &:= \sum_{x \in X} E(x,t). \end{aligned}$$

We show that the discrepancy in an interval of length L is $O(\log L)$, and this is sharp. We need the following facts about H.

Lemma 6. *For all $x \in \mathbb{Z}$, $H(x,\cdot) : \{t \in \mathbb{N}_0 \,|\, x \sim t\} \to \mathbb{R}; t \mapsto H(x,t)$ is unimodal. $H(x,t)$ is maximal for $t = x^2$. We have $H(x,x^2) = \Theta(|x|^{-1})$.*

Proof. Since $H(x,t-2) - H(x,t) = \frac{t-x^2}{t^2-t} H(x,t)$, we conclude that $H(x,t)$ is unimodal and for $|x| \ge 2$ it has exactly two maxima, namely $t = x^2 - 2$ and $t = x^2$, while for $|x| \le 1$ the latter is the only maximum. A standard estimate gives the claimed order of magnitude. $\square$

Theorem 7. *For any even initial configuration, any time t and any interval X of length L,*

$$|f(X,t) - E(X,t)| = O(\log L).$$

For every $L > 0$ there is an even initial configuration, a time t and an interval X of length L such that

$$|f(X,t) - E(X,t)| = \Omega(\log L).$$

Proof. Using that the discrepancy of a single position is bounded we can assume X ends at an even position, and then by symmetry we may assume it ends at 0, i.e., $X = [-L+1..0]$. Fix any even initial configuration. By (4), we have

$$f(X,t) - E(X,t) = \sum_{y \in \mathbb{Z}} \textsc{arr}(y,0) \sum_{x \in X} \sum_{i \geq 0} (-1)^i \textsc{inf}(y-x, t-s_i(y)).$$

Note that the summation here can be restricted to values $x \sim t$, the other values contribute zero.

Let us call

$$\textsc{con}(y) := \textsc{arr}(y,0) \sum_{x \in X} \sum_{i \geq 0} (-1)^i \textsc{inf}(y-x, t-s_i(y))$$

the contribution of the vertex y to the discrepancy in the interval X. The contribution of a vertex depends on its distance from the interval X. If y is $\Omega(L)$ away from X, its influences on the various vertices of X are roughly equal, and all such influences are quite small. In this case we bound its influence by L times the one we computed in Theorem 3:

Let $y > L$. By Lemmas 4 and 5,

$$\begin{aligned}
|\textsc{con}(y)| &= \left| \sum_{x \in X} \sum_{i \geq 0} (-1)^i \textsc{inf}(y-x, t-s_i(y)) \right| \\
&\leq \sum_{x \in X} \left| \sum_{i \geq 0} (-1)^i \textsc{inf}(y-x, t-s_i(y)) \right| \\
&\leq \sum_{x \in X} \max_t |\textsc{inf}(y-x,t)| \\
&\leq O\left(\sum_{x \in X} (y-x)^{-2} \right) = O(Ly^{-2}).
\end{aligned}$$

Hence the total contribution of these vertices is at most

$$\sum_{y > L} |\textsc{con}(y)| = O\left(\sum_{y > L} Ly^{-2} \right) = O(1)$$

and by symmetry the same bound applies to the contribution of vertices $y \leq -2L$.

We now turn to vertices $-2L < y \leq L$. Here mainly those vertices of X that are close to y contribute to $\textsc{con}(y)$. Hence, the approach above is too coarse. We use instead that (2) yields a collapsing sum. To simplify our formulas we introduce

$$H'(x,t) = H(x-1,t) + H(x,t).$$

Note that $H'(x,t) = H(x,t)$ for $x \sim t$ and $H'(x,t) = H(x-1,t)$ otherwise. Also note that $H'(x,t)$ is not unimodal in t, but fixing x and restricting t to only even or only odd values it becomes unimodal. As $s_i(y) \sim y$ we can still apply Lemma 4 below.

Using (2) and Lemmas 4 and 6 we have

$$\begin{aligned}
|\textsc{con}(y)| &= \left| \sum_{i \geq 0} (-1)^i \sum_{x \in X} \textsc{inf}(y-x, t-s_i(y)) \right| \\
&= \left| \tfrac{1}{2} \sum_{i \geq 0} (-1)^i \sum_{x \in X} \big[H(y-x+1, t-s_i(y)-1) \right. \\
&\qquad\qquad \left. - \ H(y-x-1, t-s_i(y)-1) \big] \right| \\
&= \left| \tfrac{1}{2} \sum_{i \geq 0} (-1)^i \big[H'(y+L, t-s_i(y)-1) \right. \\
&\qquad\quad \left. - \ H'(y, t-s_i(y)-1) \big] \right| \\
&\leq \left| \tfrac{1}{2} \sum_{i \geq 0} (-1)^i H'(y+L, t-s_i(y)-1) \right| \\
&\qquad\quad + \ \left| \tfrac{1}{2} \sum_{i \geq 0} (-1)^i H'(y, t-s_i(y)-1) \right| \\
&\leq \tfrac{1}{2} \max_{s \in \mathbb{N}} H'(y+L, s) + \tfrac{1}{2} \max_{s \in \mathbb{N}} H'(y, s) \\
&= O(1/(y+L-1/2)) + O(1/(y-1/2)).
\end{aligned}$$

Thus the vertices in $[-2L+1..L]$ contribute at most

$$\sum_{y \in [-2L+1..L]} |\textsc{con}(y)| = O\Big(\sum_{i=1}^{2L} 1/(i-1/2)\Big) = O(\log L).$$

Combining all cases, we have

$$|f(X,t) - E(X,t)| \leq \sum_{y \in \mathbb{Z}} |\textsc{con}(y)| = O(\log L).$$

For the lower bound, we just have to place the chips in a way the logarithmic contribution actually occurs. Without loss of generality, let L be odd.

Consider the following initial configuration (its existence is ensured by the parity forcing theorem): All arrows point towards the interval X (arrows of vertices in X may point anywhere). Let $t = L^2$. Choose an initial configuration of the chips such that $f(y,s)$ is odd if and only if $y \in [L]$ is even and $t - s = y^2$.

Now by construction, $\textsc{con}(y) = 0$ for all $y \in \mathbb{Z} \setminus [L]$. For $y \in [L]$, we have

$$\begin{aligned}
\textsc{con}(y) &= -\sum_{x \in X} \textsc{inf}(y - x, y^2) \\
&= \tfrac{1}{2} H(y, y^2) - \tfrac{1}{2} H(y + L + 1, y^2) \\
&\geq \tfrac{1}{2} H(y, y^2) - \tfrac{1}{2} H(y + L + 1, (y + L + 1)^2) \\
&= \Omega(y^{-1}).
\end{aligned}$$

Hence for this initial configuration,

$$f(X,t) - E(X,t) = \sum_{y \in \mathbb{Z}} \textsc{con}(y) = \sum_{y \in [L], y \sim 2} O(y^{-1}) = \Omega(\log L).$$

□

6 Intervals in Time

In this section, we regard the discrepancy in time-intervals. For $x \in \mathbb{Z}$ and finite $S \subseteq \mathbb{N}_0$, set

$$\begin{aligned}
f(x,S) &:= \sum_{t \in S} f(x,t), \\
E(x,S) &:= \sum_{t \in S} E(x,t).
\end{aligned}$$

We show that the discrepancy of a single vertex in a time-interval of length T is $O(\sqrt{T})$, and this is sharp.

Theorem 8. *The maximal discrepancy* $|f(x,S) - E(x,S)|$ *of a single vertex* x *in a time interval* S *of length* T *is* $\Theta(T^{1/2})$.

In the proof, we need the following fact that "rolling sums" of unimodal functions are unimodal again.

Lemma 9 (Unimodality of rolling sums)**.** *Let* $f : \mathbb{Z} \to \mathbb{R}$ *be unimodal. Let* $k \in \mathbb{N}$. *Define* $F : \mathbb{Z} \to \mathbb{R}$ *by* $F(z) = \sum_{i=0}^{k-1} f(z+i)$. *Then* F *is unimodal.*

Proof. Let f and $m \in \mathbb{Z}$ be such that f is non-decreasing in $\mathbb{Z}_{\leq m}$ and non-increasing in $\mathbb{Z}_{\geq m}$. We show that for some $m - k < M \leq m$ we have that $G(x) := F(x+1) - F(x)$ is nonnegative for $x < M$ and nonpositive for $x \geq M$. This implies that F is unimodal.

Since $G(x) = f(x+k) - f(x)$ for all $x \in \mathbb{Z}$, $G(x)$ is non-negative for $x \leq m - k$ and it is nonpositive for $x \geq m$. For $m - k \leq x < m$ we have $G(x+1) - G(x) = (f(x+k+1) - f(x+k)) - (f(x+1) - f(x)) \leq 0$, that is, G is non-increasing in $[m-k..m]$. Hence M exists as claimed. □

Of course, analogous statements hold for functions defined only on even or odd integers.

The following result says that a single odd split has an influence of exactly one on another vertex over infinite time.

Lemma 10. *For all* $x \in \mathbb{Z} \setminus \{0\}$, $\sum_{t \in \mathbb{N}} |\textsc{inf}(x,t)| = 1$.

Proof. W.l.o.g., let $x \in \mathbb{N}$. Then $|\textsc{inf}(x,t)| = \frac{1}{2}H(x-1,t-1) - \frac{1}{2}H(x+1,t-1)$. Consider a random walk of a single chip started at zero. Let $X_{y,t}$ be the indicator random variable for the event that the chip is on vertex y at time t. Let $Y_{y,t}$ be the indicator random variable for the event that the chip is on vertex y at time t and that it has not visited vertex x so far. Let T denote the first time the chip arrives at x.

For any $t > s > 0$ we have by symmetry that $\Pr(X_{x-1,t-1} = 1 | T = s) = \Pr(X_{x+1,t-1} = 1 | T = s)$. Clearly, for $t \leq T$, $X_{x+1,t-1} = 0$, and for $t > T$,

$Y_{x-1,t-1} = 0$. Thus

$$\begin{aligned}
\sum_{t\in\mathbb{N}} |\textsc{inf}(x,t)| &= \tfrac{1}{2}\sum_{t\in\mathbb{N}}(E(X_{x-1,t-1}) - E(X_{x+1,t-1})) \\
&= \tfrac{1}{2}\sum_{s\in\mathbb{N}} \Pr(T=s) \sum_{t\in\mathbb{N}} E((X_{x-1,t-1} - X_{x+1,t-1}) \,|\, T=s) \\
&= \tfrac{1}{2}\sum_{s\in\mathbb{N}} \Pr(T=s) \sum_{t\in[s]} E(X_{x-1,t-1} \,|\, T=s) \\
&= \tfrac{1}{2}\sum_{s\in\mathbb{N}} \Pr(T=s) E\Bigg(\sum_{t\in[s]} X_{x-1,t-1} \,|\, T=s\Bigg) \\
&= \tfrac{1}{2}\sum_{s\in\mathbb{N}} \Pr(T=s) E\Bigg(\sum_{t\in\mathbb{N}} Y_{x-1,t-1} \,|\, T=s\Bigg) \\
&= \tfrac{1}{2} E\Bigg(\sum_{t\in\mathbb{N}} Y_{x-1,t-1}\Bigg).
\end{aligned}$$

Note that $E(\sum_{t\in\mathbb{N}} Y_{x-1,t-1})$ is just the expected number of visits to $x-1$ before visiting x. This number of visits is exactly k if and only if the chip moves left after each of its first $k-1$ visits and right after the kth visit. This happens with probability 2^{-k}. Hence $E(\sum_{t\in\mathbb{N}} Y_{x-1,t-1}) = \sum_{i\in\mathbb{N}} i2^{-i} = 2$. □

Proof of Theorem 8. Fix any even initial configuration. Let $t_0 \in \mathbb{N}_0$ and $S = [t_0 \,..\, t_0 + T - 1]$. Without loss, let $x = 0$. By (4), we have

$$\begin{aligned}
f(0,S) - E(0,S) &= \sum_{t\in S}(f(0,t) - E(0,t)) \\
&= \sum_{y\in\mathbb{Z}} \textsc{arr}(y,0) \sum_{i\geq 0}(-1)^i \sum_{t\in S} \textsc{inf}(y, t - s_i(y)).
\end{aligned}$$

By unimodality of rolling sums (Lemma 9),

$$|f(0,S) - E(0,S)| \;\leq\; \sum_{y\in\mathbb{Z}} \max_{s\in\mathbb{N}} \Bigg|\sum_{t\in S} \textsc{inf}(y, t-s)\Bigg|.$$

We estimate the term $\max_{s\in\mathbb{N}} |\sum_{t\in S} \textsc{inf}(y,t-s)|$ for all y. For $1 \leq |y| \leq T^{1/2}$, we use Lemma 10 and simply estimate

$$\max_{s\in\mathbb{N}} \Bigg|\sum_{t\in S} \textsc{inf}(y,t-s)\Bigg| \leq \sum_{t\in\mathbb{N}} |\textsc{inf}(y,t)| = 1. \tag{6}$$

For $|y| > T^{1/2}$,

$$\max_{s\in\mathbb{N}} \left| \sum_{t\in S} \textsc{inf}(y, t-s) \right| \le T \max_{t\in\mathbb{N}} |\textsc{inf}(y,t)| = TO(y^{-2})$$

by Lemma 5. Hence

$$|f(0,S) - E(0,S)| \le \sum_{1\le|y|\le T^{1/2}} 1 \; + T \sum_{|y|>T^{1/2}} O(y^{-2}) = O(T^{1/2}).$$

For the lower bound, we invoke the parity forcing theorem again. By this, there is an even initial configuration such that all arrows point towards zero, and such that there is an odd number of chips on vertex $x \in \mathbb{Z}$ at time $t \in \mathbb{N}_0$ if and only if $x \in X := [\sqrt{T} \,..\, 2\sqrt{T}]$ and $t = 4T - x^2$. For this initial configuration and $S = [4T+1 \,..\, 5T]$, we compute

$$\begin{aligned} &|f(0,S) - E(0,S)| \\ &= \sum_{t\in S}\sum_{y\in X} |\textsc{inf}(y, t-4T+y^2)| \\ &\ge (1/2)T^{3/2} \min\left\{ |\textsc{inf}(y,t)| \,\middle|\, y \in X, t \in S, y \sim t \right\} \\ &= \Omega(T^{1/2}). \end{aligned}$$

□

7 Space-Time-Intervals

We now regard the discrepancy in space-time-intervals. Extending the previous notation, for finite $X \subseteq \mathbb{Z}$ and finite $S \subseteq \mathbb{N}_0$ set

$$\begin{aligned} f(X,S) &:= \sum_{x\in X}\sum_{t\in S} f(x,t), \\ E(X,S) &:= \sum_{x\in X}\sum_{t\in S} E(x,t). \end{aligned}$$

Theorem 11. *Let $X \subseteq \mathbb{Z}$ and $S \subseteq \mathbb{N}_0$ be finite intervals of lengths L and T, respectively. Then the maximal discrepancy $|f(X,S) - E(X,S)|$ (taken over all odd or even initial configurations) is $\Theta(T\log(LT^{-1/2}))$, if $L \ge 2T^{1/2}$, and $\Theta(LT^{1/2})$ otherwise.*

Proof. For the upper bound we use Theorems 7 and 8. To prove $|f(X,S) - E(X,S)| = O(LT^{1/2})$ we can simply apply Theorem 8:

$$\begin{aligned} |f(X,S) - E(X,S)| &\leq \sum_{x \in X} |f(x,S) - E(x,S)| \\ &\leq LO(T^{1/2}). \end{aligned}$$

For the other upper bound $|f(X,S)-E(X,S)| = O(T\log(LT^{-1/2}))$ we have to separate contributions of the vertices and apply the bounds in the proof of Theorem 7 for most of them and the bounds from the proof of Theorem 8 for the rest.

Fix an even initial configuration. Without loss of generality, let $X = [-L+1..0]$. Let $t_0 \in \mathbb{N}_0$ and $S = [t_0 \,..\, t_0 + T - 1]$. As in previous proofs, by (4) we have $f(X,S) - E(X,S) = \sum_{y \in \mathbb{Z}} \text{CON}(y)$ with

$$\text{CON}(y) := \text{ARR}(y,0) \sum_{i \geq 0} (-1)^i \sum_{x \in X} \sum_{t \in S} \text{INF}(y - x, t - s_i(y)).$$

Here $\text{CON}(y)$ is the sum for $t \in S$ of the contribution $\text{CON}_t(y)$ of y to the discrepancy of the interval X at a single time step t. The bound we established in the proof of Theorem 7 is $|\text{CON}_t(y)| = O(Ly^{-2})$ for $y > L$ and $y \leq -2L$ and $|\text{CON}_t(y)| = O(1/(y-1/2) + 1/(y+L-1/2))$ for $-2L < y \leq L$. Thus we have

$$|\text{CON}(y)| = O(LTy^{-2})$$

for $y > L$ and $y \leq -2L$ and

$$|\text{CON}(y)| = O(T/(y-1/2) + T/(y+L+1/2))$$

for $-2L < y \leq L$.

The above bounds are the largest for y close to 0 or $-L$. For $|y| \leq T^{1/2}$ and for $|y+L| \leq T^{1/2}$ we bound $|\text{CON}(y)|$ in a different way. Let X' be the interval $[-L+2\lceil T^{1/2}\rceil .. -2\lceil T^{1/2}\rceil]$ or empty if $-L+2\lceil T^{1/2}\rceil > -2\lceil T^{1/2}\rceil$. We express the contribution $\text{CON}(y)$ of y as the sum of contributions to different parts of X. Let $\text{CON}'(y)$ be the total contribution of the vertex y to the discrepancy in X' over the time interval S. Since y is separated from X' by at least $T^{1/2}$ the above bound gives $\text{CON}'(y) = O(T^{-1/2})$. Let $\text{CON}'_x(y)$ be the total contribution of y to the discrepancy of the single vertex $x \in X \setminus X'$

over the time interval S. To bound $\textsc{con}'_x(y)$ we apply the technique of the proof of Theorem 8: by Lemma 10 we have $|\textsc{con}'_x(y)| < 1$. Thus we have

$$|\textsc{con}(y)| \le |\textsc{con}'_x(y)| + \sum_{x \in X \setminus X'} |\textsc{con}'_x(y)| = O(T^{1/2}) + O(T^{1/2}) = O(T^{1/2}).$$

Let H_1 be the set of vertices y with $y \le -2L$ or $y > L$. The total contribution of these vertices is at most

$$\sum_{y \in H_1} |\textsc{con}(y)| = \sum_{y \in H_1} O(LTy^{-2}) = O(T).$$

Let H_2 be the set of vertices y with $|y| \le T^{1/2}$ or $|y + L| \le T^{1/2}$. The total contribution of these vertices is at most

$$\sum_{y \in H_2} |\textsc{con}(y)| = \sum_{y \in H_2} O(T^{1/2}) = O(T).$$

Let H_3 be the the set of vertices y outside H_1 and H_2. Their total contibution is bounded by

$$\begin{aligned}\sum_{y \in H_3} |\textsc{con}(y)| &= \sum_{y \in H_3} O(T/y + T/(y+L)) \\ &= O\left(T \sum_{i=\lceil T^{1/2} \rceil}^{2L} 1/i\right) = O(T \log(L/T^{1/2})).\end{aligned}$$

Finally we have

$$\begin{aligned}|f(X,S) - E(X,S)| &= |\sum_{y \in \mathbb{Z}} \textsc{con}(y)| \\ &\le O(T) + O(T) + O(T \log(L/T^{1/2})) \\ &= O(T \log(L/T^{1/2})).\end{aligned}$$

We now prove the corresponding lower bounds. Assume first that $L \ge 2T^{1/2}$. Set $Y = [T^{1/2} .. L]$. Choose an even initial configuration such that $f(x,t)$ is odd if and only if $x \in Y$ and $t = L^2 - x^2$. Direct all arrows towards zero. Let $S = [L^2 .. L^2 + T - 1]$. Then for $y \in Y$, with appropriately chosen

$\delta_t, \varepsilon_t \in \{0,1\}$ we have

$$\begin{aligned}
\textsc{con}(y) &= \sum_{x\in X}\sum_{t=L^2}^{L^2+T-1} \left|\textsc{inf}(y-x, t-(L^2-y^2))\right| \\
&\geq \tfrac{1}{2}\sum_{t=y^2}^{y^2+T-1} \left(H(y-1+\varepsilon_t, t-1) - H(y+L-1-\delta_t, t-1)\right) \\
&\geq \Omega\Bigg(\sum_{t=y^2}^{y^2+T-1} H(y-1+\varepsilon_t, t-1)\Bigg) \\
&= \Omega(Ty^{-1}).
\end{aligned}$$

For $y \notin Y$, $\textsc{con}(y) = 0$. Hence the discrepancy in this setting is

$$\sum_{y\in Y}\textsc{con}(y) = \sum_{y=T^{1/2}}^{L} \Omega(Ty^{-1}) = \Omega(T\log(LT^{-1/2})).$$

Assume now that $L \leq 2T^{1/2}$. The setting of Theorem 8 works for this lower bound, too. Choose an initial configuration such that $f(x,t)$ is odd if and only if $x \in X := [T^{1/2}..\,2T^{1/2}]$ and $t = 4T - y^2$. Then

$$\begin{aligned}
\textsc{con}(y) &= \sum_{x\in X}\sum_{t=4T}^{5T-1} |\textsc{inf}(y-x, t-(4T-y^2))| \\
&\geq LT\min\left\{|\textsc{inf}(y,t)| \;\middle|\; y \in \mathbb{Z}\cap[T^{1/2}..\,3T^{1/2}], t\in\mathbb{Z}\cap[T..5T], y\sim t\right\} \\
&= \Omega(L)
\end{aligned}$$

for all $y \in Y$. Again, $\textsc{con}(y) = 0$ for $y \notin Y$. Hence $\sum_{y\in\mathbb{Z}}\textsc{con}(y) = \Omega(LT^{1/2})$. □

8 Intervals in Space, Revisited

We stated in Theorem 7 that the discrepancy in an interval of length L is $O(\log L)$. Here we show that intervals of length L with about $\log L$ discrepancy are very rare, the root-mean-squared (i.e., quadratic) average of the discrepancies of a long contiguous set of intervals of length L is only $O(\sqrt{\log L})$, and this bound is tight.

For a set X of vertices we denote by $\textsc{disc}(X,t)$ the discrepancy of the set X at time t, i.e., we set $\textsc{disc}(X,t) = f(X,t) - E(X,t)$.

Theorem 12. *Let X be an interval of length L. For M sufficiently large,*

$$\frac{1}{M}\sum_{k=1}^{M} \textsc{disc}^2(X+k,t) = O(\log L).$$

Furthermore, for a given L and M there exists an even initial configuration, and a time t and an interval X of length L such that

$$\frac{1}{M}\sum_{k=1}^{M} \textsc{disc}^2(X+k,t) = \Omega(\log L).$$

Proof. For the first statement we need to prove an $O(\sqrt{\log L})$ bound on the quadratic average of the discrepancies $\textsc{disc}(X+k,t)$ with $k = 1,\ldots,M$. First note that by changing the individual discrepancies by a bounded amount, we change the quadratic average by at most the same amount. We use this observation to freely neglect $O(1)$ terms in the discrepancy of the intervals. In particular we can change the intervals themselves by adding or deleting a bounded number of vertices. We use this to make a few simplifying assumptions. As in Section 7 we assume that (i) the starting configuration is odd, (ii) the interval X is $X = [-L'..L']$ with $L' \sim t$, and (iii) M is even and we only consider even values of k, i.e., we consider the average of $\textsc{disc}^2(X+k,t)$ for $2 \le k \le M$, k even (this can be justified by considering $X+k+1$ instead of $X+k$ for odd k).

First we show that discrepancies caused by odd piles at time $t - L^2$ or before can be neglected. We start with (4) for the individual discrepancies

$\textsc{disc}(x,t)$.

$$\begin{aligned}
\textsc{disc}(x,t) &= \sum_{y\in\mathbb{Z}} \textsc{arr}(y,0) \sum_{i\geq 0} (-1)^i \textsc{inf}(y-x, t-s_i(y)) \\
&= \textsc{disc}_1(x,t) + \textsc{disc}_2(x,t); \\
\textsc{disc}_1(x,t) &= \sum_{y\in\mathbb{Z}} \textsc{arr}(y,0) \sum_{s_i(y)>t-L^2} (-1)^i \textsc{inf}(y-x, t-s_i(y)); \\
|\textsc{disc}_2(x,t)| &= \left| \sum_{y\in\mathbb{Z}} \textsc{arr}(y,0) \sum_{s_i(y)\leq t-L^2} (-1)^i \textsc{inf}(y-x, t-s_i(y)) \right| \\
&\leq \sum_{y\in\mathbb{Z}} \max_{u\geq L^2} |\textsc{inf}(y-x,u)|.
\end{aligned}$$

We have seen that $|\textsc{inf}(z,u)|$ is unimodal for fixed z and its maximum is at $u = \lceil z^2/3 \rceil$, so we have

$$\begin{aligned}
|\textsc{disc}_2(x,t)| &\leq 2\sum_{z=1}^{L} |\textsc{inf}(z,L^2)| + 2\sum_{z>L} |\textsc{inf}(z, \lceil z^2/3 \rceil)| \\
&\leq 2\sum_{z=1}^{L} \frac{z}{L^2} H(z,L^2) + 2\sum_{z>L} O(z^{-2}) \\
&\leq 2\sum_{z=1}^{L} H(z,L^2)/L + O(1/L) = O(1/L).
\end{aligned}$$

Therefore the total contribution of $\textsc{disc}_2$ to the discrepancy of an interval $X+k$ is small. For

$$\textsc{disc}_1(X+k,t) := \sum_{x\in X+k} \textsc{disc}_1(x,t)$$

we have

$$|\textsc{disc}_1(X+k,t) - \textsc{disc}(X+k,t)| = \left| \sum_{x\in X+k} \textsc{disc}_2(x,t) \right| = O(1).$$

We continue as in Section 7 collapsing a sum using $\textsc{inf}(z,u) = \frac{1}{2}H(z+1,u-1) - \frac{1}{2}H(z-1,u-1)$. We also use that $\textsc{disc}(x,t) = \textsc{disc}_1(x,t) = 0$ for $x \sim t$

as the starting configuration is odd.

$$
\begin{aligned}
& \textsc{disc}_1(X+k,t) \\
= & \sum_{x\in X+k, x\sim t+1}\sum_{y\in\mathbb{Z}} \textsc{arr}(y,0) \sum_{s_i(y)>t-L^2} (-1)^i \textsc{inf}(y-x, t-s_i(y)) \\
= & \sum_{y\in\mathbb{Z}} \textsc{arr}(y,0) \sum_{s_i(y)>t-L^2} (-1)^i \sum_x \bigg(\frac{1}{2}H(y-x+1, t-s_i(y)-1) \\
& -\frac{1}{2}H(y-x-1, t-s_i(y)-1)\bigg) \\
= & \frac{1}{2}\sum_{y\in\mathbb{Z}} \textsc{arr}(y,0) \sum_{s_i(y)>t-L^2} (-1)^i \big(H(y-k+L', t-s_i(y)-1) \\
& -H(y-k-L', t-s_i(y)-1)\big).
\end{aligned}
$$

We separate the two terms in this last expression. With

$$
D(m) := 2\sum_{y\in\mathbb{Z}} \textsc{arr}(y,0) \sum_{s_i(y)>t-L^2} (-1)^i H(y-m, t-s_i(y)-1)
$$

we have

$$
\textsc{disc}_1(X+k,t) = \frac{1}{4}D(k-L') - \frac{1}{4}D(k+L').
$$

Our original goal was to prove an $O(\sqrt{\log L})$ bound on the quadratic average of $\textsc{disc}(X+k,t)$. As $\textsc{disc}_1(X+k,t)$ differs from $\textsc{disc}(X+k,t)$ by $O(1)$ it is clearly enough to prove the same bound for the quadratic average of $\textsc{disc}_1(X+k,t)$. By the last displayed formula it is enough to prove the $O(\sqrt{\log L})$ bound on the two parts $D(k-L')$ and $D(k+L')$ separately, both for $0<k\le M$ even. It is therefore enough to bound the quadratic average of $D(m)$ for an arbitrary interval I of length M. Here we consider only values $m\sim t$, for other values of m we have $D(m)=0$.

Let $t_0=\max(0, t-L^2+1)$ be the first time-step considered. For $y\in\mathbb{Z}$ and $u\sim y+1$ we have an odd pile at y if and only if $\textsc{arr}(y,u)\neq\textsc{arr}(y,u+2)$ and in this case $\textsc{arr}(y,u)=(-1)^i\textsc{arr}(y,0)$ for the index i with $s_i(y)=u$. We estimate the contribution $D(m,y)$ of a fixed value y to the sum defining

$D(m)$. For $m \sim t$ we have

$$
\begin{aligned}
& D(m,y) := 2\textsc{arr}(y,0) \sum_{s_i(y)>t-L^2} (-1)^i H(y-m, t-s_i(y)-1) \\
= & \sum_{t_0 \le u < t, u \sim y+1} (\textsc{arr}(y,u) - \textsc{arr}(y,u+2)) H(y-m, t-u-1) \\
= & \sum_{t_0+2 \le u < t, u \sim y+1} \textsc{arr}(y,u)(H(y-m,t-u-1) - H(y-m,t-u+1)) \\
& + \textsc{arr}(y,t_1) H(y-m, t-t_1-1) - \textsc{arr}(y,t_2) H(y-m,t-t_2+1),
\end{aligned}
$$

where $t_1 = t_1(y)$ is either t_0 or t_0+1, whichever makes $t_1 \sim y+1$ and similarly $t_2 = t_2(y)$ is either t or $t+1$, so that $t_2 \sim y+1$. We have

$$
D(m) = \sum_{y \in \mathbb{Z}} D(m,y)
$$

and with

$$
D'(m) := \sum_{y \in \mathbb{Z}} \sum_{\substack{t_0+2 \le u < t-2 \\ u \sim y+1}} \textsc{arr}(y,u)(H(y-m,t-u-1) - H(y-m,t-u+1))
$$

we have

$$
\begin{aligned}
|D(m) - D'(m)| = & \left| \sum_{y \in \mathbb{Z}} (\textsc{arr}(y,t_1(y))) H(y-m, t-t_1(y)-1) \right. \\
& \left. - \ \textsc{arr}(y,t_2(y)) H(y-m, t-t_2(y)+1)) \right| \\
\le & \sum_{u \in \{0,1,t-t_0-2,t-t_0-1\}} \sum_{y \in \mathbb{Z}} H(y-m,u) \le 4.
\end{aligned}
$$

As before, we ignore the small difference and will prove the $O(\sqrt{\log L})$ bound on the quadratic average of $D'(m)$ instead of $D(m)$. Computing the square and summing over m we get the following. The summations are taken for $m \in I$, $m \sim t$, for $y, y_2 \in \mathbb{Z}$, and for $u_1, u_2 \in [t_0+2, t-3]$, $u_1 \sim u_2 \sim y+1$, respectively.

$$
\begin{aligned}
\sum_m D'^2(m) \quad = \quad & \sum_{y_1,y_2} \sum_{u_1,u_2} \textsc{arr}(y_1,u_1) \textsc{arr}(y_2,u_2) \\
& \cdot \sum_m (H(y_1-m, t-u_1-1) - H(y_1-m, t-u_1+1)) \\
& \cdot (H(y_2-m, t-u_2-1) - H(y_2-m, t-u_2+1)).
\end{aligned}
$$

Let us estimate the contribution to this sum coming from a fixed y_1, u_1, and u_2. Disregarding the signs and extending the summation for all m (even outside I) the contribution of each of the four terms we get from the multiplication is exactly 1. As u_1 and u_2 can take at most $L^2/2$ values each, the total contribution coming from a single value of y_1 is at most L^4.

Let us obtain the intervals I' and I'' from I by extending or shortening it at both ends by L^2 respectively, i.e., if $I=[a,b]$, then $I'=[a-L^2,b+L^2]$, $I''=[a+L^2,b-L^2]$. If y_1 is outside I' we have $H(y_1-m,t-u_1-1)=H(y_1-m,t-u_1+1)=0$ for all $m\in I$, therefore such y_1 has zero contribution to $\sum_m D'^2(m)$. The contribution for fixed y_1, y_2, u_1, and u_2 can usually be written in closed form using the identity

$$\sum_m H(y_1-m,v_1)H(y_2-m,v_2)=H(y_1-y_2,v_1+v_2).$$

This identity is valid if we sum over all possible values of m, but for $y_1\in I''$ the contribution of the values $m\notin I$ is zero. Therefore the contribution to $\sum_m D'^2(m)$ of the fixed terms $y_1\in I''$, y_2, u_1, and u_2 is

$$\begin{aligned}\textsc{arr}(y_1,u_1)\textsc{arr}(y_2,u_2)\sum_m &(H(y_1-m,t-u_1-1)-H(y_1-m,t-u_1+1))\\ &\cdot(H(y_2-m,t-u_2-1)-H(y_2-m,t-u_2+1))\\ =\textsc{arr}(y_1,u_1)\textsc{arr}(y_2,u_2)&(H(y,v-2)-2H(y,v)+H(y,v+2)),\end{aligned}$$

where $y=y_1-y_2$ and $v=2t-u_1-u_2$.

To estimate these contributions we first calculate

$$H(y,v-2)-2H(y,v)+H(y,v+2)=O(y^4/v^4+1/v^2)H(y,v+2).$$

The same $y=y_1-y_2$ value arises exactly once for every $y_1\in I''$, a total of $M-2L^2$ possibilities. The largest possible value of v is less than $2L^2$ and any single value v can be the result of at most v pairs u_1, u_2. There are $4L^2$ possible values of y_1 outside I'' but inside I' contributing at most $4L^6$.

Summing for all these contributions we estimate

$$\begin{aligned}\sum_m D'^2(m) &\leq 4L^6 + O\left(\sum_{v=1}^{2L^2} Mv \sum_{y\in\mathbb{Z}} (y^4/v^4 + 1/v^2)H(y, v+2)\right) \\ &= 4L^6 + O\left(M\sum_{v=1}^{2L^2}\sum_{y\in\mathbb{Z}} (y^4/v^3 + 1/v)H(y, v+2)\right) \\ &= 4L^6 + O\left(M\sum_{v=1}^{2L^2} 1/v\right) = O(L^6 + M\log L).\end{aligned}$$

Here we used the estimate on the fourth moment of the random walk:

$$\sum_{y\in\mathbb{Z}} y^4 H(y, v+2) = O((v+2)^2) = O(v^2).$$

To finish the proof we set the threshold $M > L^6$ for sufficiently large M. We did not make an effort to optimize for this threshold. This ensures that $\sum_m D'^2(m) = O(M\log L)$, so the quadratic average of $D'(m)$ (and therefore of $\textsc{disc}(X + k, t)$) is $O(\sqrt{\log L})$ as claimed.

It remains to construct a starting configuration where the quadratic average of discrepancies in the intervals of length L is large. For our construction we do not even use the value L. For a given (even) parameter t, we define a probability distribution on starting positions, such that for all $L < t$ and all intervals X of length L the expectation of $\textsc{disc}^2(X, t) = \Omega(\log L)$.

We let $r(a, b)$ stand for independent random ± 1 variables for all integers a and $b \geq 1$. We look for an even starting configuration (guaranteed by the Arrow-Forcing Theorem), such that $\textsc{arr}(x, u) = r(a, b)$ for all even x and u satisfying $4^b < u \leq 4^{b+1}$ and $a2^b < x \leq (a+1)2^b$. For simplicity we set $\textsc{arr}(x, u) = 1$ for all u and all odd x and we also set $\textsc{arr}(x, u) = 1$ for all x and $u \leq 4$.

A simple calculation similar to the one in Section 7 shows that for an interval $X = [c, d]$ we have

$$\textsc{disc}(X, t) = \sum_{a,b} h(a, b) r(a, b),$$

where the coefficients $h(a,b)$ depend on X. Further analysis shows that all coefficients are bounded and $\Theta(\log L)$ of them are above a positive absolute constant for each interval of length L. This implies that the expectation of $\textsc{disc}^2(X,t)$ is $\Omega(\log L)$, and therefore the expectation of the average $\frac{1}{M}\sum_{k=1}^{M}\textsc{disc}^2(X+k,t)$ is also $\Omega(\log L)$. This proves the second statement of the theorem. $\square$

References

[CS04] J. Cooper and J. Spencer. Simulating a Random Walk with Constant Error. `arXiv:math.CO/0402323`.

[CS05] J. Cooper and J. Spencer. Simulating a random walk with constant error. *Combinatorics, Probability and Computing*. To appear.

[Kle05] M. Kleber. Goldbug variations. *Mathematical Intelligencer*, 27:55–63, 2005.

[LP05] L. Levine and Y. Peres. Spherical Asymptotics for the Rotor-Router Model in $\mathbb{Z}^d$. `arXiv:math.PR/0503251`.